Part III

MULTIDIMENSIONAL CLASSICAL AND QUANTUM COSMOLOGY AND GRAVITATION: EXACT SOLUTIONS AND VARIATIONS OF CONSTANTS

V. Melnikov

Chapter 11

Introduction: Fundamental Physical Constants

In any physical theory we meet with constants which characterize the stability properties of different types of matter: of objects, processes, classes of processes and so on. These constants are important because they arise independently in different situations and have the same value, at any rate within accuracies we gained nowadays. That is why they are called fundamental physical constants (FPC) [1]. To define strictly this notion is not possible. It is because the constants, mainly dimensional, are present in definite physical theories. In the process of scientific progress some theories are replaced by more general ones with their own constants, some relations between old and new constants arise. So, we may talk not about absolute choice of FPC, but only about the choice corresponding to the present state of the physical sciences.

Really, quite recently (before the creation of the electroweak interaction theory and some Grand Unification Models it was considered that this *choice* is the followings:

$$c,\ \hbar,\ \alpha,\ G_F,\ g_s,\ m_p (or\ m_e),\ G,\ H,\ \rho,\ \Lambda,\ k,\ I,$$

where α, G_F, g_s and G are constants of electromagnetic, weak, strong and gravitational interactions, H, ρ and Λ are cosmological parameters (Hubble constant, mean density of the Universe and cosmological constant), k and I are the Boltzmann constant and the mechanical equivalent of heat which play the role of conversion factors between temperature from one side, energy and mechanical units from another side. After adoption in 1983 of a new definition of the meter ($\lambda = ct$ or $\ell = ct$) this role is partially played also by the speed of light c. It is now also a conversion factor between units of time (frequency) and length, it is defined with absolute (null) accuracy.

Now, when the theory of electroweak interactions has a firm experimental basis and we have some good models of strong interactions the more prefarable choice is as follows:

$$\hbar,\ (c),\ e,\ m_e,\ \theta_w,\ G_F,\ \theta_c,\ \Lambda_{QCD},\ G,\ H,\ \rho,\ \Lambda,\ k,\ I$$

and, possibly, three angles of Kobayashi-Maskawa - θ_2, θ_3 and δ. Here θ_w is the Weinberg angle, θ_c is the Cabibbo angle and Λ_{QCD} is a cut-off parameter of quantum chromodynamics. Of course, if the theory of four known now interactions will be created then we probably will have another choice. As we see the macroconstants remain the same though in some unified models, i.e. in multidimensional ones, they may be related in some manner (see below).

All these constants are known with different *accuracies*. The most precisely defined constant was and remain the speed of light c: its accuracy was 10^{-10} and now it is defined with null accuracy. Atomic constants, $e, \hbar, m$ and others are defined with accuracies $10^{-6} \div 10^{-7}$, G-with the accuracy 10^{-4}, θ_w-with accuracy 10%; the accuracy of H is also 10% though several groups give values differing by the factor of 2. Even worse situation is now with other cosmological parameters (FPC): mean density estimations vary within an order of magnitude; for Λ we have limits above and below, in particular zero value is also acceptable.

As to the *nature* of FPC, we may mention several approaches. One of the first hypotheses belongs to J.A. Wheeler: in each cycle of the Universe evolution FPC arise anew along with physical laws which govern its evolution. Thus, the nature of FPC and physical laws is connected with the origin and evolution of our Universe.

Less global approach to the nature of dimensional constants suggests that they are needed to make physical relations dimensionless or they are measures of asimptotic states. Really, the speed of light appears in relativistic theories in factors like v/c, at the same time velocities of usual bodies are less than c, so it plays also the role of an asymptotic limit. The same sense have some other FPC: $\hbar$ is the minimal quantum of action, e is the minimal observable charge (if we do not take into account quarks which are not observable in a free state) etc.

Finally, FPC or their combinations may be considered as natural scales defining basic units. If earlier basic units were chosen more or less arbitrarily, i.e. the second, meter and kilogram, than now first two are based on stable (quantum) phenomena. Their stability is ensured by well established physical laws which include FPC.

Exact knowledge of FPC and precision measurements are necessary for testing main physical theories, extention of our knowledge of nature and, in the long run, for practical applications of fundamental theories. Within this, such theoretical problems arise: 1) development of models for confrontation of a theory with experiment in critical situations (i.e. for verification of GR, QED, QCD or GUT), 2) setting limits for spacial and temporal variations of FPC.

As to *classification* of FPC we may set them now into four groups due to their generality:
1) Universal constants such as $\hbar$ which divides all phenomena into quantum and nonquantum (micro and macro worlds) and to a certain extent c, which divides all motions into relativistic and nonrelativistic, 2) constants of interactions like α, θ_w, Λ_{QCD} and G; 3) constants of elementary constituencies of matter like m_e, m_w, m_x, etc., and 4) transformation multipliers such as k, I and partially c. Of course, this division into classes is not absolute. Many constants shifted from one class to another. For example, e was a charge of a particular object-electron, class 3, then it became a characteristic of a class 2 (electromagnetic interaction, $\alpha = \frac{e^2}{\hbar c}$ in combination with $\hbar$ and c), speed of light c was nearly in all classes: from 3 it moved into 1, then also into 4. Some of the constants ceased to be fundamental (i.e. densities, magnetic moments, etc.) as they are calculated via other FPC.

As to the *number* of FPC, there are two opposite tendencies: number of "old" FPC is usually diminishing when a new, more general theory is created, but at the same time new fields of science arise, new processes are discovered in which new constants appear. So, in the long run we may come to some minimal choice which is characterized by one or several FPC, may be, connected with the so called Planck parameters-combinations of c, $\hbar$ and G:

$$L = \left(\frac{\hbar G}{c^3}\right)^{1/2} \sim 10^{-33} cm, \quad m_L = (c\hbar/2G)^{1/2} \sim 10^{-5} g\ ,$$
$$\tau_L = L/c \sim 10^{-43} s.$$

The role of these parameters is important as m_L characterizes the energy of unification of four known fundamental interactions: strong, weak, electromagnetic and gravitational ones and L is a scale where classical notions of space-time loose their meaning.

The problem of the gravitational constant G measurement and stability is a part of a very much developing field, called gravitational-relativistic metrology. It appeared due to the growth of a measuring technique precision, spread of measurements over large scales and tendency to the unification of fundamental physical interaction (see [2]).

Absolute value measurements of G. There are several laboratory determinations of G with precisions of 10^{-3} and only 4 at the level of 10^{-4}. They are (in $10^{-11} m^3 kg^{-1} s^{-2}$);

1. Facy, Pontikis, France, 1972 - 6,6714 ± 0,0006
2. Sagitov et al., USSR, 1979 - 6,6745 ± 0,0008
3. Luther, Towler, USA, 1982 - 6,6726 ± 0,0005
4. Karagioz, USSR, 1988 - 6,6731 ± 0,0004

From this table it is seen that first three experiments contradict each other (they do not overlap within their accuracies). And only the fourth experiment is in accord with the third one.

The official CODATA value of 1986

$$G = (6,67259 \pm 0,00085) \cdot 10^{-11} \cdot m^3 \cdot kg^{-1} \cdot s^{-2}$$

is based on the Luther and Towler determination. One should make a conclusion that the problem is still open and we need further experiments on the absolute value of G. Many groups are preparing and doing them using different types of technique, among them is the Karagioz group (Russia) which has the installation operating already for two years continuously [3].

There exist also some satellite determinations of G (namely $G \cdot M_{earth}$) at the level of 10^{-8} and several geophysical determinations in mines. The last give usually much higher G values than the laboratory ones.

The precise knowledge of G is necessary for the evaluation of mass of the Earth, planets, their mean density and in the end for the construction of Earth models; for transition from mechanical to electromagnetic units and back; for evaluation of other constants through relations between them given by unified theories; for finding new possible types of interactions and geophysical effects.

The knowledge of constants values has not only a fundamental meaning but also the metrological one. Modern system of standards is based mainly on stable physical phenomena. So, the stability of constants plays a crucial role. As all physical laws were established and tested during last 2-3 centuries in experiments on the Earth and in the near space, i.e. at a rather short space and time intervals in comparison with the radius and age of the Universe the possibility of slow *variations* of constants (i.e. with the rate of the evolution of the Universe) cannot be excluded a priori.

So, the supposition about the absolute stability of constants is an extrapolation and each time we must test it.

The problem of variations of FPC arose with the attempts of explanation of relations between micro and macroworld fenomena. Dirac was the first to introduce [4] the so called "Large Numbers Hypothesis" which relates some known very big (or very small) numbers with the dimensionless age of the Universe $T \sim 10^{40}$ (age of the Universe in seconds 10^{17}, divided by the characteristic elementary particle time 10^{-23} seconds). He suggested that the ratio of the gravitational to strong interactions strengths, $Gmp^2/\hbar c \sim 10^{-40}$, is inversely proportional to the age of the Universe: $Gmp^2/\hbar c \sim T^{-1}$. Then, as the age varies some constants or their combinations must vary also. Atomic constants seemed to Dirac more stable so he've chosen the variation of G as T^{-1}.

After original Dirac *hypothesis* some new ones appeared and also some generalized *theories* of gravitation admitting the variations of an effective gravitational coupling. We may single out two stages in the development of this field:

1. Study of theories and hypotheses with variations of FPC, their predictions and confrontation with experiments (1937-1977).
2. Creation of theories admitting variations of an effective gravitational constant in a particular system of units, analyses of experimental and observational data within these theories [5-7] (1977-present).

Within the development of the first stage from the analysis of the whole set of existed astronomical, astrophysical, geophysical and laboratory data the conclusion was made [6,1] that variations of atomic constants are excluded, but variations of the effective gravitational constant in atomic system of units do not contradict available experimental data at the level $10^{-11} \div 10^{-12} year^{-1}$. Moreover, in [5-7] the conception was worked out that variations of constants are not absolute but depend on the system of measurements (choice of standards, units and devices using this or that fundamental interaction). Each fundamental interaction through dynamics, described by the corresponding theory, defines the system of units and the system of basic standards.

Now we review shortly some hypotheses on variations of FPC and experimental tests [1]:

Following Dyson (1972) we may introduce dimensionless combinations of micro and macroconstants:

$$\alpha = e^2/\hbar c = 7,3.10^{-3} \ , \gamma = Gm^2/\hbar c = 5.10^{-39} \ ,$$
$$\beta = G_F \ m^2 c/\hbar^3 = 9.10^6 \ , \ \delta = H\hbar/mc^2 = 10^{-42} \ ,$$
$$\varepsilon = \rho G/H^2 = 2.10^{-3} \ , \ t = T/(e^2/mc^3) \approx 10^{40}$$

We see that α, β and ε are of order 1 and γ and δ are of the order 10^{-40}. Nearly all existing hypotheses on variations of FPC may be represented as:

Hyposesis 1 (standard):
α, β, γ are constant, $\delta \sim t^{-1}, \varepsilon \sim t$.

Here we have no variations of G and δ and ε are defined via cosmological solutions.

Hyposesis 2 (Dirac):

α, β, ε are constant, $\gamma \sim t^{-1}$, $\delta \sim t^{-1}$.

Then $\dot{G}/G = 5.10^{-11} year^{-1}$ if the age of the Universe is taken as $T = 2.10^{10}$ years.

Hyposesis 3 (Gamow):

$\gamma/\alpha = Gm^2/e^2 \sim 10^{-37}$, so e^2 or α are varied, but not G, $\beta, \gamma, \varepsilon = const$, $\alpha \sim t^{-1}$, $\delta \sim t^{-1}$.

Then $\dot{\alpha}/\alpha = 10^{-10} year^{-1}$.

Hyposesis 4 (Teller): trying to account also for deviations of α from 1 he suggested $\alpha^{-1} = \ell n \gamma^{-1}$.

Then β, ε are constants, $\gamma \sim t^{-1}$, $\alpha \sim (\ell n t)^{-1}$, $\delta \sim t^{-1}$

$$\dot{\alpha}/\alpha = 5.10^{-13} year^{-1}$$

The same relation for α and γ was used also by Landau, DeWitt, Staniucovich, Terasawa and others, but in different approaches in comparison with Teller.

Some other variants may be also possible, e.g. Brans-Dicke theory with $G \sim t^{-r}$, $\rho \sim t^{r-2}$, $r = [2 + \frac{3w}{2}]^{-1}$, the combination of Gamov's approach and Brans-Dicke's etc. [1].

There are different astronomical, geophysical and laboratory *data* on possible variations of FPC.

astrophysical data:

a) from comparison of fine structure ($\sim \alpha^2$) and relativistic fine structure ($\sim \alpha^4$) shifts in spectra of radiogalaxies Bahcall and Schmidt (1967) obtained

$$|\dot{\alpha}/\alpha| \leq 2.10^{-12} year^{-1}$$

b) comparing lines in optical ($\sim Ry = me^4/\hbar^2$) and radio bands of the same sources in galaxies Baum and Florentin-Nielsen got the estimate

$$|\dot{\alpha}/\alpha| \leq 10^{-13} year^{-1},$$

and for extragalactic objects

$$|\alpha/\alpha| \leq 10^{-14} year^{-1}$$

c) from observations of superfine structure in H-absorption lines of the distant radiosource Wolf et al. (1976) obtained that

$$| \alpha^2(m_e/m_p)g_p |< 2.10^{-14} ;$$

from these data it is seen that hyposesis 3 and 4 are excluded. The same conclusion is done on the bases of *geophysical data*. Really,

a) α-decay of $U_{238} \rightarrow Pb_{208}$. Knowing abundancies of U_{238} and P_{238} in rocks and independently the age of these rocks the limit

$$|\dot{\alpha} /\alpha |\leq 2.10^{-13} year^{-1}$$

was obtained,

b) from spontaneous fission of U_{238} such estimation was done: $|\dot{\alpha} /\alpha |\leq 2,3.10^{-13} year^{-1}$.

c) finally, from β-decay of Re_{187} to Os_{187}

$$|\dot{\alpha} /\alpha |\leq 5.10^{-15} year^{-1}$$

was obtained.

We must point out that all astronomical and geophysical estimations are strongly model-dependant. So, of course, it is always desirable to have *laboratory tests* of variations of FPC.

a) such a test was first done by the Russian group in the Committee for Standards (Kolosnitsyn, 1975). Comparing rates of two different types of clocks, one based on the Cs standard and another on the beam molecular generator they found that $|\dot{\alpha} /\alpha |\leq 10^{-10} year^{-1}$.

b) from similar comparison of Cs standard and $SCCG$ (Super Conducting Cavity Generator) clocks rates Turner (1976) obtained the limit

$$|\dot{\alpha} /\alpha |\leq 4.1 \cdot 10^{-12} year^{-1}$$

All these limits were placed on the fine structure constant variations. From the analysis of decay rates of K_{40} and Re_{187} the limit on the possible variations of the weak interaction constant was obtained (see approach for variations of β, e.g. in [8]).

$$|\dot{\beta} / \beta| \leq 10^{-10} year^{-1}.$$

But the most strict data were obtained by A. Schlyachter (USSR) from the analysis of the ancient natural nuclear reactor data in Gabon, Oklo, because the event took place 2.10^9 years ago. They are the following:

$$|\dot{G}_s / G_s| < 5.10^{-19} year^{-1}, |\dot{\alpha} / \alpha| < 10^{-17} year^{-1}$$
$$|\dot{G}_F / G_F| < 2.10^{-12} year^{-1}$$

So, we really see that all existing hyposeses with variations of atomic constants are excluded.

So, now we still have no unified theory of all four interactions. There is a good theory of electroweak interactions, models of GUT which include the strong interaction and also some attempts to create a theory of everything (TOE). As we have no such a theory it is possible to construct systems of measurements based on any of these four interactions. But practically it is done now on the basis of the mostly worked out theory – on electrodynamics (more precisely on QED). Of course, it may be done also on the basis of the gravitational interaction (as it was partially earlier). Then, different units of basic physical quantities arise based on dynamics of the given interaction, i.e. the atomic (electromagnetic) second, defined via frequency of atomic transitions or the gravitational second defined by the mean Earth motion around the Sun (ephemeris time).

It doesn't follow from anything that these two seconds are always synchronized in time and space. So, in principal they may evolve relatively each other, for example with the rate of the evolution of the Universe or some other rate.

That is why in general variations of the gravitational constant are possible in atomic system of units ($c, \hbar, m$ are constant) and masses of all particles - in gravitational system of units (G, $\hbar$, c are constants by definition). Practically we can test only the first variant as modern basic standards are defined in atomic system of measurements. Possible variations of FPC had to be tested experimentally but for this it is necessary to develop corresponding theories admitting such variations and their definite effects.

Mathematically these systems of measurement may be realized as two conformally related metric forms. Arbitrary conformal transformations give us a transition to an arbitrary system of measurements.

One of the ways to describe variable gravitational coupling is the introduction of a *scalar field* as an additional variable of the gravitational interaction. It may be done by different means (e.g. Jordan, Brans-Dicke, Canuto and others). We prefare the variant of gravitational theory with conformal scalar field (Higgs-type field [9]) where Eintein's general relativity may be considered as a result of spontaneous symmetry breaking of the conformal symmetry (Domokos, 1976). In our variant spontaneous symmetry breaking of the global gauge invariance leads to nonsingular cosmology [10]. Besides, we may get variations of the effective gravitational constant in the atomic system of units when m, c, $\hbar$ are constant and variations of all masses in the gravitational system of units (G, c, $\hbar$ are constant). It is done on the basis of approximate [11] and exact cosmological solutions with local inhomogenity [12].

The effective gravitational constant is calculated using equations of motions. Postnewtonian expansion is also used in order to confront the theory with existing experimental data. Among postnewtonian parameters the parameter f describing variations of G is included. It is defined as

$$\frac{1}{GM}\frac{d(GM)}{dt} = fH \ . \tag{0.1}$$

According to Hellings data [13] from the Viking mission

$$\tilde{\gamma} - 1 = (-1,2 \pm 1,6) \cdot 10^{3}, \quad f = (4 \pm 8) \cdot 10^{-2} \tag{0.2}$$

In the theory with conformal Higgs field [11] we obtained the following relation between f and $\tilde{\gamma}$:

$$f = 4(\tilde{\gamma} - 1) \ . \tag{0.3}$$

Using Hellings data for $\tilde{\gamma}$ we may calculate in our variant f and compare it with f from [13]. Then we get $f = (-9,6 \pm 12,8) \cdot 10^{-3}$ which agrees with (1.2) within its accuracy.

We used here only Hellings data of variations of G. But the situation with experiment and observations is not so simple. Along with [13] there are some other data [1]:

1. From the growth of corals, pulsar spin down, etc. on the level $|\dot{G}/G| < 10^{-10} \div 10^{-11} year^{-1}$.

2. Van Flandern's positive data from the analysis of a lunar mean motion around the Earth and ancient eclipses data (1976, 1981):

$$|\dot{G}/G| = (6 \pm 2)10^{-11} y^{-1}.$$

3. Reasenberg's estimates of the same Viking mission as in [13] (1987):

$$|\dot{G}/G| < (0 \pm 2) \cdot 10^{-11} y^{-1}$$

4. Hellings result in the same form is

$$|\dot{G}/G| < (2 \pm 4) \cdot 10^{-12} y^{-1}$$

As we see there is a vivid contradiction in these results, so, of course, further experiments are necessary for solving the problem of temporal G variations. The most promising are the planned future missions to Mars (1994).

According to Hellings estimations [13] after several years of observations of spacecrafts on and around the Mars one may have the improvement of the order of magnitude in a testing of $\dot{G}/G$.

As we saw different theoretical schemes lead to temporal variations of the effective gravitational constant:

1. Empirical models and theories of Dirac's type, were G is replaced by $G(t)$.

2. Numerous scalar-tensor theories of Jordan-Brans-Dicke type where G depending on the scalar field $\sigma(t)$ appears.

3. Gravitational theories with the conformal scalar field arising in different approaches [6,7,14,15]. And as we see later:

4. Multidimensional unified theories in which there are dilaton fields and effective scalar fields appear in our 4-dimensional spacetime from additional dimensions [16]. They may help also in solving the problem of changing cosmological constant from Planckian to present values.

As it was shown in [16,17] temporal variation of FPC are connected with each other in *multidimensional models* of unification of interactions. So, experimental tests on $\dot{\alpha}/\alpha$ may be at the same time be used for estimation of $\dot{G}/G$ and vice versa. Moreover, variations of G are related also to the cosmological parameters ρ, Ω and q that gives opportunities of raising the precision of their determination.

As variations of FPC are closely connected with the behaviour of internal scale factors it is a direct probe of properties of extra dimensions and corresponding theories.

Other windows for testing hidden dimensions are opening when one is studying multidimensional models in spherically-symmetrical case. Then, as we shall see, some deviations from the Newton and Coulomb laws are possible.

And at last quantum multidimensional models may help in solving such problems as the creation of the Universe, its singular state, λ-term, etc.

Thus, our main aim here is to investigate different multidimensional models of gravitation and cosmology based on exact solutions. Chapter 2 is devoted to multidimensional and multicomponent classical cosmology. In 2.1 we study Friedmann-Calabi-Yau cosmology and prove that time variations of the Newton's gravitational constant is an unavoidable one. In 2.2 we obtain exact solutions for $(4+N)$-dimensional cosmology and find relations between cosmological parameters and the time variation of G. In 2.3 we study multicomponent cosmology with Ricci-flat internal spaces and a perfect fluid matter when pressures in these spaces are proportional to the density and get exact solutions also. In 2.4 we prove that Gibbons-Maeda reduction of two-component cosmology to the Toda lattice can not be generalized for the n-component case.

Chapter 3 is devoted to the quantum multidimensional cosmology. In 3.1 the Wheeler-DeWitt equation for multidimensional cosmology with n spaces of constant curvature is proposed and some integrable cases are pointed out. In 3.2. the Wheeler-DeWitt multidimensional equation for the gravitational theory with cosmological constant is solved and quantum wormhole solutions are found. In 3.3 the WDW-equation for multidimensional cosmology with perfect fluid is solved in simplest cases.

In Chapter 4 we analyse spherically-symmetric solutions. In 4.1 we give the extension of Schwarzschild solution for a multidimensional case. Solution with a scalar field is also obtained. In 4.2 we obtain the multicomponent Tangherlini solution. 4.3 is devoted to solutions of system of multidimensional Einstein and Maxwell equations and 4.4 – to the system of Einstein-scalar-eletomagnetic fields. And in 4.5 we give the solutions for interacting scalar and electromagnetic fields, study their stability and single out BH solutions.

