LECTURES ON SEMICLASSICAL QUANTUM GRAVITY

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1. INTRODUCTION

1.1 MOTIVATION OF THE THEORY

While strong weak and electromagnetical interactions have been unified, with more or less satisfactory theories, in the last years, gravity stands appart as the more difficult interaction to be treated in a unified frame and also to be understand. The answers to this mistery lies in the fact that gravity seems a quiet different interactions than the three others. In fact, normaly we assume that it is the compensating field of an external local simmetry, while all the others are produced by gauge theories of internal simmetries, it has a dimentional coupling constant while all the other have dimensionless coupling constants, and even at the classical level, the Equivalence Principle shows that gravity is different: the inertial and gravitational mass are equal, while e.g. for electromagnetism the inertial mass is different than the electric charge.

Moreover the fact that its coupling constant is dimensional makes Quantum Gravity a non renormalizable theory. Perhaps the solution of all these problems is Supergravity (V. Nieuwenhuizen (1981)) or the theories where gravity is produced by the other interactions via an Effective Action (cfr. Adler (1982)). Meanwhyle we have not a reliable Quantum Gravity, thus, if we, some how, want to investigate the influence of gravity in quantum phenomena, at least on a first aproximation, we must rely on a semiclassical theory, like those used at the first epoch of Quantum Electrodynamics, where the electromagnetic field was considered as a classical backgroun field.

Analogously we shall develop, in this lectures a formalism where the gravitational field will be considered as a classical background field, that satisfy the field equations of General Relativity while all others field are quantized.

In paragraph 2, we shall see if a regime exist such that this approximation has a physical sense and in what extend we can consider the gravitational field as a classical one. And in the last of the lectures we shall show that this formalism yields interesting results useful for cosmology and astrophysic.

1.2 Two Words of History

The first work on the subject is perhaps the one written by Schrödinger (1932). Much later in the sixteen the systematic treatment of the subject began with Lichnerowicz (1961), (1964a), (1964b) who formulate quantum field theory in curved space time and Parker (1968) and Zel'dovich (1970) who studied the creation of particles by gravitational fields. In this first period Quantum Field Theory in De-Sitter space was also studied (Nachtman (1967a,b), Chernikov and Tagirov (1969) and Castagnino (1969), (1970), (1972)) and people try to find a particle model in curved space time (a problem that we shall face in paragraph 5). Later on, the attention was concentrated in the renormalization of the energy momentum tensor, $T_{\mu\nu}$, i.e. the computation of its renormalized vacuum expectation value $<0|T_{\mu\nu}|0>_{ren} = <T_{\mu\nu}>$, that can be used as right hand side of the Einstein equation in a semiclassical treatment. We shall develop this problem in paragraph 4 and give the principal authors names that work in this subject.

Then, in January 1974 Hawkings published his celebrated paper on the emission of thermal radiations by the black holes, using quantum field theory in curved space-time. This thermal radiation satisfies the Thermodinamical Theory of Black Holes (Davies 1977, 1978, Hawking, 1976 and 1977) and it was a first prove that the formalism is basically right. This success produced a great number of papers and alternative interpretation of the Hawking effect and put the subject at the first line of Theoretical Physic research for some years.

In the eghties the generalization to non-free quantum theories (self interacting fields , or several fields in interaction) began and it is still under research, (Birrell & Ford (1979), Nelson & Panangaden (1982), Leen (1983), Toms (1982) and (1983)) and the theory was used in more elaborated models as Grand Unified Theory (Parker & Toms (1983)) and to solve the back-reaction problem (Grib, Mamayev & Mostepanenko (1980a,b, 1984), Gunzig (1984)) but we cannot dwell in these subjects in these short lectures

In 1982 the first book on Quantum Fields in Curved Space by N.B. Birrell and P.C.W. Davies appears presenting, for the first time, the subject almost complete. The aim of these lectures is to give a more systematic and synthetic treatment, than those given by Birrell and Davies, focused in the two main problems: the vacuum definition and the Energy-Momentum Tensor renormalization, and in the internal relation between these problems.

2. SEMICLASSICAL QUANTUM GRAVITY[(1)]

2.1 THE SEMICLASSICAL REGIME

In this paragraph we will define the semiclassical regime of a completelly quantized theory in a general way. Then we will use this definition in a model theory of Quantum Gravity, a non satisfactory non renormalizable theory, hopping that the semi-classical formalism would have, anyhow, some physical sense. We will adopt the notation of De Witt (1965).

Let S_{tot} be the total action of a field theory, a functional of a field operator ϕ^i (in De Witt notation i stands for the coordinate x^μ and a generic set of indices I that, eventually, label the field components) obtained from the classical action S by the addition of an external source J_i:

(1) The paragraph is based in a paper by Castagnino & Paul (1984). Below eq. (2.1.10) we shall only sketch the demonstrations because we are only looking for a criterium of semiclassicity.

(2.1.1) $$S_{tot} = S + J_i\phi^i \quad ,$$

(in De Witt notation repetition of indices means integration in all the space-time and summation of the indices).

The generating functional is:

(2.1.2) $$W[J] = e^{iz[J]} = \langle B|A\rangle = \\ = \int \mathcal{D}\phi e^{iS_{tot}} \quad ,$$

usually $|A\rangle = |0in\rangle$; $|B\rangle = |0\ out\rangle$. We define:

(2.1.3$_1$) $$\phi^i = Z_{,}^{\ i} = \frac{\delta Z}{\delta J_i} \quad ,$$

as the "classical field" (the $\delta/\delta J_i$ is the functional derivative that will be symbolized by a comma). We can prove: that

$$\phi^i = -\frac{\langle B|\phi^i(0)|A\rangle}{\langle B|A\rangle} = -\langle\phi^i\rangle \quad ,$$

and that,

(2.1.3$_n$) $$G^{i_1\ldots i_n} = \frac{\delta^n Z}{\delta J_{i_1}\ldots\delta J_{i_n}} \quad ,$$

are the generalized Green Functions.

Then we can define the Effective Action as:

(2.1.4) $$\Gamma[\phi] = Z[J] - J_i\phi^i \quad .$$

The Effective Action most important property is that:

(2.1.5) $$\Gamma,i = \langle S,i\rangle \quad ,$$

i.e. the classical field equations are $S_{,i} = 0$ thus their quantum version is $\Gamma_{,i} = 0$.

It can be proved that the knowledge of Γ allows to calculate the quantum corrections of the system, the scattering matrix S and also to obtain the spectra and the band widths of the bound state. In short all the Quantum Theory is in Γ. If we define the base propagator of S, G_s, as the solution of:

$$(2.1.6) \qquad S_{,ik}G_s^{kj} = - \delta_i^{\ j} \quad ,$$

It can be proved that we can expand Γ in terms of S and G_s as:

$$(2.1.7) \qquad \Gamma[\phi] = S[\phi] + \Sigma = S[\phi] - \frac{1}{2} \qquad - \frac{1}{12}$$

$$- \frac{1}{8} \qquad + \frac{i}{48} \qquad + \frac{i}{8} \qquad +$$

$$+ \frac{i}{16} \qquad + \frac{i}{24} \qquad + \frac{i}{12}$$

$$+ \frac{i}{8} \qquad + \frac{i}{16} \qquad + \frac{i}{48}$$

In these diagrams the vertex of n legs correspond to:

$$(2.1.8) \qquad S_{,i_1 \ldots i_n} = \frac{\delta^n S}{\delta\phi^{i1} \ldots \delta\phi^{in}} \quad ,$$

and the lines ------ between vertices correspond to the bare propagator G_s.

The first loop of eq. (2.1.7) can be interpretated as:

$$\Sigma = -\frac{i}{2}\,\mathrm{Tr}\,\ln\,(-G^{0}S) \quad , \tag{2.1.9}$$

(where G^0 is G_s computed to the 0^{th} order in the ϕ^i) and the other terms are obtained by the usual Feynman rules using the bare propagator G_s and the vertices (2.1.3), e. g.:

$$= S_{ijk}\, G_s^{ii'} G_s^{jj'} G_s^{kk'}\, S_{i'j'k'} \quad .$$

In every quantum theory where there are, at least, two kind of particles, and where we can neglect the lines of one of the particles in the Feynman graph of the effective action in some circumstances, we can use a semiclassical regime, because we can consider this last particle as classical.

We shall clarified the issue with two very simple examples:

1 - Let us consider the action:

$$S = \int d^4x \,\{ \tfrac{1}{2}\,\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{m_1^2}{2}\,\phi_1^2 + \tfrac{1}{2}\,\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{m_2^2}{2}\,\phi_2^2 + \frac{\alpha}{4!}\,\phi_1^2\phi_2^2 + \frac{\lambda}{4!}\,\phi_1^4 \,\} \quad , \tag{2.1.10}$$

where ϕ_1 and ϕ_2 are two real scalar fields m_1 and m_2 their masses and α and λ are dimensionless coupling constants. If we compute the second term of the r.h.s. of eq. (2.1.7) to one loop order we have:

(2.1.11) $\quad = \quad + \quad + \quad +$

$+ \quad + \quad + \quad +$ etc.

(where — is the propagator of ϕ_1 and -.-.-.-.-.- the one of ϕ_2) If ϕ_2 would be not quantized all these graphs vanish with the exception of

and

Therefore the semiclassicity criterion for the field ϕ_2 is, in this case, that the mixed diagrams (which always contain at least one factor α) could be neglected with respect to the diagram which depend only on ϕ_1 (where there is always at least a factor λ). To do this it is obviously sufficient that $\alpha << \lambda$, if the masses and energies of the field quanta associated to ϕ_1 and ϕ_2 are of comparative order. In this case we shall have a semiclassical regime because in our computation we can consider ϕ_2 as a classical field.

