VACUUM IN PLANE AND CURVED SPACE-TIMES

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FIELD QUANTIZATION FOR ACCELERATED FRAMES IN FLAT AND CURVED SPACE-TIMES

Abstract :

We formulate Q.F.T. for a wide class of accelerated space-times in four dimensions and describe its thermal properties in terms of analytic mappings. We demonstrate that four dimensional Rindler space uniquely satisfies the condition of global thermal equilibrium, while spaces which are asymptotically Rindler have thermal equilibrium asymptotically. We discuss the renormalization of the quantum energy momentum tensor with application to situations in two, four and ν dimensions and specifically refer to the general result for conformally flat two dimensional space times. Covariant and non covariant regularisation schemes are presented and compared, and we make an improvement to proposals for covariantly regularizing the stress tensor by point splitting.

I - INTRODUCTION

In previous publications [1,2], analytic mappings have been used to define a wide class of accelerated space times preserving the light cone structure of flat Minkowski space time. Two dimensional Q.F.T. was then formulated in these accelerated spaces and Bogoliubov coefficients, relating a positive frequency description for accelerated and inertial bases, were given explicitely in terms of the mappings. In this paper, we extend to four dimensions a series of results stemming from this approach : four dimensional Q.F.T. and its thermal properties are analyzed in terms of analytic mappings relating some manifold to its global analytic extension. Following a description of the relevant formalism (in Section II), we give a specific demonstration (in Section III) that the four dimensional Rindler space (described by the exponential mapping) uniquely satisfies the condition of global thermal equilibrium. Spaces which are asymptotically Rindler have thermal equilibrium asymptotically. In Section IV we give a general discussion of renormalization in a curved space time where there may be no unique state singled out as a ground state, and where (even for a free scalar field) coupling to the curvature may force us to regard gravity as interacting. We examine several properties of some specific regularizing schemes which indicate how the renormalization might be effected, noting of course, that final results must be independent of any scheme we use to obtain it. A refinement to proposals for covariantly regularizing the stress tensor by point splitting is given in Section V.

Generalizations of this work including rotating accelerated frames and even non analytic mappings are given elsewhere [3].

A. Contextual background

Before presenting our work in the next sections, we pause here to survey the context in which it has been developed.

While a full quantum theory of gravity is still non-existent, continuous effort over the last quarter of a century has demonstrated the many difficulties encountered in repeated attempts to construct such a theory and have also indicated some of the particular properties which an eventually complete theory will have to possess. Complementary to these approaches, there are investigations for problems in flat space time which can throw light on both classical and quantum results in curved space time : Quantum field theory developed for curvilinear (accelerated) coordinates in flat space in a way which can be directly generalized to curved space time, may be useful for a physical and mathematical discussion of the full theory.

The genuine coordinate independence which is so familiar in the classical theory of general relativity is not a particular property of gravity but a fundamental principle prevalent in all descriptions of physical laws. On the other hand, the apparent difference which results from the treatment of a quantum field theory in a variety of coordinate systems (in either curved or flat space time) is not a coordinate effect at all, but is a consequence of the fact that physically different quantum states are correctly described by the quantum theory as being physically distinct "Canonical" states for different coordinate systems are physically different (each timelike vector field leads to a separate indication of what constitutes a definition of positive frequency).

It seems difficult to give a sensible meaning to the question : how should we describe quantum field theory for an (accelerated) *observer following a particular world line* ? Even for an uniformly accelerated observer, some subtle assumptions go into the handling of this question. Firstly, although one might use a coordinate system giving the world lines for an infinite family of uniformly accelerated observers, this system actually refers only to *accelerated frames*, since there would need to be collusion (space-like correlations) between observers in order to have and maintain them as uniformly accelerating. In addition, to use standard quantum field theory techniques, one would impose boundary conditions such that, asymptotically, modes appear as free waves in these coordinates. But for an *arbitrary accelerating observer*, both the coordinate system of his connected region, and the asymptotic boundary conditions we might assign for him can actually be chosen in an infinite variety of ways. However, a perfectly well-posed problem is : what description is there for a quantum field theory in accelerated *frames* ? We associate a physical meaning to this question by describing the boundary conditions of the quantum theory in terms of the asymptotic behaviour of those frames. This automatically specifies the quantum state to be examined. In addition it allows us to use appropriate coordinates in examining physical consequences of the quantum theory for the chosen state. Of course, these consequences are completely independent of the coordinates used to evaluate them.

Further considerations arise for curved space, in which acceleration and gravitation are locally indistinguishable but globally inequivalent. On the one hand, we have just imposed a connection between the boundary conditions of the space time and a quantum state in which to discuss physics. We have done it by using the asymptotic behaviour of the accelerated frames. On the other hand, in curved space, we shall have to decide whether there may be some link between the global properties of the gravitational field and some specific state which may be regarded (gravitationally) as a global ground state in our space time. The alternative to such a normalization is to regard gravity as essentially interacting with (otherwise free) fields propagating in a curved space time. These questions become unavoidable when one tries to handle the infinities, inevitably arising in theories of field quantization.

The usual purpose for examining Q.F.T. in flat space is to obtain an understanding about the particles which we use the fields to represent. In curved space time, this same reason exists, with the added fact that we can look at (linearized) curvature effects on the theory. But a more interesting reason in the context of general relativity is to be able to determine some non-linear effects of the coupling between matter fields and geometry through the stress energy of the quantum fields. Of course, one should include the quantum effects of gravitation itself but techniquely this is rather difficult and it has often been felt that dealing simply with a scalar field source first would give a guide to the treatment of some of the difficulties. Even to consider the back reaction of the scalar field, progress is not altogether straigthforward, the problem being that the stress tensor for a quantum field is a formally divergent operator, just as it is in flat space. Whereas one has a clear idea of what the vacuum (i.e. zero energy) state is in flat space, and therefore can give a well defined procedure for rendering quantum operators finite, in a curved space time this is not the case, and there has arisen some discussion over whether one should use a normalization procedure, equivalent to defining some specific state as having zero energy or whether a renormalization of the theory is necessary, equivalent to regarding the curvature as introducing an interaction between the scalar field and the geometry. With regard to normalization, it is not at all clear how to determine which state might naturally be regarded as having zero energy, or even whether there is any such state. On the other hand, renormalization of a theory is usually considered in the context of a perturbative expansion. However, in the case of gravitation, apart from the infinite changes which will be introduced into the quantities originally appearing in the Lagrangian, new counterterms will be required at each level of a loop expansion about some classically valued geometry (say Minkowski space). Because of this the full quantum theory of gravity is often described as being "unrenormalizable". It has also been argued that if only the matter fields are quantized, even an expansion to one loop may not make sense since the quantum fluctuations of the geometry should also be considered at exactly that level where back reaction effects for the matter fields become significant. Nevertheless there is ample reason to believe that knowledge gained from a treatment of the "semi-classical" Einstein equations will be useful in any perturbative discussions of the "full" theory. In this context, some form of the "absolute" renormalization would seem to be required, and we shall refer to methods which have been used for carrying this out. However, it simply is not clear that geometry will respond to quantum matter in the same way as does an observer in a laboratory which is accelerating. Thus, at least in any discussion of quantum fields on a fixed space time background, some form of normalization may be more in order. We will make further reference to the choices which arise here in section IV and V.

II - QUANTUM STATES AND VACUUM SPECTRA IN ACCELERATED FRAMES

The extension of earlier results to four dimensions is embodied in the following coordinate transformations :

$$
\begin{aligned}
x - t &= f(x' - t') \\
x + t &= g(x' + t') \\
y &= y' \\
Z &= Z'
\end{aligned}
\qquad (1)
$$

so that the metric takes the form

$$dS^2 = f'(x' - t')\, g'(x' + t') \quad dx'^2 - dt'^2 \quad + dy^2 + dz^2$$

where f(g) is a strictly monotonic function, unprimed coordinates refer to all of inertial Minkowski space and primed coordinates to an accelerated space time.(Below, we shall use $u = x - t$, $v = x + t$, $u' = x' - t'$, $v' = x' - t'$, for convenience). Singularities of the inverse mapping $u' = F(u)$ ($v' = G(v)$) at u_+ and u_- (v_+ and v_-) give the (x', t') boundaries of the accelerated space, thus

$$\begin{aligned} u_\pm &= f(\pm\infty) \\ v_\pm &= g(\pm\infty) \end{aligned} \tag{2}$$

Here $u_\pm$ ($v_\pm$) can take finite or infinite values. Future and past boundaries at $u = u_-$ and $v = v_-$ are defined by different types of singularities of f and g, respectively and they can have (as we will see) different associated temperatures. Analogously, for future and past boundaries at $v = v_+$ and $u = u_+$. Boundaries can be horizons or infinites. For finite $u_\pm$, $v_\pm$ the accelerated coordinates cover a bounded region (a parallelogram)

$$u - < |x - t| < u_+$$

$$v_- < |x + t| < v_+$$

of Minkowski space-time ; $u = u_\pm$ and $v = v_\pm$ represent two event horizons. (If $f = g$, the bounded region is a rhombus). There can be horizons on u but no horizons on v and vice versa, in which cases the coordinates cover an infinite strip at 45° angle with the x axis. If $u_\pm = \pm\infty$ and $v_\pm = \pm\infty$, there are no horizons. Conditions (2) guarantee that the accelerated coordinates (x', t') range over all values from $-\infty$ to $+\infty$ (light rays take an infinite time t' to reach the boundaries of the accelerated space). For $t' \to \pm\infty$ the world lines of the accelerated observers defined by x' = constant tend asymptotically to the characteristic lines $u = u_\pm$ ($v = v_\pm$) where its velocity given by $V = [g'(v') - f'(u')] / [g'(v') + f'(u')]$ reaches the values $\pm c$. These conditions ensure that one can formulate Q.F.T. in these accelerated spaces in a consistent way. In accelerated spaces for which the mappings f(g) do not satisfy conditions (2), self-adjointess of propagation equations, completeness and orthogonality of their solutions cease to hold, unless additional assumptions on the wave functions are imposed. This can be clearly illustrated by comparing the mappings

$$f_1 = \frac{\alpha u' + \beta}{\gamma u' + \delta}\ ,\ (\alpha\delta - \beta\gamma \neq 0) \text{ and } f_2 = \beta e^{\alpha u'}$$

(α , β , γ, δ) being real constants. Both mappings describe uniform accelerations (f_2 gives the Rindler frame). However, f_1 does not satisfy conditions (2) and then solely quantum effects of Casimir type can be described in terms of f_1.

In four dimensions as in the two dimensional massive case, knowledge of $f'(\pm\infty)$ $[g'(\pm\infty)]$ is also required to proceed with a discussion of Q.F.T.. Different choices are possible. For definiteness in our discussion here we will take $f = g$, $u_- = 0$ (then $u' = -\infty$ is a critical point) and $u_+ = +\infty$, so that the accelerated space covers the right-hand wedge of Minkowski space-time. Then

$$f'(-\infty) = 0 \tag{3.a}$$

and we choose

$$f'(+\infty) = +\infty \tag{3.b}$$

since the Rindler space is included by this as are also non-uniformly accelerated space-times which are asymptotically Rindler. The cases with two or zero event horizons can be easily solved from the discussion given below.

In the accelerated space-time, the minimally coupled scalar field-equation

$$\Box\, \hat{\Psi} = m^2\, \hat{\Psi} \tag{4}$$

becomes

$$\left[\frac{1}{\Lambda}(-\partial_{t'}^2 + \partial_{x'}^2) + \partial_{y'}^2 + \partial_{z'}^2 - m^2\right]\hat{\Psi} = 0 \tag{5}$$

where $\Lambda = f'(x' - t')\, f'(x' + t')$

The substitution

$$\hat{\Psi} = \frac{1}{2\pi}\, e^{i(\lambda_2 y + \lambda_3 z)}\, \phi(x', t') \qquad -\infty \leqslant \lambda_2, \lambda_3 \leqslant +\infty \tag{6}$$

yields

$$\left[-\partial^2_{t'} + \partial^2_{x'} - \Lambda M^2 \right] \phi(x', t') = 0 \text{ with } M^2 = \lambda_2^2 + \lambda_3^2 + m^2 \tag{7}$$

Conditions (3) mean that the effective mass ΛM^2 is zero on the horizon and infinite at infinity preventing particle escape there. Thus we can choose as a complete set of in-basis solutions, the functions ϕ^{in} satisfying :

$$\lim_{v' \to -\infty} \phi^{in}_{\lambda_1} = \frac{1}{2\sqrt{\pi\lambda_1}}\, e^{i\lambda_1 u'}$$

$$\lim_{u' \to +\infty} \phi^{in}_{\lambda_1} = 0 \qquad , \quad \lambda_1 > 0 \tag{9}$$

and given completely for any mapping by [2]

$$\phi^{in}_{\lambda_1} = \frac{i}{2}\sqrt{\frac{\lambda_1}{\pi}} \int_{+\infty}^{u'} d\xi\, e^{i\lambda_1 \xi}\, J_0\left(M\sqrt{v[f(\xi) - u]} \right) \tag{10}$$

where J_0 stands for the Bessel function.
This can be also written as

$$\phi^{in}_{\lambda_1} = \frac{1}{2\sqrt{\pi\lambda_1}} \left\{ e^{i\lambda_1 F(u)} + M\sqrt{v} \int_{+\infty}^{u} d\eta\, e^{i\lambda_1 F(\eta)} \frac{J_1\left[M\sqrt{v(\eta - u)}\right]}{\sqrt{\eta - u}} \right\},$$

in terms of the inverse mapping $F = f^{-1}$.

Near the horizon, for $uv \to o$, $\phi^{in}_{\lambda 1}$ behaves as

$$\phi^{in}_{\lambda_1\,(uv \to o)} = -\frac{1}{2\sqrt{\pi\lambda_1}} \left[e^{i\lambda_1 u'} - e^{2i\delta(\lambda)}\, e^{-i\lambda v'} \right] \tag{11}$$

where $\delta(\lambda)$ is real. (Due to the infinite potential barrier existing at $uv \to +\infty$, the waves coming from the past horizon leave out through the future horizon).

The functions Ψ_λ^{in} are orthonormal with respect to the scalar product

$$\langle \Psi_\lambda, \Psi_{\lambda'} \rangle = i \int {}^*\Psi_\lambda \overleftrightarrow{j_\mu} \Psi_{\lambda'} \, d\Sigma^\mu$$

$$\left(\overleftrightarrow{j_\mu} = \sqrt{-g}\, \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu} \sqrt{-g} \right)$$

With two event horizons, conditions (3) are modified to

$$f'(\pm\infty) = 0$$

and besides the solutions $\overrightarrow{\phi}_\lambda^{in} \equiv \phi_\lambda^{in}$, we would also need the solutions $\overleftarrow{\phi}_\lambda^{in}$ satisfying

$$\lim_{u' \to +\infty} \overleftarrow{\phi}^{in}_{\lambda 1} = \frac{1}{2\sqrt{\pi\lambda_1}} e^{-i\lambda_1 v'}$$

$$\lim_{v' \to -\infty} \overleftarrow{\phi}^{in}_{1} = 0 \qquad , \quad \lambda_1 > 0 \tag{12}$$

to form a complete basis.

Alternatively, with no event horizons, one could choose

$$f'(\pm\infty) = \text{finite constant } C_\pm \neq 0,$$

i.e. asymptotically inertial frames, and then $\left\{ \overrightarrow{\phi}_\lambda^{in} , \overleftarrow{\phi}_\lambda^{in} \right\}$ would describe asymptotically massive plane waves (with effective mass not necessarily equal in the left and right asymptotic regions).

The solutions ϕ_{λ_1} are described for frequencies, λ_1, which are positive with respect to the accelerated time t'. As is well known a complete set of solutions describing positive frequencies ω with respect to the inertial time t is given by

$$\varphi_k = \frac{1}{4\pi\sqrt{\pi E_k}} e^{i(k_1 x + k_2 y + k_3 z - E_k t)}$$

$$\text{with } -\infty < k_1,\ k_2,\ k_3 < +\infty \quad , \quad E_k = +\sqrt{k_1^2+k_2^2+k_3^2+m^2} > 0 \tag{13}$$

We recall that in the formulation of Q.F.T. in accelerated spaces, the dynamical operators are defined in terms of the accelerated creation and annihilation operators C_λ , $C_\lambda{}^+$ associated with the accelerated modes ϕ_λ . The vacuum state of the theory ($|0\rangle$) is defined by the inertial operators a_k associated with the inertial modes φ_k, i.e.

$$a_k\ |0\rangle = 0 \quad , \forall\, k$$

The state $|^{in}0'\rangle$ such that $C_\lambda^{in}\ |^{in}0'\rangle = 0\ , \forall \lambda$, is an excited state with respect to the true vacuum $|0\rangle$. A Bogoliubov transformation relates $C_\lambda{}^{in}$ to a_k and $a_k{}^+$,

$$C_\lambda{}^{in} = \int_{-\infty}^{\infty} d^3k\ [\, A_\lambda\ (k)\ a_k + B_\lambda\ (k)\ a_k{}^+\ ] \tag{14}$$

$$\text{where} \quad A_\lambda\ (k) = \langle\, \phi_\lambda^{in}\ ,\ \varphi_k \rangle\ , \quad B_\lambda\ (k) = \langle\, \phi_\lambda^{in}\ ,\ \varphi_k^* \rangle \tag{15}$$

Alternatively to the solutions ϕ_λ^{in} , one can define solutions ϕ_λ^{out} by fixing the positive frequency boundary condition at the future rather than at the past, thus

$$\lim_{u' \to -\infty} \phi_\lambda^{out} = \frac{1}{2\sqrt{\pi\lambda}}\ e^{-i\lambda v'} \tag{16}$$

$$\lim_{v' \to +\infty} \phi_\lambda = 0$$

They satisfy $\phi_\lambda^{out}\ (u',\ v') = \phi_\lambda^{in\,*}\ (v',\ u')$

Analogously, we could define $C_\lambda^{\,out}$ and $|'0^{out}\rangle$ such that $C_\lambda^{\,out}|0'^{out}\rangle = 0$. Note that $|0'^{out}\rangle \neq |0'^{in}\rangle$. (Only in the Rindler frame $|0'^{out}\rangle = |0'^{in}\rangle$ up to a phase factor). As we always deal with the in-formulation we omit in what follows the superscript "in".