Chapter 12

Classical Multidimensional Cosmology

1 Variations of G in 10-Dimensional Cosmology of Superstring Origin [27]

The idea of time variation of the Newton's gravitational constant originally proposed by Dirac assumed a great importance with the appearance of superstring theories. Predictions of these theories about the time variation of G must obey the present observational upper bound.

$$| \dot{G} / G | \lesssim 10^{-11} \div 10^{-12} \ ,$$

which is a gravitational test for these theories.

Here we consider the "Friedman-Calabi-Yau" (FCY) cosmology based on the ten-dimensional SO(32) – or $E_8 \times E_8$ – Yang-Mills-supergravity theory [18] with Lorentz Chern-Simons three-form, introduced by Green and Schwarz [19] for anomaly cancellations, and with the Gauss-Bonnet term, introduced in [20]. These additional terms have a superstring origin [21]: they appear as the next to leading terms in the α'-decomposition (α' is the string parameter) of the Fradkin-Tseytlin effective action [22] for a heterotic string [23] (see [24])). The supergravity action is a leading term in this decomposition.

We prove that in the FCY cosmology with the dilation field $\varphi = \varphi(t)$ the solution of equations of motion with the constant radius of an internal space ($a_6(t) = const$) does not exist for all equations of state of ten-dimensional matter.

It should be noted that, in the open-universe case of the FCY cosmology with $p_3 = p_6 = 0$ (p_3, p_6 are pressures, see (2.1.9)) and $\varphi(t) = const$, Wu and Wang calculated the present value of $\dot{G}/G$ [25] and got the estimate

$$(\dot{G}/G)_0 \approx -1.10^{-11\pm1}(y^{-1}) \, .$$

We take the action of the model as [25]

$$S = \int d^{10}x\sqrt{-g}\{\frac{1}{2\kappa^2}R - \frac{3}{4}\kappa^2\varphi^{-3/2}H^2_{MNP} - \frac{9}{16\kappa^2}(\varphi^{-1}\partial_M\varphi)^2 - -\frac{1}{4}\varphi^{-3/4}[\frac{1}{30}trF^2_{MN} + (R^2_{MNPQ} - 4R^2_{MN} - R^2)]\} + S_F \, , \tag{1.0.1}$$

where g_{MN} and φ are the metric and dilation fields, F_{MN} and H_{MNP} are the Yang-Mills and Kolb-Ramond field strengths:

$$F = \frac{1}{2}F_{MN}dx^M \wedge dx^N = dA + A \wedge A \, ,$$

where $A = A_M dx^M$ is the one-form with the value in the Lie algebra ad g, $g = SO(32)$, $e_8 \oplus e_8$ (ad g is the image of the adjoint representation of g, ad $g \approx g$ for any semi-simple Lie algebra g);

$$H = \frac{1}{3!}H_{MNP}dx^M \wedge dx^N \wedge dx^P = dB - \omega_{3Y} + \omega_{3L} \, , \tag{1.0.2}$$

where $B = \frac{1}{2}B_{MN}dx^M \wedge dx^N$ is a two-form, ω_{3Y} is the Yang-Mills Chern-Simons three-form:

$$\omega_{3Y} = \frac{1}{30} tr(A \wedge F - \frac{1}{3} A \wedge A \wedge A) \; , \tag{3.a}$$

and ω_{3L} is the Lorentz Chern-Simons three-form:

$$\omega_{3L} = tr(\omega \wedge \Omega - \frac{1}{3} \omega \wedge \omega \wedge \omega) \; . \tag{3.b}$$

In (2.1.3b) $\omega = \omega_M dx^M$ is the spin connection, which is the one-form with the value in $SO(1,9)$:

$$\omega_M = \| \omega^A_{BM} \| = \| e^A_N \nabla_M e^N_B \| \subset SO(1,9) \; ,$$

e^A_N is the basis (zehnbein) which diagonalizes the metric

$$g_{MN} = e^A_M e^B_N \eta_{AB} \; ,$$

$\| \eta_{AB} \| = diag(-1, \; +1, \; ..., \; +1)$, Ω is the curvature two-form:

$$\Omega = d\omega + \omega \wedge \omega$$

S_F in (2.1.1) is the Fermi part of the action [18], which is not essential for us because we are interested in solutions with zero Fermi fields.

The action (2.1.1) and the energy-momentum tensor T_{MN} lead to the following equations of motions [25]:

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R = \frac{9}{2} \kappa^4 \varphi^{-3/2} (H_{MPQ} H_N^{PQ} - \frac{1}{6} g_{MN} H^2_{PQS}) + \\ +9\kappa^4 \nabla^S (\varphi^{-3/2} H_{MPQ} R^{PQ}_{SN}) + \frac{9}{8} \varphi^{-2} [\partial_M \varphi \partial_N \varphi - \frac{1}{2} g_{MN} (\partial_P \varphi)^2] + \\ + \frac{1}{30} \kappa^2 \varphi^{-3/4} (tr \; F_{MP} F^P_N - \frac{1}{4} g_{MN} \; tr \; F^2_{PQ}) - \\ - \frac{1}{2} \kappa^2 \varphi^{-3/4} [\frac{1}{2} g_{MN} (R^2_{PQST} - 4 R^2_{PQ} + R^2) - 2 R R_{MN} + \\ +4 R_{MP} R^P_N + 4 R_{MPNQ} R^{PQ} - 2 R^{PQS}_M R_{NPQS}] + \kappa^2 T_{MN} \; , \end{aligned} \tag{4}$$

$$\nabla_M (\varphi^{-3/2} H^{MNP}) = 0 \; , \tag{5}$$

$$D_M (\varphi^{3/4} F^{MP}) + 9\kappa^2 (\varphi^{-3/2} F_{MN} H^{MNP}) = 0 \; , \tag{6}$$

$$\begin{aligned} 6 \nabla_M (\varphi^{-2} \partial^M \varphi) + 6 \varphi^{-3} (\partial_M \varphi)^2 + 6 \kappa^4 \varphi^{-5/2} H^2_{MNP} + \\ + \kappa^2 \varphi^{-7/4} [\frac{1}{30} tr \; F^2_{MN} + (R^2_{MNPQ} - 4 R^2_{MN} + R^2)] = 0 \; . \end{aligned} \tag{7}$$

Let us consider the ten-dimensional manifold

$$M^{10} = R \times M^3_K \times K \ , \tag{8}$$

where $M^3_K = S^3,\ R^3,\ L^3$ for $k = +1, 0, -1$, respectively, and K is the Calaby-Yau manifold, i.e. the compact, complex three-dimensional Kähler Ricci-flat manifold with the $SU(3)$ holonomy group.

Let the energy-momentum tensor be

$$T = T_{MN} dx^M \otimes dx^N = \rho(t) dt \otimes dt + p_3(t) a_3^2(t) g^{(3)} + p_6(t) a_6^2(t) g^{(6)} \ , \tag{9}$$

where $g^{(3)}$ and $g^{(6)}$ are metrics on M^3_K and K, $\rho(t)$ is an energy density in a three-space, $p_3(t)$ and $p_6(t)$ are pressures corresponding to M^3_K and K.

The system (2.1.4)-(2.1.7) on the manifold (2.1.8) with the source (2.1.9) and the following ansatz:

$$g^{(10)} = -dt \otimes dt + a_3^2(t) g^{(3)} + a_6^2(t) g^{(6)} \ , \tag{10}$$

$$H = 0 \ , \tag{11}$$

$$\varphi = \varphi(t) \ , \tag{12}$$

$$A = ad(\dot{\tau}\ (\omega^{(6)})) \tag{13}$$

leads to a cosmology model, which we call the "Friedman-Calaby-Yau" (FCY) cosmological model. In (2.1.13) $\omega^{(6)}$ is the spin connection on K corresponding to the basis $e^{(6)a}$, which diagonalizes $g^{(6)}$; $\dot{\tau}$: $SO(6) \to g$ is the enclosure of the Lie algebra $SO(6)$ (in the case $g = e_8 \oplus e_8$, $\dot{\tau}$ may be defined, for example, with the aid of the decomposition [23]: $e_8 = SO(6) \oplus V_{128}$). It follows from (2.1.13) that

$$F = ad(\dot{\tau}\ (\Omega^{(6)})), \tag{2.1.13a}$$

where $\Omega^{(6)} = d\omega^{(6)} + \omega^{(6)} \wedge \omega^{(6)}$. From (2.1.13) and the trace identity (which is not difficult to prove)

$$\frac{1}{30} tr(ad(\tau(X)) ad(\dot{\tau}\ (Y))) = tr(XY) \tag{14}$$

for all $X, Y \in SO(6)$, we have

$$\omega_{3Y} = \frac{1}{30} tr(\frac{2}{3} A \wedge F + \frac{1}{3} A \wedge dA) = tr(\frac{2}{3}\omega^{(6)} \wedge \Omega^{(6)} + \frac{1}{3}\omega^{(6)} \wedge d\omega^{(6)}) = \omega_{3L}^{(6)} \ . \quad (15)$$

In the basis $(e^{(10)A}) = (dt,\ a_3(t)e^{(3)a},\ a_6(t)e^{(6)a})$, where $e^{(3)a}$ is the basis on M_K^3 diagonalizing $g^{(3)}$, it is easy to check that

$$\omega_{3L} = \omega_{3L}^{(3)} + \omega_{3L}^{(6)} + f_3 \ , \quad (16)$$

where $df_3 = 0$ and $\omega_{3L}^{(3)} = tr(\omega^{(3)} \wedge \Omega^{(3)} - \frac{1}{3}\omega^{(3)} \wedge \omega^{(3)} \wedge \omega^{(3)}), \omega^{(3)}$ is the spin connection on M_K^3 corresponding to $e^{(3)a}$. From (2.1.2), (2.1.15) and (2.1.16) we have

$$H = dB + \omega_{3L}^{(3)} + f_3 \ . \quad (17)$$

It follows from (2.1.17) and $d\omega_{3L}^{(3)} = df_3 = 0$ that for every domain $\Omega \subset M^{10}$ with $H^3(\Omega, R) = 0$ there is some B such that $H = 0$.

The spin connection $\omega^{(6)}$ on K obeys the identity

$$D_m(\omega^{(6)})\Omega^{(6)mn} = 0 \ . \quad (18)$$

In (2.1.18) $D_m(\omega) = \nabla_m + [\omega_m, .]$. The identity (2.1.18) is equivalent to

$$\nabla_m^{(6)} R^{(6)mnpq} = 0$$

and is valid for any Kähler Ricci-flat manifold [26]. Equation (2.1.6) is satisfied identically due to (2.1.18), (2.1.10)-(2.1.13) and (2.1.13a) ($D_M = D_M(A) = \nabla_M + [A_M, .]$); (2.1.5) is satisfied owing to (2.1.11).

Equations (2.1.4) and (2.1.7) in the ansatz (2.1.10)-(2.1.13) may be rewritten in the following manner:

$$3a_3^{-2}(k + \dot{a}_3^2) = (9/16)\varphi^{-2}\dot{\varphi}^2 + \kappa^2\rho + A_6 \ , \quad (2.1.4a)$$

$$a_3^{-2}(k + \dot{a}_3^2 + 2a_3\ddot{a}_3) = -(9/16)\varphi^{-2}\dot{\varphi}^2 - \kappa^2 p_3 + B_6 \ , \quad (2.1.4b)$$

$$a_3^{-2}(k + \dot{a}_3^2 + a_3\ddot{a}_3) = -(3/16)\varphi^{-2}\dot{\varphi}^2 - (1/3)p_6 +$$
$$2\kappa^2\varphi^{-3/4}a_3^{-3}\ddot{a}_3\,(k + \dot{a}_3^2) + C_6 \ , \quad (2.1.4c)$$

$$\ddot{\varphi} - \varphi^{-1}\dot{\varphi}^2 + 3a_3^{-1}\dot{a}_3\dot{\varphi} - 4\kappa^2\varphi^{1/4}a_3^{-3}\ddot{a}_3\,(k + \dot{a}_3^2) + D_6 = 0 \ . \quad (2.1.7a)$$

In (2.1.4a)-(2.1.4c), (2.1.7a) $A_6 = B_6 = C_6 = D_6 = 0$ when $a_6(t) = const$. Equations (2.1.4a)-(2.1. 4c) and (2.1.7a) are obtained from (2.1.4) and (2.1.7) using the Ricci flatness of K and the equality

$$R^{(6)}_{pqmn}\ \dot{R}^{(6)qp'mn} = \frac{1}{30} tr\ F_{mn} F_{pq} g^{(6)mp'} g^{(6)nq} \ ,$$

which follows from (2.1.13a), (2.1.14) and the relation

$$R^{(6)p}_{qmn} = e^{(6)p}_{\alpha} e^{(6)\beta}_{q} \Omega^{(6)x}_{\beta mn} \ .$$

It follows from (2.1.4)-(2.1.7) that

$$\nabla_M T^{MN} = 0 \ . \tag{19.a}$$

Relation (2.1.19a) in the substitution of (2.1.9), (2.1.10) is equivalent to

$$\dot{\rho} + 3a_3^{-1}\, \dot{a}_3\, (\rho + p_3) + 6a_6^{-1}\, \dot{a}_6\, (\rho + p_6) = 0 \ . \tag{19.b}$$

In order to close system (2.1.4a)-(2.1.4c), (2.1.7a) we add two equations of state of the ten-dimensional matter:

$$F_i(t; \rho, p_3, p_6) = 0 \ , \qquad i = 1, 2 \ . \tag{20}$$

It is naturally to demand that for an ordinary matter

$$\rho > 0 \ , \ p_3 \geq 0 \tag{21}$$

Suppose that there is a solution of (2.1.4a)-(2.1.4c), (2.1.7a) with $a_6(t) =$ *const*.

Case 1: $\varphi(t) = const$ for all t. In this case it follows from (2.1.7a) that

$$\ddot{a}_3(k + \dot{a}_3^2) = 0 \ . \tag{22}$$

But from (2.1.4a) and (2.1.21) we get

$$k+\dot{a}_3^2 > 0 \; . \tag{23}$$

Then we find from (2.1.22) and (2.1.23) that $\ddot{a}_3 = 0$ and, using (2.1.4b) and (2.1.21), we get the inequality

$$k+\dot{a}_3^2 \leq 0 \; ,$$

which is in contradiction with (2.1.23).

Case 2: $\dot{\varphi}(t_0) \neq 0$ for some t_0. From continuity of $\dot{\varphi}\,(t)$ it follows that $\dot{\varphi}\,(t) \neq 0$ at some interval $(a,b) \ni t_0$. Differentiating (2.1.4a) we get

$$-6a_3^{-3}\,\dot{a}_3\,(k+\dot{a}_3^2) + 6a_3^{-2}\,\dot{a}_3\,\ddot{a}_3 - (9/8)\varphi^{-2}\,\dot{\varphi}\,(\ddot{\varphi} - \varphi^{-1}\,\dot{\varphi}^2) = \kappa^2\,\dot{\rho} \; . \tag{24}$$

The subtraction of (2.1.4b) from (2.1.4a) leads to

$$2a_3^{-2}(k+\dot{a}_3^2 - a_3\ddot{a}_3) - (9/8)\varphi^{-2}\,\dot{\varphi}^2 = \kappa^2(\rho + p_3) \; . \tag{25}$$

Multiplying (2.1.25) on $3a_3^{-1}\,\dot{a}_3$, adding the result to (2.1.24) and using (2.1.19b) we obtain

$$-(9/8)\varphi^{-2}\,\dot{\varphi}\,(\ddot{\varphi} - \varphi^{-1}\,\dot{\varphi}^2 + 3a_3^{-1}\,\dot{a}_3\dot{\varphi}) = 0 \; . \tag{26}$$

At the interval (a,b), $\dot{\varphi}\,(t) \neq 0$, so at this interval

$$\ddot{\varphi} - \varphi^{-1}\,\dot{\varphi}^2 + 3a_3^{-1}\,\dot{a}_3\dot{\varphi} = 0 \; .$$

This equality and (2.1.7a) result in the relationship (2.1.22) for all $t \in (a,b)$. Repeating the subsequent arguments of case 1 we come to a contradiction.

Thus in the FCY cosmology the solution of the equations of motion with $a_6(t) = const$ does not exist. But

$$G = const\; a_6^{-6}$$

(it follows from (2.1.1)). So we prove that the time variation G is an unavoidable one in 10-dimensional cosmology of the superstring origin [27]. Of course, in other models we may find solutions with $a_{int}(t) \neq const$ (see below).

If we do not apply conditions (2.1.21) then we may obtain the result $a_6 = const$. These cases are:

$$1.\phi = const, \quad a_3 = const, \quad p_6 = const, \quad p_3 = 0, \quad \rho = 0, \quad k = 0,$$
$$2.\phi = const, \quad p_6 = p_3 = \rho = 0, \quad k = -1, \quad a_3 = t + c,$$
$$3.\phi \neq const, \quad \rho < 0, \quad p_3 < 0, \quad p_6 < 0.$$

We see that all these variants are unreasonable in the modern epoch from observational point of view.

2 Solutions with Perfect Fluid in (4+N)-Dimensional Cosmology [28]

Support for time variations of fundamental physical constants especially of the gravitational constant G comes from many modern theoretical schemes: unification theories, modified theories of gravitation, e.g., scalar-tensor theories, etc. [1]. Corresponding experimental data on G are still controversial. Evidently, more tests of G variations are needed, both astronomical and laboratory ones. Much is expected from joint missions of space crafts to Mars in 1994.

As we saw multidimensional cosmological models [27] also provide a possibility of time variation of G. In these models G is not a fundamental constant since it depends on the internal space scale. Its time variation leads to varying G.

When the contribution to the gravitational field equations from the Gauss-Bonnet term (see [25]) is negligible compared with that from the Einstein one, then gravity is governed by the multidimensional Einstein equations and the cosmological problems are reduced to ordinary 4-dimensional equations with a contribution from a Ricci-flat internal space.

We consider here a more general problem, namely a $(4+N)$-dimensional cosmology with an isotropic 3-space and an arbitrary Ricci-flat internal space. The Einstein equations provide a relation between $\dot{G}/G$ and other cosmological parameters. In particular, for a spatially flat universe ($k=0$) the present observational upper bounds on $\dot{G}/G$ taken in the form

$$|\dot{G}/G| \lesssim 1.10^{-11}(y^{-1})$$

lead to the following bounds upon the density parameter Ω:

$$0.8 \lesssim \Omega \lesssim 1.2 \ .$$

Some $(4+N)$-dimensional theory is considered in an epoch when all the higher corrections to the action of gravity are negligible. It is described by the standard expression

$$S_g = \frac{1}{2\kappa^2}\int d^{4+N}x\sqrt{-g}R \ , \tag{1}$$

where κ^2 is the fundamental gravitational constant. Then the gravitational field equations are

$$R^M_P = -\kappa^2(T^M_P - \delta^M_P\frac{T}{N+2}) \ , \tag{2}$$

where T^M_P is a $(4+N)$-dimensional energy-momentum tensor, $T=T^M_M$, $M,P=0,...,N+3$. For the $(4+N)$-dimensional manifold we assume the structure

$$M^{4+N} = R\times M^3_k \times K^N \ , \tag{3}$$

where M^3_k is a 3-dimensional space of constant curvature, $M^3_k = S^3,\ R^3,\ L^3$ for $k=+1,0,-1$, respectively, and K^N is a N-dimensional compact Ricci-flat Riemann manifold. The metric is taken in the form

$$g_{MN}dx^M dx^N = dt^2 - a^2(t)g^{(3)}_{ij}(x^k)dx^i dx^j - b^2(t)g^{(N)}_{mn}(y^p)dy^m dy^n \ , \tag{4}$$

where $i, j, k = 1, 2, 3$; $m, n, p = 4, ..., N+3$; $g^{(3)}_{ij}$, $g^{(N)}_{mn}$, $a(t)$ and $b(t)$ are, respectively, the metrics and scale factors for M^3_k and K^N. For T^M_P we adopt the expression

$$(T^M_P) = diag(\rho(t), -p_3(t)\delta^i_j, -p_N(t)\delta^m_n) . \tag{5}$$

Under these assumptions the Einstein equations (2.2.2) take the form

$$\frac{3\ddot{a}}{a} + \frac{N\ddot{b}}{b} = \frac{\kappa^2}{N+2}[-(N+1)\rho - 3p_3 - Np_N] , \tag{6}$$

$$\frac{2k}{a^2} + \frac{\ddot{a}}{a} + \frac{2\,\dot{a}^2}{a^2} + \frac{N\,\dot{a}\dot{b}}{ab} = \frac{\kappa^2}{N+2}[\rho + (N-1)p_3 - Np_N] , \tag{7}$$

$$\frac{\ddot{b}}{b} + (N-1)\frac{\dot{b}^2}{b^2} + \frac{3\,\dot{a}\dot{b}}{ab} = \frac{\kappa^2}{N+2}[\rho - 3p_3 + 2p_N]. \tag{8}$$

The 4-dimensional density is

$$\rho^{(4)}(t) = \int_K d^N y \sqrt{g^{(N)}} b^N(t)\rho(t) = \rho(t)b^N(t), \tag{9}$$

where we have normalized the factor $b(t)$ by putting

$$\int_K d^N y \sqrt{g^{(N)}} = 1.$$

On the other hand, to get the 4-dimensional gravity equations one should put $8\pi G(t)\rho^{(4)}(t) = \kappa^2\rho(t)$. Consequently, the effective 4-dimensional gravitational "constant" $G(t)$ is defined by

$$8\pi G(t) = \kappa^2 b^{-N}(t) \tag{10}$$

whence its time variation is expressed as

$$\dot{G}\,/G = -N\,\dot{b}\,/b. \tag{11}$$

Some inferences concerning the observational cosmological parameters can be extracted just from the equations without solving them. Indeed, let us define the Hubble parameter H, the density parameter Ω and the deceleration parameter q referring to a fixed instant t_0 in the usual way

$$H = \dot{a}/a, \qquad \Omega = 8\pi G\rho^{(4)}/3H^2 = \kappa^2\rho/3H^2, \qquad q = -a\,\ddot{a}/\dot{a}^2 \ . \tag{12}$$

Besides, instead of G let us introduce the dimensionless parameter

$$g = \dot{G}/GH = -Na\,\dot{b}/\dot{a}\,b \ . \tag{13}$$

Then, excluding b from (2.2.6) and (2.2.8) we get

$$\frac{N-1}{3N}g^2 - g + q - A_{N,\nu}\Omega = 0 \tag{14}$$

with

$$A_{N,\nu} = \frac{1}{N+2}[2N + 1 + 3(1-N)\nu_3 + 3N\nu_N] \ ,$$

where

$$\nu_3 = p_3/\rho, \qquad \nu_N = p_N/\rho, \qquad \rho > 0 \ .$$

If $q << 1$ and $\Omega << 1$, then either g is also small and equals

$$g \approx q - A_{N,\nu}\Omega \tag{15}$$

or (if $N > 1$) it is comparatively large and is described by another root of the quadratic equation (2.2.14), namely $g \approx 3N/(N-1)$. Note that (2.2.15) for $N = 6$, $\nu_3 = \nu_6 = 0$ (so that $A_{N,\nu} = 13/8$) coincides with the corresponding relation of Wu and Wang [25] obtained for large times in case $k = -1$.