2 - Let us now consider the action:

$$S = \int d^4x \, \{ \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{m_1^2}{2} \phi_1^2 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 + \frac{m_2^2}{2} \phi_2^2 + \frac{\lambda}{4!} \phi_1^4 + \frac{\alpha}{3!} \phi_2^3 \} \quad , \tag{2.1.12}$$

where all is the same but the α-interaction is different now, and α is a dimensional coupling constant in this case. Up to *one loop order we must compare now graphs like*[2]

and

where the number of vertex α is twice the number of vertex λ. (This fact can be proved to any number of loops). The degrees of divergence are 0 and 2 respectively, and in order to make the integrals finite we can introduce a cut off ℓ and integrate over an hypercube of side ℓ (ℓ may be the characteristic length of the system and also we can Fourier transform and take a more familiar momentum cut off, the characteristic momentum of *the system*).

As Γ and its diagrams are dimentionless

will be of order λ and

will be of order $\alpha^2 \ell^2$

because α has dimension $(\text{length})^{-1}$.

Then ϕ_2 is semiclassical with respect to ϕ_1 *when* $\alpha^2 \ell^2 << \lambda$ or $\ell << \lambda \alpha^{-1/2}$ and this argument can be used with diagrams with an arbitrary number of loops with the same result. Thus we have a semiclassical regime if the characteristic length of the system is smaller than $\lambda \alpha^{-1/2}$. We can see that if one of the coupling constant is dimensional the definition of semi-classicity depends on the characteristic length (or momentum) of

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(2) We must always compare graphs with the same number of loops i.e. of the same power in $\hbar$ (Nambu 1966).

the system. We have not a semiclassical regime for every size.

We shall use this method for quantum gravity.

2.2 FEYNMAN RULES FOR QUANTUM GRAVITY

Quantum Gravity deals with gravitons, that can be considered as quantized propagating gravitational waves in a curved background space-time. Thus gravitons are quantized small disturbances, that we can studie if we separate the metric $g_{\mu\nu}$ in two pieces:

$$g_{\mu\nu} = g^{c}_{\mu\nu} + \bar{g}_{\mu\nu} \quad , \tag{2.2.1}$$

where $\bar{g}_{\mu\nu}$ represents the wave to be quantized and $g^{c}_{\mu\nu}$ the background space-time. This is the starting point of De Witt's (1967 a,b) "background field method" (see also Jackiw (1974), Abbot (1983)). The action is the ordinary Einstein action:

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\, R\, d^4x \quad , \tag{2.2.2}$$

where $\kappa^2 = 8\pi G$, being G the Newton's constant. If we use natural units ($c = \hbar = 1$) κ is the Planck length: $\sim 10^{-33}$cm. If, e.g., we use the new variable $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ and we use flat space-time as background[3] like Faddeev and Popov (1973) we have:

$$h^{\mu\nu} = \eta^{\mu\nu} + \kappa u^{\mu\nu} \quad , \tag{2.2.3}$$

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(3) In curved space-time Feynman graph would be the same.

and we can expand the action (2.2.2) in powers of $u^{\mu\nu}$:

$$S = S_2 + \sum_{n=0}^{\infty} \kappa^n S_{n+2} \quad , \tag{2.2.4}$$

where S_n is a form of n^{th}-degree in variables $u^{\mu\nu}$ and their derivatives. From this action we can deduce the Feynman graphs of Quantum Gravity. But the action S possess an invariance gauge (the group of general coordinates transformation) and the bare propagator G_s is not unique. The correct theory was obtained by Faddeev and Popov adding a gauge fixing and a "ghost" term to the free Lagrangian. Thus, the free graviton propagator is in momentum space:

$$G_{\mu\nu,\rho\sigma} = \frac{1}{p^2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}) \quad , \tag{2.2.5}$$

and the ghost propagator simply is:

$$G^{\mu\nu} = - \frac{\eta^{\mu\nu}}{p^2} \quad , \tag{2.2.6}$$

whyle the ghost vertex is:

(2.2.7) k_2^{μ} $\rho\sigma$ k_1^{ν} k_3

$$= - \frac{\kappa}{2}[\delta^{\mu}_{\nu}(k_{1\rho}k_{2\sigma} + k_{1\sigma}k_{2\rho}) - k_{1\nu}(\delta^{\mu}_{\sigma}k_{3\rho} + \delta^{\mu}_{\rho}k_{3\sigma})] \quad ,$$

$$k_1 + k_2 + k_3 = 0 \quad ,$$

where refer to ghost and to gravitons the third-order graviton vertex is:

(2.2.8)

$$= \frac{\kappa}{2} \{ \frac{k_1^2}{2} (\eta_{\mu\lambda}\eta_{\rho\sigma}\eta_{\tau\nu} +$$

$$+ \eta_{\mu\tau}\eta_{\nu\lambda}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}\eta_{\rho\tau} + \eta_{\nu\sigma}\eta_{\mu}\ \eta_{\rho\tau} +$$

$$+ \eta_{\mu\tau}\eta_{\nu\rho}\eta_{\mu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\eta_{\lambda\tau} + \eta_{\nu\tau}\eta_{\mu\rho}\eta_{\lambda\sigma} + \eta_{\nu\sigma}\eta_{\mu\lambda}\eta_{\lambda\sigma})$$

$$+ k_1^2\eta_{\mu\nu}(\eta_{\rho\sigma}\eta_{\lambda\tau} + \eta_{\rho\tau}\eta_{\lambda\sigma}) + (k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu})\eta_{\lambda\rho}\eta_{\sigma\tau}$$

$$+ (k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu})(\eta_{\lambda\sigma}\eta_{\rho\tau} + \eta_{\lambda\rho}\eta_{\sigma\tau}) -$$

$$- k_{1\nu}k_{1\tau}(\eta_{\mu\lambda}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\lambda\sigma}) - k_{1\mu}k_{1\sigma}|\eta_{\mu\lambda}\eta_{\rho\tau} +$$

$$+ \eta_{\mu\rho}\eta_{\lambda\tau}) - k_{1\mu}k_{1\tau}|\eta_{\nu\lambda}\eta_{\rho\sigma} + \eta_{\nu\rho}\eta_{\lambda\sigma}) -$$

$$- k_{1\mu}k_{1\sigma}\ (\eta_{\nu\lambda}\eta_{\rho\tau} + \eta_{\nu\rho}\eta_{\lambda\tau}) - k_{2\nu}k_{3\rho}\eta_{\lambda\sigma}\eta_{\mu\tau} -$$

$$- k_{2\nu}k_{3\lambda}\eta_{\rho\sigma}\eta_{\mu\tau} - k_{2\mu}\ k_{3\rho}\ \eta_{\lambda\sigma}\eta_{\nu\tau} -$$

$$- k_{2\mu}k_{3\lambda}\ \eta_{\rho\sigma}\eta_{\nu\tau} - k_{2\nu}k_{3\rho}\eta_{\lambda\tau}\eta_{\mu\sigma} -$$

$$- k_{2\nu}k_{3\lambda}\eta_{\rho\tau}\eta_{\mu\sigma} - k_{2\mu}k_{3\rho}\eta_{\lambda\tau}\eta_{\nu\sigma} -$$

$$- k_{2\mu}k_{3\lambda}\eta_{\rho\tau}\eta_{\nu\sigma}] +$$

+ the sum over permutation of the pairs $(\mu,\nu),(\lambda,\rho),$ $(\sigma,\tau)\}$.

All other details could be found in Fadeev-Popov paper.

2.3 SEMICLASSICAL REGIME IN QUANTUM GRAVITY

Let us face the definition of a semiclassical regime for a quantum self interacting scalar real field ϕ in a gravitational background field $g^c_{\mu\nu}$ for the metric and zero for ϕ. The action is:

$$(2.3.1) \qquad S = \frac{1}{2x^2} \int \sqrt{-g}\, R\, d^4x \; + \\ - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right] .$$

To one loop order we get for the effective action T using eq. (2.1.8):

$$(2.3.2) \qquad T = S\,(g_{\mu\nu} = g^c_{\mu\nu};\ \phi = \phi^c = 0) \\ - \frac{i}{2}\,\mathrm{Tr}\,\ln(-G\phi) - \frac{i}{2}\,\mathrm{Tr}\,\ln(-Gg) \quad .$$

These last terms are zero, in paragraph 2.2., because we are flat space-time but now we are in a generic curves spacetime of metric $g^c_{\mu\nu}$ and these terms are non vanishing but correspond to the loops:

where $G\phi$ and Gg are respectively the bare propagator of the matter field ϕ (propagator ——) and the gravitational field (propagator).

These loops do not depend on κ as we can see from equation (2.2.5). Thus both loop are of the same order, i.e., of order one. This statement is in complete acordance with the

Equivalence Principle, according to which all form of matter and energy couple with the same strength to gravity. Being the graviton a form of matter or energy, like the particle of the field ϕ, it is as well, a source of gravity. It follows that the we cannot have a semiclassical theory at any scales of distance or momentum. Thus we cannot eliminate the quantum effects of gravitons, at least completely. Then, what can we neglect ? The graviton graphs that follow the term (2.3.2), like

or ,

have in general n vertices and their integrals are of the type:

(2.3.3) $$\kappa^n \int d^4x^4 \ldots d^4x^L \ldots\ldots \quad .$$

These integrals are divergent, like all other integral of quantum gravity, but they may be made finite by the integration outside a hypercube of side ℓ, the cutoff, or characteristic length of the system. Integral (2.3.3) will therefore be of the order $\kappa^n \ell^{-n}$ because it must be dimensionless. Then if we want that the L loop graviton diagrams be negligible with respect the 1 loop graviton diagram of order 1 we must have $\kappa^n \ell^{-n} << 1$ i.e.:

(2.3.4) $$\ell >> \kappa \quad .$$

Thus we can neglect these loops if the characteristic lenght of our system is greather than the Planck lenght.

It is of fundamental importance to realize that the condition (2.3.4) do not allows to neglect the graviton 1-loop with respect to the matter 1-loop because none of both depends on κ. Thus we have not a semiclassical theory if we neglect the L-graviton-loop (L > 1) but a "truncated" theory, that we shall anyhow call "semiclassical" to avoid the invention of a new name. This semiclassical Quantum Gravity is Quantum Gravity truncated at 1-loop graviton level and can be use if the characteristic lenght of the system is greater than the Planck lenght or the characteristic times or periods grater than Planck time ($\sim 10^{-44}$ s). If ones regards 10^{-13} cm and 10^{-23} s as the lenght and time scales of important quantum processes (i.e. Compton's lenght and time respectively) it seams that there is enough place to develop a semiclassical theory.