Before proceeding further it will be useful, as in the two dimensional case, to introduce the functions

$$N(\lambda, \lambda') \equiv \langle 0|C_\lambda \, C_{\lambda'}^{+} \, |0\rangle = \int_{-\infty}^{\infty} d^3k \, B_\lambda(k) \, B^*_{\lambda'}(k) \quad (17)$$

$$R(\lambda, \lambda') \equiv \langle 0|C_\lambda \, C_{\lambda'} \, |0\rangle = \int_{-\infty}^{\infty} d^3k \, A_\lambda(k) \, B_{\lambda'}(k) \quad (18)$$

$N(\lambda, \lambda')$ and $R(\lambda, \lambda')$ describe interferences between the created modes with different frequencies λ, λ'. $N(\lambda, \lambda')$ is the production function. For $\lambda = \lambda'$ it gives the number $N(\lambda)$ of λ-quanta in the vacuum $|0\rangle$ on the total volume. The number $N_V(\lambda)$ of λ-quanta per unit volume is obtained by introducing wave packets, ie

$$N_V(\lambda) = \lim_{\xi \to \infty} \iint_0^\infty d\lambda' \, d\lambda'' \, \mathcal{W}_\xi(\lambda, \lambda') \, \mathcal{W}^*_\xi(\lambda', \lambda'') \, N(\lambda', \lambda'')$$

$\mathcal{W}_\xi$ is such that $\int_0 d\lambda \; |\mathcal{W}_\xi(\lambda, \lambda')|^2 = 1$. For instance

$$\mathcal{W}_\xi(\lambda, \lambda') = \sqrt{2\xi/\pi} \; \exp[-\xi(\lambda - \lambda')^2]$$

From eqs (10) (13) and (15), we find

$$B_\lambda(k) = B_{\lambda_1}(k_1, M) \, \delta(k_2 + \lambda_2) \, \delta(k_3 + \lambda_3) \quad , \; M^2 = \lambda_2^2 + \lambda_3^2 + m^2 \quad (19.a)$$

where

$$B_{\lambda_1}(k_1, M) = \frac{-(k_1 + E_k)}{4\pi\sqrt{\lambda_1 E_k}} \int_{u_-}^{u_+} du\, e^{-i\lambda_1 F(u) - \frac{i}{2}(k_1 + E_k)u} \quad (19.b)$$

Then, we get

$$N(\lambda_1, \lambda_1') = \int_{-\infty}^{\infty} dk_1 \, B_{\lambda_1}(k_1) B^*_{\lambda_1'}(k_1) = -\frac{1}{4\pi^2}\frac{1}{\sqrt{\lambda_1\lambda_1'}} \int_0^\infty du \, du_1 \, \frac{e^{i\lambda_1 F(u + i\varepsilon) - i\lambda_1' F(u_1 - i\varepsilon)}}{(u - u_1 + i\varepsilon)^2} \, , \; \varepsilon > 0 \quad (20.a)$$

$$R(\lambda_1,\lambda_1')=\int_{-\infty}^{\infty}dk_1\,A_{\lambda_1}(k_1)B^*_{\lambda_1'}(k_1)=+\frac{1}{4\pi^2}\frac{1}{\sqrt{\lambda_1\lambda_1'}}\int_0^{\infty}du\,du_1\,\frac{e^{-i\lambda_1F(u+i\varepsilon)-i\lambda_1'F(u_1-i\varepsilon)}}{(u-u_1+i\varepsilon)^2}\,,\ \varepsilon>0 \tag{20.b}$$

The $B_\lambda(k)$ coefficient factorizes in the product of a two dimensional massive coefficient of effective mass M and two delta functions. $N(\lambda,\lambda')$ and $R(\lambda,\lambda')$ also factorize in this way but, because of boundary conditions (3), they are independent of the mass of the field. Conversely, given $N(\lambda,\lambda')$ we reconstruct the mapping, i.e.

$$\begin{cases} f(u') = f(u'_0)\exp\left(-4\pi i\,\mathrm{Re}\int_0^{\infty}\frac{d\lambda}{\lambda}\,e^{i\lambda u'}\left[\sqrt{\lambda\lambda'}\,N(\lambda,\lambda')\right]_{\lambda'=0}\right) \\ y' = y \\ z' = z \end{cases} \tag{21}$$

where $f(u'_0)$ is an integration constant (scale factor of the transformation).

From eq. (21) we get the relation

$$\frac{1}{4\pi}\frac{d}{du'}\ln f(u') = \mathrm{Re}\int d\lambda\; e^{i\lambda u'}\left[\sqrt{\lambda\lambda'}\,N(\lambda,\lambda')\right]_{\lambda'=0} \tag{22}$$

III - UNICITY OF THE EXPONENTIAL MAPPING AND THERMAL PROPERTIES OF THE ACCELERATED FRAMES

From the above results, we prove the following theorem : each one of the following statements implies the two others

(i) The functions $N(\lambda,\lambda')$ and $R(\lambda,\lambda')$ have the form

$$N(\lambda,\lambda') = N_V(\lambda_1)\,\delta(\lambda_1-\lambda_1')\,\delta(\lambda_2-\lambda_2')\,\delta(\lambda_3-\lambda_3')$$

$$R(\lambda,\lambda') = 0$$

(ii) The Bogoliubov transformation can be decomposed as a two term one,

$$C_\lambda = [1+N_V(\lambda_1)]^{1/2}\,\tilde{C}_{\lambda(+)} - [N_V(\lambda_1)]^{1/2}\,\tilde{C}_{\lambda(-)}$$

(iii) The accelerated space is

$$f(u') = e^{2\pi T u'} \quad , \text{ i.e. } F(u) = \frac{1}{2\pi T} \ln u \tag{23}$$

where $T = [\lambda_1 N_{\gamma}(\lambda_1)]_{\lambda_1} = 0$ (24)

The equivalence of statements (i) and (iii) follows from eq. (16).

The equivalence of statements (i) and (ii) follows from the relation

$$A_{\lambda}(k) = \left[\frac{1 + N_{\gamma}(\lambda_1)}{N_{\gamma}(\lambda_1)} \right]^{\frac{1}{2}} B_{\lambda}(k)$$

which is necessary and sufficient condition for the Bogoliubov transformation being decomposable. This condition allows us to define a basis

$$\tilde{C}_{\lambda(+)} = \int_{-\infty}^{\infty} d^3k \; \frac{A_{\lambda}(k)}{[1 + N_{\gamma}(\lambda_1)]^{1/2}} a_k$$

$$\tilde{C}_{\lambda(-)} = \int_{-\infty}^{\infty} d^3k \; \frac{B_{\lambda}^{*}(k)}{[N_{\gamma}(\lambda_1)]^{1/2}} a_k^{+}$$

such that

$$[\tilde{C}_{\lambda(+)}, \tilde{C}^{+}_{\lambda'(+)}] = \int_{-\infty}^{\infty} d^3k \; A^{*}_{\lambda}(k) A_{\lambda'}(k) = [1 + N_{\gamma}(\lambda_1)] \; \delta^{(3)}(\lambda - \lambda') \tag{25.a}$$

$$[\tilde{C}_{\lambda(-)}, \tilde{C}^{+}_{\lambda'(-)}] = \int_{-\infty}^{\infty} d^3k \; B^{*}_{\lambda}(k) B_{\lambda'}(k) = N_{\gamma}(\lambda_1) \; \delta^{(3)}(\lambda - \lambda') \tag{25.b}$$

$$[\tilde{C}_{\lambda(+)}, \tilde{C}^{+}_{\lambda'(-)}] = \int_{-\infty}^{\infty} d^3k \; A_{\lambda}(k) B_{\lambda'}(k) = 0 = [\tilde{C}^{+}_{\lambda(+)}, \tilde{C}_{\lambda'(-)}] \tag{25.c}$$

We see that eqs (25.a) (25.b) give statement (i).

It should be noted that the Theorem defines f(u') as given by eq. (21) up to a bilinear transformation such that

$$\left(\frac{u - u_-}{u_+ - u}\right) = e^{2\pi T u'} \quad , u_- \leqslant u \leqslant u_+ \tag{26}$$

y' and Z' are defined up to a linear transformation on t', namely

$$y = y' + \sigma t'$$

$$z = z' + \gamma t'$$

i.e. up to a drifting (or uniform rotation) in y and Z ; σ and γ are constants.

A Corollaire of the Theorem is the following : If $N(\lambda, \lambda')$ satisfies the statement (i), then $N_V(\lambda)$ is given by

$$N_V(\lambda) = \frac{1}{(e^{\lambda/T} - 1)} \quad , \tag{27}$$

but the converse is not true.

We see that the parameter T as defined by eq. (24) plays the role of a temperature. For any of the statements (i), (ii), (iii), eq. (22) is equal to a constant of value $2\pi T$. The theorem characterizes a situation of global thermal equilibrium over the whole accelerated space. This situation implies the presence of events horizons. The Rindler frame has one event horizon (eq. 23) or two event horizons at the same temperature (eq. 26). Note that the presence of event horizons is a necessary (but not a sufficient) condition for global thermal equilibrium.

A local (or asymptotic) thermal equilibrium situation is described by the class of mappings f(u') such that

$$\lim_{u' \to \pm\infty} f(u') = e^{2\pi T_\pm u'} \quad , \tag{28.a}$$

or equivalently by

$$\lim_{\lambda \to \lambda'} N(\lambda, \lambda') = N_V(\lambda_1) \; \delta^{(3)}(\lambda - \lambda')$$

$$\lim_{\lambda \to \lambda'} R(\lambda, \lambda') = 0 \tag{28.b}$$

The accelerated spaces corresponding to eq. (28.a) or (28.b) have non-uniform acceleration but for $u' \to \pm\infty$ the acceleration becomes uniform, i.e. the systems become of Rindler type. For all these spaces $N_\gamma(\lambda)$ is given by eq. (27), or more generally by

$$N_\gamma(\lambda) = \frac{1}{2}\left[\frac{1}{(e^{\lambda/T_-} - 1)} + \frac{1}{(e^{\lambda/T_+} - 1)}\right] \quad (29)$$

The asymptotic temperature $T_\pm$ are given by

$$T_\pm = \lim_{u' \to \pm\infty} \int_0^\infty d\lambda \cos \lambda u' \left[\sqrt{\lambda\lambda'}\, N(\lambda, \lambda')\right]_{\lambda'=0} \quad (30)$$

or equivalently by

$$T_\pm = \frac{1}{2\pi}\frac{d}{du'}\left[\ln f(u')\right]_{u'=\pm\infty} = \frac{1}{2\pi}\left[\sqrt{g_{00}}\, a\right] \quad (31)$$

where $a = \frac{1}{[\Lambda(x',t')]^{1/2}} \partial_{x'} \ln \Lambda(x',t')$ is the acceleration.

A typical non-thermal situation is described by mappings f(u') such that

$$\lim_{u' \to \pm\infty} f(u') = \frac{\alpha}{u'} + \beta u' \quad (32)$$

or equivalently by

$$\lim_{\lambda \to \lambda'} N(\lambda, \lambda') = N(\lambda_1)\, \delta(\lambda_2 - \lambda_2')\, \delta(\lambda_3 - \lambda_3')$$

$$\lim_{\lambda \to \lambda'} R(\lambda, \lambda') = R(\lambda_1)\, \delta(\lambda_2 + \lambda_2')\, \delta(\lambda_3 + \lambda_3') \quad (33)$$

where $N(\lambda_1)$ and $R(\lambda_1)$ are non-vanishing finite functions of λ_1. For these spaces, the acceleration is non-uniform, particle production takes place in a non-thermal situation and confined within a finite volume of the space (the total vacuum energy is finite). All the spaces of this class have

$$T_\pm = 0 \quad \text{and} \quad N_\gamma(\lambda_1) = 0 \quad (34)$$

IV - THE ENERGY MOMENTUM TENSOR AND ITS RENORMALIZATION

When normalization with respect to some particular state which has been defined to be empty is not appropriate, it has become a practice in this field to introduce a complete renormalization of the theory, essentially by extending techniques developed for empty Minkowski space. Whatever the problems implied by that approach, we will look, for the present, at the properties of some of the schemes adopted to carry out the combined regularization and renormalization of the theory, with an emphasis on the point-separation scheme. This has the advantage for the task at hand that not only is it defined entirely in and on the original manifold (as is also zeta function regularization) but the divergences can actually be obtained from the divergent terms of the effective action given by the vacuum to vacuum amplitude. Thus, in a sense, the renormalization is immediate, with both infinite and finite subtractions being given by those terms in the effective action which lead to divergences. Dimensional regularization involves an extension of the space time with extra flat dimensions and may not be very appropriate in curved spaces where differences from other schemes have occurred for some higher spin fields. Our explicit representation of the fields means that point separation is an appropriate regularization scheme to discuss here.

We first consider the Green function $G(P_1, P_2)$ and the renormalization of $\langle 0|\Psi^2|0\rangle$. Inertial and accelerated observers define the same Green function

$$G(P_1, P_2) = \langle 0| [\Psi(P_1), \Psi(P_2)]_+ |0\rangle \qquad (35)$$

for the free fields. Inertial observers express G in terms of the modes φ_k ; accelerated observers express G in terms of the modes ϕ_λ

$$G(P_1, P_2) = \int d^3k\; \varphi_k(P_1)\, \varphi_k^*(P_2) =$$

$$= \int d^3\lambda\; \phi_\lambda(P_1)\, \phi_\lambda^*(P_2) \; +$$

$$+ 2\,\mathrm{Re} \iint d^3\lambda\, d^3\lambda' \left[N(\lambda,\lambda')\, \phi_\lambda(P_1)\, \phi^*_{\lambda'}(P_2) + R(\lambda,\lambda')\, \phi_\lambda(P_1)\, \phi_{\lambda'}(P_2) \right] \qquad (36)$$

It can be noted that

$$G'(\rho_1, \rho_2) = \langle 0' | \left[\Psi(\rho_1), \Psi(\rho_2) \right]_+ | 0' \rangle = \int d^3\lambda \, \phi_\lambda(\rho_1) \, \phi_\lambda^*(\rho_2) \tag{37}$$

is the "false" Green function defined with respect to the accelerated state $|0'\rangle$. G' is not translationally invariant. As well known, when one attempts to calculate $\langle 0 | \Psi^2 | 0 \rangle$, divergences appear :

$$\langle 0 | \Psi^2 | 0 \rangle = \int_{-\infty}^{\infty} d^3k \, |\varphi_k|^2 = \mathcal{F}' + \langle 0' | \Psi^2 | 0' \rangle \tag{38}$$

where

$$\mathcal{F}' = 2\mathrm{Re} \iint d^3\lambda \, d^3\lambda' \left[N(\lambda, \lambda') \phi_\lambda \phi_{\lambda'}^* + R(\lambda, \lambda') \phi_\lambda \phi_{\lambda'} \right] \tag{39.a}$$

$$\langle 0 | \Psi^2 | 0 \rangle = \int d^3\lambda \, |\phi_\lambda|^2 \tag{39.b}$$

The ℓ.h.s. of equation (38) is divergent. In the r.h.s. the first term ($\mathcal{F}'$) is finite whereas the second also has a divergent part. In fact, in the accelerated state, the divergences of $\langle 0' | \Psi^2 | 0' \rangle$ are exactly those which appear in $\langle 0 | \Psi^2 | 0 \rangle$, that is they are independent of the particular state chosen. This divergent behaviour is general for the expectation value of any composite operator. With respect to the eigen modes of the accelerated state, $\langle 0 | \Psi^2 | 0 \rangle$ separates into a finite term plus a term which can be recasted as the expectation value on the state $|0'\rangle$. This term contains the infinite part. The problem is then to separate the divergent and finite parts of the term given by eq. (39.b) and to justify discarding the divergences. Although usually thought of as resulting from the short distance behaviour, the divergences in eq. (38) or in the integral (39.b) are actually governed by the properties of the mode functions in the asymptotic regions of the space time . The apparent divergence dependence on the asymptotic region comes precisely from the fact that, whatever the asymptotic region, we always choose fields which, asymptotically, are "free" fields in that regions. Thus we will always have the divergence appearing exactly in the way that it occurs for the vacuum in Minkowski space. This allows us to substract the divergences in the accelerated frame by following a similar procedure to that in the inertial frames. In the inertial frame, with some regulator ϵ, we would have

$$\langle 0|\Psi^2(\varepsilon)|0\rangle = \mathrm{IFP}\langle 0|\Psi^2(\varepsilon)|0\rangle + \mathrm{IDP}\langle 0|\Psi^2(\varepsilon)|0\rangle$$

where $\lim_{\varepsilon \to 0} \mathrm{IFP}\langle 0|\Psi^2(\varepsilon)|0\rangle < \infty$

$$\lim_{\varepsilon \to 0} \mathrm{IDP}\langle 0|\Psi^2(\varepsilon)|0\rangle = \infty$$

I.D.P. and I.F.P. stand for the inertial divergent and finite parts respectively. By substracting IDP $\langle 0|\Psi^2(\varepsilon)|0\rangle$ and letting $\varepsilon \to 0$, one obtains the (Inertial) renormalized quantity

$$\mathrm{IRen}\langle 0|\Psi^2|0\rangle = \lim_{\varepsilon \to 0}\left[\langle 0|\Psi^2(\varepsilon)|0\rangle - \mathrm{IDP}\langle 0|\Psi^2(\varepsilon)|0\rangle\right] = 0 \cdot \tag{40.a}$$

In the accelerated frame we follow a similar procedure (but the inertial (ε) and accelerated (ε') regulators need not necessarily be identical

$$\langle 0|\Psi^2(\varepsilon')|0\rangle = \mathcal{F}^{-1} + \langle 0'|\Psi^2(\varepsilon')|0'\rangle$$

with

$$\langle 0'|\Psi^2(\varepsilon')|0'\rangle = \mathrm{AFP}\langle 0'|\Psi^2(\varepsilon')|0'\rangle + \mathrm{ADP}\langle 0'|\Psi^2(\varepsilon')|0'\rangle$$

$$\lim_{\varepsilon' \to 0} \mathrm{AFP}\langle 0'|\Psi^2(\varepsilon')|0'\rangle < \infty$$

$$\lim_{\varepsilon' \to 0} \mathrm{ADP}\langle 0'|\Psi^2(\varepsilon')|0'\rangle = \infty$$

Whence, we substract ADP$\langle 0'|\Psi^2(\varepsilon')|0'\rangle$ to obtain the (Accelerated) renormalized quantity

$$\mathrm{ARen}\langle 0'|\Psi^2|0'\rangle = \lim_{\varepsilon \to 0}\left[\langle 0'|\Psi^2(\varepsilon')|0'\rangle - \mathrm{ADP}\langle 0'|\Psi^2(\varepsilon')|0'\rangle\right] \tag{40.b}$$

since in the Minkowski vacuum IRen$\langle 0|\Psi^2|0\rangle = 0$. Here A.FP and A.DP stand for the accelerated finite and divergent parts respectively ; A.Ren stands for the renormalized value in the accelerated state. Clearly, renormalization in flat (Minkowski) space can be made easier than it would be in general, since vacuum expectation values of $\hat{\Psi}^2$ (and $\hat{T}_{\mu\nu}$) are zero in the inertial vacuum, and our observations here are also sufficient for a discussion of the renormalization in the interacting case.

From eqs 40.a and 40.b we see that whereas field quantization lead to identical divergences, renormalization has assigned different finite values to the same operator in different quantum states. This is irrespective of the adopted regularisation scheme. Thus $\langle \hat{T}_{\mu\nu} \rangle_{reg.}$ need not be covariant (e.g. the introduction of a cut-off in the high momenta, or ordinary "point-splitting" are not) but $\langle T_{\mu\nu} \rangle_{ren}$ should be. We refer to |4| for a critical discussion of various proposals of regularization. For later use we note the possiBility of evaluating $\langle 0|\Psi^2|0\rangle$ as the $\lim_{\rho_1 \to \rho_2} G(\rho_1, \rho_2)$ in a space time representation rather than in a mode sum . However, for arbitrary mappings we will need to use the modes ϕ_λ in calculating the divergent part (eq. 39.b) in order to obtain a finite $\langle 0'|\Psi^2|0'\rangle_{ren}$. It is only the convenient representation of quantized fields asymptotically in some particular coordinate system which has sometimes lead to the incorrect notion that the states associated with those fields are coordinate dependent : of course they are not.