If $k = 0$, then in addition to (2.2.14), one can obtain a separate relation between g and Ω, namely,

$$\frac{N-1}{6N}g^2 - g + 1 - \Omega = 0 \tag{16}$$

(this follows from the Einstein equation $R_0^0 - \frac{1}{2}R = -\kappa^2 T_0^0$, which is certainly a linear combination of (2.2.8)-(2.2.10)). Furthermore, excluding Ω from (2.2.14) and (2.2.16), we get

$$\frac{N-1}{6N}(A_{N,\nu} - 2)g^2 + (1 - A_{N,\nu})g + A_{N,\nu} - q = 0 \; . \tag{17}$$

The present observational upper bound on g is

$$| g | \lesssim 0.2 \tag{18}$$

if we take in accord with [13] $|\dot{G}/G| \lesssim 1.10^{-11}(y^{-1})$ and $H \gtrsim 5.10^{-11}(y^{-1}) \approx 50(km/s.Mpc)$. Using (2.2.16) and (2.2.18) we get the crude estimate

$$0.8 \lesssim \Omega \lesssim 1.2 \tag{19}$$

independent of N. In the case of dustlike matter ($\nu_s = \nu_N = 0$) (2.2.17) and (2.2.18) yield the following estimates on q for $N = 1, 6, \infty$, respectively:

$$q = 1, \qquad 1.5 \lesssim q \lesssim 1.75, \qquad 1.8 \lesssim q \lesssim 2.2 \; . \tag{20}$$

The above relations refer to a fixed instant, e.g., the present epoch. To answer questions concerning model evolution, it is helpful to solve the field equations. Here we consider the general solution of (2.2.6)-(2.2.10) for the case $k = \nu_3 = \nu_N = 0$ (dustlike matter)[1]. The solution is

$$\begin{cases} a(t) = & At^{\alpha}(t+T)^{\beta} \; , \\ b(t) = & Bt^{\bar{\alpha}}(t+T)^{\bar{\beta}} \; , \\ \rho(t) = & 2(N+2)/\kappa^2(N+3)T(t+T) \; , \end{cases} \tag{21}$$

[1] Equations (2.2.6)-(2.2.10) can be solved exactly at least in the following cases: i) $k = 0$, ν_3 and ν_N are arbitrary constants. ii) $k = 0, \pm 1$; $\nu_3 = 1$; ν_N is an arbitrary constant. iii) $k = 0, \pm 1$; $2\nu_N = 3\nu_3 - 1$; ν_3 is an arbitrary constant.

where $A > 0$, $B > 0$, $T \geq 0$ are constants, $0 > t > \infty$, and

$$\left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} = \frac{1}{N+3}(\pm sNr + 1), \quad \left\{ \begin{array}{c} \overline{\alpha} \\ \overline{\beta} \end{array} \right\} = \frac{1}{N+3}(\mp 3sr + 1) \ ,$$

$r = \sqrt{(N+2)/3N}, s = \pm 1$. (Equation (2.2.21) is the general solution of (2.2.6)-(2.2.8) up to the choice of a direction and a reference point of time). Solutions like (2.2.21) were first considered in [29].

Time dependences of a and b for the case $T > 0$ and $N > 1$ are plotted in fig. 1,2. One sees that the cases $s = +1$ and $s = -1$ are highly different.

For H and Ω solutions (2.2.21) give

$$H = \frac{\alpha}{t} + \frac{\beta}{t+T} \tag{22}$$

and

$$\Omega = \frac{2(N+2)t(t+T)}{3(N+3)[\alpha(t+T)+\beta t]^2} \tag{23}$$

It is easily checked that for $s = -1$ the density parameter on the expansion stage ($\dot{a} > 0$) is $\Omega \geq \Omega_* = (N+2)(N+3)/6 \geq 2$. In the case $T = 0$ which is common for the branches $s = +1$ and $s = -1, \Omega = \Omega_*$. These cases are unacceptable due to (2.2.19). See fig. 3.

So let us discuss the remaining branch $s = +1$, $T > 0$. The parameter $\Omega(t)$ monotonically increases from 0 to Ω_* while the product $H(t)t$ monotonically decreases from $(\sqrt{N(N+2)/3}+1)/(N+3)$ to $2/(N+3)$ if $N > 1$, and $H(t)t = 1/2$ if $N = 1$. For the parameters q and g we have

$$q = \frac{[\alpha(\alpha-1)(t+T)^2 + 2\alpha\beta(t+T)t + \beta(\beta-1)t^2]}{[\alpha(t+T)+\beta t]^2} \tag{24}$$

and

$$g = -N\frac{\overline{\alpha}(t+T)+\overline{\beta}t}{\alpha(t+T)+\beta t}. \qquad (25)$$

So q monotonically increases from $(\sqrt{3N(N+2)}-N-2)/(N-1)$ to $\frac{1}{2}(N+1)$ when $N>1$ and $q=1$ for $N=1 (q>0)$. On the other hand, g monotonically decreases from $6N/(3N+\sqrt{3N(N+2)})$ to $-N$. (This follows from (2.2.24) and (2.2.25)). At the moment $t=t_1=\frac{1}{2}(\sqrt{3(N/2)/N}-1)T$ we get $\Omega=1, q=(2N+1)/(N+2)$ and $g=0$.

Consequently, the model based on solution (2.2.21) with $s=+1$, $T>0$ may be considered as one of the candidates for a realistic cosmological model in the dust-dominated era. Besides, this model has also one more attractive feature. The scale factor of the internal space has a minimal value (when it is constant and so the effective gravitational constant is also constant.

3 Perfect-Fluid Type Solution in Multidimensional Multicomponent Cosmology [30]

Here we consider $(1+N_0+...+N_n)$-dimensional cosmology $(n, N_0, ..., N_n \in \mathcal{N})$ with $n+1$ Ricci-flat spaces. For the "perfect-fluid" matter with the density $\rho>0$ and the pressures

$$p_\nu = (1-h_\nu)\rho\ , \qquad (1)$$

where h_ν are constants $(\nu=0,...,n)$, satisfying

$$\Delta(h) \neq 0 \qquad (2)$$

(with $\Delta(h)$ defined in (2.3.10)), an exact solution of the Einstein equations is obtained. Note that the $n=1$ case was previously considered in ref. [31] for all h_ν, $\nu=0,1$.

Let us consider the manifold

$$M = \mathcal{R} \times M_0 \times ...M_n \ , \tag{3}$$

with the metric

$$g = -e^{2\gamma(t)} dt \otimes dt + \sum_{\nu=0}^{n} e^{2\beta_\nu(t)} g_{(\nu)} \ , \tag{4}$$

where $n \in \mathcal{N}$ and M_ν is an N_ν-dimensional Ricci-flat manifold with the metric $g_{(\nu)}$, $\nu = 0, ..., n$. For the energy-momentum tensor we adopt the expression in the "perfect-fluid" form

$$(T_N^M) = diag(-\rho(t), p_0(t)\delta_{\ell_0}^{k_0}, ..., p_n(t)\delta_{\ell_n}^{k_n}) \ , \tag{5}$$

where $k_\nu, \ell_\nu = 0, ..., N_\nu$; $\rho > 0$ and the equations of state (2.3.1) are assumed.

We put $\gamma = \sum_{\nu=0}^{n} N_\nu \beta_\nu$ in (2.3.4) (the harmonic time is used). Then the Einstein equations $R_N^M - \frac{1}{2}\delta_N^M R = k^2 T_N^M$ for the metric (2.3.4) on the manifold (2.3.3) with the energy-momentum tensor (2.3.5) and the state equations (2.3.1) have a rather simple form and are equivalent to the following system:

$$\left(\sum_{\nu=0}^{n} N_\nu \dot{\beta}_\nu\right)^2 - \sum_{\nu=0}^{n} N_\nu \dot{\beta}_\nu^2 = 2k^2 \rho \ exp \ \left(2\sum_{\nu=0}^{n} N_\nu \beta_\nu\right) \ , \tag{6}$$

$$\ddot{\beta}_\mu = k^2 \rho \exp\left(2\sum_{\nu=0}^{n} N_\nu \beta_\nu\right) b_\mu(h) \left(\sum_{\nu=0}^{n} N_\nu - 1\right)^{-1} \ , \tag{7}$$

$\mu = 0, ..., n$; where $\rho > 0$, k^2 is the gravitational constant and

$$b_\mu = b_\mu(h) = \sum_{\nu=0}^{n} N_\nu h_\nu + h_\mu \left(1 - \sum_{\nu=0}^{n} N_\nu\right) \ . \tag{8}$$

Let us introduce new variables χ_μ:

$$\chi_0 = \sum_{\nu=0}^{n} h_\nu N_\nu \beta_\nu \ , \chi_i = b_0(h)\beta_i - b_i(h)\beta_0 \ , \tag{9}$$

$i = 1, ..., n$. Eqs. (2.3.9) may be written as $\chi = S\beta$, where the matrix $S = S(h)$ is implicitly defined in (2.3.9). A straightforward calculation gives

$$\Delta \equiv det\ S(h) = [b_0(h)]^{n-1}\Delta'(h)\ , \tag{10}$$

where

$$\Delta' = \Delta'(h) = \sum_{\nu=0}^{n} b_\nu(h) h_\nu N_\nu \tag{11}$$

and $b_\nu(h)$ are defined in (2.3.8). From eqs. (2.3.7) and (2.3.9) we have $\ddot{\chi}_i = 0$ or equivalently

$$\chi_i = C_i t + D_i\ , \tag{12}$$

where C_i, D_i are constants, $i = 1, ..., n$. The conservation law $\nabla_M T^M_N = 0$ gives

$$\rho = A\ exp(\sum_{\nu=0}^{n} N_\nu (h_\nu - 2)\beta_\nu)\ , \tag{13}$$

$A > 0$ is a constant. In the non-exceptional case (2.3.2) considered here, the map (2.3.9) may be inverted,

$$\beta = S^{-1}\chi\ . \tag{14}$$

It may be checked that the matrix $S^{-1} = S^{-1}(h)$ in (2.3.14) has the following components:

$$\begin{aligned}
&S^{-1}_{00} = \frac{b_0}{\Delta'}, \quad S^{-1}_{0j} = -\frac{h_j N_j}{\Delta'}, \quad S^{-1}_{i0} = \frac{b_i}{\Delta'}\ , \\
&S^{-1}_{ij} = \frac{b_i h_j N_j}{b_0 \Delta'}, \quad i \neq j, \ \ n > 1\ , \\
&S^{-1}_{ij} = \left(\sum_{\substack{\mu=0 \\ \mu\neq i}}^{n} N_\mu h_\mu b_\mu\right) \frac{1}{b_0 \Delta'}\ ,
\end{aligned} \tag{15}$$

$n > 1$ and $S_{11}^{-1} = N_0 h_0/\Delta'$ for $n = 1; i,j = 1,...,n$.

The quadratic form in the left-hand side of eq. (2.3.6) may be expressed as

$$\sum_{\nu=0}^{n} K_{\mu\nu}\, \dot{\beta}_\mu \dot{\beta}_\nu = \sum_{\mu=0}^{n} \overline{K}_{\mu\nu}\, \dot{\chi}_\mu \dot{\chi}_\nu \ , \tag{16}$$

where $K = (K_{\mu\nu}) = (N_\mu N_\nu - N_\mu \delta_{\mu\nu})$ and

$$\overline{K} = (S^{-1})^T K S^{-1} \tag{17}$$

Calculations give for (2.3.17)

$$\overline{K}_{00} = \frac{1}{\Delta'}\left(\sum_{\nu=0}^{n} N_\nu - 1\right), \ \overline{K}_{0i} = 0,$$
$$\overline{K}_{ij} = \frac{1}{b_0^2}\left[N_i N_j - N_i \delta_{ij} + \frac{1}{\Delta'}\left(1 - \sum_{\nu=0}^{n} N_\nu\right) h_i N_i h_j N_j\right] \ , \tag{18}$$

$n > 1$ and $\overline{K}_{11} = -N_0 N_1/\Delta'$ for $n = 1; i,j = 1,...,n$. Then from (2.3.2), (2.3.6), (2.3.12), (2.3.13), (2.3.16) and (2.3.18) it follows that

$$\dot{\chi}_0^2 = C + D\ exp(\chi_0), \quad \chi_0 \neq const \tag{19}$$

(the assumption $\chi_0 = const$ leads to $\Delta(h) = 0$), where

$$C = -\Delta'\left(\sum_{i,j=1} C_i \overline{K}_{ij} C_j\right)\left(\sum_{\nu=0}^{n} N_\nu - 1\right)^{-1},$$
$$D = 2\Delta' k^2 A \left(\sum_{\nu=0}^{n} N_\nu - 1\right)^{-1}. \tag{20}$$

The system of eqs. (2.3.6) and (2.3.7) is equivalent to the system of eqs. (2.3.12), (2.3.13) and (2.3.19) (this is not difficult to prove). Solving (2.3.19) we get

$$\chi_0 = \ell n\{C/D\ sh^2[\frac{1}{2}\sqrt{C}(t - t_0)]\}, \quad \Delta', C > 0 \ ,$$
$$= \ell n[4/D(t - t_0)^2], \qquad \Delta' > 0, C = 0 \ ,$$
$$= \ell n\{-C/D\ ch^2[\frac{1}{2}\sqrt{C}(t - t_0)]\}, \quad \Delta' < 0, C > 0 \ , \tag{21}$$

where t_0 is a constant. Note that the quadratic form in (2.3.20) $C = C(C_i)$ is positively defined if $\Delta' > 0$ and has the signature $(+,-,...,-)$ for $\Delta' < 0$. This follows from (2.3.18), (2.3.20) and from the fact that the matrix K (and hence $\overline{K}$) has the signature $(+,-,...,-)$. The latter follows from the relations

$$\sum_{\mu,\nu=0}^{n} K_{\mu\nu}\beta_\mu\beta_\nu = z_0^2 - \sum_{i=1}^{n} z_i^2 \ , z_0 = \left[\left(\sum_{\nu=0}^{n} N_\nu - 1\right) / \sum_{\nu=0}^{n} N_\nu\right]^{1/2} \sum_{\nu=0}^{n} N_\nu \beta_\nu,$$

$$z_i = \left[N_{i-1} / \left(\sum_{\nu=i-1}^{n} N_\nu\right)\left(\sum_{\nu=i}^{n} N_\nu\right)\right]^{1/2} \times \sum_{\mu=i}^{n} N_\nu(\beta_\mu - \beta_{i=1}). \tag{22}$$

$i = 1,...,n$. So the solution obtained here is given by eqs. (2.3.4) and (2.3.13) with $\gamma = \sum_{\nu=0}^{n} N_\nu \beta_\nu$ and $\beta = (\beta_\nu) = S^{-1}\chi$, where S^{-1} is defined in (2.3.15) and $\chi = (\chi_\mu)$ has the components (2.3.12) and (2.3.21) with C and D defined in (2.3.20). The integration constants $A > 0, t_0$ and D_i are arbitrary, the constants C_i are arbitrary, when $\Delta' > 0$, and obey the restriction $C = C(C_i) > 0$ (see (2.3.20)), if $\Delta' < 0, i = 1,...,n$ (Δ' is defined in (2.3.11)).

To illustrate the general solution let us consider the dust case: $h_\nu = 1$, $\nu = 0,...,n$. In this case the solution in the proper-time parametrization with $0 < \tau < +\infty$ has the following form,

$$g = -d\tau \otimes d\tau + \sum_{\nu=0}^{n} a_\nu^2(\tau) g_{(\nu)},$$

$$a_\nu = A_\nu[\tau(\tau+T)]^{1/\sum_{\nu=0}^{n} N_\nu}[\tau/(\tau+T)]^{\alpha_\nu},$$

$$\rho(\tau) = 2\left(\sum_{\nu=0}^{n} N_\nu - 1\right) / \left(\sum_{\nu=0}^{n} N_\nu\right) k^2 \tau(\tau+T), \tag{23}$$

where $A_\nu > 0$, $T \geq 0$ are constants and α_ν are constants, satisfying the constraints

$$\sum_{\nu=0}^{n} N_\nu \alpha_\nu = 0, \ \sum_{\nu=0}^{n} N_\nu \alpha_\nu^2 = 1 - \left(\sum_{\nu=0}^{n} N_\nu\right)^{-1}, \tag{24}$$

$\nu = 0, ..., n$; $n \in \mathcal{N}$. (One may also consider the case $n = \infty$ in (2.3.23)). Note that special cases of solution (2.3.23) were considered in ref. [29] ($N_0 = ... = N_n = 1$) and ($n = 2$).

4 On Reduction of Multicomponent Cosmology to Toda Lattice [35]

Toda lattice equations [32] occur in many areas of physics and in gravitation as well [33,34]. Gibbons and Maeda [34] suggested a reduction of a multidimensional cosmology with two spaces of constant curvature to an open Toda lattice. Here we try to apply their approach to n spaces.

Let us briefly review the relations of ref. [34]. For the metric

$$g = -exp[2\sum_{i=1}^{2} N_i\beta_i(t)]dt \oplus dt + \sum_{i=1}^{2} e^{2\beta_i(t)} g^{(i)}$$

(the harmonic time is used; we slightly change the notations of ref. [34]) on the manifold

$$M = R \times M_1 \times M_2 \ ,$$

where the manifold M_i($dim\ M_i = N_i$) with the metric $g^{(i)}$ is the space of constant curvature ($i = 1, 2$). The vacuum Einstein equations reduce to the system of Lagrange equations for the "Toda lattice"-type Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{3} \mu_i \dot{q}_i^2 + \sum_{i=1}^{2} \theta_i e^{q_i - q_{i+1}} \qquad (1)$$

and two constraints

$$\sum_{i=1}^{3} \mu_i q_i = 0 \ , \qquad (2)$$

$$E = \frac{1}{2}\sum_{i=1}^{3}\mu_i\,\dot{q}_i^2 - \sum_{i=1}^{2}\theta_i e^{q_i - q_{i+1}} = 0\ . \tag{3}$$

In eqs. (2.4.1)-(2.4.3)

$$\mu_1 = -\frac{N_1}{2(2N_1 - 1)};\quad \mu_2 = \frac{1}{2};\quad \mu_3 = -\frac{N_2}{2(2N_2 - 1)} \tag{4}$$

(θ_i is scalar curvature of g^i). The identification between two systems is reached by the use of the following relations:

$$q_{i+1} - q_i = 2\beta_i - 2\sum_{j=1}^{n} N_j\beta_j,\quad i = 1, 2. \tag{5}$$

Let us try to generalyse the relations (2.4.1)-(2.4.5) for n-component case, $n \geq 2$. Consider the metric

$$g = -e^{2\sum_{i=1}^{n} N_i\beta_i(t)}dt \otimes dt + \sum_{i=1}^{n} e^{2\beta_i(t)}g^{(i)} \tag{6}$$

on the manifold

$$M = R \times M_1 \times \ldots \times M_n\ , \tag{7}$$

where $n \geq 2$, $dim M_i = N_i$, and $(M_i, g^{(i)})$ is the space of constant curvature, $(i = 1, \ldots, n)$. The vacuum Einstein equations for the metric (2.4.6) on (2.4.7) are equivalent to the system

$$E = \sum_{i=1}^{n} N_i\,\dot{\beta}_i^2 - \left(\sum_{i=1}^{n} N_i\,\dot{\beta}_i\right)^2 + \sum_{j=1}^{n}\theta_j e^{-2\beta_j + 2\sum_{i=1}^{n} N_i\beta_i} = 0, \tag{8}$$

$$\theta_i e^{-2\beta_i + 2\sum_{j=1}^{n} N_j\beta_j} + N_i\ddot{\beta}_i = 0. \tag{9}$$

where $i = 1, \ldots, n$. The system (2.4.9) is a Lagrange one; the corresponding Lagrangian is

$$L = \sum_{i=1}^{n} N_i \, \dot{\beta}_i^2 - \left(\sum_{i=1}^{n} N_j \, \dot{\beta}_j \right)^2 + \sum_{i=1}^{n} \theta_i e^{-2\beta_i + 2\sum_{j=1}^{n} N_j \beta_j} , \qquad (10)$$

Here $\theta_i = R[g^{(i)}], i = 1, ..., n$. In the variables

$$\alpha_i = 2\beta_i - 2\sum_{j=1}^{n} N_j \beta_j , \qquad (11)$$

the Lagrangian (2.4.10) may be rewritten as

$$L = \frac{1}{4}\sum_{i=1}^{n} N_i \, \dot{\alpha}_i^2 - \frac{1}{4(\sum_{j=1}^{n} N_j - 1)} \left(\sum_{i=1}^{n} N_i \, \dot{\alpha}_i \right)^2 + \sum_{j=1}^{n} \theta_j e^{-\alpha_j}. \qquad (12)$$

In α-variables the system (2.4.9) is equivalent to the system of Lagrange equations for (2.4.12) and (2.4.8) is equivalent to the constraint

$$E = \frac{1}{4}\sum_{i=1}^{n} N_i \, \dot{\alpha}_i^2 - \frac{1}{4\left(\sum_{j=1}^{n} N_j - 1\right)} \left(\sum_{i=1}^{n} N_i \, \dot{\alpha}_i \right)^2 - \sum_{i=1}^{n} \theta_i e^{-\alpha_i} = 0. \quad (2.4.8*)$$

Consider the hyperplane V_μ in R^{n+1}:

$$V_\mu = \{q \mid q \in R^{n+1}, \ \sum_{i+1}^{n+1} \mu_i q_i = 0\}. \qquad (13)$$

$$\alpha_i = q_{i+1} - q_i; \quad i = 1, ..., n; \qquad (14)$$

from the hyperplane V_μ (2.4.13) to R^n is a biective one if and only if

$$\sum_{j=1}^{n+1} \mu_j \neq 0. \qquad (15)$$

If there exists a set $\mu = (\mu_1, ..., \mu_{n+1})$ satisfying (2.4.15) such that for all $u \in V_\mu$

$$\sum_{i=1}^{n} N_i (u_{i+1} - u_i)^2 - \frac{1}{(\sum_{j=1}^{n} N_j - 1)} \left[\sum_{i=1}^{n} N_i (u_{i+1} - u_i) \right]^2 = 2\sum_{j=1}^{n+1} \mu_j u_j^2 ; \qquad (16)$$

then using (2.4.8*), (2.4.11), (2.4.12)-(2.4.14), it is not difficult to check that the system (2.4.8)-(2.4.9) is equivalent to the system of Lagrange equations for

$$\overline{L} = \frac{1}{2}\sum_{i=1}^{n+1}\mu_i\,\dot{q}_i^2 + \sum_{i=1}^{n}\theta_i e^{q_i - q_{i+1}} \tag{17}$$

with two constraints: (2.4.13) and $\overline{E} = 0$ ($\overline{E}$ is the energy functional for $\overline{L}$).