Beside graviton loops we have also matter loops like

that must be neglected. Can we make a more refine theory keeping these graphs and neglected the graviton loop ? Yes, we can, because graviton loops will be proportional to (2.3.3) and matter loops to:

$$\lambda^m \int d^4x_1 \ldots d^4x_L \quad .$$

For a given order in $\hbar$ we must compare diagrams with the same number of loops, then an easy calculation gives that we must compare n = 2m and that the condition to neglect graviton loops with respect to matter loops is:

$$\ell >> \kappa \lambda^{-1/2} \quad ,$$

a more restrictive condition than (2.3.4) if $\lambda < 1$

As we shall see in paragraph 4 the semiclassical (truncated) quantum gravity is renormalizable.

But the 1-graviton-loop appears because we are working in a curve background. Can we get reed of this graph if we work on a flat background $g^{c}_{\mu\nu} = \eta_{\mu\nu}$? But as Duff proved (1975) only in a generic curved background renormalization can be carried on. Also other difficulties (Christensen & Duff (1930)) make necesary the consider fields propagation in a background with arbitrary metric $g^{c}_{\mu\nu}$.

More over. Duff (1981) proved the inconsistency of the semiclassical quantum gravity (in the old version with no 1-graviton-loop) by field redefinition. On the contrary semi-classical Quantum Gravity in the new version (truncated at 1-graviton-loops level) is completely consistent under field redefinition as Quantum Gravity itself (which is consistent under field redefinition as all completely quantized theories) because it is only Quantum Gravity where a regime exists that allows to neglect L-graviton-loops $(L > 1)$ in the computation of the Effective Energy (ctr. Castagnino & Paul (1984)).

Finally we must mention that we have completely forgotten the ghost loops in this paragraph, but as they have same vertex structure as gravitons loop (ctr. eqs. (2.2.7) and (2.2.8)) the conclusion for ghost loops would be exactly the same.

2.4 THE MAIN PROBLEMS OF SEMICLSSSICAL QUANTUM GRAVITY

We shall demonstrate (cfr. eq. (4.3.8)) that the $g_{\mu\nu}$

field equation of Semiclassical Quantum Gravity (with a non self interacting field ϕ, i.e. $\lambda = 0$ in eq. (2.2.1) and a little more general geometric action), is:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H^{(1)}_{\mu\nu} + \beta H^{(2)}_{\mu\nu} = - 8\pi G \langle\psi|T_{\mu\nu}|\psi\rangle_{ren} \quad , \tag{2.4.1}$$

(cfr. Birrell & Davies (1982) eq. (6.95)) where the l.h.s. is the one of Einstein Equation with cosmological constants with two new terms quadratic in the curvature, that we shall later define, and the r.h.s. is the renormalized expectation value of the Energy-Momentum tensor in some quantum state $|\psi\rangle$.

Equation (2.4.1) is what we could expect for a quantum version of the Einstein Equation. Except from the two new terms $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$, that are necessary to make the theory a renormalizable one, that could be eliminated taking $\alpha = \beta = 0$ as "dressed" experimental coupling constants, it says that the quantum source is just the expectation value of the classical source $T_{\mu\nu}$.

Eq. (2.4.1) allows us to solve the reaction back problem, i.e. to see how the universe evolves under the action of the quantum forces, produced by the quantum field, that we take in consideration and we use to compute $T_{\mu\nu}$. In this case, for simplicity it is only the non self-interacting, scalar real field ϕ, but it could also be any other field of spin 1/2, 1, and 3/2[4]. And if we want that the theory be consistent it must also contain the graviton $\bar{g}_{\mu\nu}$ as was explain in paragraph 2.3.

(4) In this case we must use a supersymmetric Lagrangian.

Equation (2.4.1) is also valid only for times higher than the Planck's time, because only there eq. (2.3.4) is fulfilled. Thus, the principal goal of our theory is to compute the r.h.s. of eq. (2.4.1) and to solve this equation. To do this we have two main problems:

1 - $<\psi|T_{\mu\nu}|\psi>$ is a divergent quantity in general. We shall see, in paragraph 4,how to regularize this quantity and how we can put the infinitie into the coupling constant of the l.h.s. of eq. (2.4.1), obtaining in this way a renormalized equation with "dressed" coupling constant and a finite $<\psi|T_{\mu\nu}|\psi>_{ren}$ in the r.h.s.

2 - The quantum state $|\psi>$ is hill-defined in curved spacetime. We shall see in paragraph 3 that the vacuum definition in curved space-time is not trivial and thus the construction of a Fock space is not a simple problem. Using the results of paragraph 4, we shall see, in paragraph 5, that the problem can be solve only under certain circumstances and in those cases we shall find the solution. We shall also realize that the vacuum definition is observer dependent.

3 Quantum Field Theory in Curved Space-Time

3.1 Reference System in Curved Space-Time

We shall begin by the study of problem 2 of the preceeding paragraph that will be solve only in paragraph 5. As the notion of particles in curved space-time is observed dependent we shall introduce the Reference Systems right from the beginning.

Let the space-time be a C^∞ four-dimensional riemannian *manifold* V_4 endowed with metric $g_{\mu\nu}$ with signature $(+---)$ $(\mu,\nu,\ldots\alpha,\beta, = 0,1,2,3)$. We shall use the same convention than those of *Birrell and Davis (1982) i.e. (---) in the terminology* of Misner, Thorne & Wheeler (1973): $R^\alpha{}_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha{}_{\beta\gamma} - \ldots;$ $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. Now let us consider a set of infinite observer in V_4, each one is just a geometrical point P, with an space-time path $x^\mu = x^\mu(\tau)$, where τ is the proper time. We can consider this set of observers as an ideal reference fluid. Let u^μ be the unit time-like vector tangent to the paths, then if the fluid is defined in all V_4 (or in a space-time patch of V_4) we have a vector field u^μ define in al V_4 (or in a patch of V_4). Of course:

$$(3.1.1) \qquad u^\mu = \frac{dx^\mu}{d\tau} \quad ,$$

$$(3.1.2) \qquad g_{\mu\nu} u^\mu u^\nu = 1 \quad .$$

We can use, if we wish, a system of coordinates "adapted" to the fluid. We can use as coordinate x^0 any parameter of the fluid space-time paths and has coordinates x^1, x^2, x^3 any system of coordinates in a spatial hypersurface that intercept all the fluid paths , then each one of these lines is lable by (x^1,x^2,x^3); the coordinates of the interception point, and each point of *space-time by these parameters* and x^0. The components of u^μ in adapted coordinates are

$$(3.1.3) \qquad u^0 = (g_{00})^{-1/2} \quad u^i = 0 \ ; \ \ u_\mu = (g_{00})^{-1/2}\, g_{\mu 0}$$

The tangent space T_x at x can now be decompose in a one-dimentional space θ_x, in the direction of u^μ, and a three-

-dimentional space Σ_x, orthogonal to θ_x. Therefore:

$$T_x = \theta_x \oplus \Sigma_x \quad , \tag{3.1.4}$$

and we can define the projector on θ_x and on Σ_x as:

$$P_\theta(V^\mu) = u^\mu u_\nu V^\nu \quad , \tag{3.1.5}$$

$$P_\Sigma(V^\mu) = V^\mu - u^\mu u_\nu V^\nu = \tag{3.1.6}$$

$$= -\gamma^\mu{}_\nu V^\nu \quad ,$$

where:

$$-\gamma_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \quad , \tag{3.1.7}$$

and thus we have also a decomposition of the metric tensor:

$$g_{\mu\nu} = -\gamma_{\mu\nu} + u_\mu u_\nu \quad , \tag{3.1.8}$$

We can also define projector on tensors e.g.:

$$\begin{aligned} P_{\Sigma\Sigma}(t^{\mu\nu}) &= \gamma^\mu{}_\alpha \gamma^\nu{}_\beta t^{\alpha\beta} \\ P_{\Sigma\theta}(t^{\mu\nu}) &= -\gamma^\mu{}_\alpha u^\nu u_\beta t^{\alpha\beta} \\ P_{\theta\Sigma}(t^{\mu\nu}) &= -u^\mu u_\alpha \gamma^\nu{}_\beta t^{\alpha\beta} \\ P_{\theta\theta}(t^{\mu\nu}) &= u^\mu u_\alpha u^\nu u_\beta t^{\alpha\beta} \quad \text{etc.} \end{aligned} \tag{3.1.9}$$

Actually from (3.1.6) we have:

$$\begin{aligned} P_{\Sigma\Sigma}(g_{\mu\nu}) &= -\gamma_{\mu\nu} \\ P_{\Sigma\theta}(g_{\mu\nu}) &= P_{\theta\Sigma}(g_{\mu\nu}) = 0 \\ P_{\theta\theta}(g_{\mu\nu}) &= u_\mu u_\nu \quad . \end{aligned} \tag{3.1.10}$$

We can also define two "projection-derivatives". The transversal derivative

(3.1.11) $$\tilde{\partial}_\mu \phi = -\gamma_\mu{}^\nu \partial_\nu \phi \quad .$$

and the longitudinal derivative:

(3.1.12) $$\bar{\partial}_\mu \phi = u_\mu u^\nu \partial_\nu \phi \quad .$$

Thus from (3.1.6):

(3.1.13) $$\partial_\mu \phi = \tilde{\partial}_\mu \phi + \bar{\partial}_\mu \phi \quad .$$

Also the covariant derivative can be projected. We define the transversal derivative as :

(3.1.14) $$\begin{aligned} \tilde{\nabla}_\mu S_\nu &= P_{\Sigma\Sigma}(\nabla_\mu S_\nu) = \\ &= \tilde{\partial}_\mu S_\nu - (\mu \tilde{,} \nu;\lambda) S^\lambda \quad , \end{aligned}$$

where:

(3.1.15) $$\begin{aligned} (\mu \tilde{,} \nu;\lambda) &= \frac{1}{2} \left(\tilde{\partial}_\mu \gamma_{\nu\lambda} + \tilde{\partial}_\nu \gamma_{\mu\lambda} - \tilde{\partial}_\lambda \gamma_{\mu\nu}\right) = \\ &= P_{\Sigma\Sigma\Sigma}(\mu\nu;\lambda) \quad , \end{aligned}$$

Of course this derivative could be generalized to tensor of arbitrary ranck. The intrinsic properties of the reference fluid can be studied using the introduced notation. From u^μ we define:

(3.1.16) $K_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu$ the Killing tensor ,

(3.1.17) $\Omega_{\mu\nu} = \nabla_\mu u_\nu - \nabla_\nu u_\mu$ the Vortex tensor , (3.1.18) $$C_\mu = u^\nu \nabla_\nu u_\mu$$ the <u>curvature</u> vector .