We now consider the energy momentum operator (for a minimally coupled field) given by

$$\hat{T}_{\mu\nu} = \partial_\mu \hat{\Psi} \partial_\nu \hat{\Psi} - \frac{1}{2} g_{\mu\nu} \left(\partial^\sigma \hat{\Psi} \partial_\sigma \hat{\Psi} + m^2 \hat{\Psi}^2 \right) \tag{32}$$

With respect to eigen modes for the accelerated state, the vacuum expectation value $\langle 0| \hat{T}_{\mu\nu} |0\rangle$ can be expressed as

$$\langle 0| \hat{T}_{\mu\nu} |0\rangle = 2\mathrm{Re} \iint d^3\lambda\, d^3\lambda' \left[N(\lambda\lambda') T_{\mu\nu}(\phi_\lambda, \phi^*_{\lambda'}) + R(\lambda,\lambda')\, T_{\mu\nu}(\phi_\lambda, \phi_{\lambda'}) \right] + \\ + \quad \langle 0'| \hat{T}_{\mu\nu} |0'\rangle \tag{41}$$

where

$$\langle 0'| \hat{T}_{\mu\nu} |0'\rangle = \int d^3\lambda \;\; T_{\mu\nu} (\phi_\lambda , \phi^*_\lambda) \tag{42}$$

and

$$T_{\mu\nu} (\phi, \varphi) = \partial_\mu \phi\, \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \left(g^{\sigma\rho} \partial_\sigma \phi\, \partial_\rho \varphi + m^2 \phi \varphi \right)$$

Derivatives and indices generally will refer to primed (accelerated) coordinates. In our coordinates defined by eq. (1), $T_{\mu\mu}$ and T_{0i} components can be concisely expressed as

$$T_{\mu\mu}(\phi,\varphi) = \partial_{t'}\phi\,\partial_{t'}\varphi + \alpha_\mu\,\partial_{x'}\phi\,\partial_{x'}\varphi + \Lambda\beta_\mu\,(\partial_{y'}\phi\,\partial_{y'}\varphi + \\ + \alpha_\mu\,\partial_{z'}\phi\,\partial_{z'}\varphi + \gamma_\mu\, m^2\,\phi\varphi)$$

$$T_{0i}(\phi,\varphi) = \partial_{t'}\phi\,\partial_{t'}\varphi$$

where

$$\left.\begin{array}{ccccc} \alpha_0 & = & \beta_0 & = & \gamma_0 \\ \alpha_1 & = & -\beta_1 & = & \gamma_1 \\ -\alpha_2 & = & \beta_2 & = & -\gamma_2 \\ -\alpha_3 & = & -\beta_3 & = & \gamma_3 \end{array}\right\} = 1$$

The vacuum energy and momentum densities are given by $\langle 0'|\,\hat{T}_{00}|0'\rangle$ and $\langle 0'|-\hat{T}_{0i}|0'\rangle$ respectively. By using eqs (19.c) and (19.d) and performing the integrals in λ'_2 and λ'_3 we see that the diagonal oo and oi components can be written directly in terms of their two dimensional counterparts with the modified M^2 and the 22 and 33 in terms of a modified ~ ~rator which we can write for all cases as

$$T^{(2)}_{\mu\mu}(\phi,\varphi\,;M_\mu) = \partial_{t'}\phi\,\partial_{t'}\varphi + \alpha_\mu\,\partial_{x'}\phi\,\partial_{x'}\varphi + \Lambda\,M_\mu^2\,\phi\varphi$$

with $$M_\mu^2 = \beta_\mu(\lambda_2^2 + \alpha_\mu\lambda_3^2 + \gamma_\mu m^2)\quad ,\ (M_0^2 \equiv M^2)$$

Then

$$\langle 0|\,\hat{T}^{(4)}_{\mu\mu}(x',t';m)|0\rangle = \iint_{-\infty}^{\infty} d\lambda_2\,d\lambda_3\,\langle 0|\,\hat{T}^{(2)}_{\mu\mu}(x',t';M_\mu)|0\rangle$$

where

$$\langle 0|\,\hat{T}^{(2)}_{\mu\mu}(x',t';M_\mu)|0\rangle = 2\mathrm{Re}\iint_0^\infty d\lambda_1\,d\lambda_1'\,[N(\lambda_1,\lambda_1')\,T^{(2)}_{\mu\mu}(\phi_{\lambda_1},\phi^*_{\lambda_1'};M_\mu) + R(\lambda_1,\lambda_1')\,T^{(2)}_{\mu\mu}(\phi_{\lambda_1},\phi_{\lambda_1'};M_\mu)] \\ + \int_0^\infty d\lambda_1\,T^{(2)}_{\mu\mu}(\phi_{\lambda_1},\phi^*_{\lambda_1};M_\mu)\ .$$

Analogously for $\langle 0|\,\hat{T}_{0i}\,|0\rangle$; $\langle\hat{T}_{00}{}^{(2)}\rangle$ is just the two dimensional energy density with M^2 instead of m^2.

For the purposes of computation , it is useful to introduce the quantity

$$G_{\mu\nu} = \langle 0|\,\partial_\mu\hat{\Psi}\,\partial_\nu\hat{\Psi}\,|0\rangle = 2\mathrm{Re}\iint d^2\lambda\,d^2\lambda'\,[N(\lambda,\lambda')\,\partial_\mu\psi_\lambda\,\partial_\nu\psi^*_{\lambda'} + R(\lambda,\lambda')\,\partial_\mu\psi_\lambda\,\partial_\nu\psi_{\lambda'}] + \\ + \int d^2\lambda\,\partial_\mu\psi_\lambda\,\partial_\nu\psi^*_\lambda \tag{43}$$

and give explicitly

$$H = \langle 0 | \hat{T}_{00} | 0 \rangle = \frac{1}{2} \left\{ G_{00} + G_{11} + \Lambda \left(G_{22} + G_{33} + m^2 G \right) \right\}$$

$$P_i = \langle 0 | \hat{T}_{0i} | 0 \rangle = - G_{0i}$$

$$\langle 0 | \hat{T}_{11} | 0 \rangle = \frac{1}{2} \left\{ G_{00} + G_{11} - \Lambda \left(G_{22} + G_{33} + m^2 G \right) \right\}$$

$$\langle 0 | \hat{T}_{22} | 0 \rangle = \frac{1}{2\Lambda} \left\{ G_{00} - G_{11} + \Lambda \left(G_{22} - G_{33} - m^2 G \right) \right\}$$

$$\langle 0 | \hat{T}_{33} | 0 \rangle = \frac{1}{2\Lambda} \left\{ G_{00} - G_{11} - \Lambda \left(G_{22} - G_{33} + m^2 G \right) \right\}$$

where $G = \langle 0 | \hat{\Psi}^2 | 0 \rangle$ is given by eq. (36)

According to the renormalization prescription given by eq. (40), we have

$$\begin{aligned} A.Ren \langle 0 | \hat{T}_{\mu\nu} | 0 \rangle &= \langle 0 | \hat{T}_{\mu\nu} | 0 \rangle - A.DP \langle 0' | \hat{T}_{\mu\nu} | 0' \rangle \\ &= 2 \mathrm{Re} \iint d^3\lambda \, d^3\lambda' \left[N(\lambda, \lambda') T_{\mu\nu} (\phi^*_\lambda, \phi_{\lambda'}) + R(\lambda \lambda') T_{\mu\nu} (\phi_\lambda, \phi_{\lambda'}) \right. \\ &\quad \left. + A.FP \int d^3\lambda \, T_{\mu\nu} (\phi^*_\lambda, \phi_{\lambda'}) \right] \end{aligned} \tag{44}$$

It is implied here that a regulator (ε') is introduced and the $\varepsilon' \rightarrow 0$ limit is taken after substraction.

A - Applications

By way of application, we consider the explicit evaluation of our formalism for the example of Rindler space ; which as we have shown, has unique thermal properties globally. The mapping which gives rise to Rindler space is given by

$$f(x' \pm t') = l \, e^{\alpha (x' \pm t')}$$

(ℓ and α^{-1} are unrelated length scales)
so that

$$\Lambda = \ell^2 f'(x'+t')\, f'(x'-t') = \ell^2\alpha^2 e^{2\alpha x'}$$

The explicit solution of (8) satisfying boundary conditions (9) can be found by evaluating (10), or in this case, directly : i.e.

$$\psi_\lambda = \frac{1}{2\pi} e^{i(\lambda_2 y' + \lambda_3 z')} \phi_{\lambda_1}(x', t') , \qquad (45.a)$$

$$\phi_{\lambda_1}(x',t') = \frac{(\ell M/2)^{-i\lambda_1/\alpha} e^{-i\lambda_1 t'}}{\sqrt{\pi\lambda_1}\;\Gamma(-i\lambda_1/\alpha)} K_{-i\lambda_1/\alpha}(\ell M e^{\alpha x'}) \qquad (45.b)$$

where $K_\nu(z)$ is a modified Bessel function. Then, from eqs (19) and (20) (or equivalently (19) and (18)) we find

$$B_{\lambda_1}(k, M) = \frac{1}{2\pi\alpha}\sqrt{\frac{\lambda_1}{E}}\,\Gamma(-i\lambda_1/\alpha)\, e^{-\pi\lambda_1/2\alpha} \left(\frac{k_1 + E}{2}\right)^{i\lambda_1/\alpha}$$

$$A_{\lambda_1}(k, M) = e^{\lambda\alpha\pi} B_{\lambda_1}(k, M)$$

$$N(\lambda, \lambda') = \frac{\delta^{(3)}(\lambda - \lambda')}{(e^{2\pi\lambda_1/\alpha} - 1)} \qquad (47.a)$$

$$R(\lambda, \lambda') = 0 \qquad (47.b)$$

Since the full field is given by (45.a) we have immediately $\partial_{t'}\psi_\lambda = -i\lambda_1\psi_\lambda$, $\partial_{y'}\psi_\lambda = -i\lambda_2\psi_\lambda$, $\partial_{z'}\psi_\lambda = i\lambda_3\psi_\lambda$. Thus, for example, in G_{02} we have

$$2\,\mathrm{Re}\iint d^3\lambda\, d^3\lambda' \frac{\delta^{(3)}(\lambda-\lambda')}{(e^{2\pi\lambda_1/\alpha} - 1)}(i\lambda_1\psi_\lambda^*)(i\lambda_2'\psi_{\lambda'}) = 2\,\mathrm{Re}\int_0^\infty d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2\, d\lambda_3 \frac{(-\lambda_1\lambda_2)\,\psi_\lambda^*\psi_\lambda}{(e^{2\pi\lambda_1\alpha} - 1)} =$$

$$2\,\mathrm{Re} \int_0^\infty d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2\, d\lambda_3 \frac{(-\lambda_1\lambda_2)e^{-\pi\lambda_1/\alpha}}{8\pi^4\alpha} K_{i\lambda_1/\alpha}(Me^{\alpha x'})\, K_{-i\lambda_1/\alpha}(Me^{\alpha x'}) = 0 .$$

This last result following since the λ_1 and λ_3 integrals are finite and the λ_2 integral is odd ; recall from (8) that M depends on λ_2 and λ_3. Similarily, we find that the corresponding term in the expressions for G_{03} and G_{23} also vanish. And, since $K_{\nu} = K_{-\nu}$ we have in G_{01}

$$2\,\mathrm{Re} \iint d\lambda\, d\lambda' \frac{\delta(\lambda-\lambda')}{(e^{2\pi\lambda_1/\alpha}-1)} (i\lambda_1 \psi_\lambda^*)(\partial_{x'}\psi_{\lambda'}) = 2\,\mathrm{Re} \int d\lambda \frac{i\lambda_1}{(e^{2\pi\lambda_1/\alpha}-1)} (\partial_{x'}\psi_\lambda)\psi_\lambda = 0,$$

where the integrals are all finite but the integration result is purely imaginary. The same result follows for this term in G_{12} and G_{13}. Thus we have shown in particular that each $\langle \hat{T}_{0i} \rangle$ (component of the Poynting vector) is zero, in accordance with the global thermal equilibrium properties enjoyed by Rindler space time. The same is true for the vacuum expectation value of the angular momentum operator $\hat{L}^{\mu\nu} = \int_\Sigma dv \,(x^\mu \hat{T}^{o\nu} - x^\nu \hat{T}^{o\mu})$. This cannot be considered, however, as characterizing an isotropic thermal radiation for the Rindler vacuum. Rindler observers have a preferred direction namely their spatial direction of acceleration. In Rindler space, $\langle \hat{T}_{00} \rangle$ is a constant up to a dependence in x' through $\sqrt{g_{00}} = \alpha e^{\alpha x'}$. One could regard the gradient $\partial_{x'} \langle \hat{T}_{00} \rangle \check{e}_{x'}$ as defining a preferred direction of the thermal vacuum. On the other hand, the response of a uniformly accelerated detector model with a directional discrimination has been found to be non-isotropic [5-7]. It is not possible to construct an accelerated flat space time for which the vacuum is spherically symmetric and in global thermal equilibrium. The mapping

$$r \pm t = e^{\alpha (r' \pm t')} \quad , \quad \varphi = \varphi' \quad , \quad \Theta = \Theta' \qquad (48)$$

(r, φ , Θ , being spherical type coordinates) yields to

$$ds^2 = \alpha^2 e^{2\alpha r'} (dr'^2 - dt'^2) + e^{2\alpha r'} \cosh^2 \alpha t' \, d\Omega^2(\theta,\varphi) .$$

The vacuum spectrum $N_v(\lambda)$ is Planckian with asymptotic temperature $\alpha/2\pi$, but there is an asymptotic (rather than global) thermal equilibrium situation in this case.

It is useful at this point to consider the case of two dimensions, for which the solutions to the wave equation and Bogoliubov coefficients are given directly by eqs (45.b) and (46) with M=m ; N and R are given by eqs. (47) with $\lambda = \lambda_1$. Thus, we have for G

$$G = AF\rho \int_0^\infty d\lambda_1 \coth \frac{\pi\lambda_1}{\alpha} \phi^*_{\lambda_1} \phi_{\lambda_1}$$

$$= AF\rho \int_0^\infty \frac{d\lambda_1}{\alpha\pi^2} \cosh \frac{\pi\lambda_1}{\alpha} K_{i\lambda_1/\alpha}(me^{\alpha x'}) K_{-i\lambda_1/\alpha}(me^{\alpha x'}) \quad (49)$$

The point of reconsidering the massive two dimensional case is the following : we find that a naive regularization scheme suggests a renormalization which (surprisingly) leads to a sensible physical result. We can examine this outcome rigorously only in the two dimensional massless case but particular computations for four dimensions show that a class of mappings exists for which a similar result might be proved. The special features we have used in two dimensions apparently need only be partially present in four dimensions. A simple evaluation of $\langle \hat{T}_{00} \rangle$ for any f(g) in the massless case can be obtained from a space time representation rather than from a mode sum for G. Point splitting is a natural choice of regularization in that case. For the purpose of this section, we will use ordinary point-splitting ; a fully covariant regularization is given in the next section.

The same kind of separation in eq. (38) between finite and infinite terms occurring in the accelerated frame can be observed if we consider $\langle 0|\Psi^2|0\rangle$ as $\lim_{P_1 \to P_2} G(P_1,P_2)$ in a space time representation. The square length

$$\sigma^2 = (x_1 - x_2)^2 - (t_1 - t_2)^2 = \Delta u \, \Delta v \ , \quad (\Delta u \equiv u_1 - u_2 \ , \quad \Delta v \equiv v_1 - v_2)$$

is expressed in terms of the mapping f,g as $\sigma^2 = [f(u_1') - f(u_2')][g(v_1') - g(v_2')]$.
In the inertial frame, when $\Delta u \to 0$, $\Delta v \to 0$, σ^{-2} diverges as $(\Delta u \, \Delta v)^{-1}$. In the accelerated frame, when $\Delta u' \to 0$, $\Delta v' \to 0$ ($\Delta u' \equiv u'_1 - u'_2$, $\Delta v' \equiv v'_1 - v'_2$) :

$$f(u_1') = f(u_2') + f'(u_2')\,\Delta u' + f''(u_2') \frac{\Delta u'^2}{2!} + f''' \frac{\Delta u'^3}{3!} + \cdots$$

$$g(v_1') = g(v_2') + g'(v_2')\,\Delta v' + g''(v_2') \frac{\Delta v'^2}{2!} + g''' \frac{\Delta v'^3}{3!} + \cdots$$

and then

$$\sigma^{-2} = \frac{1}{\Delta u' \, \Delta v' \, f' g'} \left[1 - a_f \Delta u' + b_f \Delta u'^2 \right] \left[1 - a_g \Delta v' + b_g \Delta v'^2 \right]$$

where

$$a_f = \frac{1}{2}\,\frac{f''}{f'} \qquad , \qquad b_f = \frac{1}{2}\left[\,\frac{1}{2}\left(\frac{f''}{f'}\right)^2 - \frac{1}{3}\,\frac{f'''}{f'}\,\right]$$

$$a_g = \frac{1}{2}\,\frac{g''}{g'} \qquad , \qquad b_g = \frac{1}{2}\left[\,\frac{1}{2}\left(\frac{g''}{g'}\right)^2 - \frac{1}{3}\,\frac{g'''}{g'}\,\right]$$

In the limit $\Delta u' \to 0$, $\Delta v' \to 0$, σ^{-2} separates in a finite plus an infinite term as

$$\sigma^{-2} = \frac{1}{f'g'}\left[\frac{1}{\Delta u'\Delta v'} - \frac{a_f}{\Delta v'} - \frac{a_g}{\Delta u'}\right] + \frac{1}{f'g'}\left[b_f\frac{\Delta u'}{\Delta v'} + b_g\frac{\Delta v'}{\Delta u'} + a_f a_g\right]$$

Even to have a well determined finite part here, one must be careful to specify how the limits $\Delta u' = (\Delta x' - \Delta t') \to 0$, $\Delta v' = (\Delta x' + \Delta t') \to 0$ should be taken. But this simple calculation shows how a purely divergent quantity acquires a different finite term depending on the "renormalization" scheme chosen (here expressed in curvilinear coordinates). Particularly illustrative of this is the evaluation of $\langle 0|\hat{T}_{\mu\nu}|0\rangle$ in the massless case. In two dimensions we have

$$\begin{aligned} G(\rho_1,\rho_2) &= -\frac{1}{4\pi}\ln\left[(u_1-u_2)(v_1-v_2)-i\varepsilon\right] \\ &= -\frac{1}{4\pi}\ln\left[(f_1-f_2)(g_1-g_2)-i\varepsilon\right] \end{aligned} \tag{50}$$

and

$$\begin{aligned} \mathcal{D}_{12}\,G &= \left(\partial_{u'_1}\partial_{u'_2} + \partial_{v'_1}\partial_{v'_2}\right)G = \\ &= -\frac{1}{4\pi}\left[\frac{f'_1 f'_2}{(f_1-f_2)^2} + \frac{g'_1 g'_2}{(g_1-g_2)^2}\right] \end{aligned} \tag{51}$$

here $\quad f_i \equiv f(u'_i) \quad , \quad g_i \equiv g(v'_i) \quad , \quad i = 1,2.$

In the limit $u'_1 \to u'_2$, we have

$$\frac{f'_1 f'_2}{(f_1-f_2)^2} = \frac{1}{\Delta u'^2} + \frac{1}{6}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]$$

Then

$$\mathcal{D}_{12}\, G = -\frac{1}{4\pi}\left\{ \frac{1}{\Delta u'^2} + \frac{1}{\Delta v'^2} + \frac{1}{6}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right] + \frac{1}{6}\left[\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right]\right\} + O(\Delta u') + O(\Delta v'). \tag{52}$$

On the other hand,

$$\mathcal{D}_{12}\, G' = \frac{1}{4\pi}\int_0^\infty \frac{d\lambda}{\lambda}\left(e^{i\lambda\Delta u'} + e^{i\lambda\Delta v'}\right) = -\frac{1}{4\pi}\left(\frac{1}{\Delta u'^2} + \frac{1}{\Delta v'^2}\right) \tag{53}$$

and we see that the divergent part appearing above is the whole of the operator for the accelerated state. Thus a substraction of the divergent part is equivalent to a normalization with respect to the accelerated state. So we have

$$\text{AFP}\,\langle 0|\hat{T}_{00}|0\rangle \equiv \langle 0|\hat{T}_{00}|0\rangle - \langle 0'|\hat{T}_{00}|0'\rangle =$$

$$= -\frac{1}{24\pi}\left\{\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right] + \left[\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right]\right\}. \tag{54}$$

In the Rindler case : $f = e^{\alpha u'}$, $g = e^{\alpha v'}$, each $f(g)$ term gives $-\frac{\alpha^2}{2}$, so

$$\text{AFP}\,\langle 0|\hat{T}_{00}|0\rangle = +\frac{\alpha^2}{24\pi} = \frac{\pi T^2}{6}$$

corresponding to the energy density of a Planckian gas at temperature $T = \alpha/2\pi$. We can also show how this result emerges from a mode sum representation. If we take point splitting in the time direction for simplicity, we obtain :

$$\langle 0|\hat{T}_{00}|0\rangle = \frac{1}{2\pi}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty d\lambda\, \lambda\, e^{i\lambda\varepsilon'\alpha/\pi} \coth\lambda =$$

$$= \frac{1}{2\pi}\left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{2}\,\zeta\left(2, -\frac{i\alpha\varepsilon'}{\pi}\right) + \frac{\pi^2}{\alpha^2\varepsilon'^2}\right]$$

where

$$\zeta\left(2, -\frac{i\alpha\varepsilon'}{2\pi}\right) = \sum_{n=0}^{\infty} \frac{1}{(n - i\alpha\varepsilon'/2\pi)^2} = -\frac{4\pi^2}{\alpha^2\varepsilon'^2} + \zeta(2)$$