Thus, we have a reduction of n-component cosmology to the Lagrange system of "Toda lattice"-type with two constraints when there exists the set of "masses" satisfying (2.4.15)-(2.4.16). Unfortunately, this takes place only for $n = 2$. When $n > 2$ there is no $\mu = (\mu_1, ..., \mu_{n+1})$ satisfying (2.4.15)-(2.4.16). The outline of the proof of this proposition is the following. Suppose that there exists $\mu = (\mu_1, ..., \mu_{n+1})$ satisfying (2.4.15)-(2.4.16), then from (2.4.16) $\mu_1 \neq 0$. If we put in (2.4.14) $u_3 = ... = u_{n+1}$ and

$$u_1 = -\frac{1}{\mu_1}\sum_{i=2}^{n+1}\mu_j u_j\ ; \tag{18}$$

then we get from (2.4.14) the system of three equations on μ_1, μ_2 and $\overline{\mu}_3 = \sum_{i=3}^{n+1}\mu_i$. Solving it by the condition of (2.4.15), we find

$$\mu_1 = \frac{N_1(\overline{N}_3 - 1)}{2(2N_1 + \overline{N}_3 - 1)}, \quad \mu_2 = \frac{1 - \overline{N}_3}{2}, \quad \mu_3 = \frac{N_2(\overline{N}_3 - 1)}{2(2N_2 + \overline{N}_3 - 1}), \tag{19}$$

where $\overline{N}_3 = \sum_{i=3}^{n+1} N_i \neq 1$; in the $\overline{N}_3 = 1$ case the solution is absent (note that for $\overline{N}_3 = 0$ we have (2.4.4); this solution is unique). Putting $u_4 = ... = u_{n+1}$, and u_1 from (2.4.18) in (2.4.16) we reduce (2.4.16) to the system of six equations on $\mu_1, \mu_2, \mu_3, \overline{\mu}_4 = \sum_{i=4}^{n+1}\mu_i$. One of these equations (the equality on coefficients at u_2u_4-terms) is the following

$$N_1\left(1 + \tfrac{\mu_2}{\mu_1}\right)\tfrac{\overline{\mu}_4}{\mu_1} - \tfrac{1}{\left(\sum_{j=1}^{n} N_j - 1\right)}\left[N_1\left(1 + \tfrac{\mu_2}{\mu_1}\right) - N_2\right]$$
$$\times\left(N_1\tfrac{\overline{\mu}_4}{\mu_1} + N_3\right) = \tfrac{2\mu_2\overline{\mu}_4}{\mu_1}. \tag{20}$$

But eq. (2.4.20) is in contradiction with eq. (2.4.19). This contradiction proves our proposition [35].

Thus, it is impossible to generalize the Gibbons-Maeda reduction prescription [34] for n-component case with $n > 2$, and therefore another approach (or some modification of ref. [34] is needed for studying integrability of multicomponent cosmology.

Chapter 13

Quantum Multidimensional Cosmology

1 On Wheeler-De Witt Equation in Multidimensional Cosmology [52]

Recently a growth of interest in investigations of Wheeler-De Witt (WDW) equation [36-38] for multidimensional cosmology is denoted. (We should note that the Hartle and Hawking paper [39] played an essential role in stimulating the new activity in quantum cosmology. This fact is connected, on one side, with a great attention to more-dimensional field theories: supergravity and superstrings, and, on the other side, with the consideration of the quantum cosmological models, containing more than one scale factors [40-42].

In quantum multidimensional cosmology we hope to find answers to the same problems as in 4-dimensional one: singular state, creation of the Universe, cosmological term nature and value, possible "seeds" of structure formation, variations of constants etc.

Besides, quantum cosmological models may open the way to the "third quantization" scheme which allows us to come from quantum mechanical approach to the quantum field theory one.

And, finally, quantum cosmological scheme is adequate mainly for the description of the Early Universe and as far as we believe in some unified theories of fundamental interactions then we had to use some multidimensional variants of quantum cosmology.

There are some technical problems of quantum cosmology such as boundary conditions and operator ordering.

It is well known that one of the main problems of quantization is the operator ordering one. In [43-47] the operator ordering problem is solved in favour of the following covariant form of WDW equation.

$$\left(-\frac{1}{2}\Delta[G] + aR[G] + V\right)\Psi = 0, \tag{1}$$

where Ψ is the wave function of the Universe, G is the metric on the superspace [36], $\Delta[G]$ and $R[G]$ are the Laplace-Beltrami operator and the scalar curvature, respectively, constructed from G, V is a potential and a is a constant. The term $aR[G]$ in (3.1.1) is responsible for an operator ordering ambiguity (and for renormalization, when it is needed, as well). The Laplace-Beltrami form of WDW equation was considered previously in [36,48].

Here we consider multidimensional cosmology with $n > 1$ spaces of constant curvature and obtain the WDW equation (3.1.1) with

$$a = (n-2)/8(n-1). \tag{2}$$

In this case eq. (3.1.1) is invariant under the conformal transformations of G and Ψ, induced by the choice of gauge. Such form of WDW equation was discussed earlier by Misner [48].[1]. In our case the WDW equation has the simplest form in the harmonic-time gauge (H), because in this gauge the minisuperspace metric G_H is flat. The metric G_H is diagonalized. It has a pseudo-Euclidean signature. In the case of Ricci-flat spaces the WDW equation is reduced to the d'Alembert equation. When $n = 2$ and one of the spaces is Ricci-flat, the WDW equation is reduced to the Klein-Gordon one.

Let us consider the metric

[1]Such form of WDW equation was also discussed in [49]

$$g_{(0)} = -exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} exp[2\beta^i(t)]g_{(i)}, \tag{3}$$

on the manifold

$$M = \mathbf{R} \times M_1 \times ... \times M_n, \tag{4}$$

where the manifold M_i with metric $g_{(i)}$ is a N_i-dimensional compact space of constant curvature. Substituting the metric (3.1.3) into Einstein's action

$$S = \frac{1}{2\kappa^2}\int_M dx \mid g \mid^{1/2} R + \frac{1}{\kappa^2}\int_{\partial M} dy \mid h \mid^{1/2} K,$$

where the second term is the standard Gibbons-Hawking boundary term [50], we get

$$S = \int dtL \tag{5.a}$$

with

$$L = \frac{\mu}{2}exp\left[-\gamma + \sum_{i=1}^{n} N_i\beta^i\right]\left[\sum_{j=1}^{n} N_j(\dot{\beta}^j)^2 - \left(\sum_{j=1}^{n} N_j\,\dot{\beta}^j\right)^2\right] +$$
$$+\frac{\mu}{2}exp\left[\gamma + \sum_{i=1}^{n} N_i\beta^i\right]\sum_{j=1}^{n}\theta_j exp[-2\beta^j]. \tag{5.b}$$

In (3.1.5b) $\theta_i \equiv R[g_{(i)}]$ is the scalar curvature of $g_(i)$,$i = 1, ..., n$ (usually, the metric $g_{(i)}$ is normalized in such way that $\theta_i = k_i N_i(N_i - 1), k_i = 0, \pm 1$) and

$$\mu = \left(\prod_{i=1}^{n} V_i\right)/\kappa^2,$$

where V_i is the volume of M_i. The system of Einstein's equations for the metric (3.1.3) on the manifold (3.1.4) is equivalent to the system of Lagrange equations for L (3.1.5b).

The Lagrangian (3.1.5b)may be written in the following manner:

$$L = \mathcal{N}\left(\frac{\mu}{2}\mathcal{N}^{-2}G_{ij}\,\dot{\beta}^i\dot{\beta}^j\, - V\right), \tag{6}$$

where

$$V = -\frac{\mu}{2} exp\left[2\sum_{j=1}^{n} N_j\beta^j\right]\sum_{j=1}^{n}\theta_i exp[-2\beta^i] \tag{7}$$

is the potential, $G_{ij} = N_i\delta_{ij} - N_iN_j$ are the components of the metric

$$G = G_{ij}d\beta^i \otimes d\beta^j \tag{8}$$

on the minisuperspace $\mathbf{R}^n$ and $\mathcal{N}$ is a Lagrange multiplier

$$\mathcal{N} = exp\left[\gamma - \sum_{i=1}^{n} N_i\beta^i\right].$$

The Lagrangian (3.1.6) is a degenerate one. Its degeneracy is connected with an invariance of the action under the gauge transformations

$$\beta^i(t) \mapsto \beta^i(h(t)), \quad \mathcal{N}(t) \mapsto \mathcal{N}(h(t))\frac{dh}{dt},$$

where h is an arbitrary reparametrization of time ($h \in Diff\mathbf{R}$).

Let us fix the gauge in (3.1.6):

$$\mathcal{N} = exp[-2f], \tag{9}$$

where $f = f(\beta)$ is an arbitrary smooth function on the minisuperspace $\mathbf{R}^n$. Such a gauge we call the f-gauge. Then it is not difficult to check that the system of Lagrange equations for L in the gauge (3.1.9) is equivalent to the system consisting of the Lagrange equations for

$$L^f = \frac{\mu}{2}G^f_{ij}(\beta)\,\dot{\beta}^i\dot{\beta}^j - V^f(\beta) \tag{10}$$

and a constraint

$$E^f = \frac{\mu}{2}G^f_{ij}(\beta)\,\dot{\beta}^i\dot{\beta}^j + V^f(\beta) = 0, \tag{11}$$

where

$$G^f = exp[2f]G, \quad V^f = exp[-2f]V.$$

Introducing generalized momenta $\pi_\nu = \frac{\partial L^f}{\partial \dot\beta^\nu} = \mu G^f_{\nu\rho}\, \dot\beta^\rho$ one may see that this system is equivalent to the system of Hamiltonian equations for the Hamiltonian

$$H^f = \frac{1}{2\mu}(G^f)^{ij}(\beta)\pi_i\pi_j + V^f(\beta) \tag{12}$$

with the constraint

$$H^f = 0, \tag{13}$$

where $(G^f)^{ij}(\beta) = exp[-2f(\beta)]G^{ij}$, and

$$G^{ij} = \frac{\delta_{ij}}{N_i} + \frac{1}{1 - \sum_{i=1}^{n} N_i}. \tag{14}$$

In (3.1.12) π_i are the canonical conjugate momenta.

At a quantum level the constraint (3.1.13) transforms into the WDW equation

$$\hat H^f \Psi^f = 0, \tag{15}$$

where Ψ^f is the wave function in the f-gauge and $\hat H^f$ is an operator corresponding to (3.1.12). Standard quantization procedure using change from π_ν to $\hat\pi_\nu = -i\partial/\partial\beta^\nu(\hbar = 1)$ leads to nonsingle-valued definition of $\hat H^f$ due to factor ordering problem. As it was shown in [44] for quantization of a simple Lagrange system with Lagrangian

$$\ell = \frac{1}{2}\gamma_{\alpha\beta}(x)\, \dot x^\alpha \dot x^\beta$$

and correspondingly with Hamiltonian

$$h = \frac{1}{2}\gamma^{\alpha\beta}(x)p_\alpha p_\beta,$$

where $\gamma = \gamma_{\alpha\beta}(x)dx^\alpha \otimes dx^\beta$ is a metric in a configuration manifold, in order to ensure the independance of quantization procedure from the choice of coordinates one should use the following form of the operator $\hat{h}$:

$$\hat{h} = -\frac{1}{2}\Delta(\gamma) + aR(\gamma),$$

where $\Delta(\gamma)$ and $R(\gamma)$ are the Laplace operator and scalar curvature for metric γ.

Then, in our case the demand of covariance of $\hat{H}^f[G^f]$ under general coordinate transformations in the minisuperspace leads to the following quantization prescription:

$$\hat{H}^f = -\frac{1}{2\mu}\Delta[G^f] + \frac{a^f}{\mu}R[G^f] + V^f, \tag{16}$$

where the constant a^f fixes the operator ordering ambiguity in the f-gauge.

It is natural to claim that WDW equation (3.1.15) has gauge-covariant form, i.e. the WDW equations (3.1.15) in f_1-and-f_2-gauges are equivalent for all f_1 and f_2. This takes place if and only if (3.1.15) is equivalent to

$$\hat{H}\Psi = 0 \tag{17}$$

for all f, where

$$\hat{H} = \hat{H}^f\ |_{f=0}, \Psi = \Psi^f\ |_{f=0}\ .$$

We call the f-gauge with $f = 0\,(\gamma = \sum_{i=1}^n N_i\beta^i)$ the harmonic-time gauge. (In this case $\Delta[g_{(0)}]\varphi = 0$, where $\varphi(t,y) = t$). It is natural to put

$$\Psi^f = exp[bf]\Psi, \quad a^f = a, \tag{18}$$

where b is a constant. Then, using (3.1.18), it is easy to prove that the equivalence of (3.1.15) and (3.1.17) takes place only when

$$b = 1 - \frac{n}{2} \tag{19}$$

and a is given by (3.1.2) ($n > 1$). For these values of a and b

$$\hat{H} = exp[-2f]exp[bf]\hat{H}\; exp[-bf].$$

The coefficients a and b are the well-known ones in the conformally covariant theory of a scalar field [51]. (Note that the $n = 1$ case is an anomalous one: there is no b, satisfying the gauge-covariance condition when $n = 1$).

So, we obtained the WDW equation in the conformally covariant (gauge covariant) form. Note that in the $n = 2$ case $a = 0$ and $\Psi^f = \Psi$ for all f: the Ψ function is gauge invariant.

We may also consider the case $n = 1$ cosmology with homogeneous scalar field. It is not difficult to show that in this case eq. (3.1.15) is equivalent to the WDW equation considered in [42], where the ambiguity parameter [39] is $p = 1$.

Let us consider the harmonic-time gauge. In this gauge the metric $G_H = G^f \mid_{f=0} = G$ is flat and the WDW equation (3.1.15) becomes

$$\left(-\frac{1}{2\mu}G^{ij}\frac{\partial}{\partial\beta^i}\frac{\partial}{\partial\beta^j} + V\right)\Psi = 0, \tag{20}$$

where G^{ij} and V are given by (3.1.14) and (3.1.7), respectively.

It is easy to show that

$$G = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i,$$

where

$$\begin{cases} z^0 = \left[\left(\sum_{j=1}^n N_j - 1\right) / \sum_{j=1}^n N_j\right]^{1/2} \sum_{j=1}^n N_j\beta^j, \\ z^i = \left[N_i / \left(\sum_{j=i}^n N_j\right)\left(\sum_{j=i+1}^n N_j\right)\right]^{1/2} \sum_{j=i+1}^n N_j(\beta^j - \beta^i), \end{cases} \tag{21}$$

$i = 1, ..., n-1$. Thus G has a pseudo-Euclidean signature, z^0 is a "time" coordinate.

The WDW equation (3.1.20) is an integrable one at in the following two cases:

1) $\theta_1 = ... = \theta_n = 0$. In coordinates (3.1.21) eq. (3.1.20) has d'Alembert's form

$$\left[-\left(\frac{\partial}{\partial z^0}\right)^2 + \sum_{i=1}^{n-1}\left(\frac{\partial}{\partial z^i}\right)^2\right]\Psi = 0. \tag{22}$$

Equation (3.1.22) describes, for example, the evolution of Bianchi type-I Universe ($n = 3,\ N_i = 1, i = 1, 2, 3$), which was studied in [42].

2)$n = 2;\ \theta_1 \neq 0,\ \theta_2 = 0$. In coordinates $x = x(z(\beta))$ where $z = z(\beta)$ is given in (3.1.21) and $x = x(z)$ is defined in the following manner:

$$x^0 \pm x^1 = exp[a_\pm(z^0 \pm z^1)]/a_\pm,$$

where

$$a_\pm = (N_1 + N_2)^{-1/2}\left[(N_1 + N_2 - 1)^{1/2} \pm \left(\frac{N_2}{N_1}\right)^{1/2}\right],$$

the WDW equation (3.1.20) takes a Klein-Gordon's form

$$\left[-\left(\frac{\partial}{\partial x^0}\right)^2 + \left(\frac{\partial}{\partial x^1}\right)^2 + \mu^2\theta_1\right]\Psi = 0, \tag{23}$$

with mass $m^2 = -\mu^2\theta_1$. Here x^0 is a "time" coordinate. For $\theta_1 > 0$ we have a tachyon. Equation (3.1.23) is relevant not only in the case of the Kaluza-Klein cosmologies, but it describes, for example, the Bianchi type-III Universe [42] ($N_1 = 2,\ N_2 = 1,\ \theta_1 < 0$) and the Kantowski-Sachs Universe [41,42] ($N_1 = 2,\ N_2 = 1,\ \theta_1 > 0$).

Here we obtained the WDW equation for multidimensional cosmology [52]. The WDW equation has superspace covariant form and the demand of the covariance of the WDW equation under the gauge transformations fixes the operator ordering ambiguity parameter uniquely. The WDW equation has the simplest form in the harmonic-time gauge, because in this gauge the minisuperspace metric G_H is flat. The diagonalization of G_H shows that G_H has a pseudo-Euclidean signature. Two integrable cases were considered. If all spaces are Ricci-flat the WDW equation is reduced to the d'Alembert equation. When $n = 2$ and one of the spaces is Ricci-flat the WDW equation is reduced to the Klein-Gordon one. The solution of these equations may be found anywhere.

Let us introduce the scalar field into this model: $\varphi = \varphi(t)$.

Then, the total action will be $\overline{S} = S + S_\varphi$ with

$$S_\varphi = \int dx \mid g \mid^{1/2} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - u(\varphi) \right], \tag{24}$$

where $u(\varphi)$ is a potential of a φ-field.

Putting (3.1.3) in (3.1.24) we obtain $S_\varphi = \int dt\ L_\varphi$ with

$$L_\varphi = \frac{1}{2}\kappa^2 \mu\ exp\left(-\gamma + \sum_{\nu=0}^{n} N_\nu \beta^\nu\right) \dot{\varphi}^2 - \kappa^2 \mu u(\varphi) exp\left(\gamma + \sum_{\nu=0}^{n} N_\nu \beta^\nu\right). \tag{25}$$

Combining (3.1.5) and (3.1.25) we get

$$\bar{L} = L + L_\varphi = \mathcal{N}\left[\frac{1}{2}\mu\mathcal{N}^{-2}(G_{\mu\nu}\ \dot{\beta}^\mu \dot{\beta}^\nu + \kappa^2\ \dot{\varphi}^2) - \left(V + \mu\kappa^2 u\ exp\left\{2\sum_{\nu=0}^{n} N_\nu \beta^\nu\right\}\right)\right]$$

with lapse function $\mathcal{N}$ defined earlier. So, we see that introduction of a scalar field changes the minisuperspace, its metric and a scalar field potential in such a way:

$$R^{n+1} \to R^{n+2}, \quad G \to \bar{G} = G + \kappa^2 d\varphi \otimes d\varphi,$$
$$V \to \bar{V} = V + \kappa^2 \mu u(\varphi) exp\left(2\sum_{\nu=0}^{n} N_\nu \beta^\nu\right).$$

Now, if we take $\bar{f} = \bar{f}(\beta, \varphi)$ on a minisuperspace R^{n+2} we obtain in $\bar{f}$-gauge:

$$\hat{\bar{H}}^{\bar{f}} \psi^{\bar{f}} = 0,$$

where $\hat{\bar{H}}^{\bar{f}}$ is obtained from $\hat{H}^f$ substituting $f \to \bar{f}$ and $n \to n+1$. In harmonic ($\bar{f} = 0$) gauge we have:

$$\left[-\frac{1}{2\mu}\left(G^{\mu\nu}\frac{\partial}{\partial\beta^\mu}\frac{\partial}{\partial\beta^\nu} + \frac{1}{\kappa^2}\frac{\partial^2}{\partial\varphi^2}\right) + \bar{V}\right]\Psi = 0$$

For all $\theta_i = 0$, $u = 0$ (minimally coupled scalars field) this Eq. reduces to the d'Alembert equation in coordinates $(z^\mu, \kappa\varphi)$. For $n = 0$, $u = 0$, $\theta_0 \neq 0$, it leads to a Klein-Gordon form:

$$\left[-\left(\frac{\partial}{\partial y^0}\right)^2 + \left(\frac{\partial}{\partial y^1}\right)^2 + \mu^2\theta_0\right]\Psi = 0$$

with $y^0 \pm y^1 = [N_0/(N_0-1)]^{1/2} exp[\beta^0(N_0 - 1) \pm \kappa\varphi]$, $y^0 > | y^1 |$.

Here we present the explicit form of the solution of WDW-Eq. for $\theta_1 \neq 0$, $\theta_i = 0$, $i = 2, ..., n$ and a scalar field as a source of Einsteins eqs. [53].