Their transverse and longitudinal projections are (cfr. Cattaneo (1961)):

(3.1.19) $$\Omega_{\mu\nu} = \tilde{\Omega}_{\mu\nu} + C_\mu u_\nu - C_\nu u_\mu \quad ,$$

(in adapted coordinates we have:

(3.1.20) $$\tilde{\Omega}_{\mu\nu} = u_0 \, [\tilde{\partial}_\mu (\frac{u_\nu}{u_0}) - \tilde{\partial}_\nu (\frac{u_\mu}{u_0})] \quad , \quad)$$

(3.1.21) $$K_{\mu\nu} = \tilde{K}_{\mu\nu} - C_\mu u_\nu - C_\nu u_\mu \quad ,$$

(in adapted coordinates we have

(3.1.22) $$\tilde{K}_{\mu\nu} = u^0 \partial_0 \gamma_{\mu\nu} \quad , \quad)$$

and C_μ is transversal i.e. $C_\mu u^\mu = 0$.

The properties of the reference fluid could be studied using these tensors i.e.:

i) If $\tilde{\Omega}_{\mu\nu} = 0$ the fluid is irrotational or curl-free. These fluid has normal hypersurfaces and the vector u^μ can be written as $u_\mu = \psi \partial_\mu \phi$ where ψ and ϕ are two scalar functions. In this case it is possible to choose a system of "adapted" coordinates with normal hypersurfaces where:

(3.1.23) $$u_\mu = (u_0, 0, 0, 0) \quad ,$$

instead of (3.1.3).

From (3.1.20) we can see that, if adapted coordinates, with normal hypersurfaces, exist, in fact it is $\tilde{\Omega}_{\mu\nu} = 0$.

ii) If $\tilde{K}_{\mu\nu} = 0$ the fluid is Born rigid (Born (1909)) because using eq. (3.1.21) we can see that $\tilde{K}_{\mu\nu}$ could be considered the spacial deformation tensor.

iii) If $C_\mu = 0$ the fluid is a geodesic one.

iv) If $\Omega_{\mu\nu} = 0$, then the fluid is irrotational and geodesic, and u_μ could be written as $u_\mu = \partial_\mu\phi$ where ϕ is a scalar function that could be considered as a time coordinate. Using this time coordinate we obtain a "completelly adapted" or "synchronic" system of coordinates with normal hypersurfaces where:

(3.1.24) $$u^\mu = u_\mu = (1, 0, 0, 0) \quad .$$

v) If $K_{\mu\nu} = 0$ the fluid is rigid and geodesic and u^μ is a unit-<u>Killing</u> vector.

vi) If there is a generic time-like <u>Killing-vector</u> K_μ such that:

(3.1.25) $$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \quad ,$$

we can take u_μ as the unit vector parallel to K_μ :

(3.1.26) $$u^\mu = K^\mu/K \quad , \quad K = (g_{\mu\nu}K^\mu K^\nu)^{1/2} \ .$$

Then:

(3.1.27) $$K_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu = K_\nu \partial_\mu K^{-1} + K_\mu \partial_\nu K^{-1} \quad .$$

Thus from (3.1.20):

$$\tilde{K}_{\mu\nu} = 0 \qquad C_\mu = \partial_\mu \log K . \tag{3.1.28}$$

Thus the fluid is Born rigid but not a geodesic one.

Now if we define a time coordinate t such that $K^\mu = dx^\mu/dt$, then

$$K^0 = \frac{dx^0}{dt} = \frac{dt}{dt} = 1 \tag{3.1.29}$$

Contracting (3.1.25) with K^μ and K^ν we obtain:

$$\frac{dK}{dt} = 0 \quad . \tag{3.1.30}$$

From eqs. (3.1.29) and (3.1.26):

$$\frac{du^0}{dt} = 0 \quad , \tag{3.1.31}$$

and, at the point considered, we can, choose u^μ as in eq. (3.1.3) then at that point:

$$\frac{du^\mu}{dt} = 0 \ , \tag{3.1.32}$$

and from eqs. (3.1.7) and (3.1.21) we have:

$$\frac{d}{dt} g_{\mu\nu} = 0 \quad , \tag{3.1.33}$$

i.e., space-time is stationary using the time coordinate t (see an alternative demostration in Hawking & Ellis (1973)).

Two useful equation are:

$$\nabla_\mu u_\nu = \frac{1}{2} (\tilde{K}_{\mu\nu} + \tilde{\Omega}_{\mu\nu}) - u_\mu C_\nu \ , \tag{3.1.34}$$

and

$$(3.1.35)\qquad (\mu,\nu;\lambda) = (\mu \overset{\sim}{,} \nu;\ \lambda) + \frac{1}{2}\{u_\mu(\tilde{K}_{\nu\lambda}+\Omega_{\nu\lambda}) + u_\nu(\tilde{K}_{\mu\lambda}+\Omega_{\mu\lambda}) + u_\lambda(Q_{\mu\nu}-\tilde{K}_{\mu\nu})\quad ,$$

where:

$$(3.1.36)\qquad Q_{\mu\nu} = \partial_\mu u_\nu + \partial_\nu u_\mu \quad .$$

3.2 QUANTUM FIELD THEORY IN CURVED-SPACE TIME AND ARBITRARY SYSTEM OF REFERENCE

We must realize that ordinary Quantum Field Theory is developed in unbounded flat Minkowski space-time and in an inertial system of reference. If any of this three conditions change we are faced with an unconventional quantum field theory. We shall studied the case of unbounded curved space-time with arbitrary (but irrotational) reference system.

We shall quantize only a scalars neutral field $\phi(x)$. The Lagrangian density is:

$$(3.2.1)\qquad L(x) = \frac{1}{2}\,[-g(x)]^{1/2}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-[m^2+\xi R(x)]\phi^2\,\}\ .$$

where m is the mass, and the term $\xi R\phi^2$, where ξ is a dimensionless coupling constant an R(x) de Ricci scalar curvature, is the only possible local coupling with the correct dimension. If $\xi = 0$ we will have the so call "Minimal Coupling", instead if $\xi = 1/6$ we shall speak of "Conformal Coupling", because the formalism turns out to be invariant under conformal transformations if m=0, in this case.

The action is:

$$S = \int L(x) d^4x \quad , \tag{3.2.2}$$

and the field equation:

$$[\Box_x + m^2 + \xi R(x)]\phi(x) = 0 \quad , \tag{3.2.3}$$

where $\Box_x = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplace operator. We can define an Hermitian inner product (;) in the space of solutions of eq. (3.2.3):

$$(\phi_1,\phi_2) = -i \int_\Sigma \phi_1^*(x)\overleftrightarrow{\partial}_\mu\phi_2(x)[-g_\Sigma(x)]^{1/2}\, d\Sigma^\mu = (\phi_2,\phi_1)^* \tag{3.2.4}$$

where $d\Sigma^\mu = n^\mu d\Sigma$ and n^μ is the future-directed time-like unit vector orthogonal to the space-like surface Σ . If Σ is a Cauchy surface (we suppose that V_4 is globally hyperbolic) we can use the Green Theorem and show that (ϕ_1,ϕ_2) is independent of Σ .

In general we can also demonstrate that there exists sets of solutions $u_i(x)$ and $u_i^*(x)$ (where the index i is the set of quantities necessary to label de modes) of eq. (3.2.3) which are orthogonal in the inner product (3.2.4), precisellly:

$$\begin{aligned}(u_i,u_j) &= \delta_{ij}\\ (u_i^*,u_j^*) &= -\delta_{ij}\\ (u_i,u_j^*) &= 0\end{aligned} \tag{3.2.5}$$

and this set is a basis of the space of solutions of eq. (3.2.3).

Thus we can expand the field $\phi(x)$ as:

(3.2.6) $$\phi(x) = \sum_i [a_i u_i(x) + a_i^+ u_i^*(x)] \quad .$$

If now we consider $u_i(x)$ and $u_i^*(x)$ as negative and positive frequency solutions and a_i and a_i^+ as creation and annihilation operators, covariant quantization can be implemented adopting the commutation relations:

(3.2.7) $$[a_i, a_j] = [a_i^+, a_j^+] = 0 \quad ,$$

$$[a_i, a_j^+] = \delta_{ij} \quad .$$

The vacuum state is the obvious generalization of the flat space motion:

(3.2.8) $$a_i |0\rangle = 0 \quad , \quad \forall i \quad ,$$

and the construction of a Fock space proceeds exactly as in Minkowski space e.g. the particle number operator in mode i is:

(3.2.9) $$N_i = a_i^+ a_i \quad , \quad \text{etc.}$$

The problem is however that while in Minkowski space-time and in intertial coordinates (t,x,y,z) we have a natural basis: the plane-wave solutions of eq. (3.2.3) $\exp(ik_\mu x^\mu)$, in curved space time and in an arbitrary system of coordinates we have not such a basis . We shall see below which are the properties of basis $\exp(ik_\mu x^\mu)$ obviously related with the existence of a invariance group in Minkowski space-time: the Poincaré Group, but for the moment let us suppose that we have a second basis

$u_j(x)$ and see what problems appear if we have two unprivileged different basis. The field $\phi(x)$ may be expanded in the new basis:

$$(3.2.10) \qquad \phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^+ \bar{u}_j^*(x)] \quad ,$$

a new vacuum state $|\bar{0}>$ could be defined:

$$(3.2.11) \qquad \bar{a}_j |\bar{0}> = 0 \qquad \forall j$$

and a new Fock space could be constructed.