Then $$\langle 0|\hat{T}_{00}|0\rangle = -\frac{1}{2\pi}\frac{1}{\varepsilon'^2} + \frac{\alpha^2}{24\pi}$$

and

$$AFP\langle 0|\hat{T}_{00}|0\rangle = \frac{\pi T^2}{6} \quad .$$

The same calculation in four dimensions starting from

$$G(\rho_1,\rho_2) = \frac{1}{4\pi}\frac{1}{(\sigma^2+\Delta y^2+\Delta z^2)} = \frac{1}{4\pi}\frac{1}{\Delta}$$

$$\Delta = \Delta f\,\Delta g + \Delta y^2 + \Delta z^2 \quad , \qquad \begin{aligned}\Delta f\,\Delta g &= (f_1-f_2)(g_1-g_2)\\ \Delta y &= (y_1-y_2)\\ \Delta z &= (z_1-z_2)\end{aligned}$$

leads to

$$\mathcal{D}_{12}\,G = \frac{1}{2\pi^2\Delta^3}\left[\Delta g^2 f_1' f_2' + \Delta f^2 g_1' g_2'\right] + \frac{\Lambda}{\pi^2\Delta}\left[\Delta y^2 + \Delta z^2 - \frac{1}{4\Delta^2}\right]$$

Now

$$\mathcal{D}_{12} \equiv \partial_{u_1'}\partial_{u_2'} + \partial_{v_1'}\partial_{v_2'} + \frac{\Lambda}{2}\left(\partial_{y_1}\partial_{y_2} + \partial_{z_1}\partial_{z_2}\right)$$

and

$$\langle \hat{T}_{00}\rangle = \Lambda \lim_{1\to 2} \mathcal{D}_{12}\,G \quad .$$

Taking the limit $\Delta y' \to 0$, $\Delta z' \to 0$ first :

$$\mathcal{D}_{12}\,G = \frac{1}{2\pi^2}\frac{1}{\Delta f\,\Delta g}\left[\frac{f_1' f_2'}{\Delta f^2} + \frac{g_1' g_2'}{\Delta g^2}\right] - \frac{\Lambda}{4\pi^2}\frac{1}{(\Delta f\,\Delta g)^2} \tag{55}$$

and then $u'_1 \rightarrow u'_2$ $(v'_1 \rightarrow v'_2)$, straight-forward but lengthy calculations [expansions to the 4th and 5th derivatives of $f(g)$ must be now included] , give

$$ADP \langle 0|T_{00}|0\rangle = -\frac{1}{2\pi^2}\,\frac{1}{\Delta u'^4}\left[1+(a-\tilde{B}-\tilde{A}^2)\Delta u'^2-(b+a\tilde{A}+\tilde{C}-\tilde{A}\tilde{B})\Delta u'^3\right]$$

$$+ \text{ (same term in } \Delta v' \text{)}$$

$$AFP\langle 0|T_{00}|0\rangle = \frac{1}{2\pi^2}\left[(a+\frac{\tilde{B}}{2})\tilde{B} + (b-\frac{\tilde{C}}{2})\tilde{A} + (c-\tilde{D})\right] \tag{56}$$

where

$$a = \frac{1}{6}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]$$

$$b = \frac{1}{(12)^2}\left[\frac{f^{IV}}{f'} - 4\frac{f''f'''}{f'^2} + 9\left(\frac{f''}{f'}\right)^3\right]$$

$$c = \frac{1}{(24)^2}\left[\frac{f^{V}}{f'} - 3\frac{f''f^{IV}}{f'^2} + 21\left(\frac{f''}{f'}\right)^2\frac{f'''}{f'} - \left(\frac{f'''}{f'}\right)^2\right]$$

$$\tilde{A} = A \quad , \quad \tilde{B} = (A^2 - B) \quad , \quad \tilde{C} = 2AB - C \quad , \tilde{D} = (B^2 - 2AC - D)$$

$$A = \frac{1}{2}\left[\frac{g''}{g'} - \frac{f''}{f'}\right]$$

$$B = \frac{1}{6}\left[\frac{g'''}{g'} + \frac{f'''}{f'} - \frac{3}{2}\frac{f''g''}{f'g'}\right]$$

$$C = \frac{1}{24}\left[\frac{g^{IV}}{g'} - \frac{f^{IV}}{f'} - 2\left(\frac{f''g''' - g''f'''}{f'g'}\right)\right]$$

$$D = \frac{1}{120}\left[\frac{g^{V}}{g'} + \frac{f^{V}}{f'} - \frac{5}{2}\left(\frac{f^{IV}}{f'} - \frac{g^{IV}}{g'}\right)\left(\frac{f''}{f'} - \frac{g''}{g'}\right) + \frac{10}{3}\frac{f'''g'''}{f'g'}\right]$$

In the Rindler case, these eqs give

$$a = \alpha^2/12 \quad , \quad b = \alpha^3/3 \quad , \quad c = \alpha^4/32$$

$$A = 0 \quad , \quad B = \alpha^2/12 \quad , \quad C = 0 \quad , \quad D = \frac{2}{45}\alpha^4$$

$$\tilde{A} = 0 \quad , \quad \tilde{B} = -\alpha^2/12 \quad , \quad \tilde{C} = 0 \quad , \quad \tilde{D} = \frac{3}{80}\alpha^4$$

$$AFP\langle 0|\hat{T}_{00}|0\rangle = \frac{1}{2\pi^2}\left(\frac{3}{2}\tilde{B}^2 + c - \tilde{D}\right)$$

$$= \frac{1}{2\pi^2}\,\frac{\alpha^4}{240} = \frac{\pi^2 T^4}{30} \quad , \qquad (57)$$

corresponding to a Planckian density energy at temperature $T = \dfrac{\alpha}{2\pi}$ in four dimensions.

In the two dimensional case, the finite part of D_{12} G is

$$-\frac{1}{24\pi}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right] - \frac{1}{24\pi}\left[\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right]$$

, where each factor is related to a Schwarzian derivative which, like the measure in eq. (20) is invariant under homographic transformations : $f \to (af+b)(cf+d)^{-1}$. But, from Davies and Fulling [8] or our covariant point splitting result later in this paper, this is just the negative of $A\,\mathrm{ren.}\langle 0'|\hat{T}_{00}|0'\rangle =$ $= \langle 0'|\hat{T}_{00}|0'\rangle - \langle 0|\hat{T}_{00}|0\rangle$. Thus we know that the subtracted divergence is exactly an expectation value with respect to some state, which happens to be (homographically equivalent to) the accelerated state, and that we have therefore normalized the Minkowski vacuum with respect to this state. In four dimensional case we stumble upon a similar result for the exponential mapping without being able to find the general class for which it is true.

B - ν Dimensions :

For massive fields, calculation by dimensional regularization is often very instructive. We shall first calculate $\langle \hat{T}_{\mu\lambda} \rangle$ in generic space time dimensions and then consider dimensionalities of interest. We analyze now $\langle \hat{T}_{\mu\lambda} \rangle$ as a function of the space time dimension.

The Green function in ν dimensions is given by

$$G(X) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{e^{ikX}}{k^2+m^2+i\varepsilon} = \frac{m^{\frac{\nu}{2}-1}}{(2\pi)^{\nu/2}} \frac{K_{\frac{\nu}{2}-1}[m\sqrt{X^2+i\varepsilon}]}{[\sqrt{X^2+i\varepsilon}]^{\frac{\nu}{2}-1}} \tag{58}$$

where

$$X^2 = -X_0^2 + X_1^2 + \ldots\ldots + X_{\nu-1}^2 \quad \text{and} \quad K_\lambda(z) = \frac{\pi}{2} \frac{I_{-\lambda}(z) + I_\lambda(z)}{\sin \pi\lambda}, \quad \text{Re}\lambda \neq 0$$

By using

$$I_\lambda(z) = \frac{1}{\Gamma(\lambda+1)} \left(\frac{z}{2}\right)^\lambda \left[1 + \frac{1}{(\lambda+1)}\left(\frac{z}{2}\right)^2 + \frac{1}{2(\lambda+1)(\lambda+2)}\left(\frac{z}{2}\right)^4 + O(z^6)\right.$$

we have

$$G(X) = \frac{m^{\nu/2}}{(4\pi)^{\nu/2}} \Gamma\left(1-\frac{\nu}{2}\right)\left[1 + \frac{m^2X^2}{2\nu} + O(X^4)\right] + \tag{59}$$

$$+ \frac{m^{2-\nu}}{4\pi^{\nu/2}} \Gamma\left(\frac{\nu}{2}-1\right)\left[1 + \frac{m^2X^2}{2(4-\nu)} + \frac{m^4X^4}{8(4-\nu)(6-\nu)} + O(X^6)\right.$$

By computing

$$\partial_\mu \partial_\lambda G(x) = \frac{m^\nu}{(4\pi)^{\nu/2}} \Gamma\left(1-\frac{\nu}{2}\right)\frac{g_{\mu\lambda}}{\nu} + \frac{\Gamma\left(\frac{\nu}{2}-1\right)(x^2)^{-\nu/2}}{4\pi^{\nu/2}} \cdot \Big\{$$

$$\cdot \Big\{ (2-\nu)g_{\mu\lambda} - \nu(2-\nu)\frac{X_\mu X_\lambda}{X^2} + \frac{m^2}{2}\left[X^2 g_{\mu\lambda} + (2-\nu)X_\mu X_\lambda\right]$$

$$+ \frac{m^4}{8}\left[\frac{X^4 g_{\mu\lambda}}{(4-\nu)} + X^2 X_\mu X_\lambda\right] \Big\} \quad ,$$

$$\partial_\mu \partial^\mu G(X) = \frac{m^\nu}{(4\pi)^{\nu/2}} \Gamma\left(1-\frac{\nu}{2}\right) + \frac{\Gamma\left(\frac{\nu}{2}-1\right)(X^2)^{1-\frac{\nu}{2}}}{4\pi^{\nu/2}} m^2 \left[1 + \frac{m^2 X^2}{2(4-\nu)} + O(X)\right]$$

we obtain

$$\langle T_{\mu\lambda}(X) \rangle = \lim_{X\to 0} \left[\partial_\mu \partial_\lambda - \frac{g_{\mu\lambda}}{2} (\partial_\alpha \partial^\alpha - m^2) \right] G(X)$$

$$= \langle T_{\mu\lambda}(x=0) \rangle + \langle \widetilde{T}_{\mu\lambda}(X) \rangle \qquad (60)$$

here

$$\langle T_{\mu\lambda}(X=0) \rangle = \frac{m^\nu}{(4\pi)^{\nu/2}} \Gamma\left(1-\frac{\nu}{2}\right)\left(\frac{1}{\nu}-1\right) g_{\mu\lambda} \qquad (61)$$

$$\langle \widetilde{T}_{\mu\lambda}(X) \rangle = \frac{(X^2)^{1-\nu/2}}{4\pi^{\nu/2}} \Gamma\left(\frac{\nu}{2}-1\right) \left\{ \frac{(2-\nu)}{X^2} \left(g_{\mu\lambda} - \nu \frac{X_\mu X_\lambda}{X^2} \right) - \right.$$
$$\left. - \frac{m^2}{2} \left[g_{\mu\lambda} - (2-\nu)\frac{X_\mu X_\lambda}{X^2} - \frac{m^2}{4} X_\mu X_\lambda + \frac{3}{4} \frac{m^2 X^2}{(4-\nu)} g_{\mu\lambda} + \frac{m^2 X^4}{8(4-\nu)(6-\nu)} g_{\mu\lambda} \right] \right\} \qquad (62)$$

Let be now $\nu \to d = 2,4$. We have

$$\nu = d + (\nu - d)$$

$$\frac{X^{-\nu}}{4\pi^{\nu/2}} = \frac{\beta^{d-\nu}}{4\pi^{d/2}} \frac{1}{X^d} \left[1 - (\nu - d) \log \frac{\sqrt{\pi} X}{\beta} + O(\nu - d)^2 \right]$$

$$\Gamma\left(1-\frac{\nu}{2}\right) = (-1)^{1-d/2} \left[\frac{2}{d-\nu} - C \right] + O(\nu - d)$$

$$\left(\frac{1}{\nu} - 1\right) = -\frac{1}{d} \left[d - 1 + \frac{1}{d}(\nu - d) + O(\nu-d)^2 \right]$$

We get

$$\langle T_{\mu\lambda}(X=0)\rangle_{\nu\to 2} = \frac{\beta^{\nu-2}}{4\pi}\frac{m^2}{2}g_{\mu\lambda}\left[\frac{2}{\nu-2}+\log\left(\frac{ma}{2\sqrt{\pi}\beta}\right)^2\right],\ a=e^{\frac{1}{2}(1+C)} \tag{63.a}$$

$$\langle T_{\mu\lambda}(X=0)\rangle_{\nu\to 4} = \frac{\beta^{\nu-4}}{(4\pi)^2}\frac{m^4}{2}g_{\mu\lambda}\left[-\frac{6}{\nu-4}-\log\left(\frac{m\beta}{2\sqrt{\pi}b}\right)^6\right],\ b=e^{-1/12} \tag{63.b}$$

and

$$\langle \tilde{T}_{\mu\lambda}(X)\rangle_{\nu\to 2} = \frac{\beta^{\nu-2}}{4\pi}\frac{m^2}{2}g_{\mu\lambda}\left[-\frac{2}{\nu-2}+\log a\pi\left(\frac{X}{\beta}\right)^2\right] - \tag{64.a}$$
$$-\frac{\beta^{\nu-2}}{4\pi}\left[2\frac{g_{\mu\lambda}}{X^2}-4\frac{X_\mu X_\lambda}{X^4}+m^2\frac{X_\mu X_\lambda}{X^2}\right].$$

$$\langle \tilde{T}_{\mu\lambda}(X)\rangle_{\nu\to 4} = \frac{\beta^{\nu-4}}{(4\pi)^2}\frac{m^4}{4}g_{\mu\lambda}\left[\frac{6}{\nu-4}-\log\left(\frac{\sqrt{\pi}X}{\beta}\right)^6\right] - \tag{64.b}$$
$$-\frac{\beta^{4-\nu}}{(4\pi)^2}\left[\left(\frac{8}{X^4}+2\frac{m^2}{X^2}\right)g_{\mu\lambda}-\left(\frac{32}{X^6}+\frac{4m^2}{X^4}+\frac{m^4}{2X^2}\right)X_\mu X_\lambda\right]$$

Now, if ν is used as a regulator, we take ν negative and $X = 0$. Thus, in dimensional regularisation $\langle T_{\mu\lambda}\rangle$ is given by eq. (61) and its expressions for $\nu \to 2, 4$ are eqs (63.a) (63.b). If we use X as a regulator, then $\langle T_{\mu\lambda}(x)\rangle$ is given by eq. (60) and its expressions for $\nu \to 2, 4$ are

$$\langle T_{\mu\lambda}(X)\rangle_{\nu\to 2} = \frac{\beta^{2-\nu}}{4\pi}\left\{\frac{m^2}{2}g_{\mu\lambda}\log\left(\frac{mXa}{2}\right)^2-2\frac{g_{\mu\lambda}}{X^2}+4\frac{X_\mu X_\lambda}{X^4}-m^2\frac{X_\mu X_\lambda}{X^2}\right\} \tag{65.a}$$

$$\langle T_{\mu\lambda}(X)\rangle_{\nu\to 4} = \frac{\beta^{4-\nu}}{(4\pi)^2}\left\{\frac{m^4}{4}g_{\mu\lambda}\log\left(\frac{mX}{2b}\right)^6-8\frac{g_{\mu\lambda}}{X^4}+\frac{32X_\mu X_\lambda}{X^6}-\frac{2m^2g_{\mu\lambda}}{X^2}+\right.$$
$$\left.+4m^2\frac{X_\mu X_\lambda}{X^4}+\frac{m^4}{X^2}\frac{X_\mu X_\lambda}{X^2}\right\}. \tag{65.b}$$

The poles at $\nu = 2, 4$ appearing in the $X = 0$ and $X \neq 0$ terms (63) (64) mutually cancel .

It can be noted that eq. (61) is only valid for $m \neq 0$. $\langle T_{\mu\lambda}\rangle$ as given by dimensional regularisation does not include the m = 0 limit. This is so because for X = 0, the only term contributing to G(X) comes from $I_{\nu/2-1}$ (X). The m = 0 contribution is contained in the $I_{\frac{\nu}{2}-1}$ (X) term which vanish at X = 0. To include the m = 0 term, $\langle T_{\mu\lambda}\rangle$ must be defined for X ≠ 0 and point splitting is required.

V - COVARIANT POINT SPLITTING REGULARIZATION

There is a very natural form of the regularized operator which suggests itself here. In keeping with the efforts of other authors, it embodies numerous properties expected from the classical definition. It can be easily built up piece by piece to reflect the symmetry, conservation and conformal tracelessness of the classical expression and by construction has the property of being parallelly transportable between the separated points which give rise to its definition. Somewhat surprisingly, in one way or another, it differs from the corresponding quantity used by other authors in the field, but only by terms which are odd in the separation, i.e. whose sign changes under parallel transport. Whatever the value of using it for a renormalization may be, it would seem worthwhile to record our regularized tensor here and to give some arguments for its adoption.

Constructing the tensor

From the point splitting point of view, just as the Green function is the fundamental object from which to construct quantities such as $\langle\psi^2\rangle$, we seek some analogous object for $\langle T_{\mu\nu}\rangle$ which preserves the Lorentz covariance and, where applicable, the conformal invariance : also, we seek an object which is formally conserved in the coincidence limit, the final result automatically being conserved, since renormalization takes place in the Lagrangian. However it is a little difficult to implement conservation for our regularized tensor unless the tensor itself is regular in the coincidence limit.