The WDW equation in the harmonic time gauge $\gamma = \sum_{i=1}^n N_i\beta^i$ and the minisuperspace metric G takes the form:

$$-2\mu\hat{H}\Psi = \left(-\frac{\partial^2}{\partial\nu^{0^2}} + \frac{\partial^2}{\partial\nu^{1^2}} + ... + \frac{\partial^2}{\partial\nu^{n-1^2}} + \frac{\partial^2}{\partial\varphi^2} + \mu^2\theta_1 e^{2q\nu^0}\right)\Psi = 0 \qquad (26)$$

where $\varphi \equiv \kappa\,\tilde{\varphi}$ and $\tilde{\varphi}$ is a minimally coupled massless scalar field, $\mu = \prod_{i=1}^n V_i/\kappa^2$, V_i is the volume of M_i, κ^2 is the gravitational constant (without losing generality, we can put $\mu = 1$),

$$q^2 = (N_1 - 1)/N_1$$

and we use the coordinate transformation

$$q\nu^0 = (N_1 - 1)\beta^1 + \sum_{i=2}^n N_i\beta^i,$$

$$q\nu^1 = \left(\frac{\sum_{j=1}^n N_j - 1}{N_1 \sum_{j=2}^n N_j}\right)^{1/2} \sum_{j=2}^n N_j \beta^j,$$
$$q\nu^i = \left(\frac{(N_1 - 1)N_i}{N_1 \sum_{j=1}^n N_j \sum_{j=i+1}^n N_j}\right)^{1/2} \sum_{j=i+1}^n N_j(\beta^j - \beta^i) \qquad i = 2, ..., n-1.$$

It was shown in [52] that equation (3.1.26) is covariant under the coordinate transformation in the minisuperspace and conformal covariant under the conformal transformation

$$G \to G^f = e^{2q\nu^0} G$$
$$\Psi \to \Psi^f = e^{(1-n/2)q\nu^0} \Psi$$
$$U \to U^f = e^{-2q\nu^0} U = -\frac{1}{2}\theta_1 \mu$$

The conformal minisuperspace metric G^f, is the Milne-type one

$$d\tilde{s}^2 = e^{2q\nu^0}[-(d\nu^0)^2 + (d\nu^1)^2 + ... + (d\nu^n)^2] = -(d\tilde{t})^2 + q^2\tilde{t}^2 \sum_{i=1}^n (d\tilde{x}^i)^2 \qquad (27)$$

where

$$\varphi \equiv \nu^n \quad \tilde{t} = q^{-1} e^{q\nu^0} \quad \tilde{x}^i = \nu^i \quad i = 1, ..., n.$$

The equation for the scalar field ϕ with the mass $\mu(-\theta_1)^{1/2}$ in the Milne universe (3.1.27) coincides with the WDW equation (3.1.15) and reduces to equation (3.1.26) after the conformal transformation $\phi = [exp(1 - n/2)q\nu^0]\Psi$. One may use this fact to investigate the WDW equation since the theory of the scalar field in the Milne universe is well known. Solutions of the WDW equation (3.1.26) are obtained by the separation of variables

$$\Psi = \Psi_0(\nu^0)...\Psi_{n-1}(\nu^{n-1})\Psi_n(\varphi) \qquad (28)$$

where

$$\Psi_i(\nu^i) = e^{iu_1\nu^i} \qquad i = 1, ..., n-1$$
$$\Psi_n(\varphi) = e^{iu_n\varphi}$$

and Ψ_0 satisfy the equation

$$\left(-\frac{d^2}{d\nu^{02}} + \theta_1 e^{2q\nu^0}\right)\Psi_0 = \varepsilon\Psi_0 \tag{29}$$

where ε and arbitrary numbers v_i are related to each other

$$\varepsilon = \sum_{i=1}^{n} v_i^n.$$

The solutions of the equation (3.1.29) are

$$C_{i\sqrt{\varepsilon}/q}\left(\frac{\sqrt{|\theta_1|}}{q}e^{q\nu^0}\right)$$

where C is the modified Bessel function I or K in the case $\theta_1 > 0$, or the Bessel function of the first, second or third kind in the case $\theta_1 < 0$.

The solutions (3.1.28) are the eigenstates of the quantum-mechanical operators $\prod_{\nu^i} = -(i/\ell)\partial/\partial\nu^i$, $i = 1, ..., n-1$; $\prod_{\nu^n} \equiv \prod_{\varphi} = -(i/\ell)\partial/\partial\varphi$ with eigenvalues $(1/\ell)\nu_n$, where $\ell = 1$ for the Lorentzian spacetime region and $\ell = i$ for the Euclidean one.

2 Exact Solutions for Models with Cosmological Constant

1.In this and the next section we investigate several models of classical and quantum multidimensional cosmology with the aim of finding exact solutions and their applications to main problems of cosmology. We start from the study of the cosmological constant role in multidimensional scheme and find classical and quantum solutions of the wormhole and tunnelling types. The scalar field generalization of the solutions is also obtained.

The quantum wormholes were defined by Hawking and Page [54] as solutions of the Wheeler-DeWitt (WDW) equation with boundary conditions: (i) the wave function is exponentially damped for large spacial geometries; (ii) the wave function is regular when the spatial geometry degenerates.

The given approach may be considered as a quantum extension of the classical wormhole paradigma (see, for example, [55-58]). We remind that classical wormholes usually are euclidean metrics that consist of two large regions joined by a narrow throat (handle). They exist for special types of matter [55,59-64]. Macroscopic wormholes may ensure the evaporation of BH and the microscopic ones may be used in solving the problem of the cosmological constant.

In [65,66] the quantum wormhole solutions were obtained for the cosmological model with n ($n > 1$) spaces of constant curvature, when one of them has a non-zero (positive) curvature and the space-time is minimally coupled with a massless scalar field. We note, that when the scalar field is absent, the WDW-equation for this model was proposed in [52].

Here, we first consider the cosmologial model with n ($n > 1$) Ricci-flat spaces and non-zero cosmological constant Λ. For $\Lambda < 0$ we found a family of quantum wormhole solutions with a continuous spectrum similar to the approach used in [67] and also in [65-66]. Solutions of the WDW equation in four dimensions with $\Lambda \neq 0$ and conformal scalar field as well were first found in [38] and [37] correspondingly (see also [1], where the solution with a minimally coupled scalar field is also described). They also satisfy conditions [54] for quantum wormholes. Formally even DeWitt's solution with dust has a quantum wormhole behaviour though it is rather questionable to apply solution with dust at small scales.

2. The model. We consider the cosmological model with the metric

$$g = -exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} exp[2x^i(t)]g^{(i)}, \tag{1}$$

on the manifold

$$M = R \times M_1 \times \ldots \times M_n, \tag{2}$$

where the manifold M_i with the metric $g^{(i)}$ is a Ricci-flat space of dimension N_i, i.e.

$$R_{m_i n_i}[g^{(i)}] = 0, \tag{3}$$

$i = 1, \ldots, n$; $n \geq 2$. We put

$$\gamma = h \equiv \sum_{i=1}^{n} N_i x^i \tag{4}$$

in (3.2.1) (harmonic time is used). Using (3.2.3) and (3.2.4), we get the following non-zero components of the Ricci-tensor for the metric (3.2.1)

$$R_{00} = -\sum_{i=1}^{n} N_i [\ddot{x}^i - \dot{h}\dot{x}^i + (\dot{x}^i)^2], \tag{5}$$

$$R_{m_i n_i} = g^{(i)}_{m_i n_i} \ddot{x}^i exp(2x^i - 2h), \tag{6}$$

$i = 1, \ldots, n$.

The action of the model is

$$S = \frac{1}{2\kappa^2} \int d^D x |g|^{\frac{1}{2}} (R - 2\Lambda) + S_{GH}, \tag{7}$$

where κ^2 is the fundamental gravitational constant, Λ is the cosmological constant, $D = 1 + \sum_{i=1}^{n} N_i$ is dimension of M and S_{GH} is the standard Gibbons-Hawking boundary term [50]. It follows from (3.2.5), (3.2.6) that Einstein equations (corresponding to the action (3.2.7))

$$R_{MN} - \frac{1}{2} R g_{MN} = -\Lambda g_{MN} \tag{8}$$

for the metric (3.2.1) with γ from (3.2.4) are equivalent to the following set of equations

$$E \equiv \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j + V = 0, \tag{9}$$

$$u^i \equiv \ddot{x}^i - \frac{2\Lambda}{D-2} exp(2h) = 0, \tag{10}$$

$i = 1, \ldots, n$. In (3.2.9)

$$G_{ij} = N_i \delta_{ij} - N_i N_j \tag{11}$$

are the components of the minisuperspace metric,

$$V = \Lambda exp(2h) \tag{12}$$

is the potential. (We note, that $R_{00} - \frac{1}{2} R g_{00} + \Lambda g_{00} = -E$.) Equations (3.2.10) are equivalent to the Lagrange equations for the Lagrangian

$$L \equiv \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - V = 0, \tag{13}$$

This equivalence follows from the relations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = G_{ij} u^j, \tag{14}$$

$i = 1, \ldots, n$, and non-degeneracy of the minisuperspace metric [30,52].

Equations (3.2.9), (3.2.10) are easily solved [68,69]. After an appropriate redefinition of the time variable the metric (3.2.1) may be represented in the following form

$$g = -d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{15}$$

$$a_i(\tau) = A_i [\sinh(\sqrt{\varepsilon} \frac{\tau}{T}) / \sqrt{\varepsilon}]^{\nu} [\tanh(\frac{\tau \sqrt{\varepsilon}}{2T}) / \sqrt{\varepsilon}]^{\alpha_i}, \tag{16}$$

where $\nu = (D-1)^{-1}$, $A_i \neq 0$ are constants, $\varepsilon = \Lambda / |\Lambda| = \pm 1$,

$$T = [(D-2)/2|\Lambda|(D-1)]^{1/2}, \tag{17}$$

and the parameters α_i satisfy the relations

$$\sum_{i=1}^{n} N_i \alpha_i = 0, \sum_{i=1}^{n} N_i (\alpha_i)^2 = 1 - \nu. \tag{18}$$

Remark 1. In [30] Einstein equations

$$R_{MN} - \frac{1}{2} R g_{MN} = \kappa^2 T_{MN} \tag{19}$$

for the metric (3.2.1) were integrated even for the "perfect-fluid" matter, when pressures in all spaces are proportional to the density : $p_i = (1 - h_i)\rho$, $h_i = const$, $i = 1, \ldots, n$, $\rho > 0$. We note that in the case $h_i = 2, p_i = -\rho$ the solution from [30] coinsides with the solution (3.2.15)-(3.2.18) with the relations $\Lambda = \kappa^2 \rho > 0$ and $\rho = A = const$ imposed.

3. The WDW equation. The WDW equation for the model in harmonic time gauge (3.2.4) reads as follows:

$$(\frac{1}{2}G^{ij}\partial_i\partial_j - \mu^2 V)\Psi = 0, \tag{20}$$

where $\Psi = \Psi(x)$ is a wave function of the Universe, V is the potential (3.2.12) and $\partial_i = \partial/\partial x^i$. In (3.2.20)

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{21}$$

are the components of the matrix inverse to the matrix (G_{ij}) (3.2.11) and $\mu = \prod_{i=1}^{n} V_i/\kappa^2, V_i$ is the volume of M_i. (We suppose that all $V_i < +\infty$; in other cases the parameter μ should be introduced "by hand" as a parameter of the theory.)

The WDW equation (3.2.20) can be easily deduced by a procedure similar to that of Ref. [52]. It is also in an agreement with a general scheme of [49]. Without a loss of generality we put $\mu = 1$ below. The minisuperspace metric $G = G_{ij}dx^i \otimes dx^i$ (3.2.11) was diagonalized in [30] (see also [52])

$$G = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{22}$$

where

$$z^0 = q^{-1}\sum_{j=1}^{n} N_j x^j, \tag{23}$$

$$z^i = [N_i/\Sigma_i\Sigma_{i+1}]^{1/2} \sum_{j=i+1}^{n} N_j(x^j - x^i), \tag{24}$$

$i = 1, \ldots, n-1$, where

$$q = [(D-1)/(D-2)]^{1/2}, \Sigma_i = \sum_{j=i}^{n} N_j \tag{25}$$

(we remind that $D = 1 + \sum_{i=1}^{n} N_i$).

The WDW equation (3.2.20) ($\mu = 1$) in variables (3.2.23), (3.2.24) takes the following form

$$[-\frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} + \sum_{i=1}^{n-1} \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} - 2\Lambda \exp(2qz^0)]\Psi = 0. \tag{26}$$

We are seeking the solution of (3.2.26) in the form

$$\Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^0), \tag{27}$$

where $\vec{p} = (p_1, \ldots, p_{n-1})$ is a constant vector (generally from C^{n-1}), $\vec{z} = (z^1, \ldots, z^{n-1})$, $\vec{p}\vec{z} : p_i z^i$. The substitution of (3.2.27) into (3.2.26) gives

$$[-(\frac{\partial}{\partial z^0})^2 - 2\Lambda \exp(2qz^0)]\Phi = E\Phi \tag{28}$$

where $E = \vec{p}\vec{p} = \sum_{i=1}^{n-1} p_i^2$. Solving (3.2.28), we get

$$\Phi(z^0) = B_\nu(\sqrt{-2\Lambda}q^{-1}e^{qz^0}), \tag{29}$$

where $\nu = i\sqrt{E}/q = i|\vec{p}|/q$, and $B_\nu = I_\nu, K_\nu$ is modified Bessel function. We note, that

$$v = \exp qz^0 = \prod_{i=1}^{n} a_i^{N_i} \tag{30}$$

is the volume scale factor ($a_i = e^{x^i}$).

The general solution of Eq. (3.2.26) has the following form

$$\Psi(z) = \sum_{B=I,K} \int d^3\vec{p} C_B(\vec{p}) e^{i\vec{p}\vec{z}} B_{i|\vec{p}|/q}(\sqrt{-2\Lambda}q^{-1}e^{qz^0}), \tag{31}$$

where functions C_B ($B = I, K$) belong to an appropriate class.

4. Quantum wormholes[68-69]. We restrict our consideration by real values of p_i. In this case $E \geq 0$.

If $\Lambda > 0$ the wave function Ψ (3.2.27) is not exponentially damped, when $v \to \infty$, i.e. the condition (i) is not satisfied. It oscillates and may be interpreted as corresponding to the classical Lorentzian solution.

For $\Lambda < 0$, the wave function (3.2.27) is exponentially damped for large v only, when $B = K$ in (3.2.29). But in this case the function Φ oscillates an infinite number of times, when $v \to 0$. So, the condition (ii) is not satisfied. The wave function describes the transition between Lorentzian and Euclidean regions. (If $E < 0$, we have an analogous transition for $\Lambda > 0$ and the Euclidean region for $\Lambda < 0$.)

The functions

$$\Psi_{\vec{p}}(z) = e^{i\vec{p}\vec{z}} K_{i|\vec{p}|/q}(\sqrt{-2\Lambda}q^{-1}e^{qz^0}), \tag{32}$$

may be used for constructing the quantum wormhole solution. Like in [65] we consider the superpositions of singular solutions

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk \Psi_{qk\vec{n}}(z) e^{-ik\lambda}, \tag{33}$$

where $\lambda \in R$ and $\vec{n}$ is unit vector: $(\vec{n})^2 = 1$ ($\vec{n} \in S^{n-1}$). The calculation gives

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = exp[-\frac{\sqrt{-2\Lambda}}{q}e^{qz^0}\cosh(\lambda - q\vec{z}\vec{n})]. \tag{34}$$

It is not difficult to verify that the formula (34) leads to solutions of the WDW equation (3.2.26), satisfying the quantum wormholes boundary conditions.

These results can be easily generalized, when a massless scalar field minimally coupled to gravity is included. In this case the action (3.2.7) is modified by the substitution $S \mapsto S + S_{\varphi}$, where

$$S_{\varphi} = \int d^D x |g|^{1/2}[-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi]. \tag{35}$$

Then, the minisupermetric (3.2.22) of the model is changed:

$$G \mapsto G + \kappa^2 d\varphi \otimes d\varphi. \tag{36}$$

If we define $z^n = \kappa\varphi$, then all formulas of this section are valid with the substitution $n \mapsto n+1$.

Remark 2. We also note that the the functions

$$\Psi_{m,\vec{n}} = H_m(x^0)H_m(x^1)\exp[-\frac{(x^0)^2+(x^1)^2}{2}] \tag{37}$$

where

$$x^0 = (2/q)^{1/2}(-2\Lambda)^{1/4}\exp(qz^0/2)\cosh(\frac{1}{2}q\vec{z}\vec{n}),$$
$$x^1 = (2/q)^{1/2}(-2\Lambda)^{1/4}\exp(qz^0/2)\sinh(\frac{1}{2}q\vec{z}\vec{n}),$$

$m = 0, 1, \ldots$, are also the solutions of the WDW equation with the quantum wormhole boundary conditions. Solutions of such type were previously considered in [54,65,66]. (They are called discrete spectrum quantum wormholes.)

2.1 Model with a Perfect Fluid [70]

Now we consider another cosmological model with the metric (3.2.1) on the manifold (3.2.2), but in this case metrics $g^{(i)}$ are Einstein spaces of constant curvature.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i \; g^{(i)}m_i n_i \qquad i = 1, ..., n; \quad n \geq 2. \tag{1}$$

stress-energy tensor is taken in the form:

$$T_N^M = \sum_{\alpha=1}^{m} T_N^{M(\alpha)}, \tag{2}$$

where

$$T_N^{M(\alpha)} = diag(-\rho^{(\alpha)}(t), p_1^{(\alpha)}(t)\delta_{k_1}^{\ell_1}, ..., p_n^{(\alpha)}(t)\delta_{k_n}^{\ell_n}, \tag{3}$$

$\alpha = 1, ..., m$ and

$$\nabla_M T_N^{M(\alpha)} = 0 \tag{4}$$

with $\alpha = 1, ..., m-1$ (when $m = 1$ relation (3.3.4) is absent).

So, the material content of the model is a multicomponent perfect fluid with a conserved stress-energy tensor of each component. We remark that (3.3.4) is also valid when $\alpha = m$. It follows from $\nabla_m T_N^m = 0$ due to multidimensional Einstein Equations:

$$R_N^m - \frac{1}{2}\delta_N^M R = \kappa^2 T_N^M. \tag{5}$$

Equations of state for each component are taken as

$$p_i^\alpha(t) = [1 - h_i^\alpha(\chi(t))]\rho^\alpha(t), \tag{6}$$

with

$$h_i^{(\alpha)}(\chi) = \frac{1}{N_i}\frac{\partial}{\partial \chi_i}\phi^{(\alpha)}(\chi), \tag{7}$$

$i = 1, ..., n$, functions $\phi^{(\alpha)}(\chi)$ are smooth,

$$\alpha = 1, ..., m. \tag{8}$$

Now, non null components of Ricci-tensor for metric (2.1) are the following:

$$R_{00} = -\sum_{i=1}^{n} N_i[\ddot{\chi}^i - \dot{\gamma}\dot{\chi}^i + (\dot{\chi}^i)^2], \tag{9}$$

$$R_{m_i n_i} = g_{m_i n_i}^{(i)}[\lambda^i + (\ddot{\chi}^i + \dot{\chi}\,(\sum_{i=1}^{n} N_i \chi^i - \dot{\gamma}))exp(2\chi^i - 2\gamma) \tag{10}$$

$i = 1, ..., n.$

Here we also use the harmonic time gauge with $\gamma = \gamma_0$.

Using (3.2.4) and (3.3.9-10) Einstein equations for metric (3.2.1) and stress-tensor (3.2.6-7) are equivalent to the following system:

$$\frac{1}{2} G_{ij} \dot{\chi}^i \dot{\chi}^j + V = 0, \quad (11.a)$$

$$\lambda^i + \ddot{\chi}^i exp(2\chi^i - 2\gamma_0) =$$

$$\kappa^2 \; exp \; 2\chi^i \cdot \sum_{\alpha=1}^{m} [p_i^{(\alpha)} + (\mathcal{D}-2)^{-1}(p^{(\alpha)} - \sum_{i=1}^{n} N_i p_i^{(\alpha)})], \quad (11.b)$$

$$i = -1, ..., n.$$

Here $\mathcal{D} = 1 + \sum_{i=1}^{n} N_i$ is the total dimension,

$$G_{ij} = N_i \delta_{ij} - N_i N_j$$

are components of a supermetric and

$$V = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \; exp(-2\chi^i + 2\gamma_0) + \kappa^2 \sum_{\alpha=1}^{m} \rho^{(\alpha)} exp(2\gamma_0)$$

is the potential.

Relations (3.3.4) may be written in the form

$$\dot{\rho}^{(\alpha)} + \sum_{i=1}^{n} N_i \; \dot{\chi}^i \; (\rho^{(\alpha)} + p_i^{(\alpha)}) = 0 \quad (12)$$

for $\alpha \in \{1, ..., m\}$ and due to (3.3.6)-(3.3.7) are easily integrated:

$$\rho^{(\alpha)}(t) = A_\alpha exp[-2N_i \chi^i(t) + \phi^{(\alpha)}(\chi(t))], \quad (13)$$

where $A_\alpha = const.$.

Using (3.3.13) it is easy to see that Eqs. (3.3.11b) are the Lagrange Eqs. corresponding to the Lagrangian

$$L = \frac{1}{2} G_{ij} \dot{\chi}^i \dot{\chi}^j - V \tag{14}$$

where the potential is of the form

$$V = V(\chi) = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \, exp[-2\chi^i + 2\gamma_0(\chi)] + \kappa^2 \sum_{\alpha=1}^{n} A_\alpha exp(\phi^{(\alpha)}(\chi)). \tag{15}$$

Eq. (3.3.11a) plays the role of the constraint:

$$E = \frac{1}{2} G_{ij} \dot{\chi}^i \dot{\chi}^j + V = 0 \tag{16}$$

We see that as usual the energy is equal to zero. In quantum approach the multidimensional Wheeler-DeWitt equation in the harmonic gauge is as in [52]:

$$\frac{1}{2}(G^{ij}\partial_i\partial_j + \mu^2 V)\Psi = 0, \tag{17}$$

where

$$G^{ij} = \frac{\delta_{ij}}{N_i} + \frac{1}{2 - \mathcal{D}}. \tag{18}$$

and μ^2 is a dimensional parameter.

For simplicity let us consider a particular case

$$\lambda_i = 0, \quad i = 1, ..., n; \quad m = 1, h_i^{(1)}(\chi) = h_i = const.$$

We introduce the following notations:

$$u_i = N_i h_i, \quad u^i = G^{ij} u_j, \tag{19}$$

$$u^2 = G^{ij} u_i u_j = \sum_{i=1}^{n} N_i (h_i)^2 + \frac{1}{2 - \mathcal{D}} (\sum_{i=1}^{n} N_i h_i)^2 \tag{20}$$

Let $u^2 < 0$.

Remark 1. In isotropic case $h_i = h : u^2 = h^2\frac{D-1}{D-2} \leq 0$. When $h = 0$ we have $p_i = \rho$ that is a stiff equation of state. For $h = 1$ we get $p_i = 0$, i.e. dust; for $h = 2$ one obtains $p_i = -\rho$ - the case of the cosmological constant.