The basis are related by:

$$(3.2.12) \qquad \bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*) \quad ,$$

or conversely:

$$(3.2.13) \qquad u_i = \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*) \quad .$$

These relations are known as Bogoliubov transformations (Bogoliubov 1958), and we obviously have:

$$(3.2.14) \qquad \alpha_{ij} = (\bar{u}_i, u_j) \ ; \ \beta_{ij} = -(\bar{u}_i u_j^*) \quad ,$$

(3.2.13) follows from (3.2.12) because both basis are orthonormal, thus the coefficients must satisfy some relations:

$$(3.2.15) \qquad \sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij} \quad ,$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0 \qquad (3.2.16)$$ [5]

From expansions (3.2.6) and (3.2.10) and eq. (3.2.12) and (3.2.13) one obtain:

$$a_i = \sum_j (\alpha_{ji}\bar{a}_j + \beta_{ji}\bar{a}_j^{+} \quad , \qquad (3.2.17)$$

$$\bar{a}_j = {}_i \; ({}^{*}_{ji} a_i - {}^{*}_{ji} a_i^{\dagger}) \quad . \qquad (3.2.18)$$

It follows from (3.2.12) that the two basis are physically different if and only if $\beta_{ji} \neq 0$, i.e. if $\beta_{ji} = 0$ (3.2.12) could only be considered as a redefinition of the negative frequency among themselves, the real problem only appears when there is a mixture of positive and negative frequencies. For example in such a case $|\bar{0}\rangle$ will not be annihilated by a_i:

$$a_i|\bar{0}\rangle = \sum_j \beta^{*}_{ji}|\bar{1}_j\rangle \neq 0 \quad , \qquad (3.2.19)$$

and the expectation value of N_i in state $|\bar{0}\rangle$ is:

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \quad , \qquad (3.2.20)$$

which says that the vacuum of, modes $\bar{u}_j$, contain $\sum_j |\beta_{ji}|^2$ particles in modes u_i. Thus a empty state in one basis is not empty in the

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[5] This equation means that the matrices α and β commute, thus we can diagonalize them both. We then obtain an obvious simplification of the formulae

$$\alpha_{ij} = \alpha_i\delta_{ij}$$
$$\beta_{ij} = \beta_i\delta_{ij}$$
$$|\alpha_i|^2 - |\beta_i|^2 = 1$$

In fact, matrices α_{ij} and β_{ij} are hermitian (cfr. (3.2.4)).

others. This "creators of particles" process may be explained: the space-time curvature, or the detector, if we use a non inertial system, give energy to the field, that produces particles. The following formulae related the state vector of the Fock space based on $|0\rangle$ to the one based on $|\bar{0}\rangle$:

(3.2.21)
$$|{}^1n_{i_1}\ {}^2n_{i_2},\ldots\rangle =$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1\ldots j_k} |\bar{1}_{j_1}, \bar{1}_{j_2},\ldots,\bar{1}_{j_k}\rangle \times$$
$$\times \langle \bar{1}_{j_1},\bar{1}_{j_2},\ldots,\bar{1}_{j_k}|{}^1n_{i_1},{}^2n_{i_2},\ldots\rangle \quad ,$$

where:

(3.2.22)
$$|{}^1n_{i_1}\rangle = |1_{i_1},1_{i_1},\ldots,1_{i_1}\rangle/({}^1n_i!)^{1/2} \quad ,$$

so "1_{i_1}" is repeated ${}^1n_{i_1}$ times . $\langle\,|\,\rangle$ is a S matrix element that can be also written in terms of Bogoliubov coefficients e.g.:

(3.2.23)
$$\langle\bar{0}|1_{j_1},1_{j_2},\ldots,1_{j_k}\rangle =$$
$$=\begin{cases} i^{k/2}\langle\bar{0}|0\rangle \sum_{\rho} \Lambda_{\rho_1\rho_2}\ldots\Lambda_{\rho_{k-1}\rho_k} & k \text{ even} \\ 0 & k_{\text{odd}} \end{cases} \quad ,$$

or

(3.2.24)
$$\langle\bar{1}_{j_1},\bar{1}_{j_2},\ldots,\bar{1}_{j_k}|0\rangle =$$
$$=\begin{cases} i^{k/2}\langle\bar{0}|0\rangle \sum_{j} \nu_{\rho_1\rho_2}\ldots\ \nu_{\rho_{k-1}\rho_k} & k \text{ even} \\ 0 & k_{\text{odd}} \end{cases} \quad ,$$

where ρ represent all distinct permutations of $\{j_1 \ldots j_k\}$ and:

$$\Lambda_{ij} = -i \sum_k \beta_{kj}\alpha^{-1}_{ik} \tag{3.2.25}$$

$$\nu_{ij} = i \sum_k \beta^*_{jk}\alpha^{-1}_{ki}$$

Another many-particles to many-particles amplitude may be found in Birrell & Taylor (1980). From eq. (3.2.23-24) we can see that particles are always produce in pairs, as a consequence of the conservation of the momentum tensor.

3.3 The Particle Modes in the Trivial Case and two Examples

In the preceding paragraph we learn that at Quantum Field Theory formalism could translated almost unchanged from flat space-time and inertial system to curved space-time and general system with the exception of the flat plane wave basis exp i $(k_\mu x^\mu)$ i.e. the particle model. We have also learn that for every basis of positive and negative frequencies there is a well define vacuum and a particle model. Thus the problem is how to generalize the equivalent notions of positive and negative frequency basis, vacuum, or particle model. The generalization could be very easy under certain circumstances, and completelly impossible in other cases (e.g. if V_4 is not globally hyperbolic). Let us begin by the easiest one.

From now on we only consider irrotational reference systems, in fact it is a set large enough for our purpose and the normal surfaces to the fluid paths are, in general usefull, then

$\tilde{\Omega}_{\mu\nu} = 0$.

Using eq. (3.1.28) we have in this case that eq. (3.2.3) reads:

(3.3.1)
$$(u^0)^2\ \partial_{00}\ \phi - \tilde{\Delta}\ \phi + C_\mu \partial^\mu \phi$$
$$+ \frac{1}{2}\ g^{\mu\nu} Q_{\mu\nu} u_\lambda \partial^\lambda \phi - \frac{1}{2}\ \gamma^{\mu\nu} \tilde{K}_{\mu\nu} u_\lambda \partial^\lambda \phi$$
$$+ (m^2 + \xi R)\phi = 0 \quad ,$$

where $\tilde{\Delta} = \gamma^{\mu\nu} \tilde{\nabla}_\mu \tilde{\partial}_\nu$ is the transversal Laplace operator.

Let us suppose now that the V_4 has a Killing vector and that the fluid is the one defined by this Killing vector, then from eq. (3.1.23) $\tilde{K}_{\mu\nu} = 0$ and using the time coordinate t we have:

(3.3.2)
$$\frac{\partial}{\partial t}\ u_0 = 0 \quad ,$$

(3.3.3)
$$\frac{1}{2}\ g^{\mu\nu} Q_{\mu\nu} = g^{00} \partial_0 u_0 = 0 \quad ,$$

and eq. (3.3.1) becomes:

(3.3.4)
$$(u^0)^2\ \partial_{00}\phi - \tilde{\Delta}\phi + C_\mu \partial^\mu \phi$$
$$+ (m^2 + \xi R)\phi = 0 \quad .$$

From (3.1.33) we also have that $\frac{d}{dt} R = 0$ and as C_μ is transversal (i.e. $u_\mu C^\mu = 0$) we can solve eq. (3.3.4) by separation writting:

(3.3.5)
$$\phi(x^\mu) = T(t) E(x^i) \quad .$$

Then we have:

$$(3.3.6) \qquad E(x^i)(u^0)^2(x^i)\ddot{T}(t) - T(t)\tilde{\Delta}E(x^i) +$$

$$+ C_j(x')T(t)\partial^j E(x^i) +$$

$$+ (m^2 + \xi R(x^i))T(t)E(x^i) = 0 \quad ,$$

i.e.

$$\frac{\ddot{T}(t)}{T(t)} = \frac{\tilde{\Delta}E(x^i) - C_j(x^i)\partial^j E(x^i) - (m^2 + \xi R(x^i)E(x^i)}{(u^0)^2(x^i)E(x^i)}$$

$$= \omega^2 = \text{const.} \quad ,$$

Therefore the time factor is:

$$(3.3.7) \qquad T(t) \sim \exp(\pm i\omega t) \quad ,$$

and using the + (-) sign we can define the positive (negative) frequency solution in a natrual way: the one with time factor $\exp(i\omega t)$ ($\exp(-i\omega t)$). A vacuum, and a particle model, are thus defined, we shall call it a "trivial vacuum" (of course the Minkowski vacuum is a trivial one). We shall see below, using another method, that this vacuum has all the properties requires to be considered a good vacuum. The transversal part of eq. (3.3.6) yields:

$$(3.3.8) \qquad (u^0)^{-2}[\tilde{\Delta}E - C_j\partial^j E - (m^2 + \xi R)E] = \omega^2 E \quad ,$$

Thus E is an eigenfunction of the operator $(u^0)^{-2}[\tilde{\Delta} - C_j\partial^j - (m^2+\xi R)]$ and ω^2 is the corresponding eigenvalue.

Thus when we have a Killing vector field we have a rivial vacuum and the problem is solve.

From eq. (3.3.7) we see that in the case of trivial acuum the positive frequency modes satisfy:

$$(3.3.9) \qquad i \frac{\partial}{\partial t} u_j = \omega u_j \qquad \omega > 0 \quad ,$$

ınd, which is the same thing, if there is a Killing vector:

$$(3.3.10) \qquad iL_k u_j = k\omega u_j \qquad \omega > 0 \quad ,$$

where L_k is the Lie derivative with respect to the Killing vector.