The classical expression for the stress-energy tensor of a scalar field has terms like $a_{\mu\nu}\,\psi(X)\,\psi(X)$, $\psi_{;\mu}(X)\,\psi_{;\nu}(X)$ and $\psi_{;\mu\nu}(X)\,\psi(X)$ occurring in it, where the (regular) $a_{\mu\nu}$ may be $g_{\mu\nu}$, $R_{\mu\nu}$, $R_{\mu\nu}$ $g_{\mu\nu}$

(contractions of these terms also appear). In the quantum theory, since these terms represent products of field operators (or their derivatives) at a point, they are divergent (ill-defined). However, terms such as $\psi(X)\psi(X')$, $\psi_{;\mu}(X)\ \psi_{;\nu'}(X')$, $\psi_{;\mu'}(X')\ \psi(X)$, $\psi_{;\mu\nu}(X)\ \psi(X')$, $\psi_{;\mu'\nu'}(X')\ \psi(X)$ are clearly well defined and will be suitable for use in building up a regularized quantity. In addition, there is an (essentially) unique object, the geodetic bivector of parallel displacement, $g_{\mu\nu'}$, which allows the indices expressing dependence at one point to be translated to the other, so that a properly tensorial quantity can be defined. Whatever other properties the stress-energy may have, since this parallel transport is needed for the definition of our regularized tensor, it would seem most useful to have a definition which is invariant under such transport between the separated points : i.e. $\hat{T}_{\mu\nu}(X,X') = g_\mu{}^{\mu'} g_\nu{}^{\nu'} \hat{T}_{\mu'\nu'}(X,X')$

We can think of this as a "symmetry" under $X \leftrightarrow X'$; terms lacking this symmetry would vanish in the classical expression obtained from the coincidence limit). Thus

$$\psi_{;\mu}(X)\,\psi_{;\nu}(X) \longrightarrow \frac{1}{2}\left[g_\mu{}^{\mu'}\psi_{;\mu'}(X')\psi_{;\nu}(X) + g_\nu{}^{\nu'}\psi_{;\mu}(X)\psi_{;\nu'}(X')\right]$$

$$\psi_{;\mu\nu}(X)\,\psi(X) \longrightarrow \frac{1}{2}\left[\psi_{;\mu\nu}(X)\psi(X') + g_\mu{}^{\mu'}g_\nu{}^{\nu'}\psi_{;\mu'\nu'}(X')\psi(X)\right]$$

and

$$a_{\mu\nu}(X)\,\psi(X)\psi(X) \rightarrow \frac{1}{2}\left[a_{\mu\nu}(X) + g_\mu{}^{\mu'}g_\nu{}^{\nu'}a_{\mu'\nu'}(X')\right]\psi(X)\psi(X)$$

In the first two cases, the symmetry under $\mu \leftrightarrow \nu$ is preserved, in the third case no symmetry whatever is used in the construction. The contraction terms in the regularized tensor follow unambiguously from these definitions, which lead to an explicit preservation of the vanishing of the trace in the conformally invariant theory. Thus, in a curved space time, with $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$, $G = [\psi(x_1), \psi(x_2)]_+$ and conformal coupling ξ , we will consider

$$\hat{T}^{\mu\nu} = \left(\tfrac{1}{2}-\xi\right)\left(g^{\nu}{}_{\nu'}\,G^{;\mu\nu'} + g^{\mu}{}_{\mu'}\,G^{;\nu\mu'}\right) + 2\left(\xi-\tfrac{1}{4}\right) g^{\mu\nu} g_{\rho\sigma'}\,G^{;\rho\sigma'} - \xi\left[G^{;\mu\nu} + g^{\mu}{}_{\mu'}\,g^{\nu}{}_{\nu'}\,G^{;\mu'\nu'}\right] + \xi\, g^{\mu\nu}\left[G_{\rho}{}^{\rho} + G_{\rho'}{}^{\rho'}\right] + \tfrac{1}{2}\xi\left[\mathcal{G}^{\mu\nu} + g^{\mu}{}_{\mu'}\,g^{\nu}{}_{\nu'}\,\mathcal{G}^{\mu'\nu'}\right]G - \tfrac{1}{2}m^2 g^{\mu\nu} G \qquad (66)$$

In the expression given by Brown and Ottewill [9], the operators occurring in the first and third terms of eq. (66) are not distinguished. In ref. [10], Wald does not apply point splitting to the Einstein tensor. Dowker and Critchley[11], split the Ricci curvature and $\nabla^{\mu}\nabla^{\nu}$, but not the scalar curvature and $\nabla^{\mu}\nabla^{\nu'}$. Christensen[12], uses the conformally coupled field equations

$$\psi(X)_{;\mu}{}^{\mu} = \left[\xi R(X) + m^2\right]\psi(X)\,,\quad \psi(X')_{;\mu'}{}^{\mu'} = \left[\xi R(X') + m^2\right]\psi(X')$$

to partially remove these second order operators (so that tracelessness in the conformally invariant case becomes independent of the field equations) but he makes no distinction between the scalar curvature at the two points. Candelas[13] fully removes the $\Box G$ (and $\Box' G$) terms. Davies and Fulling[7] in fact remark that some asymmetry between X and X' may be introduced into the regularized tensor by use of the field equations. In his examination of the scheme proposed by Adler et al.[14], Wald shows that a certain regular, boundary dependent Green function used in their analysis, is not symmetric and does not satisfy the wave equation in both X and X'. Using his proposal for the regularized energy momentum tensor, he finds that their renormalized tensor is not conserved and suggests a modification to correct this fault at the expense of introducing an anomalous trace. Although our proposal alters a number of the features used by Wald in his analysis, it will not spoil his results concerning conservation since he applies his regularized operator only to a non-divergent quantity, and it turns out that the difference we would introduce vanishes identically in the coincidence limit. However, our proposal introduces additional direction dependent finite terms which would in general lead to additional $\Box R$ terms in the trace (even with respect

to Wald's work, but not that of Christensen) and would spoil conservation if there is anyone who has genuinely established conservation for his renormalized result ! Ultimately, the residual effect of our proposal partly depends on one's handling of direction dependent terms.
In flat space time, with $\xi = 0$, from eq. (66) we have

$$\hat{T}^{\mu\nu} = \frac{1}{2}\left\{ g^{\nu}_{\ \nu'}\, G^{;\mu\nu'} + g^{\mu}_{\ \mu'}\, G^{;\mu'\nu} - g^{\mu\nu} g_{\rho\sigma'}\, G^{;\rho\sigma'} - m^2 g^{\mu\nu} G \right\}$$

In the coordinate system defined by eq.(1), we have

$$g_{\mu\nu} = (-\Lambda,\ \Lambda\ ,\ 1,\ 1\ )$$

$$g_{\mu\nu'} = \left(\begin{array}{cc|cc} -r & s & & \\ -s & r & \multicolumn{2}{c}{0} \\ \hline & & 1 & \\ \multicolumn{2}{c|}{0} & & 1 \end{array}\right), \qquad g^{\mu}_{\ \nu'} = \frac{1}{\Lambda_1}\left(\begin{array}{cc|cc} r & -s & & \\ -s & r & \multicolumn{2}{c}{0} \\ \hline & & \Lambda_1 & \\ \multicolumn{2}{c|}{0} & & \Lambda_1 \end{array}\right)$$

$$g^{\mu\nu'} = \frac{1}{\Lambda_1\Lambda_2}\left(\begin{array}{cc|cc} -r & -s & & \\ s & r & \multicolumn{2}{c}{0} \\ \hline & & \Lambda_1\Lambda_2 & \\ \multicolumn{2}{c|}{0} & & \Lambda_1\Lambda_2 \end{array}\right), \qquad g_{\mu}^{\ \nu'} = \frac{1}{\Lambda_2}\left(\begin{array}{cc|cc} r & s & & \\ s & r & \multicolumn{2}{c}{0} \\ \hline & & \Lambda_2 & \\ \multicolumn{2}{c|}{0} & & \Lambda_2 \end{array}\right)$$

with

$$r = \frac{1}{2}\left(f'_1 g'_2 + g'_1 f'_2 \right), \qquad \Lambda_1 = f'_1 g'_1, \qquad f_i \equiv f(u'_i),$$

$$s = \frac{1}{2}\left(f'_1 g'_2 - g'_1 f'_2 \right), \qquad \Lambda_2 = f'_2 g'_2, \qquad g_i \equiv g(v'_i),$$

$$, \quad i = 1, 2 .$$

Then

$$\hat{J}_{oo} = \frac{1}{2\Lambda_2}\left\{ r\left(G_{00'} + G_{11'}\right) + s\left(G_{01'} + G_{10'}\right) + \Lambda_1 \Lambda_2 \left(G_{22'} + G_{33'} + m^2 G\right)\right\}$$

Of course, we could take the expectation of this with respect to any state. If we choose, say, the Minkowski vacuum |0>, the operator G would become just the Minkowski Green function. If we choose an accelerated state |0'>, G would become the Green function for that state.

As an example, we consider the two dimensional massless case referred to earlier (as then, we will for convenience use the Feymann Green function rather than the symmetric one). For the accelerated state we have

$$G' = -\frac{1}{4\pi} \ln \left[\Delta u' \Delta v' \right], \qquad \left(\Delta u' = u_1' - u_2' \quad , \quad \Delta v' = v_1' - v_2' \right)$$

and then

$$\langle 0' | \hat{J}_{oo} | 0' \rangle = -\frac{1}{4\pi} \left[\frac{f_1'}{f_2'} \frac{1}{(\Delta u')^2} + \frac{g_1'}{g_2'} \frac{1}{(\Delta v')^2} \right]$$

Using $\frac{\partial}{\partial u} = \frac{1}{f'} \frac{\partial}{\partial u'}$, we expand

$$\frac{f_1'}{f_2'} = f_1' \left\{ \frac{1}{f'} + \frac{1}{f'} \partial_{u'}\left(\frac{1}{f'}\right) \Delta u + \frac{1}{f'} \partial_{u'}\left[\frac{1}{f'} \partial_{u'}\left(\frac{1}{f'}\right)\right] \frac{\Delta u^2}{2!} \right\}_{f' = f_1'} =$$

$$= 1 - \frac{f_1''}{f_1'^2} \Delta u + \left(-\frac{f_1'''}{f_1'^3} + 3 \frac{f_1''^2}{f_1'^4} \right) \frac{\Delta u^2}{2!}$$

and

$$\frac{1}{\Delta u'} = \frac{1}{\Delta u} \frac{(f_2 - f_1)}{\Delta u'} = \frac{1}{\Delta u} \left[f_1' + f_1'' \frac{\Delta u'}{2!} + f_1''' \frac{(\Delta u')^2}{3!} \right] =$$

$$= \frac{f_1'}{\Delta u} \left[1 + \frac{f_1''}{2 f_1'^2} \Delta u + \left(\frac{1}{6} \frac{f_1'''}{f_1'^3} - \frac{1}{4} \frac{f_1''^2}{f_1'^4} \right) (\Delta u)^2 \right] .$$

to obtain

$$\langle 0'|\hat{T}_{00}|0'\rangle = -\frac{1}{4\pi}\left\{\frac{f_1'^2}{\Delta u^2}+\frac{g_1'^2}{\Delta v^2}+\left[-\frac{1}{6}\frac{f_1'''}{f_1'}+\frac{1}{4}\left(\frac{f_1''}{f_1'}\right)^2\right]+\left[-\frac{1}{6}\frac{g_1'''}{g_1'}+\frac{1}{4}\left(\frac{g_1''}{g_1'}\right)^2\right]\right\}$$

From our previous expression

$$\sigma^2 = (f_2-f_1)(g_2-g_1) \quad , \quad \partial_{u_1}\sigma^2 = -f_1'\Delta v \quad , \quad \partial_{v_1}\sigma^2 = -g_1'\Delta u \quad ,$$

the divergent part can be written as

$$-\frac{1}{4\pi}\left\{\frac{(\partial_{u_1}\sigma^2)^2+(\partial_{v_1}\sigma^2)^2}{(\sigma^2)^2}\right\}$$

which can be substracted by a renormalization of the cosmological constant to give

$$\mathrm{Ren}\,\langle 0'|\hat{T}_{00}|0'\rangle = -\frac{1}{4\pi}\left\{\left[-\frac{1}{6}\frac{f_1'''}{f_1'}+\frac{1}{4}\left(\frac{f_1''}{f_1'}\right)^2\right]+\left[-\frac{1}{6}\frac{g_1'''}{g_1'}+\frac{1}{4}\left(\frac{g_1''}{g_1'}\right)^2\right]\right\}$$

As mentioned earlier, this is the negative of the result of our naive subtraction scheme for the Minkowski vacuum. Using this covariant method instead, for the Minkowski vacuum, and expressing the result with respect to accelerated coordinates we would have

$$\langle 0|\hat{T}_{00}|0\rangle = -\frac{1}{4\pi}\left\{\frac{f_1'^2}{\Delta u^2}+\frac{g_1'^2}{\Delta v^2}\right\} ,$$

and obtain

$$\mathrm{Ren}\,\langle 0|\hat{T}_{00}|0\rangle = 0$$

VI - CONCLUSIONS

We have extended, from two to four dimensions, a previous description of accelerated frames using analytic mappings. For a treatment of the divergent operator products arising in quantum field theory, we show that in a normalization with respect to a natural ground state specified in terms of these accelerated frames, the coordinate mapping (and its inverse) relating the accelerated frames to global inertial coordinates, plays a distinguished role. As in two dimensions, the exponential mapping is uniquely singled out as giving rise to a situation indicating global thermal equilibrium, and asymptotic properties of mappings are directly related to thermal properties in the asymptotic regions. We do not attempt to settle the question whether normalization may be more appropriate in curved spaces, but as a unique way of indicating which state to use for a normalization has not yet been developped, it seems that to discuss the back reaction problem a renormalization may be essential. In this context we propose a new regularized energy momentum tensor using covariant point splitting, and discuss its properties and relation to previous proposals.

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SEMICLASSICAL QUANTUM GRAVITY IN TWO AND FOUR DIMENSIONS

A full quantum theory of Gravity is still non-existent. There exists what is called a "Semiclassical Quantum Gravity". This refers to different approachs and approximations :

i) Q.F.T. in curved space-time, in which matter fields are quantized on classical gravitational backgrounds, one of the first important examples being the Hawking radiation by black-holes ; this is also of conceptual and practical interest in early Cosmology and Inflation.

ii) Semiclassical Einstein equations, in which quantized matter fields react back (through the expectation value of the energy-momentum tensor) on the geometry (the so-called "back-reaction problem"); important problems being the resolution of the last time evolution of black holes due to the reaction of Hawking radiation and the reaction of particle production in the early-time evolution of the Universe.

iii) Semiclassical approximation to path integral of gravity and matter fields, developped in the context of euclidean gravity with instanton and partition function methods (Gibbons and Hawking), recently combined with the Wheeler-Dewitt equation of canonical quantization and applied to Cosmology for the problem of initial conditions and ground state (Hartle-Hawking wave function).

One of the well known examples in previous approachs is that of the thermal properties of black-holes : first suspected at the very classical (and purely formal) level, properly found at the level of Q.F.T. in curved space-time, then recovered at the three-level from the path integral of gravity and matter fields.

Here we report our recent work in connection with the above mentioned problems :

1) We show that <u>Semiclassical Quantum Gravity in two dimensions</u> is exactly solvable. The general solution of two dimensional semiclassical Einstein equations is presented and analyzed in terms of analytic mappings. The connection with the Liouville theory is derived.

2) We investigate the role of discrete symmetries (in particular, the antipodal identification map) and the modification of the space time topology for Q.F.T. in curved space time. We analyze the implications for <u>Antipodal Identified Black Holes</u>, at the level of Green functions, Fock space and Thermal properties. (Work in collaboration with W.F. Whiting from North-Caroline University at Chapell-Hill).

3) We investigate the vacuum fluctuations $\langle\hat{\Psi}^2\rangle$ and the energy momentum tensor $\langle\hat{T}_{\mu\nu}\rangle$ of massless fields of spins 0,½ and 1 near static distorted black-holes. We give a complete geometric invariant description of these quantities near the event horizon for both the Hartle-Hawking (thermal) and the Boulware ("non-thermal") vaccua. (Work in collaboration with V.P. Frolov from Lebedev Institute of Moscow).

1. <u>SEMICLASSICAL QUANTUM GRAVITY IN TWO DIMENSIONS AND LIOUVILLE THEORY</u>.

The general solution of semiclassical two-dimensional Einstein equations is exactly found. It is given by a constant curvature metric parametrized by solutions ("Wave functions") of a zero energy Schrodinger equation. Global, thermal and topological properties of the universe are analyzed as a function of its quantum matter content including the graviton contribution.

1.1- Two dimensional field theories are useful in the understanding of more relativistic four dimensional theories and are in some cases, exactly solvable. Semiclassical approaches are helpful to get explicit (not merely formal) results and a qualitative understanding to quantization. Because of the difficulties to quantize Gravity, a semiclassical two dimensional treatment of the problem is interesting. Recently, two dimensional gravity has raised interest, mainly in connection with Polyakov's work on strings |1|. The *classical* Einstein equations do not describe the dynamics of the gravitational field in two dimensions because $\int \sqrt{g}\, R\, d_x{}^2$ is a topological invariant and because is $T_\mu^\mu = 0$ for a classical matter source. It has been proposed that Liouville theory which in geometric form reads $R+\Lambda=0$, (Λ= const.) could governs the dynamics of two dimensional gravity |2|. In this paper we consider *semiclassical* Einstein equations in two dimensions and *derive* Liouville theory as *one* of the dynamical equations of the gravitational field, the other equation involved appears to be a Schrodinger's one. The trace anomaly $\langle \hat{T}_\mu^\mu \rangle \neq 0$ for a quantum matter source allows for a non trivial dynamics of the semiclassical Einstein equations in two dimensions. Semiclassical in this context means that matter fields $\hat{\phi}$ including the graviton are quantized to one-loop level and coupled to (c-number) gravity through the equations

$$R_{\mu\nu} - 2^{-1} R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle \hat{T}_{\mu\nu}(\hat{\Phi}, g_{\mu\nu}) \rangle \quad (1.1)$$

$\langle \hat{T}_{\mu\nu} \rangle$ is the expectation value of the stress tensor operator $\hat{T}_{\mu\nu}$ of quantum matter fields, renormalized in such a way that is covariantly conserved $\nabla^\nu \langle \hat{T}_{\mu\nu} \rangle = 0$.

Eqs (1) for $g_{\mu\nu}$ are highly complicated and need to be treated within some type of self consistent framework. $\langle\hat{T}_{\mu\nu}\rangle$ depends on the geometry and on the choice of the quantum state $|\rangle$, that is on the choice of the boundary conditions of matter fields. Therefore $\langle\hat{T}_{\mu\nu}\rangle$ is not a local geometrical object. In two dimensions, the semiclassical eqs (1.1) reduce to

$$\Lambda g_{\mu\nu} = 8\pi G \langle \hat{T}_{\mu\nu} \rangle \tag{1.2}$$

which are non-trivial because $\langle\hat{T}_{\mu}^{\mu}\rangle \neq 0$. The metric can be always written in the conformally flat form

$$dS^2 = C(u,v)\, du\, dv \tag{1.3}$$

where $u=x-t$, $v=x+t$. The geometry is uniquely characterized by the curvature scalar

$$R = 4C^{-1}\partial_u\partial_v \ln C = 4C^{-3}[C\partial_u\partial_v C - \partial_u C\partial_v C] \tag{1.4}$$

$\langle\hat{T}_{\mu\nu}\rangle$ is uniquely determined by the trace anomaly value

$$\langle T_{\mu}^{\mu}\rangle = -\gamma(24\pi)^{-1} R \tag{1.5}$$

and explicitely given by [3]

$$\langle T_{\mu\nu}\rangle = \Theta_{\mu\nu} - \gamma(48\pi)^{-1} R\, \delta_{\mu\nu} + P_{\mu\nu}$$

$$\langle T_{uu}\rangle = -\gamma(12\pi)^{-1}\sqrt{C}\;\partial_u^2(\sqrt{C})^{-1} + \tilde{U}(u) \tag{1.6a}$$

$$\langle T_{vv}\rangle = -\gamma(12\pi)^{-1}\sqrt{C}\;\partial_v^2(\sqrt{C})^{-1} + \tilde{V}(v) \tag{1.6b}$$

$$\langle T_{uv}\rangle = -\gamma(48\pi)^{-1} R\, g_{uv}\,. \tag{1.6c}$$

$P_{\mu\nu}$ is any conserved traceless tensor taking into account the dependence of $\langle\hat{T}_{\mu\nu}\rangle$ on the quantum state of matter fields. It represents the non local part of $\langle\hat{T}_{\mu\nu}\rangle$: $P_{uu}=\tilde{U}(u)$, $P_{vv}=\tilde{V}(v)$, $P_{uv}=P_{vu}=0$. $\tilde{U}$ and $\tilde{V}$ are arbitrary functions of the indicated variables. The coefficient γ takes into account the spin(s) dependence and the number of degrees of freedom of the fields. The total value of γ is discussed in section I4. The semiclassical eqs (1.2) give

$$\langle T_{uu}\rangle = 0 \quad , \quad \langle T_{vv}\rangle = 0 \tag{1.7a}$$

$$R + \tilde{\Lambda} = 0 \quad , \quad \tilde{\Lambda} = 6\,\Lambda^{-1}\gamma \tag{1.7b}$$

Eq.(1.7b) is the Liouville equation in geometrical form. In terms of the conformal factor C it reads

$$4\,\partial_u\partial_v \ln C + \tilde{\Lambda}\,C = 0 \tag{1.8}$$

or $$4\,\partial_u\partial_v\,\phi + \tilde{\Lambda}\,\beta^{-1}e^{\beta\phi} = 0 \quad , \; C = e^{\beta\phi} \, , \; \beta = \text{const.}$$

As it is well known, the general solution is

$$\phi = \beta^{-1}\ln\frac{f'(u)\,g'(v)}{\left[1 + (\tilde{\Lambda}/8)\,f(u)\,g(v)\right]^2} \tag{1.9}$$

Here f and g are not totally arbitrary functions but determined in terms of $\tilde{U}$ and $\tilde{V}$ by eqs (1.7a) :

$$\sqrt{f'}\;d_u^2\left(\sqrt{f'}\right)^{-1} - 12\pi\gamma^{-1}\,\tilde{U}(u) = 0 \tag{1.10a}$$

$$\sqrt{g'}\;d_v^2\left(\sqrt{g'}\right)^{-1} - 12\pi\gamma^{-1}\,\tilde{V}(v) = 0 \tag{1.10b}$$