The minisuperspace metric $G_{ij}dx \otimes dx^i$ is diagonalized by the linear transformation

$$z^a = v_i^a x^i, \tag{21}$$

So,

$$\eta_{ab}v_i^a v_j^b; = G_{ij} \quad \text{with} \quad \eta^{ab} = diag(-1, 1, ..., 1) \tag{22}$$

It is easy to check that for $u^2 < 0$ there exist a matrix (V_i^a) satisfying (3.3.22) which has the form

$$v_i^0 = u_i/\sqrt{-u^2}. \tag{23}$$

Then in z-variables the Wheeler-DeWitt Eq. transforms to

$$[\Box_z - 2\bar{A}exp(2qz^0)]\Psi = 0, \tag{24}$$

where

$$\bar{A} = (\kappa\mu)^2 A_1,\ 2q = \sqrt{-u^2},\ \Box_z = \eta^{ab}\frac{\partial}{\partial z^a}\frac{\partial}{\partial z^b}, \tag{25}$$

We search the solution of (3.3.24) as

$$\Psi(z) = exp(i\bar{p}\bar{z})\phi(z^o) \tag{26}$$

Here $\bar{p} = (p_1, ..., p_n)$, $\bar{z} = (z^1, ..., z^{n-1})$, $\bar{p}\bar{z} = \sum_{i=1}^{n-1} p_i z^i$.

Substituting (3.3.26) into (3.3.24) we get

$$\left[-(\frac{\partial}{\partial z^0})^2 - 2\kappa^2 A\ exp(2qz^0)\right]\phi = 2\varepsilon\phi \tag{27}$$

where $\varepsilon = \sum_{i=1}^{n-1} p_i^2$.

Its solution is

$$\phi(z^0) = B_\nu(\sqrt{-2A}q^{-1}e^{qz^0}),$$

$\nu = i\sqrt{2\varepsilon}/q = i \mid \bar{p} \mid /q; B_\nu = I_\nu, K_\nu$ are modified Bessel functions.

General solution of the WDW-equation has the form

$$\Psi(z) = \sum_{B=K,I} \int d^{D-1}pC_B(p)e^{i\bar{p}\bar{z}}B_{i\nu(p)}(\sqrt{-2A}q^{-1}e^{qz^0}) \qquad (28)$$

As in the previous case of Λ-term we also may single out solutions of the quantum wormholes type:

$$\Psi_{\lambda,\bar{n}} = exp\left[-\frac{\sqrt{-2A}}{2}e^{qz^0}ch(\lambda - q\bar{z}\bar{n})\right], \qquad (29)$$

where $\bar{n} \in s^{n-1}$

Note: In a classical case the corresponding solution has the form (see also [39]):

$$g = -(\prod_{i=1}^{n}(a_i(\tau))^{2N_i-u_i})d\tau \oplus d\tau + \sum_{i=1}^{n} a_i^2(\tau)g^{(i)},$$

where

$$a_i(\tau) = A_i\left(\frac{sh(\tau r/T)}{r}\right)^{\beta_i}\left(\frac{th(\tau r/2T)}{r}\right)^{\alpha^i},$$

with

$$r = \sqrt{A/ \mid A \mid} = \sqrt{\pm 1}, \quad \beta^i = 2u^i/u^2$$

and parameters α^i satisfy relations:

$$\sum_{i=1}^{n} u_i \alpha^i = 0,$$
$$\sum_{i,j=1}^{n} G_{ij} \alpha^i \alpha^j = -\frac{4}{u^2}. \qquad (30)$$

Now we change from the multidimensional cosmological solutions to multidimensional spherically symmetric ones.

Chapter 14

Classical and Quantum Spherically-Symmetrical Solutions in Multidimensional Gravitation

In previous chapters we studied mainly cosmological solutions of multidimensional models. Their basic feature was the prediction of variations of the effective gravitational constant with time. Only in very particular cases it is possible to have $G = const$.

Here we give extentions of some spherically-symmetric solutions of GR to the multidimensional case and see that extra dimensions lead to cardinal physical effects – to deviations from the Newton and Coulomb laws, to variations of the effective gravitational constant with range [72,73].

Scalar and electromagnetic fields and also their interaction will be studied within these models. We analyse the stability properties of obtained exact solutions and show that only multidimensional BH solutions are stable.

Quantum analogues of these solutions are also obtained, wormhole solutions are singled out. So, we shall see that there are several manifestations of extra dimensions properties which in principle may be tested in our 4-dimensional space-time.

1 Generalized Schwarzschild Solution in Multidimensional Gravitation

Similar methods used in the previous section may be used also to obtain exact solutions in a spherically symmetrical case when all the internal spaces are Ricci-flat.

So, the problem is to find solutions for the metric of the form

$$g = -e^{2\gamma(u)} dt \otimes dt + e^{2\alpha(u)} dR \otimes dR + + e^{2\beta_0(u)} d\Omega^2 + \sum_{i=1}^{n} e^{2\beta_i(u)} g_{(i)} \tag{1}$$

on the manifold

$$M = R \times R \times S^2 \times M_1 \times \cdots \times M_n, \tag{2}$$

satisfying vacuum Einstein Eqs., where M_i are Ricci-flat manifolds of dimension N_i with metrics $g_{(i)}$, $i = 1, ..., n, d\Omega^2$ is a canonical metric on S^2, u is a radial-type variable connected with r by the relation $r = e^{\beta_0(u)}$. Denote $\gamma = \beta_{-1}$, $N_{-1} = 1$, $N_0 = 2$. Let $\alpha = \alpha_0 \equiv \sum_{\nu=-1}^{n} \beta_\nu N_\nu$ (u is a harmonic radial variable). Then Einstein Eqs. $R_{MN} = 0$ will be ($A' \equiv \frac{d}{du} A$):

$$\begin{aligned} &\sum_{\nu=-1}^{n} [-\beta_\nu'' + \alpha_0' \beta_\nu' - (\beta_\nu'^2] N_\nu = 0, \\ &\beta_i'' = 0, \quad i = -1, 1, \cdots, n, \\ &\beta_0'' = e^{2\alpha_0 - 2\beta_0}. \end{aligned} \tag{3}$$

Solving (4.1.3) in variables $x = \beta_0 - \alpha_0$ we get:

$$\begin{aligned} &\beta_i = A_i \overline{u} + D_i, \quad i = -1, 1, \cdots, n. \\ &\beta_0 = -\ell n\ f - \sum_{\nu \neq 0} (A_\nu \overline{u} + D_\nu) N_\nu, \\ &\alpha_0 = -2\ell n\ f - \sum_{\nu \neq 0} (A_\nu \overline{u} + D_\nu) N_\nu, \end{aligned} \tag{4}$$

where

$$f = f(\overline{u}, B) = \begin{cases} \frac{sh(\sqrt{B}\overline{u})}{\sqrt{B}}, & B > 0 \\ \overline{u}, & B = 0. \end{cases} \tag{5}$$

In (4.1.4) $\overline{u} = \varepsilon(u + u_0), \varepsilon = \pm 1, u_0, A_i, D_i$ are arbitrary constants, $i = -1, 1, \cdots, n$. B is defined by

$$2B = \left(\sum_{\nu \neq 0} A_\nu N_\nu\right)^2 + \sum_{\nu \neq 0} N_\nu A_\nu^2 \tag{6}$$

($\sum\limits_{\nu \neq 0}$ means summation over $\nu : \nu = -1, 1, \cdots, n$). If we redenote constants

$$\begin{aligned} &c_i = e^{2D_i}, \quad a_i\sqrt{B} = -A_i, \quad i = 1, \cdots, n; \\ &c = e^{D_{-1}}, \quad a\sqrt{B} = -A_{-1}, \\ &L = 2\sqrt{B}\left(-\sum_{\nu \neq 0} D_\nu N_\nu\right) \end{aligned} \tag{7}$$

and introduce a new variable R:

$$R = e^{-\sum_{\nu \neq 0} D_\nu N_\nu} \times \begin{cases} \frac{2\sqrt{B}}{1 - e^{-2\overline{u}\sqrt{B}}}, & B > 0 \\ 1/\overline{u}, & B = 0 \end{cases} \tag{8}$$

then (4.1.1) and (4.1.4) will give more familiar form of a spherically symmetric metric:

$$\begin{aligned} g = &-c^2 dt \otimes dt \left(1 - \frac{L}{R}\right)^a + dR \otimes dR \left(1 - \frac{L}{R}\right)^{-a - \sum_{i=1}^n a_i N_i} + \\ &+ d\Omega^2 R^2 \left(1 - \frac{L}{R}\right)^{1 - a - \sum_{i=1}^n a_i N_i} + \sum_{i=1}^n c_i g_{(i)} \left(1 - \frac{L}{R}\right)^{a_i}, \end{aligned} \tag{9}$$

$R > L$, where constants $L \geq 0, c, c_1, \cdots, c_n > 0$ are arbitrary and $a, a_1, \cdots, a_n$ obey the relation:

$$\left(a+\sum_{i=1}^{n} a_i N_i\right)^2 + a^2 + \sum_{i=1}^{n} a_i^2 N_i = 2. \tag{10}$$

solution (4.1.10) for $n = 1$ was considered earlier in [74,75]. When $L = 0$ solution (4.1.10) is trivial: 4-section is flat and $g_{(i)}$ are constant. For $L > 0$ and

$$a - 1 = a_1 = \cdots = a_n = 0 \tag{11}$$

this solution is a sum of Schwarzschild solution with gravitational radius L and tensor field $\sum_{i=1}^{n} c_i g_{(i)}$. If $L > 0$ then $a > 0$ corresponds to attraction and $a < 0$ describes repulsion.

Now let us study the problem of a horizon in this solution considering the 4-section of the metric. For $L > 0$ the horizon exists at $R = L$ only when (4.1.11) holds.

Really, for a light radial geodesic $ds_4^2 = 0$ we have:

$$c(t - t_0) = \int_R^{R_0} dx \left(1 - \frac{L}{R}\right)^{-a-\frac{1}{2}\sum_{i=1}^{n} a_i N_i} . \tag{12}$$

Relation (4.1.10) is equivalent to the identity:

$$\left(a + \frac{1}{2}\sum_{i=1}^{n} a_i N_i\right)^2 = 1 - \frac{1}{2}\sum_{i=1}^{n} a_i^2 N_i - \frac{1}{4}\left(\sum_{i=1}^{n} a_i N_i\right)^2 . \tag{13}$$

If not all $a_i = 0 (i = 1, \cdots, n)$ then due to (4.1.13)

$$\mid a + \frac{1}{2}\sum_{i=1}^{n} a_i N_i \mid < 1, \tag{14}$$

and so the integral (4.1.12) is convergent for $R = L$, i.e. radial light ray reaches surface $R = L$ at a finite time. If $a_1 = \cdots = a_n = 0$ then due to (4.1.10) $a = \pm 1$. When $a = 1, a_1 = \cdots = a_n$ metric g_4 coincides with the Schwarzschild solution having a horizon at $R = L$. If $a = -1, a_1 = \cdots = a_n = 0$ then integral (4.1.12) is also finite for $R = L$ and so the horizon is absent. So, $R = L$ is a horizon only when scale factors of internal spaces are constant and 4-section of the total metric coincides with the Schwarzschild solution.

Solution (4.1.9) is easily generalized when a scalar field (minimaly coupled) is taken into account.

Then the action of the model is

$$S = \frac{1}{2}\int dx \mid g \mid^{\frac{1}{2}} \left(\frac{R}{\kappa^2} - g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right). \tag{15}$$

which leads to equations of motion

$$R_{MN} = \kappa^2\partial_M\varphi\partial_N\varphi \tag{16}$$

$$\Delta\varphi = 0, \tag{17}$$

with Δ-Laplace operator for metric g.

Solution of Eq. (4.1.17) in u-coordinate is:

$$\Phi = \overline{Q}u + D, \tag{18}$$

where $\overline{Q}$ and D are constants. In r-coordinate we have

$$\phi = \frac{1}{2}q\ \ell n\left(1 - \frac{L}{R}\right) + D, \tag{19}$$

where q is a constant scalar charge, metric g is given by the same formula (4.1.9) and instead of (4.1.10) we have the following relation between constants $a, a_1, \cdots a_n, q, \kappa$:

$$\left(a + \sum_{i=1}^{n} a_i N_i\right)^2 + a^2 + \sum_{i=1}^{n} a_i^2 N_i + \kappa^2 q^2 = 2, \tag{20}$$

It is easy to prove that if the scalar field is present the horizon for $R = L$ exists only when

$$q = a_1 = \cdots = a_n = 0,\ a = 1. \tag{21}$$

So, when a scalar field is present there is no BH solutions in multidimensional gravitation.

One may get restrictions on constants, or properties of extra dimensions, if we use the postnewtonian approximation and compare it with the known data from experiments in the Solar system.

It is known that postnewtonian metric may be represented via PPN parameters as:

$$\begin{aligned} g_{oo} &= 1 - 2U + 2\beta U^2 + O(U^3), \\ g_{oi} &= 0, \\ g_{ij} &= -\delta_{ij}(1 + 2\gamma U + O(U^2)), \; i,j = 1,2,3. \end{aligned} \tag{22}$$

where β and γ are taken from classical GR tests or Viking data analysis.

In order to compare our metric with (4.1.22) we make transformations to isotropic coordinates in 4-section. Than the spacial part will be conformally-flat: $R = r(1 + L/4r)^2$.

$$\begin{aligned} ds^2_{(4)} &= A(r)dt^2 - B(r)(dt^2 + r^2 d\Omega^2), \\ A(r) &= \left(\frac{1 - L/4r}{1 + L/4r}\right)^{2a}, \; B(r) = (1 + L/4r)^4 \left(\frac{1 - L/4r}{1 + L/4r}\right)^{2-2a-2b}. \end{aligned} \tag{23}$$

$b = \sum_{i=1}^{n} a_i N_i$.

Expanding (4.1.23) into series over L/r at large r and comparing (4.1.22) with (4.1.23) we get:

$$\beta = 1, \;\; \gamma = 1 + b/a, \;\; 2Gm = aL. \tag{24}$$

Using data of [13] on γ we obtain:

$$b/a = \gamma - 1 = (-0,7 \pm 1,7) \cdot 10^{-3}. \tag{25}$$

It is seen that (4.1.25) is satisfed when $a_1, \cdots, a_n$ are rather small. For $n = 1$ we have:

$$a_1 N_1 = (-0,7 \pm 1,7) \cdot 10^{-3}, \quad a \approx 1 - \frac{1}{2} a_1 N_1. \tag{26}$$

Another generalization of the Schwarzschild solution may be obtained if one introduces presures in internal dimensions ($n = 1$ for simplicity):

$$\begin{aligned} R_M^N - \frac{1}{2}\delta_M^N R &= -\kappa^2 T_M^N, \\ T_M^N &= diag(\rho, 0, 0, 0, , -p, \cdots - p). \end{aligned} \tag{27}$$

in the metric

$$ds^2 = e^{2\gamma(u)} du^2 - e^{2\alpha(u)} du^2 - e^{2\beta(u)} d\Omega^2 - e^{2\mu(u)} ds_N^2. \tag{28}$$

As it was shown in [16] BH may exist in this case with changing internal scale factors and nonull pressures. The horizon takes place for $u \to \infty$ and is characterized by:

$$\beta \to const, \ \mu \to const, \ e^{\mu} \sim e^{-uk}, \ e^{\gamma} \sim e^{-uk}. \tag{29}$$

and constants k, m and μ_1 are connected by:

$$(-Gm + N\mu_1/2)^2 + N(N+2)\mu_1^2/4 = k^2. \tag{30}$$

m is a total mass.
at infinity ($u \to 0$):

$$e^{\beta} \sim 1/u, \ e^{\alpha} \sim 1/u^2, \ \mu \approx \mu_1 u, \ \gamma \approx -Gmu, \ u \to 0. \tag{31}$$

2 On Black Holes in Multidimensional Theory [76]

In a previous section the Schwarzschild solution was generalized for the case of n internal Ricci-flat spaces [77]. It was shown that a horizon in the four-dimensional section of the metric exists only when the internal space scale factors are constant.

Here we consider exact static, spherically symmetric solutions of the Einstein equations in $(2+d+N_1+\cdots+N_n)$-dimensional gravity ($d \geq 2$) with a chain of n Ricci-flat internal spaces. We show that as in the case $d=2$ a horizon is absent in all nontrivial cases. Finally, we consider a formal analog of the solution for the case of p-adic numbers [79].

We consider the Einstein equations

$$R_{MN} = 0 \tag{1}$$

on the D-dimensional manifold

$$M = M_0 \times M_1 \times \cdots M_n, \tag{2}$$

where

$$\dim\ M_i = N_i, \quad D = 2 + d + \sum_{j=1}^{n} N_j, \quad i > 0,$$

M_0 is $(2+d)$-dimensional space-time ($d \geq 2$) and M_i are Ricci-flat manifolds with the metrics $g_{(i)}, i = 1, \cdots n$. We seek solutions of (4.2.1) such that M_0 is static, spherically symmetric ($O(d+1)$-symmetric), while all the scale factors exp (β_i) of the internal spaces M_i depend on the radial coordinate u, i.e., the D-metric is

$$\begin{aligned} g = & -exp[2\gamma(u)]dt \otimes dt + exp[2\alpha(u)]du \otimes du \\ & + exp[2\beta(u)]d\Omega_d^2 + \sum_{i=1}^{n} exp[2\beta_i(u)]g_{(i)}, \end{aligned} \tag{3}$$

where $d\Omega_d^2 = g_{(0)}$ is the standard S^d metric.

If we denote $\gamma = \beta_{-1}, N_{-1} = 1$ and $\beta = \beta_0, N_0 = d$ and choose the harmonic radial coordinate u such that $\alpha = \sum_{k=-1}^{n} \beta_i N_i$ then the Einstein equations (4.2.1) can be written in the form

$$\begin{aligned} R_{11} &= \sum_{i=-1}^{n} (-\beta_i'' + \alpha'\beta_i' - \beta_i'^2)N_i = 0, \\ R_{00} &= exp(2\beta_{-1} - 2\alpha)\beta_1'' = 0, \\ R_{kl} &= g_{(o)kl}[d - 1 - \beta_o'' exp(2\beta_o - 2\alpha)] = 0. \\ R_{m_i n_i} &= -g_{(i)m_i n_i}\beta_i'' exp(2\beta_i - 2\alpha), \ i = 1, \cdots, n. \end{aligned} \tag{4}$$

This set of equations is easily solved, so that the metric $g_{(3)}$ after and appropriate redefinition of the radial coordinate ($u \Rightarrow R = R(u)$) may be written in the following way,

$$\begin{aligned} g = & -c^2[1-\varepsilon(L/R)^{d-1}]^a dt\otimes dt \\ & + [1-\varepsilon(L/R)^{d-1}]^{(a+b+d-2)/(1-d)} dR\otimes dR \\ & + [1-\varepsilon(L/R)^{d-1}]^{(a+b-1)/(1-d)} R^2 d\Omega_d^2 \\ & + \sum_{i=1}^{n} c_i [1-\varepsilon(L/R)^{d-1}]^{a_i} g_{(i)}, \\ \varepsilon = & \pm 1, \end{aligned} \tag{5}$$

where $L \geq 0, R > 0$ and $R > L$ for $\varepsilon = +1$; $L, c \neq 0$ and $c_i \neq 0$ are constants,

$$b = \sum_{i=1}^{n} a_i N_i, \tag{6}$$

and the constants $a, a_1, \cdots, a_n$ satisfy the relation

$$(a+b)^2 + (d-1)\left(a^2 + \sum_{i=1}^{n} a_i^2 N_i\right) = d. \tag{7}$$

In the case $d = 2$ and $\varepsilon = +1$ this solution coincides with that of ref. [77].

Let us consider the $(2+d)$-dimensional section of the metric (4.2.5). In the case $L = 0$ the metric is flat, while for $L > 0$ and

$$a - 1 = a_1 = \cdots = a_n = 0 \tag{8}$$

it coincides with the Tangherlini solution [80].

Now let us prove that a horizon at $R = L (L > 0)$ takes place only in the case (4.2.8) for $\varepsilon = +1$. Indeed, for the light propagating along a radius from a place with $R = R_0$ towards the center the coordinate time interval is

$$t - t_0 = \frac{1}{c}\int_R^{R_0} dx[1-(L/R)^{d-1}]^{\lambda}, \tag{9}$$

where,

$$\lambda = \frac{1}{2}\left(\frac{a+b+d-2}{1-d} - a\right) \tag{10}$$

Relation (4.2.7) is equivalent to the identity

$$(a+b/d)^2 = 1 - \frac{d-1}{d}\sum_{i=1}^{n} a_i^2 N_i - \frac{b^2}{d^2}(d-1). \tag{11}$$

Let us $\varepsilon = +1$. If some $a_i (i = 1, \cdots, n)$ are nonzero, then by (4.2.11) $a + b + /d < 1$ and from (4.2.10) $\lambda > -1$, hence, the integral ((4.2.9) converges at $R = L$. This means that a radial light beam reaches the surface $R = L$ in a finite time interval, i.e. it is not a horizon. When $a_i = 0$, $i = 1, \cdots, n$, then $a = \pm 1$. For the Tangherlini case $a = +1$ we have a horizon ($\lambda = -1$), and for $a = -1$ ($\lambda = 1/(d-1)$) the horizon at $R = L$ is absent. Evidently, for $\varepsilon = -1$ the horizon at $R = L$ is absent too. This completes the proof.

At present there is more interest in considering the physical models with p-adic numbers [79] instead of real ones. This interest was stimulated mainly by the pioneering works on p-adic strings [81,82]. Recently a p-adic generalization of the classical and quantum gravitational theory was defined [83] and some solutions of the Einstein equations were considered [83,84]. In this section we consider the p-adic analog of the solution (4.2.5). Let us briefly recall the definition of p-adic numbers [79,85]. Let p be a prime number. Any rational number $a \neq 0$ can be represented in the form $a = p^k m/n$, where the integer numbers m and n are not divisible by p. Then the p-adic norm is defined as follows: $| a |_p = p^{-k}$. This norm is non-Archimedean: $| a + b |_p \leq max(| a |_p, | b |_p)$. The completion of Q with this norm is the p-adic number field Q_p. Any nonzero p-adic number a $\in Q_p$ can be uniquely represented as the series

$$a = p^k(a_0 + a_1 p + a_2 p^2 + \cdots), \tag{12}$$

where $a_0 = 1, \cdots, p-1$, and $a_i = 0, \cdots, p-1$ for $i > 0$.

The definitions of derivatives, manifold and tensor analysis in the p-adic case are similar to those of the real case. The power p-adic function is defined as follows,

$$(1+x)_p^\alpha \equiv exp_p\{\alpha[log_p(1+x)]\}, \tag{13}$$

where $| x |_p < 1$ and $| \alpha |_p | x |_p < \delta_p$. Here $\delta_p = 1$ for $p \neq 2$ and $\delta_2 = \frac{1}{2}$. The definition is correct, for the functions exp_p and log_p are well defined on the discs $\{| x |_p < \delta_p\}$ and $\{| x - 1 |_p < 1\}$ respectively [79].