The easiest examples of all the formalism, of this and the preceeding paragraphs, is the study of Quantum Field Theory in Minkowski space, but in Rindler coordinates (Fulling 1973). Let us consider the two-dimensional Minkowski space with metric

$$(3.3.11) \qquad ds^2 = dt^2 - dx^2$$

and the following coordinates transformation:

$$(3.3.12) \qquad t = a^{-1} e^{a\xi} \sinh a\eta$$

$$x = a^{-1} e^{a\xi} \cosh a\eta \quad ,$$

a = const. > 0; $-\infty < \xi,\eta < \infty$. The metric becomes:

$$(3.3.13) \qquad ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2) \quad .$$

The coordinates (η,ξ) cover only a quadrant of Minmowski space i.e. $x > |t|$ in fig. 1. Lines η = const. are straight lines while lines ξ = const. are hyperbolae:

$$x^2 - t^2 = a^{-2} e^{2a\xi} = \text{const.} \qquad (3.3.14)$$

We can consider all these hyperbolae like a non-inertial reference, fluid each one of their points being an uniformly accelerated observer, because they have a proper acceleration $ae^{-a\xi}$. The observers proper time is:

$$\tau = e^{a\xi}\eta . \qquad (3.3.15)$$

The system $(\eta;\xi)$ is known as the Rindler coordinates system (Rindler 1966, see also Born (1909)) and the portion $x > |t|$ as the Rindler wedge (R). A second Rindler wedge $x < |t|$ may by obtained reflecting the first one (L).

If we consider η as the Rindler time we see that metrics (3.3.11) and (3.3.13) are two static metrics belonging to two diferent fluid, an inertial one, coordinates (t,x), and an accelerated one, coordinate (η,ξ).

But we must realize that η could be considered as a time only in the wedge R and we must change its sign in the region L, if we want that the tangent vector to the fluid lines, for grouwing η, would point towards the future. Also if we compute the tangent vector to the reference fluids we can find that both are Killing vector fields, thus both fluids yield trivial vacua.

Thus we have the ordinary Minkowski basis:

$$\bar{u}_{\underline{k}} = (4\pi\omega)^{-1/2}\, e^{i(kx-\omega t)} \quad , \qquad (3.3.16)$$

where, in the $m = 0$ case that we shall study, $\omega = |k|$, $-\infty < k < \infty$ and the Rindler basis:

$$u_k = (4\pi\omega)^{-1/2}\, e^{ik\xi \pm i\omega\eta} \quad , \tag{3.3.17}$$

where the upper sign is valid in the region L and the lower sign in the R region, due to the "time reversal" in L with respect to R. Therefore really the positive frequency base is:

$$\begin{aligned} {}^{R}u_k &= (4\pi\omega)^{-1/2}\, e^{ik\xi - i\omega\eta} && \text{in R} \; , \\ &= 0 && \text{in L} \; , \end{aligned} \tag{3.3.18}$$

$$\begin{aligned} {}^{L}u_k &= (4\pi\omega)^{-1/2}\, e^{ik\xi + i\omega\eta} && \text{in L} \; , \\ &= 0 && \text{in R} \; . \end{aligned} \tag{3.3.19}$$

The first set is complete in R while the second set is complete in L, so both sets form a complete basis, and this basis could be analytically continued into the regions F and R[(6)].

The field may be expanded in either basis:

$$\phi = \sum_{k=-\infty}^{\infty} (\bar{a}_k \bar{u}_k + \bar{a}_k^{+} \bar{u}_k^{*}) \quad , \tag{3.3.20}$$

or

$$\begin{aligned} \phi = \sum_{k=-\infty}^{\infty} (a_k^{(L)}\, {}^{L}u_k + a_k^{+(L)}\, {}^{L}u_k^{*} + \\ + a_k^{(R)}\, {}^{R}u_k + a_k^{+(R)}\, {}^{R}u_k^{*}) \quad , \end{aligned} \tag{3.3.21}$$

yielding two different Fock spaces and two different vacuum states. The Minkowski vacuum $|0M\rangle$ such that:

$$\bar{a}_k |0M\rangle = 0 \qquad \forall k \quad , \tag{3.3.22}$$

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(6) Using its Cauchy data and continuing the solution to all Minkowski space--time.

and the Rindler vauum $|OR\rangle$ such that:

(3.3.23) $$a_k^{(L)}|OR\rangle = a_k^{(R)}|OR\rangle = 0 \qquad \forall k \quad .$$

Using the formalism of paragraph 3.2 we can them reach to eq. (3.2.20), that in this case, it turns out to be:

(3.3.24) $$\langle OM|\overset{+}{a}{}_k^{(L,R)} \; a_k^{(L,R)}|OM\rangle = = (e^{2\pi\omega/a} - 1)^{-1} \quad .$$

Therefore Rindler observers will detect particles of mode k in Minkowski vacuum, with precisely the Planck radiation spectrum at a temperature:

(3.3.25) $$T = a/2\pi k_3 \quad .$$

So we see that on accelerated observer detect a thermal bath of radiation in empty flat space-time. The mecahnical agency that moves the particle detectors of the accelerated Rindler fluid gives the necessary energy to produce this radiation bath. We refer to Birrell and Davis (1982) for a complete discussion of particle detectors, that we can not give here, that, in fact, proves that temperature (3.3.25) is the one measured by the detector.

Thus we conclude that, if we pass from the inertial reference system of Minkowski space-time to a non-inertial reference system particles appear, produced by the force that move the detector of the non-inertial reference fluid.

Now, let us give a complementary example (Bernard and Duncan (1977)), both system of reference will be inertial but space-time will be curved. Let's consider a two-dimensional Robertson-Walker universe with metric:

$$ds^2 = dt^2 - a^2(t)dx^2 \quad . \tag{3.3.26}$$

Introducing a new parameter, the "conformal time":

$$\eta = \int^t \frac{dt}{a(b)} \quad , \tag{3.3.27}$$

we can writte eq. (3.3.27) as:

$$\begin{aligned} ds^2 &= a^2(\eta)(d\eta^2 - dx^2) = \\ &= C(\eta)(d\eta^2 - dx^2) \quad , \end{aligned} \tag{3.3.28}$$

where $C(\eta) = a^2(\eta)$ is the conformal scale factor that justifies the name of η . Thus the line element is conformal to the one of Minkowski space time. Suppose that:

$$C(\eta) = A + B \tanh \rho\eta, \qquad A,B,\rho = \text{const}, \tag{3.3.29}$$

and let us consider the comoving reference fluid i.e. the one with space-time path given by x = const t, η variable. In curved space-time there are not inertial reference systems, but we can consider as locally inertial, their natural generalization, the ones with geodesic fluid lines. We shall also call theses systems inertial; then the comoving fluid of our example is an inertial system. Besides:

(3.3.30) $\quad C(\eta) \to A \pm B \quad$ when $\quad \eta \to \pm\infty$,

(see fig. 2) thus the metric is asymptotically static in the far past and the far future with, Killing vector field, and we have two trivial vacua there. In fact, in this case things are completetly clear because in the far past and the far future space-time turns out to be the Minkowski one and reference system are really inertial. We can separate the variables as:

(3.3.31) $$u_k(\eta,x) = (2\pi)^{-1/2} e^{ikx} \chi_k(\eta) \quad ,$$

and substitute (3.3.31) in place of ϕ into the field equation (3.2.3) with $\xi = 0$ and metric (3.3.26), then one obtains for $\chi_k(\eta)$:

(3.3.32) $$\frac{d^2}{d\eta^2} \chi_k(\eta) + (k^2 + C(\eta)m^2)\chi_k(\eta) = 0 \quad ,$$

which has two independent normalized solutions:

(3.3.33) $$u^{in}{}_k(\eta,x) = (4\pi\omega_{in})^{-1/2}$$

$$\exp\{ikx - i\omega_+\eta - (i\,\omega_-/\rho)\ln[2\cosh(\rho\eta)]\} \times$$

$$\times \; {}_2F_1(1+(i\omega_-/\rho),\, i\omega_-/\rho;\, 1-i\omega_{in}/\rho \;;$$

$$;\, \tfrac{1}{2}\,(1+\tanh\rho\eta))_{\eta\to-\infty} \to (4\pi\omega_{in})^{-1/2} e^{ikx-i\omega_{in}\eta}$$

and

(3.3.34) $$u^{out}{}_k(\eta,x) = (4\pi\omega_{out})^{-1/2} \; \times$$

$$\exp\{ikx-i\omega_+\eta-(i\omega_-/\rho)\ln[2\cosh(\rho\eta)]\}\times$$

$$\times\ {}_2F_1(1+(i\omega_-/\rho),\ i\omega_-/\rho;\ 1+(i\omega_{out}/\rho)\quad ;$$

$$;\ \frac{1}{2}\ (1-\tanh\rho\eta))_{\eta\to+\infty}\to(4\pi\omega_{out})^{-1/2}\ e^{ikx-i\omega_{out}\eta}$$

where

$$\omega_{in}=[k^2+m^2(A-B)]^{1/2}\quad ,$$

(3.3.35)

$$\omega_{out}=[k^2+m^2(A+B)]^{1/2}\quad ,$$

$$\omega_{\pm}=\frac{1}{2}\ (\omega_{out}\pm\omega_{in})\quad .$$

While (3.3.33) satisfies eq. (3.3.9), in the far part, and therefore is the "in" basis, (3.3.34) satisfies the equation in the far future beeing the "out" basis. Clearly u_k^{in} and u_k^{out} are not equal, in fact, u^{in} can be obtained from u^{out} via a Bogoliubov transformations:

(3.3.36)

$$u_k^{in}(\eta,x)=\alpha_k\ u_k^{out}\ (\eta;x)\ +$$

$$+\ \beta_k\ u_{-k}^{out^*}\ (\eta,x)\quad ,$$

where coefficient α and β are:

(3.3.37)

$$\alpha_k=(\frac{\omega_{out}}{\omega_{in}})^{1/2}\ \frac{\Gamma(1-\frac{i\omega_{in}}{\rho})\Gamma(-\ i\frac{\omega_{out}}{\rho})}{\Gamma(-\ i\frac{\omega_+}{\rho})\Gamma(1\ -\ \frac{i\omega_+}{\rho})}$$