That is to say, the solution to the back-reaction problem in two dimensions is determined by a constant curvature metric (eq.1.3)

$$C = \frac{f'(u)\;g'(v)}{\left[1 - (R/8)\;f(u)\;g(v)\right]^2} \quad , \tag{1.11}$$

parametrized by solutions of a zero-energy Schrodinger equation

$$d_u^2\,\tilde{X}_u(u) - 12\pi\gamma^{-1}\,\tilde{U}(u)\,\tilde{X}_u(u) = 0 \tag{1.12a}$$

$$d_v^2\, \tilde{X}_v\,(v) - 12\pi\gamma^{-1}\, \tilde{V}(v)\, \tilde{X}_v(v) = 0 \qquad (1.12b)$$

By giving the "potentials" $\tilde{U}(u)$ and $\tilde{V}(v)$, i.e. by specifying the quantum state of the matter fields, eqs (1.12) determine the "wave functions"

$$\tilde{X}_u = (\sqrt{f'})^{-1} \quad , \quad \tilde{X}_v = (\sqrt{g'})^{-1} \qquad (1.13)$$

To know the geometry configuration as a function of the quantum state of matter fields, we consider the transformations

$$u_k = f(u) \quad , \quad v_k = g(v) \qquad (1.14)$$

The O(2,2) group of bilinear transformations is the invariance group for both Liouville equation (1.9) and the Schrodinger eq.(1.12). The first term of eq.(1.12) is the Schwarzian derivative (D[f]) of f : $D[f] = \sqrt{f'}\, d_u^2 \left(\frac{1}{\sqrt{f'}}\right) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$. Under the Möbius or bilinear transformations, f becomes a new function, but D[f] is invariant, determinying the same vacuum state of the fields. Eq.(1.14) can be considered as the mapping relating some manifold covered by the coordinates u, v to its global analytic extension (realized in the coordinates u_K, v_K). These are monotonic increasing functions satisfying the conditions |4|.

$$u_{K\pm} = f(\pm\infty) \quad , \quad v_{K\pm} = g(\pm\infty) \qquad (1.15)$$

$u_{K+}(u_{K-})$ can take finite or infinite values allowing for one, two or none event horizons in the space time. Same considerations hold for the mappings g. In particular, f = g. Properties of the Schrodinger eqs (1.12) can be derived from the asymptotic properties of these mappings. At an event horizon, $f'(-\infty) = 0$ and the "wave function" is $\tilde{X}=\infty$ there. On the contrary at the infinity, if for instance $f'(+\infty)=+\infty$, then $\tilde{X}(+\infty)=0$ and $\tilde{U}(+\infty)=+\infty$. In particular, the values $\tilde{U}=0, \tilde{V}=0$ in eqs(1.10), determine f(g) as

$$\tilde{X} = \text{const}, \qquad f = (\alpha u + \beta)\, /\, (\sigma u + \delta) \qquad (1.16)$$

with $(\alpha\delta-\beta\sigma) = 1$ and α, β, σ, δ, constant parameters in accordance with the invariance properties discussed above. The corresponding vacuum state ($|>_K$) can be considered as a reference or "minimal" vacuum at zero temperature, respect to which, states corresponding to non-zero potentials $\tilde{U}$ and $\tilde{V}$, appear as excited or thermal ones. A constant potential $\tilde{U}(u) = \tilde{U}_o$ such that $\tilde{U}_o/\gamma > 0$ (fig. 1) gives

$$\tilde{X} = A\, e^{-\tilde{\varkappa} u} \quad , \quad f = (2\tilde{\varkappa}A^2)^{-1}\, e^{2\tilde{\varkappa} u} \qquad (1.17)$$

where A is a normalizing constant (we will choose $A = (\sqrt{2\tilde{\varkappa}})^{-1}$) and $\tilde{\varkappa}$, is the zero-energy transmission coefficient

$$\tilde{\varkappa} = \sqrt{12\,\pi\,\gamma^{-1}\,\tilde{U}_o} \qquad (1.18)$$

The solution $\tilde{X}$ has been choosen in order to have f as an increasing function. The mapping

$$\begin{aligned} u_k &= e^{2\tilde{\varkappa} u} \qquad 0 \leq u_K \;,\; v_K \leq +\infty \;, \\ v_K &= e^{2\tilde{\varkappa} v} \qquad -\infty \leq u \;,\; v \leq +\infty \end{aligned} \qquad (1.19)$$

defines an event horizon at $u_K\, v_K = 0$ ($uv = -\infty$) and carries an intrinsic temperature

$$T = \pi^{-1}\tilde{\varkappa} = \sqrt{12\,(\pi\gamma)^{-1}\,\tilde{U}_o} \qquad (1.20)$$

as it can be seen by putting $t = i\tau$ ($u = x - i\tau$) and so $0 < \tau < \pi/\tilde{\varkappa}$. On the contrary, if $(\tilde{U}_o/\gamma) < 0$, there is no transmission coefficient ($\tilde{\varkappa}$ becomes imaginary) and no event horizon is formed. The geometry does not carry an intrinsic temperature in this case. More generally, each positive discontinuity in the "effective" potential $\tilde{U}_o/\gamma$ gives rise to an event horizon in the space time, the transmission coefficient $\tilde{\varkappa} = \left|\frac{\tilde{X}'}{\tilde{X}}\right|_{\text{horizon}}$ playing the role of the "surface gravity" $\varkappa = 2\tilde{\varkappa}$ of the horizon.

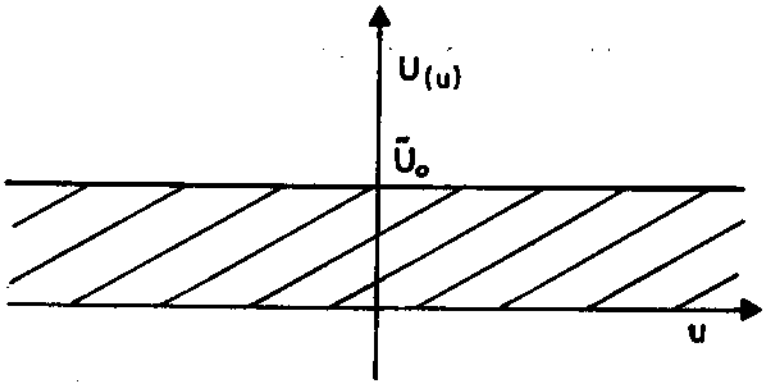

Fig.1.a

Constant effective potential $(\tilde{U}_0/\gamma)>0$ corresponding to the zero-energy Schrodinger eq(12). The wave function $Ae^{-\tilde{\varkappa}u}$, $\tilde{\varkappa}=\sqrt{12\pi\gamma^{-1}\tilde{U}_0}$, determines a mapping $u_K=(2\tilde{\varkappa}A^2)^{-1}e^{2\tilde{\varkappa}u}$. γ is the trace anomaly factor.

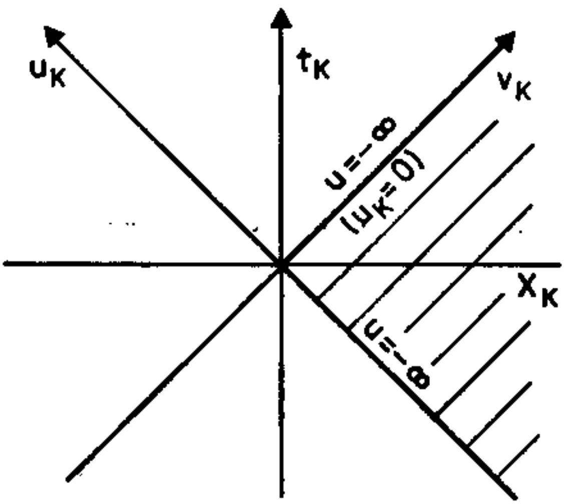

Fig. 1.b

Space time diagramm corresponding to the potential of fig.(1.a).. The zero-energy transmission coefficient ($\tilde{\varkappa}$) is twice the surface gravity of the horizon ($\varkappa$); $T=\pi^{-1}\tilde{\varkappa}$ the associated temperature. $u_K=x_K-t_K$, v_K+t_K are "Kruskal" (global) type coordinates, $u=v-t$, $v=x+t$ are of "Schwarzschild's" type.

1.2- Global properties of the semiclassical geometry.

It is convenient to rescale coordinates ($q=R/\theta$)

$$U = \sqrt{|q|}\,u_k \quad , \qquad U = f(u)$$
$$V = \sqrt{|q|}\,v_k \quad , \qquad V = g(v) \tag{1.21}$$

such that
$$dS^2_{(\mp)} = \frac{1}{|q|}\;\frac{1}{(1\mp UV)^2}\,dU\,dV \tag{1.22}$$

$$= \frac{1}{|q|}\;\frac{f'(u)\;g'(v)}{\left[1\mp f(u)g(v)\right]^2}\,du\,dv \tag{1.23}$$

The sign -(+) correspond here to $q>0$ ($q<0$) respectively.The case $q>0$ describes a semiclassical de Sitter geometry. By defining

$$U = e^{\left(\frac{r^*-t}{r_H}\right)} = f(u)$$
$$V = e^{\left(\frac{r^*+t}{r_H}\right)} = f(v) \tag{1.24}$$

where $r^* = r_H\, 2^{-1} \ln\left[(r_H - r)/(r_H + r)\right]$, $r_H = \frac{1}{2}\sqrt{|q|}$, the metric (1.22) can be written in the static form

$$dS^2 = -(1 - r^2/r_H^2)\,dt^2 + (1 - r^2/r_H^2)^{-1}\,dr^2 \tag{1.25}$$

which has an event horizon at $r=r_H=\sqrt{\gamma/3\Lambda}$. (See fig.(2)).

$\xi_{a;b}\,\xi^b = \varkappa\,\xi_a$ ($\xi = \partial/\partial t$ is the Killing vector such that $|\xi|=1$ at $r=0$) defines the surface gravity as

$$\varkappa = \sqrt{\Lambda_{eff}/3} = \sqrt{3\Lambda/\gamma} \;.$$

$\varkappa$ is twice the "transmission coefficient" eq.(1.18) for $\tilde{U}_o = \Lambda/(16\pi)$. The temperature is $T=(2\pi)^{-1}\varkappa = (2\pi)^{-1}\sqrt{3\Lambda/\gamma}$ involving besides Λ the trace anomaly coefficient γ. The case $q<0$ describes a semiclassical anti-de-Sitter geometry, obtained from the above situation by the analytic continuation $r_H \to -i\, r_H$. The mapping (1.24) becomes

$$U = e^{i\left(\frac{r^*-t}{r_H}\right)} \quad , \qquad V = e^{i\left(\frac{r^*+t}{r_H}\right)} \tag{1.26}$$

for real time t and coordinate $r^*/r_H = -\mathrm{arctg}(r_H/r)$. The metric is real

$$dS^2_{(+)} = -(1+r^2/r_H^2)\ dt^2 + \frac{dr^2}{(1+r^2/r_H^2)},\ \text{without event horizon.} \quad (1.27)$$

The geometry does not carry an intrinsic (real) temperature (T becomes imaginary). The mapping eq.(1.26) in this case is not strictly increasing, which is associated to the fact that (AdS) is oscillatory in time and not globally hyperbolic.

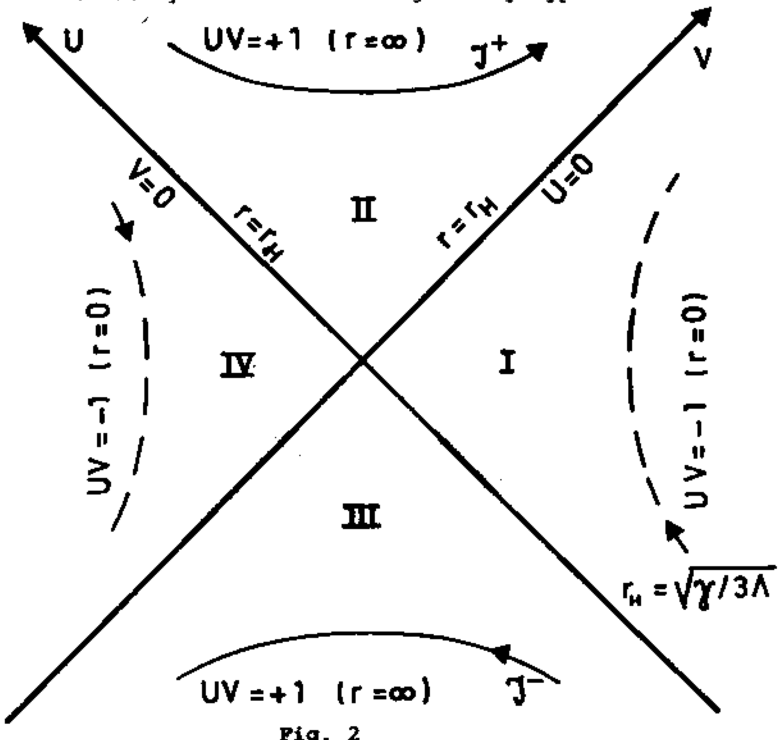

Fig. 2
Global Structure of the space-time for the case $(\Lambda/\gamma)>0$. The semiclassical geometry is of the de-Sitter type with one event horizon and intrinsic temperature $T=\sqrt{\Lambda_{eff}/3}$, $\Lambda_{eff}=9\Lambda/\gamma$. For $(\Lambda/\gamma)<0$, the geometry is anti-de-Sitter.

1.3- Instantons.

The analytic continuation $t=i\tau$ (τ real) maps the metric (1.22) onto a definite positive metric

$$dS^2_{(\mp)} = (\partial_z \partial_{\bar z} \Phi_{(\mp)})\, dz\, d\bar z \qquad (1.28)$$

Here $z = X + iT$, $\bar z = X - iT$. $\Phi_{(\mp)} = |q|^{-1} \ln [1 + z \bar z]$ is the solution of the Euclidean Liouville equation. For q>0 Eq.(1.28) is the projective complex line ($\mathbb{C}P^1$); $1/2\sqrt{|q|}$ is the curvature radius of the space.
We can consider $\mathbb{C}P^1$ as a gravitational instanton |6| of two dimensional Gravity : complete, non singular and definite positive solution of the semiclassical Einstein equations in two dimensions. The Euler number is given by

$$\chi = (4\pi)^{-1} \int_{\mathcal{M}} \sqrt{g}\, R\, d^2x + (2\pi)^{-1} \int_{\partial\mathcal{M}} \sqrt{\sigma}\, K\, dy \qquad (1.29)$$

and the euclidean action is

$$\hat{I} = 4^{-1} (\chi - 1 - \Lambda\Omega) \quad , \Omega = (2\pi)^{-1} \int \sqrt{g}\, d^2x \qquad (1.30)$$

g and σ are the determinants of the metrics over the manifold M and over its boundary $\partial\mathcal{M}$, respectively. K is the trace of the extrinsic curvature. $\chi = 1$ and $\hat{I} = 0$ for flat Minkowski space ; $\chi = 0$ and $\hat{I} = -1/4$ for flat Rindler space. For the $\mathbb{C}P^1$ instanton, $\chi = -1$ and $\hat{I} = -1/4(2+\gamma/6)$.

1.4- Cosmological configurations and "critical dimensions".

The cosmological constant $\tilde{\Lambda} = 6\Lambda/\gamma$ in the Liouville eq. (1.9) is modified with respect to the classical one by the trace anomaly factor γ of eq.(1.5). The character of the solution depends on the sign of Λ/γ. Vector fields in two dimensions do not contribute to γ. For γ fixed by eq.(1.5), the scalar contribution is positive and that of gravitons is negative. Therefore : I) If sign $\Lambda \neq$ sign γ, i.e. $\Lambda > 0$ and $\gamma < 0$ (graviton dominated universe) or $\Lambda < 0$ and $\gamma > 0$ (matter dominated universe), the geometry has R>0 with one event horizon. II) If sign Λ=sign γ, the geometry has R<0 without horizon. This means that for a given sign of Λ, the presence or absence of event horizons depends on the number of matter fields. The Universe could change from an Anti-de-Sitter to a de-Sitter phase (or vice-versa). The graviton contribution is crucial here to arise these possibilities. This contrasts with the standard classical situation (in four dimensions) in which R and the presence or not of event horizon only depends on Λ. If N (the number of matter fields)$\to\infty$ then the Hawking temperature $T\to 0$ and the semiclassical geometry is flat even if $\Lambda\neq 0$. If $\gamma=0$ the dynamics is not determined by the semiclassical Einstein equations. In ref.7 the Liouville equation has been derived in the semiclassical context but the graviton contribution so crucial to this problem has been overlooked. The total value of γ as calculated in refs. 8 and 9 (denoted $\gamma_{(GKT)}$ and $\gamma_{(CD)}$ following notation of ref. 10) is

$$\gamma_{(GKT)}=(N_0-1+N_{\frac{1}{2}}-\frac{15}{2}N_{3/2}),\ \gamma_{(CD)}=(N_0-1-N_{\frac{1}{2}}+N_{3/2}) \quad (1.31)$$

Here, the graviton interacts with N_s massless fields of spins s, $s\leq 3/2$. {The graviton contribution to γ was also obtained equal to -1 in ref.(10)} In the context of quantized strings |1|, the trace anomaly coefficient for a theory with N matter fields coupled to two dimensional gravity was obtained equal to |1|

$$\gamma_{(P)} = N - 26 \text{ for bosons,} \quad (1.32)$$
$$\gamma_{(P)} = N - 10 \text{ for fermions (with supersymmetric coupling)}$$

We denote it $\gamma_{(P)}$ because of ref.(1). (See also refs. 11-14 for a review). These values were calculated at the one loop level in the conformal gauge $g_{\mu\nu}=e^{\phi}\eta_{\mu\nu}$. The "critical dimension" 26(10) in eq.(1.32) is only the ghost part (Faddeev-Popov determinant) of the graviton contribution. It does not take into account the quantization of the conformal factor

the Liouville field ϕ) that remains fixed. This should explain the difference between the values 1 in eq.(1.31) and 26(10) in eq.(1.32). The value of γ that should be considered in the Liouville equation (9) of two dimensional gravity is that given by eq.(1.31) and not that of eq.(1.32). Understanding in connection with the quantization of the Liouville theory in this context desserves future investigation. It would be interesting to connect the results found here with those obtained from a semiclassical limit of the Hawking "wave function approach" |15| and the Jackiw model |16|.
More details about this work are given elsewhere |17|.

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2. Q.F.T. AND THE ANTIPODAL IDENTIFICATION OF BLACK HOLES.

The antipodal points (U,V,θ,φ) and $(-U,-V,\pi-\theta,\pi+\varphi)$ of the Schwarzschild-Kruskal manifold, usually interpreted as two different events (in two different worlds) are considered here as physically identified (to give one single world). This has fundamental consequences for the Q.F.T. formulated

on this manifold. The antipodal symmetric fields have (globally) zero norm. The usual particle-antiparticle Fock space definition breaks down. The antipodal symmetric Green functions have the same periodicity $\beta=8\pi M$ in imaginary (Schwarzschild) time as the usual (non-symmetric) ones. (Identification with "conical singularity" whould give a period $\beta/2$). In any case, no usual thermal interpretation is possible for $T=\beta^{-1}$ (nor for $T_0=2/\beta$ or any other value) in the theory. In the light of these results we discuss "old" ideas and more recent ones on identification.

The formulation of Quantum Field Theory in non trivial (curved or flat) space times has given new fundamental features with respect to the usual understanding of Q.F.T. in trivial (Minkowski flat) space time, i.e.:i) the possibility for a given field theory to have different alternative well defined Fock spaces (different "sectors" of the theory);(ii) the presence of "intrinsic" statistical features (temperature, entropy) arising from the non-trivial structure (geometry, topology) of the space-time and not from a superimposed statistical description of the quantum matter fields. Relevant examples are Q.F.T. on the Rindler manifold and its analytic mapping extensions (1-3), black-holes and cosmological (de Sitter) space-times (4-6).