Let us consider the p-adic manifold

$$Q_p \times Q_p \times S^d \times M_1 \times \cdots M_n, \tag{14}$$

where $(S^d, g_{(o)})$ is a space of constant curvature

$$R^o_{ijk\ell} = g^{(o)}_{ik} g^{(o)}_{j\ell} - g^{(o)}_{i\ell} g^{(o)}_{jk}$$

and $(M_i, g_{(i)})$ are Ricci-flat manifolds.

For $R \neq 0$ and

$$|\frac{L}{R}|_p^{d-1} < \min(1, 1/ | a_i |_p), \quad i = -2, -1, \cdots, n, \tag{15}$$

with

$$a_2 = \frac{a + b + d - 2}{1 - d}, \quad a_1 = a, \quad a_0 = \frac{a + b - 1}{1 - d},$$

the metric (4.2.5) on the manifold (4.2.14) is well defined. Then the Einstein equations for the metric (4.2.5), (4.2.15) on the manifold (4.2.14) are satisfied identically, when the parameters $a, a_1, \cdots, a_n \in Q_p$ obey the restriction(4.2.7).

This can be easily checked using the identity

$$[(1 + x)^\alpha]' = \alpha(1 + x)^\alpha/(1 + x),$$

$| x |_p, | \alpha |_p | x |_p < 1$ (the verification of (4.2.1) in the p-adic case is just the same as in the real one).

In the $d = 2$ case this solution was considered earlier in ref. [84]. It was pointed out that there is an infinite number or rational solutions of (4.2.7) in this case. For example, we may consider the set [84]

$$a_1 = \frac{4k}{N_1(N_1+2)k^2+1},\ a_i = 0,\ i > 1,$$
$$a = \frac{-2N_1k \pm [k^2N_1(N_1+2)-1]}{k^2N_1(N_1+2)+1},\ k \in \mathbf{Z}$$

In the p-adic case there exist pseudo-constant functions $C = C(R)$ such that $C'(R) = 0$ but $C(R)$ is not identically constant [85]. Such functions may be used in generalization of well-known solutions of differential equations. In our case there is also a possibility for the constants $c, c_1, \cdots, c_n$ and $a, a_1, \cdots, a_n$ to be replaced by the pseudo-constants (of course, the restriction (4.2.15) should be preserved).

There is another possibility to generalize the solution (4.2.5). We may suppose that the components of the metric g_{MN} belong to some extension of Q_p. It may be the quadratic extension of Q_p or even Ω_p, which is the completion of the algebraic closure of Q_p [79]. In this case the constants in (4.2.5) may belong to the extension of Q_p.

The solution (4.2.5) can be also generalized on scalar-vacuum and electrovacuum cases. The last generalization for $d = 2$ is considered in the next section.

3 On Charged Black Hole in Multidimensional Theory [86]

Let us consider the action

$$S = \int d^Dx\sqrt{-g}\left[\frac{1}{2k^2}R - \frac{1}{4}F_{MN}F^{MN}\right], \tag{1}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is the strength of the electromagnetic field A, R is a scalar curvature of the metric $g_{MN}dx^M \otimes dx^N$ and k is gravitational constant. The field equations, corresponding to Eq. (4.3.1), are

$$\nabla_M F^{MN} = 0, \tag{2}$$

$$R_{MN} - (1/2)g_{MN}R = k^2[F_{MN}F^P_N - (1/4)g_{MN}F_{PQ}F^{PQ}]. \tag{3}$$

Let us consider the D-dimensional manifold

$$M = M_0^{(4)} \times M_1 \times \cdots \times M_n, \tag{4}$$

where $M_0^{(4)}$ is 4-dimensional space-time manifold, M_i are Ricci-flat manifolds with the metrics $g_{(i)}$ and

$$\dim M_i = N_i, \quad D = 4 + \sum_{i=1}^{n} N_i, \quad i = 1, \cdots, n.$$

We are interested in static, spherically symmetric (O(3)-symmetric) solutions of Eqs. (4.3.2) and (4.3.3) on the manifold (4.3.4), and so we consider the following ansatz for the metric

$$\begin{aligned} g = & - e^{2\beta_{-1}(u)} dt \otimes dt + e^{2\sum_{\nu=-1}^{n} N_\nu \beta_\nu(u)} du \otimes du \\ & + e^{2\beta_0(u)} d\Omega^2 + \sum_{i=1}^{n} e^{2\beta_i(u)} g_{(i)}, \end{aligned} \tag{5}$$

and for A_μ

$$A_0 = \varphi(u), \quad A_i = 0, \quad i = 1, 2, 3. \tag{6}$$

In Eq. (4.3.5), $d\Omega^2$ is the standard metric on $S^2, N_{-1} = 1, N_0 = 2$ and u is a radial variable.

From Eqs. (4.3.2), (4.3.5) and (4.3.6) we have

$$\varphi' = Q e^{2\beta_{-1}}, \tag{7}$$

where Q is a constant. Using Eqs. (4.3.5) and (4.3.7) we find that Eq. (4.3.3) is equivalent to the following system of equations

$$\beta''_{-1} = k^2 Q^2 \frac{D-3}{D-2} e^{2\beta_{-1}}, \tag{8}$$

$$\sum_{\nu=-1}^{n} \mid -\beta''_\nu + \beta'_\nu \alpha' - (\beta'_\nu)^2 \mid N_\nu = -k^2 Q^2 e^{2\beta_{-1}} \frac{D-3}{D-2}, \tag{9}$$

$$1 - e^{2\beta_0 - 2\alpha}\beta_0'' = \frac{k^2 Q^2}{D-2} e^{2\beta_0 + 2\beta_{-1} - 2\alpha}, \tag{10}$$

$$\beta_i'' = k^2 \frac{Q^2}{2-D} e^{2\beta_{-1}}, \quad i = 1, \cdots, n, \tag{11}$$

where $\alpha = \sum_{\nu=-1}^{n} N_\nu \beta_\nu$. Solving Eqs. (4.3.7)-(4.3.11), we obtain

$$\begin{aligned} g = & - f_1(u) dt \otimes dt + (f_1(u))^{1/(3-D)} f_2^2(u) e^{2\sum_{i=1}^{n} N_i(A_i u + D_i)} du \otimes du \\ & + (f_1(u))^{1/(3-D)} f_2(u) e^{-2\sum_{i=1}^{n} N_i(A_i u + D_i)} d\Omega^2 \\ & + \sum_{i=1}^{n} f_1^{1/(3-D)}(u) e^{2(A_i u + D_i)} g_{(i)}, \end{aligned} \tag{12}$$

and

$$\varphi = -\frac{1}{k^2 Q}\left(C_1 \frac{D-2}{D-3}\right)^{1/2} cth\left[\left(C_1 \frac{D-3}{D-2}\right)^{1/2}(u - u_1)\right] + \varphi_o. \tag{13}$$

In Eq. (4.3.12)

$$\begin{aligned} f_1(u) &= C_1 / k^2 Q^2 sh^2\left[\left(C_1 \frac{D-3}{D-2}\right)^{1/2}(u - u_1)\right], \\ f_2(u) &= C_2 / sh^2[\sqrt{C_2}(u - u_2)], \end{aligned}$$

where D_i, φ_o, u_1, u_2 and $Q \neq 0$ are arbitrary constants and the constants C_1, C_2 and A_i obey the following relation

$$2C_2 = C_1 + \left(\sum_{i=1}^{n} N_i A_i\right)^2 + \sum_{i=1}^{n} N_i A_i^2. \tag{14}$$

For $N_i \to 0, D \to 4$ we have the well-known Reissner-Nordstrom solution

$$\begin{aligned} g = & -(1 - L/r + k^2 Q^2/2r^2) dt \otimes dt \\ & + (1 - L/r + k^2 Q^2/2r^2)^{-1} dr \otimes dr + r^2 d\Omega^2, \end{aligned} \tag{15}$$

$$\varphi = Q/r + Const. \tag{16}$$

In our case ($D > 4$) the Coulomb law (4.3.16) is modified by the presence of internal dimensions. The dependence of the potential φ on the radial variable r, where

$$r^2 = f_1^{1/(3-D)}(u) f_2(u) e^{-2\sum_{i=1}^{n} N_i(A_i u + D_i)} d\Omega^2 \tag{17}$$

(see Eq. (4.3.12)), may be found by substitution of $u = u(r)$ from Eq. (4.3.13).

Note that the solution (4.3.12)-(4.3.13) may be generalized also on p-adic and $D = d + \sum_{i=1}^{n} N_i$ ($0(d-1)$-symmetric) cases.

4 Scalar-Electrovacuum Multidimensional Solutions

Multidimensional gravity as an approach to field unification can be traced back to the famous works of Kaluza and Klein [90,91]. Today's increased interest to this field is largely stimulated by studies in superstring theories [92] whose field-theoretical limit typically contains more than four dimensions; in such theories gravity is described with reasonable accuracy by multidimensional Einstein equations. Studies of their solutions can lead to predictions of direct observational manifestations of extra dimensions. Thus, cosmological models predict variations of the gravitational constant G, so that observational constants imply certain limits on model parameters. Another possible window to the multidimensional world is opened by analysis of local effects which could be sensitive to spatial variations of extra-dimension parameters. This section discusses some effects of this sort, in particular, those connected with electric charges of isolated bodies.

We consider exact, static, spherically symmetric solutions of the Einstein-Maxwell-scalar equations in $(4 + N_1 + \cdots + N_n)$-dimensional gravity with a chain of n Ricci-flat interal spaces [89]. Our approach differs from that adopted in some papers on multidimensional black holes in that any solutions, not only black-hole ones, are sought. Consequently, the place of black holes (if any) in the whole set of solutions, as well as the properties of all spherical configurations, become clearer.

Basic equations. We consider the Einstein equations

$$R_M^N = -\kappa^2[T_M^N - \delta_M^N\, T/(D-2)] \tag{1}$$

with the energy-momentum tensor (EMT) $T_M^N(M,N=1,\cdots,D)$; $T \equiv T_M^M$) in the D-dimensional manifold

$$M = M^{(4)} \times M_1 \times \cdots \times M_n; \quad \dim M_i = N_i; \quad D = 4 + \sum_{i=1}^{n} N_i, \tag{2}$$

where $M^{(4)}$ is the ordinary space-time and M_i are Ricci-flat manifolds with the intervals $ds^2_{(i)}$, $i = 1,\cdots,n$. We seek solutions of (4.4.1) such that $M^{(4)}$ is static, spherically symmetric, while all the scale factors e^{β_i} of the internal spaces M_i depend on the radial coordinates u, i.e., the D-metric is

$$\begin{aligned} ds_D^2 &= ds_4^2 - \sum_{i=1}^{n} e^{2\beta_i(u)} ds^2_{(i)}; \\ ds_4^2 &= e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\beta(u)}d\Omega^2 \end{aligned} \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the standard S^2 metric.

If we denote $\gamma = \beta_{-1}, N_{-1} = 1, \beta = \beta_0, N_0 = 2$ and choose the harmonic radial coordinate u such that

$$\alpha = \sum_{i=-1}^{n} \beta_i N_i \tag{4}$$

the Ricci tensor components R_M^N can be written in the form

$$\begin{aligned} R_0^0 &= -e^{-2\alpha}\gamma''; \\ R_1^1 &= -e^{-2\alpha}\sum_{i=-1}^{n} N_i[\beta_i'' + \beta_i'^2 - \beta_i' \sum_{j=-1}^{n} N_j\beta_j']; \\ R_2^2 &= R_3^3 = e^{-2\beta} - e^{-2\alpha}\beta''; \\ R_\mu^N &= 0 (N > 3;\ \mu = 0,\cdots,3); \\ R_{a_j}^{b_i} &= -\delta_j^i \delta_{a_i}^{b_i} e^{-2\alpha}\beta_i'' \end{aligned} \tag{5}$$

where the indices $a_j(b_i)$ refer to the internal subspace $M_j(M_i)$.

The electromagnetic field $F_{MN} = \partial_M A_N \partial_N A_M$ with the Lagrangian $L_{em} = -(1/4)F^{MN}F_{MN}$ is assumed to be Coulomb-like: $A_M = \delta^0_M A_0(u)$. Then the D-dimensional Maxwell equations $\Delta_N F^{NM} = 0$ give:

$$F^{01} = q/\sqrt{g} = qe^{-2\alpha}, \quad q = const(charge) \tag{6}$$

$$g = |detg_{MN}| = exp(2\alpha + 2\sum_{i=-1}^{n} N_i\beta_i) = e^{4\alpha}. \tag{7}$$

The corresponding EMT is

$$\begin{aligned} T^N_{M(em)} = & -F^{NP}F_{MP} + \frac{1}{4}\delta^N_M F^{PR}F_{PR} = \\ & = \frac{1}{2}q^2 e^{2\gamma-2\alpha} diag(1,1,-1,\cdots,-1). \end{aligned} \tag{8}$$

Its trace is proportional to $(D-4)$.

Besides, we admit existence of a minimally coupled scalar field φ (or even a multiplet of such fields which would not make our task more difficult) with the Lagrangian $L_s = \varphi'^N \varphi_{'N}/2$. The field equation $\nabla^N\nabla_N\varphi = 0$ with $\varphi = \varphi(u)$ due to (4.4.4) gives

$$\varphi'' = 0, \quad \varphi' = c = \text{const} \quad \text{(scalar charge)}. \tag{9}$$

The scalar field EMT is

$$T^N_{M(s)} = \varphi_{'M}\varphi'^N - \frac{1}{2}\delta^N_M \varphi'^P \varphi_{'P} = \frac{1}{2}e^{-2\alpha}c^2\ diag(1,-1,1,\cdots,1a). \tag{10}$$

The general solution. With (4.4.5), (4.4.8) and (4.4.10) some combinations of the Einstein equations with the EMT $T^N_M = T^N_{M(em)} + T^N_{M(s)}$ are easily solvable, namely

$$R^0_0 + \kappa^2[T^0_0 - T/(D-2)] = e^{-2\alpha}[\gamma'' - Q^2 e^{2\gamma}] = 0; \tag{11}$$

$$R_0^0 + (D-3)R_{a_i}^{a_i} = e^{-2\alpha}[\gamma'' + (D-3)\beta_i''] = 0, \quad i = 1, \cdots, n; \tag{12}$$

$$G_1^1 + G_2^2 = e^{-2\alpha}(\alpha'' - \beta'' - e^{2\alpha - 2\beta}) = 0; \tag{13}$$

$$G_1^1 + \kappa^2 T_1^1 = -e^{-2\beta} + e^{-2\alpha} - \frac{1}{2}\sum_{i=-1}^{n} N_i \beta_i'^2 + + \frac{1}{2}\left(\sum_{i=-1}^{n} N_i \beta_i'\right)^2 - C^2 + \frac{D-2}{2D-6}Q^2 e^{2\gamma} = 0 \tag{14}$$

where

$$C = \kappa c/\sqrt{2}; \quad Q = \kappa q (D-3)^{1/2}/(D-2)^{1/2}; \tag{15}$$

and $G_M^N = -R\delta_M^N/2 + R_M^N$ is the Einstein tensor; in (4.4.12) there is no summation over a_i. Equations (4.4.11)-(4.4.13) form a set of $(n+2)$ equations for $(n+2)$ variables $\beta_{-1} \equiv \gamma, \beta_0 \equiv \beta, \beta_1, \cdots, \beta_n$ while (4.4.14) is their first integral leading to a relation among the emerging integration constants. We obtain:

$$(4.4.11) \to e^{-\gamma} = Qs(h, u + u_1); \quad h, u_1 = const; \tag{16}$$

$$(4.4.12) \to \beta_i = -\gamma/(D-3) + h_i u + 1_i; \quad h_i, 1_i = const; \tag{17}$$

$$(4.4.13) \to e^{\beta - \alpha} = s(k, u + u_2); \quad k, u_2 = const; \tag{18}$$

$$(4.4.14) \to k^2 sign\ k = \frac{D-2}{2D-6}h^2 sign\ h + +C^2 + \frac{1}{2}\left(B^2 + \sum_{i=1}^{n} N_i h_i^2\right) \tag{19}$$

where

$$B = \sum_{i=1}^{n} N_i h_i \tag{20}$$

and we have introduced the function

$$s(a, x) = \begin{cases} a^{-1}\sinh\ ax & for \quad a > 0; \\ x & for \quad a = 0; \\ a^{-1}\sin\ ax & for \quad a < 0. \end{cases} \tag{21}$$

The constants u_2 and l_i are inessential; one can make them equal zero shifting the origin of the coordinate u and rescaling the coordinates in the subspace M_i. The resulting form of the D-metric is

$$ds_D^2 = e^{2\gamma}dt^2 - e^{-2\gamma/(D-3)}\frac{e^{-2Bu}}{s^2(k,u)}\left[\frac{du^2}{s^2(k,u)} + d\Omega^2\right] - \\ -e^{-2\gamma/(D-3)}\sum_{i=1}^{n} e^{2h_i u} ds_{(i)}^2; \qquad (22)$$

$$e^{-\gamma} = Qs(h, u+u_1).$$

Combined with (4.4.6) and (4.4.9), (4.4.22) completely describes the static, spherically symmetric, multidimensional scalar-electrovacuum configuration. The solution contains $(n+3)$ essential integration constants: the charges q and c (or their "geometrized" versions Q and C), the extra-dimension factors h_i and the mass m which can be related to h and Q:

$$Gm^2 = M^2 = Q^2 + h^2 \; sign \; h. \qquad (23)$$

where G is Newton's gravitational constant. The constant k is determined by (4.4.19) while u_1 is found from the requirement that the time t should be the proper time for an observer at rest at spatial infinity $u = 0$:

$$e^{-\gamma(0)} = Qs(h, u_1) = 1. \qquad (24)$$

The factors $e^{2\beta_i} = 1$ at $u = 0$, so that the real physical scale of the extra dimensions is hidden in $ds_{(i)}^2$.

Special cases. Let us point out some spacial cases of our solution.

a) To "switch of" the scalar field and get a purely electrovacuum solution it is sufficient just to remove the term C^2 in the constraint (4.4.19) for the constants.

b) If, instead of φ, we "switch off" the electric field, i.e., put $Q \to 0$, we obtain:

$$e^{-\gamma} = Q \; s(h, u+u_1) \to e^{hu} \quad (h > 0) \qquad (25)$$

leading to the generalized Schwarzschild solution described in [77]. To restore its specific form given in [77] one should re-denote

$$h \to -A_{-1}; \quad h/(D-3) + h_i \to A_i \quad (i = 1, \cdots, n). \tag{26}$$

As in [77], after the further substitution

$$u = -\frac{1}{2k} \ln \left(1 - \frac{2k}{R}\right), \quad A_{-1} = -ka, \quad A_i = -ka_i \; (i = 1, \cdots, n) \tag{27}$$

the metric is brought to the following convenient form:

$$\begin{aligned} ds_D^2 = {} & (1 - 2k/R)^a dt^2 - (1 - 2k/R)^{-a-b} [dR^2 + (1 - 2k/R) R^2 d\Omega^2] \\ & - \sum_{i=1}^{n} (1 - 2k/R)^{a_i} ds_{(i)}^2; \quad b = \sum_{i=1}^{n} N_i a_i. \end{aligned} \tag{28}$$

New constants $a, a_1, \cdots, a_n$ and C satisfy the relation

$$(a + b)^2 + a^2 + \sum_{i=1}^{n} N_i a_i^2 + 2C^2/k^2 = 2 \tag{29}$$

If $k = 0$, then the 4-dimensional section of $M, M^{(4)}$ (described by the first line in (4.4.28)) is flat while for $k > 0$ and

$$a - 1 = a_1 = \cdots = a_n = 0 \tag{30}$$

it coincides with the Schwarzschild solution.

c) In the general solution with $Q \neq 0$ one cannot "freeze" the extra dimensions, i.e., make $\beta_i(u) = const (i = 1, \cdots, n)$ by a choice of integration constants. Hence the Reissner-Nordström (RN) solution and its scalar generalization [93,94] are obtained from (4.4.22) only when all the extra dimensions are eliminated ($n = 0, D = 4$). To get the familiar form of the RN solution corresponding to

$$k = h, \quad h^2 sign\ h = M^2 - Q^2, \tag{31}$$

one should just transform (4.4.22) to the curvature coordinates putting

$$r = \mid Q \mid s(h, u + u_1)/s(h, u). \tag{32}$$

Properties of the Solution. Charged black holes. Our solution is defined in the region from $u = 0$ to either $u = \infty$ (if $h \geq 0$), or $u = u_{max} = \pi/ \mid h \mid -u_1$ (if $h < 0$). The value $u = 0$ corresponds to spatial infinity or the physical space-time where the metric ds^2_{ph} is asymptotically flat since

$$e^{\gamma} \to 1; \quad e^{\beta} \to \infty; \quad \mid \beta' \mid e^{\beta - \alpha} \to 1 \tag{33}$$

(the latter condition provides the proper radius-circumference relation for remote coordinate circles).

In the case $h < 0$ the value $u = u_{max}$ corresponds to a central repulsive singularity of RN type ($e^{\beta} \to 0$, $e^{\gamma} \to \infty$), with an infinite electromagnetic field and a finite scalar field. The extra dimensions are also singular, unless h_i are chosen specially to avoid this.

For $h \geq 0$ the limiting value of $u, u = \infty$, corresponds to an attractive apparent singularity ($e^{\gamma} \to 0$) which can occur either at the centre (if $e^{\beta} \to 0$), or at a certain sphere (if $e^{\beta} \to r^* < \infty$), or in a "cavity" beyond a neck (if $e^{\beta} \to \infty$ at $u \to \infty$), depending on the values of the integration constants. Indeed, near $u = \infty, e^{\beta}$ behaves like

$$exp\{[h/(D-3) - k - B]u\} \tag{34}$$

and can tend to any nonnegative value including infinity since $B = \sum N_i h_i$ can have any value and either sign.

Let us find out whether the apparent singularity at $u = \infty$ can be an event horizon for the physical metric ds^2_4. Recall that event horizons are invisible for external static observers, hence we seek such configurations that the integral

$$t^* = \int e^{\alpha - \gamma} du \tag{35}$$

expressing a light signal travel time, diverges at $u = \infty$.