(3.3.38)

$$\beta_k=(\frac{\omega_{out}}{\omega_{in}})^{1/2}\ \frac{\Gamma(1-\frac{i\omega_{in}}{\rho})\Gamma(i\frac{\omega_{out}}{\rho})}{\Gamma(\frac{i\omega_-}{\rho})\Gamma(1\ +\ \frac{i\omega_-}{\rho})}\quad .$$

From eq. (3.2.12) we see that the Bogoliubov coefficients are:

(3.3.39)

$$\alpha_{kk'}=\alpha_k\delta_{kk'},\quad \beta_{kk'}=\beta_k\ \delta_{-kk'}\quad ,$$

and using all the formalism of paragraph 3.2 we can reach to eq. (3.2.17) that in this case becomes:

$$(3.3.40) \qquad \langle 0_{in}|N_k^{out}|0_{in}\rangle = |\beta_k|^2 = \frac{\sinh \pi\omega_-/\rho}{\sinh(\pi\omega_{in}/\rho)\sinh(\pi\omega_{out}/\rho)} .$$

Thus if the universe is empty in the far past the system state vector is $|0_{in}\rangle$. As we are working in Heisenberg picture $|0_{in}\rangle$ will be always the state of the system. Nevertheless the expectation value of the out-basis particle number operator $N_k^{out} = a_k^{+out}\, a_k^{out}$ is different from zero.

Thus in the far future an unaccelerated inertial reference system will detect the presence of particle, even if the universe is empty in the far past.

In this case the particle creation is a consequence of the curvature of space-time,and the universe expansion give the energy for this particle creation.

3.4 GREEN FUNCTIONS

In this paragraph we shall introduce the Green Functions and restudy the problem of the vacuum definition using this new language.

Lichnerowicz (1961-1964) showed that in every globally *hyperbolic manifold V_4 there exists two elementary kernels* $E^{\pm}_{x'}(x) = E^{\pm}(x,x')$ that satisfy the equation:

$$(3.4.1) \qquad (\Box_x + m^2 + \xi R)E^{\pm}_{x'}(x) = (-g)^{-1/2}\,\delta(x',x) \quad ,$$

with support in $E^+(x')$ and $E^-(x')$ respectively i.e. the casual future and the casual past of x'. These solutions are unique and $\delta(x,x')$ is the Dirac δ on the manifold. This elementary kernels have the property:

$$(3.4.2) \qquad E^+(x,x') = E^-(x',x) \quad .$$

We shall call propagator to the differences:

$$(3.4.3) \qquad G(x,x') = E^+(x',x) - E'(x',x) \quad ,$$

which is also unique, obviously antisymmetric:

$$(3.4.4) \qquad G(x,x') = -G(x',x) \quad ,$$

and from eq.(3.4.1) it satisfy:

$$(3.4.5) \qquad (\Box_x + m^2 + \xi R)G(x,x') = 0 \quad .$$

Now, let Σ be a Cauchy surface of V_4, and $\phi_\Sigma(x)$ and $\dot{\phi}_\Sigma(x)$ the Cauchy data on Σ ; then it can be proved that the solution of the Cauchy problem (eq. (3.2.3) with Cauchy data ϕ_Σ and $\dot{\phi}_\Sigma$) is:

$$(3.4.6) \qquad \phi(x') = \int_\Sigma [G(x',x)\dot{\phi}_\Sigma(x) - \phi_\Sigma(x)n^\mu\partial_\mu G(x',x)] \; d\Sigma_x \quad ,$$

where n^μ and $d\Sigma$ are defined after eq. (3.2.4). Using the inner product defined in this equation we can writte eq. (3.4.6) as:

$$\phi(x') = i(G(x',x),\phi_{\Sigma}(x))_{x'} \qquad , \tag{3.4.7}$$

where the subindex x means that we have perform the integration over this variable. From eq. (3.4.6) we can obtain the following properties by means of simple derivation and inspection of the results:

$$G(x,x') = 0 \qquad \text{if } x,x' \in \Sigma \ , \tag{3.4.8}$$

$$n^{\mu}\partial_{\mu}G(x,x') = (\gamma)^{-1/2}\,\delta_{\Sigma}(x,x') \quad \text{if } x,x' \in \Sigma \ , \tag{3.4.9a}$$

$$n^{\mu}\partial_{\mu}n^{\nu}\partial'_{\nu}G(x,x') = 0 \qquad \text{if } x,x' \in \Sigma \ , \tag{3.4.9b}$$

where $(\gamma)^{-1/2}\,\delta_{\Sigma}(x,x')$ is the Dirac δ on the Cauchy surface Σ with metric $\gamma_{\mu\nu}$. From the first equation we can see that for a x' fixed the support of $G(x,x')$ is contained in the emission of x' i.e. $E^{+}(x') \cup E^{-}(x')$ which also follows from the definition (3.4.3).

Now, it is straightforward to see that the propagator $G(x,x')$ has all the properties to be considered as the commutator of the scalar field, thus:

$$[\phi(x),\phi(x')] = -iG(x,x') \tag{3.4.10}$$

and also

$$\langle 0|[\phi(x),\phi(x')]|0\rangle = -iG(x,x') \tag{3.4.11}$$

In flat space-time $G(x,x')$ becomes the usual $\Delta(x,x')$.

Now, from (3.2.5) and (3.2.6) we have that:

$$(3.4.12) \qquad (u_j,\phi) = a_j \; ,$$

$$(u_j^*,\phi) = -a_j^+ \; ,$$

thus:

$$(3.4.13) \qquad \phi(x) = \sum_j (u_j,\phi)u_j(x) - (u_j^*\phi)u_j^*(x) =$$

$$= (\sum_j u_j(x')u_j^*(x) - u_j^*(x')u_j(x), \phi(x'))_{x'} \; .$$

Comparing with (3.4.7) we obtain:

$$(3.4.14) \qquad G(x,x') = i \sum_j u_j^*(x)u_j(x') - u_j(x)u_j^*(x') \; ,$$

i.e. the expansion of the propagator in basis u_i. Also, using this equation and eq. (3.2.6), eq. (3.4.10) follows from eq. (3.2.7). As G(x,x') is unique and $\{u_i\} \cup \{u_i^*\}$ is an arbitrary orthonormal basis expansion (3.4.13) is invariant under Bogoliubov transformations. This can also be seen by direct computation, (cfr. Castagnino (1978)) and also that we can use, in formula (3.4.11), an arbitrary vacuum and we allway obtain the same and unique G(x,x').

From the beginning of this paragraph and paragraph 3.2 we can conclude that, up to here, the generalization of Quantum Field Theory to curve space, in arbitrary reference system is simple and straightforward and we have found exactly the same formalism. In fact, we have deal only with the canonical formalisms, (embodied in equations (3.2.6), (3.2.7), (3.4.10) and (3.4.14)) that we have demonstrated it is invariant under Bogoliubov transformation and therefore unambiguous.

The real problems begins with the vacuum definition or the particle model that are related with another Green Function: $G_1(x,x')$.

As $G(x,x')$ is the generalization of the usual $\Delta(x,x')$ to curved space-time, $G_1(x,x')$ is the generalization of $\Delta_1(x,x')$. As we shall see the subspaces of positive and negative frequency solutions are determined by $G_1(x,x')$. Naturally we ask for $G_1(x,x')$ the same properties $\Delta_1(x,x')$ has, namely:

(3.4.15)

$$\text{I} - \quad G_1(x,x') \text{ must be real,}$$

$$\text{II} - \quad G_1(x,x') = G_1(x',x) \quad ,$$

$$\text{III} - \quad (\Box_x + m^2 + \xi R)G_1(x,x') = 0 \quad ,$$

$$\text{IV} - \quad G(x,x') = -i(G_1(x,y);G_1(x',y))_y \quad .$$

These equations are the natural generalization of the properties of $\Delta_1(x,x')$, of flat space-time, but it is evident that they alone do not define $G_1(x,x')$. In fact, IV may be considered a unique boundary condition, on a Cauchy surface Σ. Thus, differential equation III, with only one boundary condition, cannot define $G_1(x,x')$ entirely.

This is why we must look for other condition on $G_1(x,x')$.

Meanwhile , let ϕ_1 be a wave function define as:

$$\phi_1(x) = i(G_1(x,x'),\phi(x'))_{x'} \quad . \tag{3.4.16}$$

We shall call "ip" to this linear mapping:

$$ip\phi(x) = \phi_1(x) \quad . \tag{3.4.17}$$

From condition IV we have:

$$[\phi_1(x)]_1 = -\phi(x) \quad , \tag{3.4.18}$$

or:

$$p \circ p = I \quad . \tag{3.4.19}$$

Let us now define a new inner product:

$$\langle \phi_1, \phi_2 \rangle = (\phi_1, p\phi_2) \quad . \tag{3.4.20}$$

From II we have that this product is also Hermitian, and that p is a self-adjoint operator for both products (,) and <,>. The product <,> is the generalization to curved space-time of a simmilar product of flat space-time (cfr. Schweber (1962) p. 57) that it is positive definite, thus it is natural to ask that $G_1(x,x')$ would also satisfy:

V "The inner product <,> gives a positive norm" i.e.:

$$\langle \phi, \phi \rangle \geq 0 \quad ; \quad \langle \phi ; \phi \rangle = 0 \longrightarrow \phi = 0 \tag{3.4.21}$$

Conditions I to V were define by Lichnerowicz (1964a) and they are possibly the minimal set of conditions that $G_1(x,x')$ must satisfy. If $G_1(x,x')$ satisfy condition I to IV we can define a decomposition of the space of solutions of eq. (3.2.3) in a subspace of positive frequency solutions and a subspace of negative frequency solutions, that can be found using two projectors that can be written using operator p: If ϕ is a solution of eq. (3.2.3) its positive frequency component ϕ^+ and its negative frequency components ϕ^- are:

$$\phi^+ = \frac{1}{2}(1+p)\phi \quad , \quad \phi^- = \frac{1}{2}(1-p)\phi \quad . \tag{3.4.22}$$

If we define a basis u_i^* in the positive frequency subspace it is easy to see that u_i is a basis in the negative frequency subspace and that:

$$(3.4.23) \qquad G_1(x,x') = \sum_i u_i(x)u_i^*(x) + u_i^*(x)u_i(x') \quad ,$$

is the expansion of $G_1(x,x')$ in this basis. In fact it can be seen that this $G_1(x,x')$ satisfies eq. (3.4.16) and it also has all the properties from I to V. Thus $G_1(x,x')$ defines a decomposition in positive and negative frequency components, and viceversa.