However, it is known that each given space-time M with metric g satisfying the Einstein equations, admits, in principle, different possible associated topologies. (Einstein equations being purely differential, fix the local geometry but not the global topology of the space-time). Results i) and ii) on Q.F.T. in non-trivial manifolds refer to a particular choice of the space-time topology. (even if that choice could be considered as the "most reasonable" one). Our aim is to investigate the modifications to the above results when space time topologies, different to the usually considered ones are allowed. Important though a modification of the spatial topology is, it merely influencies the spatial part of the positive (negative) frequency basis and the value of the vacuum energy (Casimir effects) but fundamental changes to the features (i) and (ii) above referred would not be expected. Recently,we have shown that more drastic consequences appear from a modification of the space time topology. We consider space times of the form $\tilde{M}=M/\Gamma$ where Γ is a discrete subgroup of isometries of M without fixed points: $\tilde{M}$ is non-singular and it is obtained from its universal covering M by identifying points equivalent under Γ. For the relevant examples we have in mind, we will restricts ourselves to the particular interesting case in which Γ is the antipodal transformation on M :

$$J: \quad P(X) \longrightarrow \bar{P}(\bar{X}) \qquad (2.1)$$

The antipode $\bar{P}(\bar{X})$ of the point P(X) is defined as having its light-cone parallel to that of P. Thus, we are adopting the so-called "elliptic interpretation" (7) of space-time, consisting in deeming antipodes to represent the same world-point or event. Such an interpretation applies to cosmology (7,8) as well as to black-holes (9-13). We discuss Q.F.T. on Schwarzschild space time with this identification. We find that :

(i) the usual Fock space construction breaks down in the identified space because, even at the classical level,a complete set of positive (negative) frequency basis can not be constructed : eigenmodes have zero norm ! (on global sections). It means that quantum creation and annhilition can not be defined and no vacuum Fock state exists. |In the usual theory (without identification) there are different well defined possibilities (and thus an ambiguity) in choosing a positive frequency basis, but such an unity normalized basis always exist|.

(ii) the usual thermal features in this context are destroyed.The space-time identification considered here does not manifest itself as a mere changement in the value of temperature (a "naive" consideration could lead to a modification of it by a factor two) but the usual notion of temperature in this context (i.e. the inverse imaginary time periodicity of the

Green functions) doesn't apply. The zero norme fields here do not allow us to describe any thermal properties in the normal way. Similar results to ours have been recently discussed independently by Gibbons [16].

In Schwarzschild space-time, the antipodal transformation J(eq.2.1) without fixed points takes the form :

$$J:\ (U,V,\Omega) \qquad (-U,-V,\bar{\Omega}) \qquad (2.2)$$

Here $U = X - T$, $V = X + T$

$\Omega = (\theta,\varphi)$

$\bar{\Omega} = (\pi-\theta\ ;\ \varphi+\pi)$,

are global (Kruskal) type coordinates, related to local (Schwarzschild) ones by

$$\mathrm{I}\quad \begin{cases} X = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M}\cosh\left(\frac{t}{4M}\right) \\ T = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M}\sinh\left(\frac{t}{4M}\right) \end{cases}$$

$$\mathrm{II}\quad \begin{cases} X = \left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M}\sinh\left(\frac{t}{4M}\right) \\ T = \left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M}\cosh\left(\frac{t}{4M}\right) \end{cases}$$

$$\mathrm{III}\quad \begin{cases} X = -\left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M}\cosh\left(\frac{t}{4M}\right) \\ T = -\left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M}\sinh\left(\frac{t}{4M}\right) \end{cases} \qquad (2.3)$$

$$\mathrm{IV}\quad \begin{cases} X = -\left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M}\sinh\left(\frac{t}{4M}\right) \\ T = -\left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M}\cosh\left(\frac{t}{4M}\right) \end{cases}$$

The inverse transformations are $X^2-T^2=\left(\frac{r}{2M}-1\right)e^{r/2M}$ in I, II, III, IV.

$$t = \begin{cases} 4M\tanh^{-1}\left(\frac{T}{X}\right) & \text{in I, III} \\ 4M\tanh^{-1}\left(\frac{X}{T}\right) & \text{in II , IV} \end{cases}$$

Eqs.(2.3) can be written in a way as

$$\mathrm{I}\quad \begin{cases} U = e^{\varkappa u} \\ V = e^{\varkappa v} \end{cases}$$

with

$$U = X - T \quad , \quad u = r_* - t \quad , \quad \varkappa = \frac{1}{4M}$$
$$V = X + T \quad \quad \quad v = r_* + t$$

and

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \quad ,$$

in region I. Similarly, for the corresponding exponential mappings in regions II, III, IV :

$$\mathrm{II}\quad \begin{cases} U = -e^{\varkappa u} \\ V = e^{\varkappa v} \end{cases} , \; r_* = r + 2M \ln\left(1 - \frac{r}{2M}\right)$$

$$\mathrm{III}\quad \begin{cases} U = -e^{\varkappa u} \\ V = -e^{\varkappa v} \end{cases} , \; r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

$$\mathrm{IV}\quad \begin{cases} U = e^{\varkappa u} \\ V = -e^{\varkappa v} \end{cases} , \; r_* = r + 2M \ln\left(1 - \frac{r}{2M}\right)$$

Eqs(2.3) displays the "double-universe" nature of the Kruskal Schwarzschild geometry (fig.3), usually referred as an "eternal black-hole". Two Schwarzschild coordinate patches (I,II) and (III,IV) are needed to cover all of space-time. There are two (past and future) $r=0$ singularities (corresponding to $T=+\sqrt{1+X^2}$ and $T=-\sqrt{1+X^2}$) and two exterior ($r>r_H$) asymptotically flat regions corresponding to $X>|T|$ and $X<|T|$. For $M=0$, each of these Kruskal regions, becomes all of Minkowski space-time. (The dynamic space time of a black hole formed by gravitational collapse, only has one future horizon and one exterior region).

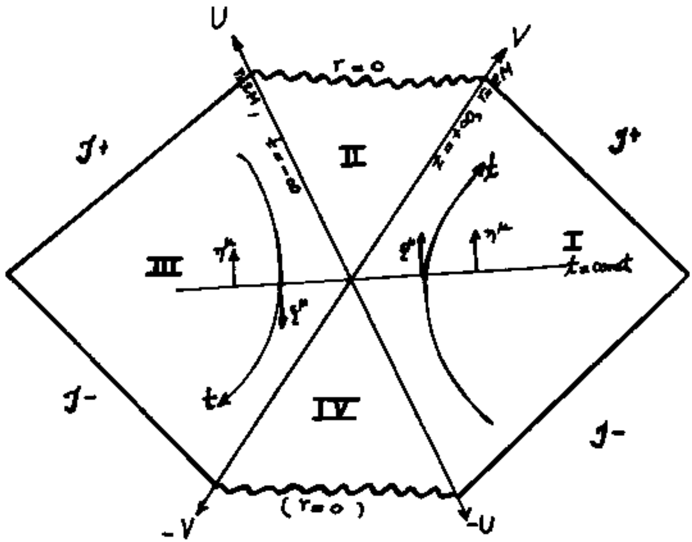

Fig. 3

Kruskal extension of the Schwarzschild manifold. It is symmetric under reflections U→-U, V→-V and consists of four regions : (I) and (III) are isometric regions in which r>2M : (the timelike Killing vector $\xi = \partial/\partial t$ is future directed in I and past directed in III). (II) and IV) are isometric but time reversed regions in which the r = 0 physical singularity evolves.

In the usual (conventional) interpretation, [19,20] antipodal points P(X) and $\bar{P}(\vec{\bar{X}})$ of Kruskal space are <u>physically distinct</u> events which are causally disconnected. Every point mass (would) splitt the universe in two : the real world (I) and its mirror (inaccesible) copy (III). A Kruskal space is a wormhole, connecting two distant regions of ordinary space.

In the elliptic interpretation, antipodal points $P(X)$ and $\bar{P}(\bar{X})$ are physically identified. They are considered as different representations in Kruskal space of <u>one and the same</u> Schwarzschild (and ultimately Minkowski) event (r, t, θ, φ). Thus, there is only a single one world with only one singularity and only one exterior region and no wormhole is needed. The price of this more "economical" picture is that M is not time orientable. There is a breakdown of the global distinction between past and future in the interior region $r<r_H$. However, no problem does not arise for $r>r_H$.

Now, we wish to consider fields on the identified space-time above discussed, which are symmetric under the action of the antipodal operator J (eq.2.2). Thus we define

$$\Psi_{JS} = \frac{1}{2}\,[\Psi(X) + \Psi(JX)] \qquad (2.4)$$

We will build up these symmetric fields from fields with arguments specified in the right hand wedge. These building blocks will be either positive or negative frequency components with an inner product given by the usual Klein-Gordon one

$$\langle \Psi, \Phi \rangle = -i \int \Psi^{*} \overleftrightarrow{j}^{\mu} \Phi \; d\Sigma_{\mu} \qquad (2.5)$$
$$(\overleftrightarrow{j}^{\mu} = \sqrt{g}\, g^{\mu\nu} \overrightarrow{\partial_\nu} - \overleftarrow{\partial_\nu}\, g^{\mu\nu} \sqrt{g}\;)$$

taken over the right wedge. Each of these fields (whether positive or negative frequency on the right wedge) can be extended to be a positive or negative frequency field on global space-like surfaces. But in the inner product defined on the original (full) Hilbert space, our symmetric fields defined by eq.(2.4) have zero norm in global space-like surfaces. This is so because on a global space like surface, time orientation has been reversed in the left hand wedge, relative to the local orientation in the right hand wedge.

Our identified manifolds have non trivial (multiply connected) space time topology. Instead of choosing symmetric fields on these manifolds we could equally well have choosen antisymmetric fields, that is fields which change sign under the antipodal map J on the global manifold [see e.g. ref.(16)] This would give a situation some what analogous to the twisted fields considered by several authors some years ago [21,22]. However, previously, it was only the <u>spatial</u> topology which was non-trivial and the problem of zero norm states never arose.

This property of the symmetric field theory can be understood in terms of its projections on separate halves of the global (e.g. Kruskal) manifold. Although our symmetric fields have zero norm on global space like sections, they have positive (negative] norm on the half space to the future [past] of the past horizon (fig.4). Particles and anti-particles can be well defined on each half space but with conjugate roles. Thus, from a global point of view, all distinction between particles and antiparticles is removed by the identification.

We define now the symmetric Green function

$$G_{JS}(X,X') = \langle \Psi_{JS}(X)\ \Psi_{JS}(X')\rangle \qquad (2.6)$$

and express it in terms of the Green function for the ordinary fields operators

$$G(X,X') = \langle \Psi(X)\ \Psi(X')\rangle \qquad (2.7)$$

Here < > stands for expectation value in the ordinary Fock states of the ordinary field, ie $|0\rangle$. Thus,

$$G_{JS}(X,X') = \frac{1}{4}\left[G(X,X')+G(X,JX')+G(JX,X')+G(JX,JX')\right] \quad (2.8)$$

It is necessary to indicate rather carefully what are the properties of the symmetric Green function G_{JS}. To do this, we use Schwarzschild's type coordinates u,v covering the right hand wedge introduced in section I, (eq.2.3), ie

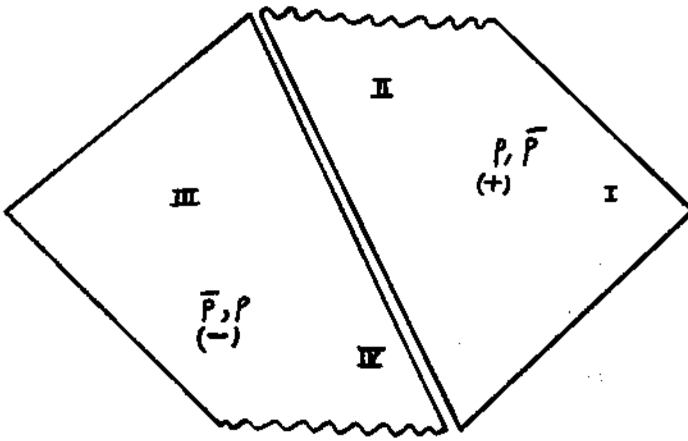

Fig. 4

A decomposition of the identified manifold into separate halves on which the symmetric fields have positive and negative norm respectively.

$$U = e^{\alpha u} \qquad\qquad u = x - t$$
$$V = e^{\alpha v} \quad , \qquad v = x + t$$

It is clear that the Green function G is unchanged if $t \rightarrow t + i\beta$ where $\beta = \frac{2\pi}{\alpha}$, and G_{JS} similarly has this property, ie

$$G(t,t'+i\beta\,;\Omega) = G(t,t';\Omega) \tag{2.9}$$

$$G_{JS}(t,t'+i\beta\,;\Omega) = G_{JS}(t,t';\Omega) \tag{2.10}$$

(For the Schwarzschild black hole, $\beta = 8\pi M$).

In addition, we have

$$G_{JS}(t,t' + \frac{i\beta}{2}\,;\bar{\Omega}) = G_{JS}(t,t';\Omega) \tag{2.11}$$

but

$$G_{JS}(t,t'+\frac{i\beta}{2};\Omega) \neq G_{JS}(t,t';\Omega) \qquad (2.12)$$

The singularity structure of G_{JS} is clearly demonstrated in fig.(5) where the orientation in the sphere must be carefully respected.

If G satisfies eq.(2.9), then G_{JS} satisfies eq.(2.10). If we had not mapped to antipodal points on the sphere under J, ie if instead of J, we were taken

$$J_o : (U,V,\Omega) = (-U,-V,\Omega) \qquad (2.13)$$

we would have had a conical singularity (at U = V = 0) in our spacetime and the corresponding symmetric Green function $[G_{JoS}]$ would be unchanged under $t \to t+i\,\beta/2$, ie

$$G_{JoS}(t,t'+\frac{i\beta}{2};\Omega) = G_{JoS}(t,t';\Omega) \qquad (2.14)$$

Usually, in (euclidean) Q.F.T. one can regards G satisfying eq.(2.9) as a thermal Green function at temperature $1/\beta$. However, even though our symmetric Green function G_{JS} is periodic with period β (and not $\beta/2$), this does NOT mean that we can interpret it as a thermal Green function at temperature β^{-1} (or at any other temperature). Our zero norm fields do not allow us to construct any thermal properties in the normal way. [And the same results apply to G_{JoS} satisfying eq.(2.14)].

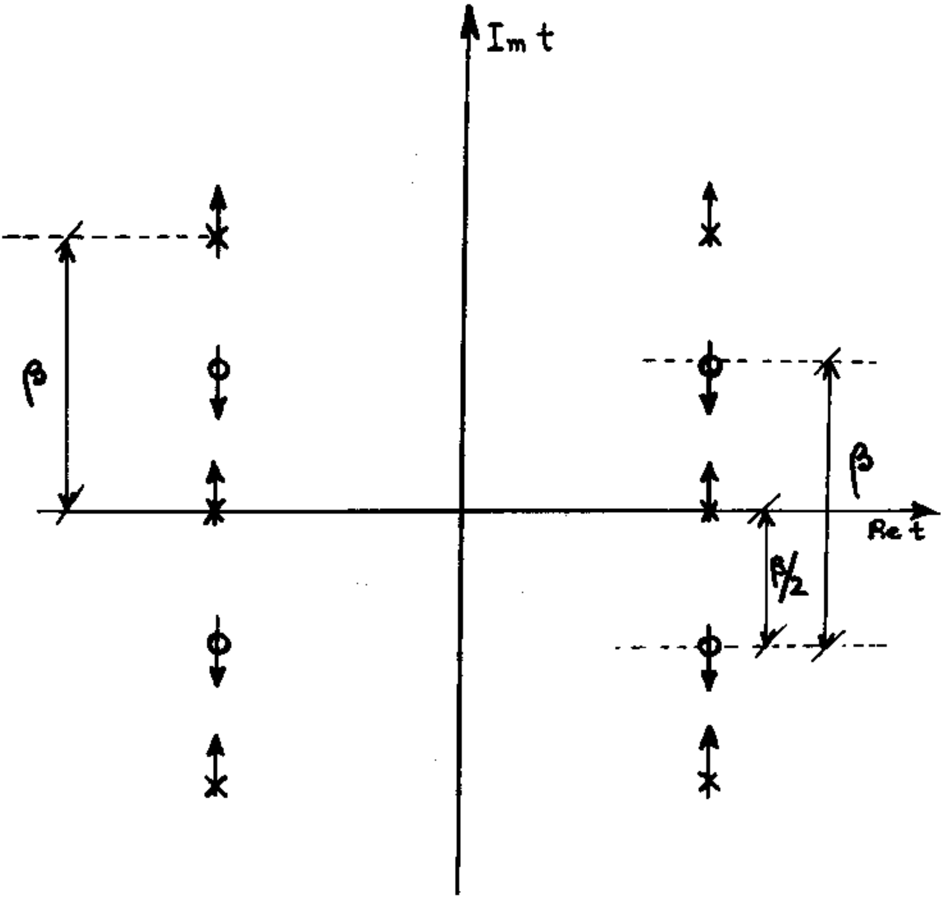

Fig. 5

The complex (t-t') plane. The crosses (x) indicate the positions of the singularities of the usual Green function G(X,X') where X and X' have the same orientation on the 2-sphere; they are repeated in the imaginary direction with period β. Circles (O) mark the singularities of G(X,JX') where in their spatial dependence, X and X' lie on opposite sides of the 2-sphere; these have the same period β, but are shifted with respect to crosses by β/2 in the imaginary direction. Crosses and circles then represent singularities of the antipodal symmetric Green function $G_{JS}(X,X')$ and the arrows indicate the different spatial orientations of the points X and X'.

The main result in this section which we would like to emphasize is that the discrete antipodal symmetry of the classical Schwarzschild-Kruskal manifold can not be implemented without problems at the level of quantum Fock space. One can not construct (local) Fock states symmetric under the antipodal transformations by a quantum operator acting on the usual (non symmetric) states. In particular, there is no antipodal symmetric vacuum state. We can however, implement this symmetry on the field operators in the space configuration and work on the usual (non-symmetric) Fock vacuum state (it is not necessary for the vacuum of a quantum theory to have a symmetry of the classical manifold). In that case, an antipodal symmetric Green function (G_{JS}) can be defined (eq.2.6) and expressed in terms of the Green functions G of the ordinary theory.

For the antipodal identification map J(eq.2.3) without fixed points, if the ordinary Green function G has period β, then G_{JS} has the same periodicity. For the antipodal map Jo (eq.2.13) with conical singularity, G_{JOS} has period $\beta/2$. In any case, the important result here, is that the zero norm states of the theory do not allow us to interpret, as in the usual way, $T=1/\beta$ (nor $To=2/\beta$) as a temperature of the theory.

From the results presented here, it is clear to us what is the precise context in which the results of refs(17,18) arise. The <u>actual</u> framework of refs.(17,18) is that of QFT in curved (and non quantum) space time. Identification of classical space time has been (implicitely or explicitely) adopted, precisely, the identification map J_o (the presence of conical singularity is not a relevant criticism here). The bras in the right hand wedge of space time were implicitely identified to kets in the left hand wedge. A factor 2 for the quantity $To=2/\beta$ (twice the usual $T=1/\beta$) was found. However, usual Fock space, non-zero norm states and an operator projection relating the usual states to the identified (symmetric) ones were <u>assumed to exist (and used) for the</u>

theory. The quantity $T_o=2/\rho$ was interpreted as a temperature. All that is not consistent. The factor 2 for T_o found in refs (17,18) is correct (within the above precise context) but its assumed derivation and interpretation there, are not correct. While such conclusions might still hold in the full theory of quantized Gravity, a framework for determining this does not exist in refs (17,18). (nor here !).

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QUANTUM FIELD THEORY AND THE ANTIPODAL IDENTIFICATION OF DE-SITTER SPACE. ELLIPTIC INFLATION.

Abstract : The elliptic version of space-time is obtained by deeming antipodes to represent the same world point or event. We investigate the consequences of this identification for Q.F.T. as formulated in de Sitter space and its implications for inflation. Antipodally symmetric and antisymmetric fields and Green functions are described. We calculate for these fields the expectation values of the square of the field operator and stress-tensors in the family of de Sitter O(1,4) invariant vacua and study limiting cases of interest. In the inflationary regime the antipodal identification gives for $\langle\varphi^2\rangle$ a value which differs in a factor 2 from the ordinary Bunch-Davies value. The modifications introduced to $\langle T_{\mu\nu}\rangle$ vanish in the conformally invariant case. The antipodally identified theory also allows a better understanding of the massless and minimally coupled ordinary theory (without identification). We found new vacuum states which are O(4), O(1,3) (and E(3)) invariant and calculate the stress-tensors for them.