It is helpful to pass from h_i to A_i by (4.4.26), so that the relation (4.4.19) among the constants takes the form

$$2k^2 = 2C^2 + (\overline{B} - h)^2 + h^2 + \sum_{i=1}^{n} N_i A_i^2, \quad \overline{B} = \sum_{i=1}^{n} N_i A_i, \tag{36}$$

or, equivalently,

$$(h - \overline{B}/2)^2 = k^2 - C^2 - \overline{B}^2/4 - (1/2)\sum_{i=1}^{n} N_i A_i^2. \tag{37}$$

On the other hand, when $u \to \infty$,

$$e^{\alpha-\gamma} \sim exp[2u(h - \overline{B}/2 - k)], \tag{38}$$

so that the integral (4.4.36) diverges if $h - \overline{B}/2 \geq k$. By (4.4.38) this is possible only if

$$C = A_i = \overline{B} = 0 \Rightarrow h_i = -h/(D-3); \quad k = h. \tag{39}$$

Thus the scalar field is excluded while the extra dimensions do not become trivial.

By (4.4.40) the condition that e^{β} has a finite limit at $u = \infty$ (see (4.4.35)) is fulfilled automatically.

With (4.4.40) and (4.4.23), using again the substitution (4.4.27) for u, one obtains the following expression for our metric (4.4.22):

$$ds_D^2 = \frac{\left(1 - \frac{2k}{R}\right) dt^2}{\left(1 + \frac{M-K}{R}\right) 2} - \left(1 + \frac{M-k}{R}\right)^{\frac{2}{D-3}} \left[\frac{dR^2}{1 - \frac{2k}{R}} + R^2 d\Omega^2 + \sum_{i=1}^{n} ds_{(i)}^2\right] \tag{40}$$

This expression generalizes the RN black-hole metric to space-times with an arbitrary set of additional Ricci-flat spaces.

In case $Q = 0$, (4.4.41) turns into the Schwarzschild metric with trivial extra dimensional since $M = k$. This confirms the conclusion [77] that Schwarzschild black holes have no nontrivial multidimensional generalization (within our choice of ds_{ph}^2).

For $D = 4$ (4.4.41) is just the RN metric; it is brought to the usual curvature coordinates by a mere shift $R = r - M + k$.

The extra-dimension scale factor in (4.4.41) is nonsingular and smoothley grows from 1 at $R = \infty$ to $\left(1 + \frac{M-k}{2k}\right)^{2/(D-3)}$ at the horizon $R = 2k$.

Concluding Remarks

It is of interest that the 2-parameter family of black holes was selected from the $(n + 3)$-parameter family of solutions (4.4.6), (4.4.19), (4.4.22) by the single requirement that the boundary $u = \infty$ should be invisible. The other essential features of the resulting metric, namely, that the boundary is a sphere of finite radius and that the extra dimensions are nontrivial but nonsingular, are obtained automatically.

The black-hole solution (4.4.41) is a very special case of (4.4.22) (2 versus $(n+3)$ parameters); the same is valid for black holes obtained under other assumptions. Thus, very strong arguments should be drawn in order to show that real collapsing bodies can form black holes if a multidimensional theory of gravity holds.

Among the observable local effects of extra dimensions there are standard post-Newtonian relativistic effects whose values can differ from those in general relativity.

Charged multidimensional solutions lead in general to the modification of the Coulomb law. Really,

$$E = |\vec{E}| = (F^{01}F_{10})^{1/2} = \frac{|q|}{r^2} exp\left(-\sum_{i=1}^{n} N_i\beta_i\right), \tag{41}$$

which for large r may be written as

$$E = \frac{|q|}{r^2}\left[1 - \frac{1}{2}\left(\frac{D-4}{D-3}M + \sum_{i=1}^{n} N_i h_i\right) + 0\left(\frac{1}{r^2}\right)\right]. \tag{42}$$

For the BH case:

$$E = \frac{|q|}{r^2}\left[1 - \frac{1}{r}\frac{D-4}{D-3}(M + \sqrt{M^2 + Q^2}) + 0\left(\frac{1}{r^2}\right)\right]. \tag{43}$$

So, we see that deviations from the Coulomb law depend on the number of dimensions, total mass and charge of the system and also on the range.

5 Multidimensional Model with Interaction of Scalar and Electromagnetic Fields. Stability of Solutions [95]

Now we pass to a more complicated system of interacting fields which arise in a field limit of superstring theories. We consider D-dimensional space-time V_D with a chain of Ricci-flat spaces $M_i (i = 1, \cdots, n)$.

The Lagrangian of the system is:

$$L = R^{(D)} + g^{MN}\varphi_{'M}\varphi_{'N} - e^{2\lambda\varphi}F^{MN}F_{MN}; \tag{1}$$

and metric

$$ds_D^2 = \overline{g}_{\omega\nu}dx^\mu dx^\nu + \sum_{i=1}^{n} e^{2\alpha_i(x)}ds_i^2, \tag{2}$$

where λ is a coupling constant and $\overline{g}_{\mu\nu}$ is a 4-metric Lagrangian (4.5.1) may be transformed to 4-dimensional form from D-dimensional metric g_{MN} to 4-dimensional metric $\overline{g}_{\mu\nu}$. Then scale factors $\alpha_i(x)$ become scalar fields in V_4. But it is more convenient to use conformal transformations

$$\overline{g}_{\mu\nu} = e^{\sum g_{\mu\nu}}, \ \sum = \sum_i N_i\alpha_i(x), \tag{3}$$

after which in the anzatz:

$$\varphi = \varphi(x^\mu), \ F_{\mu\nu} = F_{\mu\nu}(x^\alpha); \ F_{MN} = 0 \ npu \ M, N > 3$$

Lagrangian (4.5.1) is becoming as

$$\widetilde{L} = R^{(4)} + \frac{1}{2}\Sigma^{'\alpha}\Sigma_{'\alpha} + \sum_k N_k\alpha_{k,\mu}\alpha_k{}^{,\mu} + \varphi^{,\mu}\varphi_{,\mu} - e^{\sum + 2\alpha\varphi}F^{\alpha\beta}F_{\alpha\beta}, \tag{4}$$

where $R^{(4)}$ corresponds to metric $g_{\mu\nu}$. Metric $g_{\mu\nu}$ will be represented in a standard spherically symmetric form:

$$ds_4^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\beta}d\Omega^2. \tag{5}$$

We point out that static solutions is better to seek in a D-dimensional metric with Lagrangian (4.5.1). Metric (4.5.4) is more adapted to stability studies.

Field Eqs. corresponding to Lagrangian (4.5.4) are:

$$2\Box\alpha_k + \Box\sum + e^{\sum+2\lambda\phi}F^{\alpha\beta}F_{\alpha\beta} = 0, \tag{6}$$

$$\Box\varphi + \alpha e^{\sum+2\alpha\varphi}F^{\alpha\beta}F_{\alpha\beta} = 0, \tag{7}$$

$$\nabla_\alpha\left(e^{\sum+2\alpha\varphi}F^{\alpha\beta}\right) = 0, \tag{8}$$

$$G_\mu^\nu + \frac{1}{2}S_\mu^\nu(\sum) + \sum_k N_k S_\mu^\nu(\alpha_k) + S_\mu^\nu(\varphi) + e^{\sum+2\alpha\varphi}E_\mu^\nu = 0, \tag{9}$$

where G_μ^ν is 4-dimensional Einstein tensor,

$$S_\mu^\nu(\varphi) = \varphi^{,\nu}\varphi_{,\mu} - \frac{1}{2}\delta_\mu^\nu\varphi^{,\alpha}\varphi_{,\alpha}\ ;$$
$$E_\mu^\nu = -2F^{\nu\alpha}F_{\mu\alpha} + \delta_\mu^\nu F^{\alpha\beta}F_{\alpha\beta}/2.$$

Here we also suppose that $\varphi = \varphi(u), A_\mu = \delta_\mu^0 A_0(u)$, $\Box u = 0$ and

$$\alpha(u) = 2\beta(u) + \gamma(u). \tag{10}$$

We remark that u is also harmonic in D-dimensional metric ${}^D\Box u = 0$ due to (4.5.3), (4.5.4) but not in the metric $\overline{g}_{\mu\nu}$. Solution of (4.5.8) with coordinate condition (4.5.10) gives

$$F^{01} = q\ exp(-2\alpha - \sum - 2\alpha\varphi), q = const, \tag{11}$$

and (4.5.9) leads to

$$G_1^1 + G_2^2 = \beta'' + \gamma'' - e^{2\beta+2\gamma} = 0,$$

$$e^{-\beta-\gamma} = s(k,u) \equiv \begin{cases} k^{-1} shku & (k > 0), \\ u & (k = 0), \\ k^{-1}\sin ku & (k > 0), \end{cases} \tag{12}$$

where $k = const$. From (4.5.6), (4.5.7) and (4.5.9) one gets:

$$\alpha_i = h_i u - \frac{1}{AN}(\omega + \lambda C u), \tag{13}$$

$$\varphi = \frac{Cu}{A} - 2\lambda N_+ \omega, \tag{14}$$

$$2\gamma = 2N_+(\omega + \lambda C u) + Bu, \tag{15}$$

with function $\omega(u)$ defined by

$$e^{-\omega} = Qs(h, u + u_1);\ h, u_1 = const;\ Qs(h, u_1) = 1 \tag{16}$$

and h, h_i and C are constants of integration; other constants are defined by:

$$N = D - 3 = 1 + \sum_i N_i;\ B = \sum_i N_i h_i, \tag{17}$$

$A + 1 + \lambda^2(N+1)/N$, $Q^2 = q^2/N_+$, $N_+ = (N+1)/(2AN)$.

The final form of D-dimensional metric is:

$$ds_D^2 = e^{2\overline{\gamma}} dt^2 - \frac{e^{2\overline{\gamma}/N - 2Bu}}{s^2(k,u)} \left[\frac{du^2}{s^2(k,u)} + d\Omega^2 \right] - e^{-2\overline{\gamma}/N} \sum_i e^{2h_i u} ds_i^2, \tag{18}$$

where

$$\overline{\gamma} = (\omega + \lambda C u)/A. \tag{19}$$

Constants of integration are related due to $\binom{1}{1}$-component of (4.5.9) by the following equality:

$$2k^2 signk = 2N_+ h^2 signh + \frac{C^2}{A} + B^2 + \sum_i N_i h_i^2. \qquad (20)$$

This general static spherically-symmetric solution has $(n+3)$ essential integration constants: scalar charge C, electric charge q (or Q), "charges" of extra dimensions h_i and mass m which is defined by the expansion $u \to 0$ $(r \to \infty)$ and is connected with C,Q,h:

$$AGm + \lambda C = (Q^2 + h^2 signh)^{1/2}, \qquad (21)$$

Coordinate u is defined in the region $[0, \infty)$ if $h \geq 0, u_1 > 0$ or up to $u_{max} > 0$ in other cases. Scale factors $e^{\alpha_i} = 1$ for $u = 0$.

Here are some properties of the solution:

a) when $\lambda = 0$ we obtain the solution for linear scalar and electromagnetic fields, discussed in a previous section,

b) scalar field is "switched of" when $\lambda = C = 0$;

c) elimination of an electromagnetic field is done for $Q = 0$. Then we get the generalized Schwarzschild solution after transformation

$$u = -\frac{1}{2k} \ln \left(1 - \frac{2k}{R}\right); \quad \frac{-h + \lambda C}{A} = ka; \quad h_i = k\left(-a_i + \frac{a}{N}\right), \qquad (22)$$

d) when extra dimensions are absent we obtain the solution [96]

e) in a general case there is no any choice of integration constants when extra dimensions are frozen out, i.e. $\alpha_i = const$ for Q (or q)$\neq 0$. The behaviour of metric coefficients for $u \to \infty$ is:

$$\overline{\beta} \sim \left[\frac{-h+\lambda C}{AN} - B - k\right] u, \;\; \overline{\gamma} \sim \frac{-h+\lambda C}{A} u, \;\; \alpha_i \sim \left[\frac{-h+\lambda C}{AN} + h_i\right] u, \qquad (23)$$

so they may be finite or infinite. Calculations show that the solution has a naked singularity at $u = u_{max}$ or $u = \infty$ in all cases except:

$$h_i = -k/N; \; h = k; \; C = -\lambda k(N+1)/N, \qquad (24)$$

when the sphere $u = \infty$ is a horizon and the integral $\int exp(\alpha - \gamma)du$ for the light travel time is divergent. Then,, only two independent integration constants remain: m and Q; $r = e^{\beta}$ and e^{α_i} are finite. Using (4.5.22) we get more familiar form of the solution:

$$ds^2_R = \frac{(1-2k/R)dt^2}{(1+p/R)^{2/A}} - (1+p/R)^{2/AN}\left[\frac{dR^2}{1-2k/R} + R^2 d\Omega^2 + \sum_i ds_i^2\right],$$
$$p = A(Gm - k) = \sqrt{Q^2 + k^2} - k. \qquad (25)$$

It is a generalized Reissner-Nordstrom solution and is reduced to it when $D = 4, \lambda = 0$.

We point out that (4.5.25) is a special case of general metric (4.5.8) (2 parameters instead of $(n+3)$). So, there must be strong arguments that within the frames of multidimensional theory real collapse may lead to formation of BH. And such situation really arrises when we investigate stability of static solutions.

Here we shall again demonstrate one of the specific for multidimensional systems potentially observable effect-violation of the Coulomb law. Due to (4.5.11) $E = (F^{01}F_{10})^{1/2}$ and for our solution:

$$E = (\mid q \mid /r^2) exp(-\sum -2\lambda\varphi), \qquad (26)$$

So, it is seen that the deviations from the Coulomb law are due to extra dimensions and also because of a scalar-electromagnetic interaction.

For the BH case $(r \to \infty)$ we have:

$$E = \frac{|q|}{r^2}\left\{1 - \frac{1}{r}\left[(Gm-k)\frac{N-1}{N} + 2\lambda^2(Gm+k)\frac{N+1}{N}\right] + O\left(\frac{1}{r^2}\right)\right\} \qquad (27)$$

Stability Problem. Let us investigate small perturbations from static configurations:

$$\delta\varphi(u,t),\ \ \delta\alpha_i(u,t),\ \ \delta g_{\mu\nu}(u,t),\ \ \delta F_{\mu\nu}(u,t), \qquad (28)$$

which preserve spherical symmetry, i.e. monopole ones. Then dynamical degrees of freedom are restricted by the scalar field and scale factors α_i which in 4-dimensional representation behave as effective scalar fields. We take for simplicity only one internal space:

$$N_1 = D - 4 = N - 1 > 0;\ \ N_2 = N_3 = \cdots = 0. \qquad (29)$$

Perturbed metric functions $\tilde{\gamma}\,(u,t)$ and $\tilde{\alpha}\,(u,t)$ are taken in the form:

$$\tilde{\gamma}\,(u,t) = \gamma(u) + \delta\gamma(u,t);\ \ \tilde{\alpha}\,(u,t) = \alpha(u) + \delta\alpha(u,t), \qquad (30)$$

Similar relations are written for $\tilde{\varphi}\,(u,t), \tilde{\alpha}_1\,(u,t) \equiv \tilde{\mu}\,(u,t)$ and $\tilde{F}_{\mu\nu}\,(u,t)$. Perturbed Maxwell field is defined by $\tilde{A}_0\,(u,t)$.

Integrating (4.5.8) we get

$$\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta} = -2q^2 e^{-4\lambda\tilde{\varphi}-2(N-1)\tilde{\mu}}/r^4, \qquad (31)$$

where q and r are not perturbed (we analyse only dynamical perturbations but not changes of constants). From (4.5.6) and (4.5.7) we obtain Eqs. for $\delta\mu$ and $\delta\varphi$:

$$r^4\delta\ddot{\mu} - \delta\mu'' - \mu'(\delta\gamma' - \delta\alpha') + 2\mu''\delta\alpha = -\frac{2q^2e^{2\omega}}{N+1}\omega, \qquad (32)$$

$$r^4\delta\ddot{\varphi} - \delta\varphi'' - \varphi'(\delta\gamma' - \delta\alpha') + 2\varphi''\delta\alpha = 2\lambda q^2 e^{2\omega}\omega, \qquad (33)$$

$$\omega \equiv 2\lambda\delta\varphi + (N-1)\delta\mu, \qquad (34)$$

where μ', μ'', φ' and φ'' are static functions. $\delta\gamma'$ and $\delta\alpha$ are defined from (4.5.9). $\binom{1}{0}$-component of Einstein Eqs. is easily integrated over t:

$$2\beta'\delta\alpha = \frac{N^2-1}{2}\mu'\delta\mu + \varphi'\delta\varphi + F(u), \tag{35}$$

and difference of $\binom{0}{0}$ and $\binom{1}{1}$ components gives:

$$2\beta'(\delta\alpha' + \delta\gamma') = (N^2-1)\mu'\delta\mu + 2\varphi'\delta\varphi. \tag{36}$$

Taking $\delta\alpha$ and $\delta\gamma'$ from (4.5.35), (4.5.36) and putting them into (4.5.32), (4.5.33) we get coupled wave Eqs. for $\delta\mu$ and $\delta\varphi$:

$$r^4\delta\ddot{\varphi} - \delta\varphi'' + \frac{N^2-1}{2}\left(\frac{r\varphi'\mu'}{r'}\right)\delta\mu + \left(\frac{r\varphi'^2}{r'}\right)'\delta\varphi = 2\lambda g^2 e^{2\omega}\omega, \tag{37}$$

$$r^4\delta\ddot{\mu} - \delta\mu'' + \frac{N^2-1}{2}\left(\frac{r\mu'^2}{r'}\right)'\delta\mu + \left(\frac{r\varphi'\mu'}{r'}\right)'\delta\varphi = -\frac{2q^2e^{2\omega}}{N+1}\omega. \tag{38}$$

Our static system is unstable if there exist growing at $t \to \infty$ physically allowed solutions of Eqs. (4.5.37), (4.5.38). We define solutions as physically allowed if

$$\delta\mu \to 0, \quad \delta\varphi \to 0 \quad if \quad u \to 0 \tag{39}$$

at space infinity $r \to \infty$ and

$$|\,\delta\mu/\mu\,| < \infty, \quad |\,\delta\varphi/\varphi\,| < \infty \tag{40}$$

at singularities and horizons. We also eliminate energy fluxes from outside but it only limits constants of integration.

We study the stability of solution when system (4.5.37)-(4.5.38) reduces to wave Eqs. with one unknown function:

1. dilaton field is absent: $\lambda = 0, \varphi \equiv \delta\varphi \equiv 0$,
2. extra dimensions are absent: $\mu \equiv \delta\mu \equiv 0, N = 1$,
3. some combinations of (4.537)-(4.5.38) lead to equation with one unknown function.

One may show that case 3) is realized for

$$\mu' = K\varphi'; \quad K = \frac{1}{(N+1)}\lambda \ \ or \ \ K = -\frac{2\lambda}{N-1}. \tag{41}$$

that is the case of a BH solution. Let us consider each of these variants.

1. Separating variables in (4.5.38) and transforming μ and u to normal Liouville form according to:

$$\delta\mu = e^{i\Omega t}y(x)/r, \quad x = -\int r^2(u)du, \tag{42}$$

we get Schrodinger-type Eq.

$$y_{xx} + [\Omega^2 - V(x)]y = 0 \tag{43}$$

with effective potential

$$V(x) = \frac{1}{r^3}\left(\frac{r'}{r^2}\right)' + \frac{N^2-1}{r^2}\left(\frac{\mu'^2}{r'}\right)' + \frac{2Q^2N}{r^4}e^{2\omega} \tag{44}$$

Our static system is unstable if there exist physically allowed solutions of (4.5.43) with $\Omega^2 < 0$ (negative energy levels in Schrodinger Eq.). It is shown in [95] that asymptotic form of the potential $V(x)$ for $u \to u_{max}$ is

$$V(x) = -\frac{(N+1)(3N+1)}{4(2N+1)^2x^2}(1+o(1)). \tag{45}$$

i.e. have negative values. So, the system is unstable and this instability is of a catastrophic character as $\mid \Omega \mid$ is not limited from above.

2. The same result is obtained also for $N = 1$, $\mu \equiv \delta\mu \equiv 0$ [95]: also catastrophic instability. Let us now consider perturbations of multidimensional BH described by the system (4.5.37), (4.5.38) under the condition (4.5.24). Introducing linear combinations of perturbations of ω in (4.5.24) and

$$v = \lambda(N+1)\delta\mu - \delta\varphi, \tag{46}$$

it is seen that (4.5.37)-(4.5.38) leads to two independent wave Eqs.

$$r^4\ddot{v} - v'' = 0, \quad r^4\ddot{\omega} - \omega'' + H(x)u = 0, \tag{47}$$

$$H(x) = [N - 1 + 2\lambda^2(N+1)]\left[\frac{N+1}{2}\left(\frac{r\mu'^2}{r'}\right)' + \frac{2q^2e^{2\omega}}{N+1}\right]. \tag{48}$$

Now we pass to normal form similar to (4.5.42):

$$v = e^{i\Omega_1 t}y(x)/r; \quad \omega = e^{i\Omega_2 t}z(x)/r, \quad x = -\int r^2(u)du. \tag{49}$$

For $y(x)$ and $z(x)$ Schrodinger-type Eqs. are obtained:

$$y_{xx} + [\Omega_1^2 - V_1(x)]y = 0; \quad z_{xx} + [\Omega_2^2 - V_2(x)]z = 0. \tag{50}$$

with effective potentials

$$V_1 = \frac{1}{r^3}\left(\frac{r'}{r^2}\right)'; \quad V_2 = V_1 + \frac{1}{r^4}H(x). \tag{51}$$

They may be written in explicit form via R (4.5.22):

$$V_1 = \frac{R-2k}{r^4(R+p)^2}\{pN_+R(R-2k) + (R+pN_-)[(2k+pN_+)R + 2kpN_-]\}, \tag{52}$$

$$H(x) \sim \frac{R(R-2k)}{AN(R+p)^2}\left[p^2N_- + 2kp + \frac{p^2}{AN}\frac{(2R+p+pN_-)(2k+pN_-)}{(R+pN_-)^2}\right], \tag{53}$$

where nonessential constant coefficient is ommited. A, p, N_+ are defined in (4.5.17), (4.5.25) and $N_- = 1 + N_+$, so that

$$r = R(1+p/R)^{N_+} = R^{N_-}(R+p)^{N_+}. \tag{54}$$

It follows from (4.5.52) and (4.5.53) that $V_1 > 0$ and $V_2 > 0$ when $R > 2k$. Boundary conditions (4.5.39)-(4.5.40) are fulfilled so positivness of V_1 and V_2 means that solutions of (4.5.50) with $\Omega_1^2 < 0$ and $\Omega_2^2 < 0$ are absent. So, multidimensional BHs are stable against monopole perturbations. Other types of multidimensional spherically symmetric solutions are strongly unstable. This means that a lot of BHs may be present at the Early Universe if it is described by some multidimensional model.