It is known that a given local spacetime geometry with metric g satisfying the Einstein equations admits in principle different possible associated global topologies ; however most of our present understanding of Quantum Field Theory in a given non-trivial manifold refers to a particular choice of the <u>spacetime</u> topology. (Important though a modification of the <u>spatial</u> topology is, fundamental changes to the known features on Q.F.T. in non-trivial manifolds would not be expected). Spaces $\bar{\mathcal{M}}_g$, locally identical to $\mathcal{M}_g$ but with different (large scale) spacetime topology can be obtained by identifying points in $\mathcal{M}_g$ equivalent under a discrete isometry without fixed points. The field theories as formulated in the spaces $\mathcal{M}_g$ and $\bar{\mathcal{M}}_g$ are essentially different [1,2]. In de Sitter space the simplest such identification is to identify the antipodal points (the antipode $\bar{P}$ of a point P is defined as having its light-cone without intersecting that of P). This is an old idea first proposed by

Schrödinger and called the "elliptic interpretation" [3]. For black holes, it has been shown recently by Gibbons [1], Whiting and one of the authors [2], that the antipodal identification destroys the thermal features and the usual Fock space construction. Here we investigate the antipodal identification for Q.F.T. in de Sitter space and its consequences for inflation. The high symmetry of de Sitter space allows us to go further in the understanding of the antipodally identified field theory because all the relevant quantities (Green functions, vacuum expectation values of the field operator and of the energy-momentum tensor) can be exactly known. From the Antipodally symmetric theory we also obtain new results for the ordinary (non antipodally symmetric) one.

The (discrete) antipodal symmetry is implemented here at the level of the field operators on the space configuration but not at the level of the vacuum states on the Fock space (there is no Fock antipodally symmetric vacuum state). The vacua of the theory are taken as the usual Fock vacuum states (it is not necessary for the vacuum of a quantum theory to have a symmetry of the classical manifold).

In order to understand the possibility of the "elliptic interpretation" of de Sitter space, let us describe some interesting aspects of the ordinary interpretation.

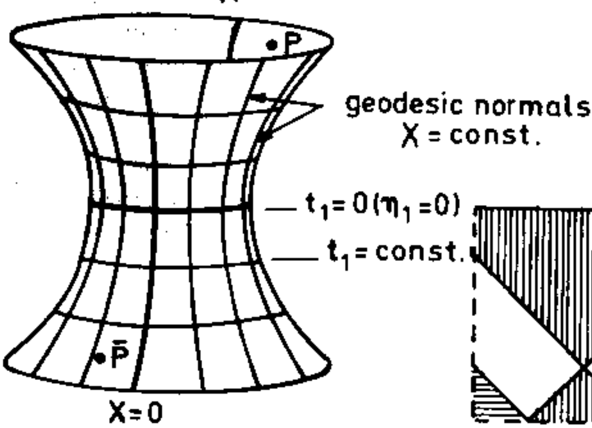


Fig. 1

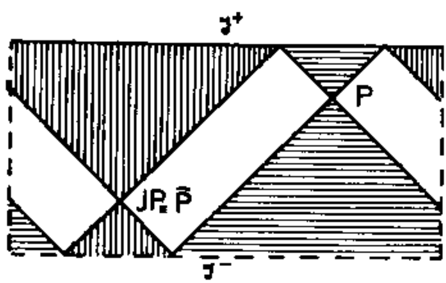


Fig. 2

de Sitter space is an hyperboloid embedded in a five-dimensional flat space (two dimensions are suppressed in fig. 1). The Penrose diagram (Fig.2) allows us to notice that :

(i) the antipodal points P and JP are spacelike separated in de Sitter space and

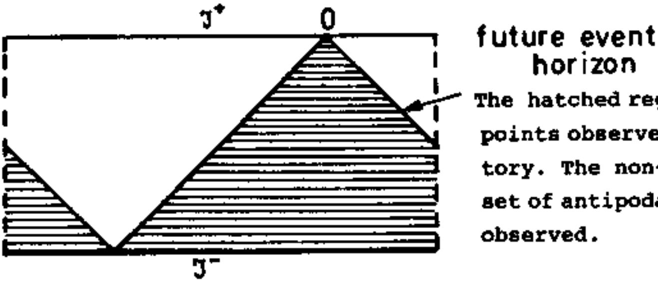
(ii) the interiors of the light cones of P and JP do not intersect.

Therefore, it is obvious that an observer $\hat{o}$ moving in de Sitter space, cannot, during its history :

(i) meet both P and JP

(ii) receive a message from P and from JP

(iii) receive a message from P and send a message to JP.



The hatched region is the set of points observed by $\hat{o}$ during its history. The non-hatched region is the set of antipodal points of the points observed.

All the events that have been observed by $\hat{o}$ during its history are the points inside the future event horizon. The set of this points cover exactly <u>one half</u> of the manifold. The other half is the set of the points outside the future event horizon. It is made up of the antipodal points of the points inside the future event horizon. Consequently, the "elliptic interpretation", destroys the notion of event horizon and an observer in de Sitter space can observe during its history all the events. The resulting manifold is non-simply-connected. Furthermore, it is space-orientable but non-time-orientable because the antipodal transformation J reverses the direction of time. If we identify P and $\bar{P}$, the future light cone in P is identified with the past cone in $\bar{P}$, but one cannot continuously do such an identification of past and light cone over the whole manifold. This is related to the fact that J is not in the same connected component of the O(1,4) group as the Identity but it is an element of the disconnected component G_T which contains time reversal.

J is an inversion in $\mathbb{R}^5$:

$$J : x^a \longrightarrow - x^a \qquad (a = 1,\ldots 5)$$

In the different known coordinate systems on the hyperboloid, J takes the form

$$J\,(t, \chi, \theta, \varphi) \;\longrightarrow\; (-t, \pi-\chi, \pi-\theta, \pi+\varphi)$$

$$J_E\,(\tau, r, \theta, \varphi) \;\longrightarrow\; (\tau+\beta/2, r, \pi-\theta, \pi+\varphi)$$

$$J\,(u, v, \theta, \varphi) \;\longrightarrow\; (-u, -v, \pi-\theta, \pi+\varphi)$$

J_E stands for the Euclidean version ($t = i\tau$) of J in "static coordinates", where $\beta = 2\pi\sqrt{3/\Lambda}$.

We consider (scalar) fields on the identified (elliptic) space discussed above, which are symmetric (or antisymmetric) under the action of J, ie

$$\phi_{JS} = \frac{1}{\sqrt{2}}\left[\phi(X) + \phi(JX)\right]$$

$$\phi_{JA} = \frac{1}{\sqrt{2}}\left[\phi(X) - \phi(JX)\right]$$

The corresponding Green functions are defined by

$$G_{\alpha_{JS}}(X,X') = \langle\alpha|\{\phi_{JS}(X)\,;\,\phi_{JS}(X')\}|\alpha\rangle =$$
$$= \frac{1}{2}\left[G_\alpha(X,X') + G_\alpha(X,JX') + G_\alpha(JX,X') + G_\alpha(JX,JX')\right]$$

(Similarly for G_{JA}), where

$$G_\alpha(X,X') = \cosh 2\alpha\; G_0(X,X') + \sinh 2\alpha\; G_0(X,JX')$$

and

$$G_\alpha(X,X') = G_\alpha(JX,JX')$$

Thus

$$G_{\alpha JS}(X,X') = e^{2\alpha}\left[G(X,X') + G(X,JX')\right]$$

$$G_{\alpha JA}(X,X') = e^{2\alpha}\left[G(X,X') - G(X,JX')\right]$$

The real parameter α labels the one parameter family of de Sitter invariant vacua. In particular we will take $\alpha = 0$ in order to have for $G_{JS}(G_{JA})$ the Hadamard form, but even in this case, G_{JS} and G_{JA} have both singularities (at $X = X'$ and $X = JX'$) and both with the same strength.
The modes $\{ \phi_{n\, {JS \atop JA}} \}$ associated to the fiedls $\phi_{JS \atop JA}$ have zero norm on global spacelike sections and as a consequence there is no Fock state vacuum $|0_{JS}\rangle$ ($|0_{JA}\rangle$) which is J-symmetric (J-antisymmetric). Therefore

$$G_{JS}(x, x') \neq \langle 0_{JS} | \{ \phi(x), \phi(x') \} | 0_{JS} \rangle$$

We calculate $\langle \phi^2_{JS \atop JA} \rangle$ and the energy momentum tensor $\langle T_{\mu\nu\, {JS \atop JA}} \rangle$
We obtain

$$\langle \phi^2_{JS \atop JA} \rangle = \langle \phi^2 \rangle \pm \langle \bar{\phi}^2 \rangle$$

$$\langle T_{\mu\nu\, {JS \atop JA}} \rangle = \langle T_{\mu\nu} \rangle \pm \langle \bar{T}_{\mu\nu} \rangle$$

where

$$\langle \bar{\phi}^2 \rangle = \frac{1}{16\pi \cos \pi\nu} \left[m^2 + (\xi - \tfrac{1}{6}) R \right]$$

$$\langle \bar{T}_{\mu\nu} \rangle = -\frac{m^2}{64\pi \cos \pi\nu} \left[m^2 + (\xi - \tfrac{1}{6}) R \right] g_{\mu\nu}$$

$\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ stand for the usual (so called Bunch-Davies or euclidean[4]) values of the ordinary theory. $\langle \bar{\phi}^2 \rangle$ and $\langle \bar{T}_{\mu\nu} \rangle$ stand for the new terms introduced by the elliptic theory. Here $\nu = (\frac{9}{4} - \frac{M^2}{H^2})^{1/2}$, $M^2 = m^2 + \xi R$, m^2 is the mass of the fields, ξ is the coupling and $R = 12 H^2$, ($H = \sqrt{\Lambda/3}$). We study limiting cases of interest

$M^2 \ll H^2$ (inflationary)
$M^2 \gg H^2$ (massive)
$M^2 = 2H^2$ (conformal invariant)

In the inflationary regime, we find

$$\langle \phi^2_{JS} \rangle = 2 \langle \phi^2 \rangle \qquad , \left(\langle \phi^2 \rangle = \frac{R^2}{384\, \pi^2 M^2} \right)$$

$$\langle T_{\mu\nu\, JS} \rangle = \frac{61}{151} \langle T_{\mu\nu} \rangle \qquad , \left(\langle T_{\mu\nu} \rangle = \frac{-61\ R^2 g_{\mu\nu}}{138240\, \pi} \right)$$

The J-symmetric theory allows good inflation. On the contrary, for the J-antisymmetric theory for $M^2 << H^2$ we find

$$\langle \phi^2_{JA} \rangle = 0$$

$$\langle T_{\mu\nu\, JA} \rangle = - \frac{29}{61} \langle T_{\mu\nu} \rangle$$

which does not satisfy the fundamental hypothesis of inflation [5]. However, the J-antisymmetric theory allows a better understanding of the massless and minimally coupled ($m^2 = 0$, $\xi^2 = 0$) ordinary theory (without identification). $G_{JA}(X,X')$ is not infrared divergent. We find new vacuum states which are O(4) and O(1,3) invariant [together with E(3) [6] these are the maximal subgroups of O(1,4)]. We obtain

$$\lim_{\substack{m^2 \to 0 \\ \xi \to 0}} \langle \phi^2 \rangle_{O(1,4)} \neq \langle \phi^2 \rangle_{E(3)} \neq \langle \phi^2 \rangle_{O(4)} \neq \langle \phi^2 \rangle_{O(1,3)}$$

and

$$\lim_{\substack{m^2 \to 0 \\ \xi \to 0}} \langle T_{\mu\nu} \rangle_{O(1,4)} \neq \langle T_{\mu\nu} \rangle_{E(3)} = \langle T_{\mu\nu} \rangle_{O(4)} = \langle T_{\mu\nu} \rangle_{O(1,3)}$$

Similar results hold for the JS and JA-theories. This is a manifestation of the fact that the O(1,4) invariant Green function is infrared divergent in the massless and $\xi = 0$ limit and there is no Fock state which is O(1,4) invariant in that case. Therefore, the correct values of $\langle T_{\mu\nu} \rangle$ for the massless minimally coupled field are those corresponding to the E(3), O(4) and O(1,3) vacua and not to the limiting case of the O(1,4) vacuum which does not exist in such limit.

More results and detailed derivations are given in ref. 8. The J-symmetric field theory in the "elliptic" de Sitter space applies equally well to other scenarios of inflation as the Starobinsky model [9] and the models of quantum creation of the universe as proposed by Vilenkin [10], Linde [5] and Hartle-Hawking [11]. This will be discussed elsewhere [12].

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3. GEOMETRICAL INVARIANT DESCRIPTION OF QUANTUM EFFECTS NEAR DISTORTED BLACK-HOLES.

We investigate the contribution of massless fields of spins $0, \frac{1}{2}$ and 1 to the vacuum polarization near the event horizon of static Ricci-flat space-times. We do not assume any particular spatial symmetry. Within the Page-Brown "ansatz" we calculate $\langle\varphi^2\rangle^{ren}$ and $\langle T_{\mu\nu}\rangle^{ren}$ near static distorted black-holes, for both the Hartle-Hawking ($|\rangle_H$) and Boulware ($|\rangle_B$) vaccua. Using Israel's description of static space-times, we express these quantities in an invariant geometric way. We obtain that $\langle\varphi^2\rangle_H^{ren}$ and $\langle T_{\mu\nu}\rangle_H^{ren}$ near the horizon depend only on the two-dimensional geometry of the horizon surface. We find $\langle\varphi^2\rangle_H^{ren}=1/48\pi^2\ K_o$, $\langle T_o{}^o\rangle_H^{ren}=(7\alpha+12\beta)K_o{}^2 - \alpha\,{}^{(2)}\Delta K_o$. K_o is the Gaussian curvature of the horizon and α, β are numerical coefficients depending on the spin of a field. The term in ${}^{(2)}\Delta K_o$ is characteristic of the distortion of the black-hole. When the event horizon is not distorted, K_o is a constant and this term disappears.

Quantum effects near black-holes are of particular interest by several reasons. Knowledge of the renormalized vacuum expectation value of the stress-energy tensor [$\langle T_{\mu\nu}(x)\rangle^{ren}$] that can be considered as a measure of the vacuum polarization, is crucial in order to determine the space-time evolution of an evaporating black-hole. As a first step, one usually considers the situation when the space-time geometry is given, that is one deals with the quantum field theory on a given spacetime background. Such an approximation is expected to be rather good when the mass M of the black-hole is much larger than the Planckian mass $m_p=(\hbar c/G)^{\frac{1}{2}}$ In this case, one can use the one-loop approximation in which the contributions of different physical fields to $\langle T_{\mu\nu}\rangle^{ren}$ are summed additively and may be considered separately. The contributions of massive fields (with mass m) contain the factor $\mathcal{E}=m_p{}^4/m^2M^2$. The presence of a small parameter $\mathcal{E}$ (for $\lambda=h/mc\ll 2GM/c^2$) and the fact that the contributions of massive fields are essentially local allow to study them in detail |1|. The contributions of massless fields which are essentially non-local are much more complicated |2|. Here we investigate the contribution of massless fields of spins

0,½ and 1 to the vacuum polarization near the event horizon of static Ricci-flat space-times. We do not assume any particular spatial symmetry for the geometry. A rather simple approach for calculating $\langle T_{\mu\nu}(x)\rangle^{ren}$ in static space-times has been proposed by Page [3]. His approximation has been shown extremely good in the external spacetime of a Schwarzschild black-hole [4-6]. Another approach which gives for a conformal scalar field in the Schwarzschild metric the same approximation expressions as Page, has been proposed by Brown [7-8]. These approaches are based on the possibility to obtain $\langle T_{\mu\nu}\rangle^{ren}$ in the spacetime of interest from that calculated in an appropriate conformally related spacetime where trace anomalies vanish. We analyze within Page's approximation, the influence of an external gravitational field on the vacuum polarization near black-holes. Such a field arises when there are massive bodies outside of the black-holes. Their gravitational field changes the metric near the event horizon and distorts the black-hole [9-11]. In the case of the scalar field φ there is also of some interest the investigation of $\langle\varphi^2\rangle^{ren}$ which describes the quantum fluctuations of this field. As it is known, these expectation values depend on the choice of the vacuum state. We deal here with the Hartle-Hawking ($|\,\rangle_H$) and the Boulware ($|\rangle_B$) vaccua corresponding to a thermal and to an empty state at large radii respectively. ($|\rangle_B$ is pathological at the horizon in the sense that $\langle T_{\mu\nu}\rangle^{ren}_B$ and $\langle\varphi^2\rangle^{ren}_B$ diverge there). We express these <u>four dimensional</u> quantities in an invariant geometric way. We obtain that $\langle\varphi^2\rangle^{ren}_H$ and $\langle T_{\mu\nu}\rangle^{ren}_H$ near the horizon depend only on the two dimensional geometry of the horizon surface. We find [12] :

$$\langle\varphi^2\rangle_H = \frac{1}{48\pi^2}\, K_o$$

$$\langle T_o{}^o\rangle_H = (7\alpha + 12\beta)K_o{}^2 - \alpha\, {}^{(2)}\Delta\, K_o$$

and

$$\langle\varphi^2\rangle_B = -\frac{\varkappa_0^{\,2}}{48\pi^2 X} + \frac{K_0}{48\pi^2} + O(X)\ , \quad (X = -\xi_\mu \xi^\mu)$$

$$\langle T_\mu{}^\nu\rangle_B = 2\,\frac{\varkappa_0^{\,4}}{X^2}\left(\beta - \frac{\alpha}{3}\right) \mathrm{diag}\,(-3,\ 1,\ 1,\ 1)_\mu^\nu$$

K_o is the Gaussian curvature of the horizon surface and α, β are numerical coefficients depending on the spin of fields The term in ${}^{(2)}\Delta$ Ko is characteristic of the distortion of the black-hole. When the event horizon is not distorted, Ko is a constant and this term disappears.

In a number of cases, the formulas obtained here coincide identically with the exact values of $\langle\varphi^2\rangle_H$ and $\langle T_{\mu\nu}\rangle_H$. In particular, it happens for $\langle\varphi^2\rangle_H$ and $\langle T_{\mu\nu}\rangle_H$ of the electromagnetic field at the horizon of the Schwarzschild's black-hole and for $\langle\varphi^2\rangle_H$ at the pole of the event horizon of the axially symmetry distorted black-hole. The reason for this as well as the reason for the remarkable accuraly of the Page's approximation in the case of the scalar field till now remain unknown.

For the Schwarzschild black hole the values of $\langle T_\mu{}^\nu\rangle_H^{(s)}$ at the horizon, in the Page's approximation are

$$\langle T_\mu{}^\nu\rangle_H = \frac{1}{1920\pi^2(2M)^4}\,\mathrm{diag}\,(3,3,1,1)_\mu^\nu \quad ,\ s = 0$$

$$\langle T_\mu{}^\nu\rangle_H = \frac{1}{960\pi^2\,(2M)^4}\,\mathrm{diag}\,(-1,-1,[illegible],[illegible])_\mu^\nu \quad ,\ s = \tfrac{1}{2}$$

$$\langle T_\mu{}^\nu\rangle_H = \frac{1}{480\pi^2\,(2M)^4}\,\mathrm{diag}\,(-41,-41,28,28)_\mu^\nu \quad S = 1$$

and the exact asymtotics of $\langle T_\mu^\nu\rangle_B$ near the horizon are

$$\langle T_\mu^\nu\rangle_B = \frac{-h(s)}{6\pi^2 X^2}\,\varkappa_0^{\,4}\,F(s)\,\mathrm{diag}\,(-3,1,1,1)_\mu^\nu\ ,$$

where

$$F(s) = \int_0^\infty \frac{dx\ x\ (x^2+s^2)}{\exp(2\pi x)-(-1)^{2s}} = \begin{cases} \dfrac{1}{240}, & s = 0 \\ \dfrac{17}{1920}, & s = \frac{1}{2} \\ \dfrac{37}{480}, & s = 1 \end{cases}$$

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