It is also easy to see that $G_1(x,x')$ is <u>not invariant</u> under a Bogoliubov transformation, thus there are infinite $G_1(x,x')$, each one corresponding to a different decomposition, because there are infinite vacua or infinite particle models.

The invariance of eq. (3.4.14) and the non-invariance of eq. (3.4.23) explain why it is easy to generalize the canonical formalism and why it is so difficult to generalize the notion of vacuum.

Finally if $|0\rangle$ is an arbitrary vacuum, using the positive and negative frequency solution corresponding to that vacuum it can be proved that:

$$(3.4.23a) \qquad G_1(x,x') = \langle 0|\{\phi(x),\phi(x')\}|0\rangle \quad ,$$

where $\{;\}$ is the anticommutator. But now $G_1(x,x')$ is not the same if we change the vacuum to $|\bar{0}\rangle \neq |0\rangle$. We have:

$$(3.4.24) \qquad \bar{G}_1(x,x') = \langle\bar{0}|\{\phi(x),\phi(x')\}|\bar{0}\rangle \quad ,$$

where $\overline{G}_1(x,x') \neq G_1(x,x')$ is a new kernel that correspond to vacuum $|\overline{0}\rangle$.

If we have a vacuum in the far past $|0_{in}\rangle$ and a vacuum in the far future $|0_{out}\rangle$, as in the last example of paragraph 3.3, we can also define a Feynman propagator:

$$(3.4.25) \qquad G_F(x,x') = \frac{\langle 0_{out}|T(\phi(x);\phi(x'))|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} \quad .$$

It can be proved that $G_F(x,x')$ satisfies:

$$(3.4.26) \qquad [\Box_x + m^2 + \xi R(x)]G_F = -\delta(x-x')(-g)^{-1/2} \quad ,$$

(for detail see De Witt and Brehme (1960) and Friedlander (1975)) as in the last example of paragraph 3.3 and that it can be expanded in the basis u_k, u_k^* as : (if $|0_{out}\rangle = |0_{in}\rangle$)

$$(3.4.27) \qquad iG_F(x,x') = \sum_k \theta(x^0-x'^0)u_k(x)u_k^*(x') + \theta(x'^0-x^0)u_k^*(x)u_k(x')$$

thus G_F is also non-invariant if we change the basis.

Thus we can say that we have:

i) $G(x,x')$ a no-vacuum Green Function because it is vacuum--independent

ii) $G_1(x,x')$ a one-vacuum Green Function because it depends on the vacuum

We choose, (cfr. eq. (3.4.23a))

iii) $G_F(x,x')$ a two-vacuum Green Function, because it depends on two vaccua if we want it to propagate positive frequency solution to the far future and negative frequency solution to the far past.

Now we are ready to rephrase the vaccum problem in Green Functions language. For example Rideau (1965) proved that conditions I to V plus Poincaré invariance in Minkowski space--time define an unique $G_1(x,x')$ precisely $\Delta_1(x,x')$. Thus plane wave model is the unique Poincaré invariant model in flat space-time.

More generally if there is a Killing vector or, what is the same, the metric in adapted coordinates, is stationary (and we suppose the fluid irrotational to make things easier) if Y_j is a basis of solution of equation (3.3.8) i.e.:

$$(3.4.28) \qquad (u^0)^{-2}[\tilde{\Delta} + C^i\partial_i - (m^2+\xi R)]Y_j = \omega_j^2 i Y_j$$

a basis of solution of eq. (3.2.8) is:

$$(3.4.29) \qquad u_j = K_j\, e^{-i\omega_j x^0}\, Y_j(x^i) \quad ,$$

where K_j a normalization coefficient. This are the functions that we have considered as positive frequency solution because they have the factor $e^{-i\omega_j x^0}$ as in the flat space case. But now the $G_1(x,x')$ that correspond to that basis reads:

$$(3.4.30) \qquad G_1(x,x') = \sum_j C_j C_j^* \times [e^{-i\omega_j(x^0-x'^0)} Y_j(x^i)\overset{*}{Y}_j(x'^i) + e^{i\omega_j(x^0-x'^0)} \overset{*}{Y}_j(x^i) Y_j(x'^i)] \quad ,$$

and the kernel turns out to be invariant by translations along the Killing vector field, i.e. $x^0 \to x^0+\Delta x^0$, $\dot{x}_0 \to \dot{x}^0+\Delta x^0$. Thus as the particles model is embodied in $G_1(x,x')$ we have proved that the trivial vacuum model is the one that correspond to a $G_1(x,x')$ invariant under translation along the Killing vector field. As this generalized the corresponding properties of $\Delta_1(x,x')$ of flat space-time we can be more confident that our trivial model is the correct one.

3.5 ADIABATIC VACUUM

Even if we are convinced that the trivial vacuum is a good physical vacuum we realize that it is too restrictive because in the general case we do not have a Killing vector field. Thus we must try to find other criteria to study the vacuum problem, the Minkowski limit is one of them. The success of the Quantum Field Theory in Minkowski space-time leads us to think that there must be some kind of approximation to that concept and that in slowly varying universes the particle motion must have some sense. In fact, we live in an expanding universe and we speak of particles in everyday Physics. Thus some sort of Minkowski limit must exist, and this limit must be a necessary condition that we must impose to the theory.

To make the ideas more precise let us first study the case of a Robertson-Walker universe, in the comoving frame of reference, and then let us generalize the results to more general geometries (paragraph 3.6).

The metric of a spatially flat[7] Robertson-Walker universe is:

$$(3.5.1) \qquad ds^2 = dt^2 - a^2(t)(dx^2+dy^2+dz^2) \quad .$$

A basis of solutions of the field equation (3.2.3) is given by the functions:

$$(3.5.2) \qquad u_k(t,\bar{x}) = (2\pi a)^{-3/2}\, e^{i\underline{k}.\underline{x}}\; f_k(t) \quad ,$$

and its complex conjugate u_k^*, where $f_k(t)$ must satisfy the equation:

$$(3.5.3) \qquad \ddot{f}_k + 3\frac{\dot{a}}{a}\,\dot{f}_k + (\frac{k^2}{a^2} + m^2 + \xi R)\, f_k = 0 \quad ,$$

and where $k^2 = \underline{k}.\underline{k}$. Now we propose the following expression for $f_k(t)$:

$$(3.5.4) \qquad f_k(t) = \frac{\exp[-i\int^t W_k(t')dt']}{[\;\Lambda_k(t'))]^{1/2}} \quad .$$

where W and Λ are real functions to be determinated by (3.5.3).

Note that this procedure involves no loss of generali we have simply written $f_{\underline{k}}(t)$ in terms of its argument and absolute value. Now if we impose the orthonormality conditions (3.2.5) to the functions u_k we obtain $W = \Lambda$; with this choice u_k, u_k^* becames an orthonormal basis.

---

(7) In the spatially curved case we would speak of static limit.

In order to satisfy eq. (3.5.3) $W_k(t)$ must be a solution of the equation:

$$W_k^2 + W_k^{1/2} [W_k^{-1/2}]'' = \tag{3.5.5}$$

$$= \frac{k^2}{a^2} + m^2 + \xi R - \frac{3}{2} (\dot{H} + \frac{3}{2} H) \quad ,$$

where $H = \dot{a}/a$ is the Hubble coefficient. In the flat or the static case (cfr. eq. (3.4.28)) a = const. and we would have:

$$W_k = \omega_k = (m^2 + \frac{k^2}{a^2})^{1/2} \quad . \tag{3.5.6}$$

If the space-time is slowly varying, then the derivative terms *will be small compared to* ω_k^2, *so we have a zeroth order approxi-*mation if we substitute:

$$W_k^{(0)} = \omega_k \quad , \tag{3.5.7}$$

in the integrand (3.5.4). The solution of (3.5.5) may be approximated by iteration using $W_k^{(0)}$ as the lowest order i.e. it can be computed with the W.K.B. method, and we obtain:

$$W_k = \omega_k [1 - \frac{1}{2} (\xi - \frac{1}{6}) \frac{R}{\omega_k^2} + \tag{3.5.8}$$

$$+ \frac{1}{6} (H^2 + \frac{R}{6}) \frac{m^2}{\omega_k^4} + \frac{5}{8} \frac{H^2 m^4}{\omega_k^6} + \ldots] \quad ,$$

where $H = a^{-1}\dot{a}$ is the Hubble coefficient and R is the curvature scalar $R = 6(\dot{H}+2H^2)$. Thus we find a solution u_k if we substitute W_k in the integrand (3.5.4) and an independent solution taking the complex cunjugate u_k^*. These solutions correspond to a vacuum that we shall call the adiabatic vacuum. We can find other solutions and other vacua by a Bogoliubov transformation i.e.:

$$(3.5.9) \qquad \overline{u}_k = \alpha_k u_k + \beta_k u_k^* \quad .$$

Is it, in some sense, the adiabatic vacuum a privilegeted one ? To see this we can take the Minkowski limit introducing an adiabatic parameter T and replacing everywhere t by t/T (we shall take, of course, T = 1 at the end of the calculation). Then the adiabatic limit or Minkowski limit is obtained when $T \to \infty$. The introduction of T changes:

$$a \to a, \quad a' \to \frac{a}{T} \quad , \quad H \to \frac{H}{T} \quad , \quad R \to \frac{R}{T^2} \quad , \quad \text{etc.}$$

Thus equation (3.5.8) becomes:

$$(3.5.10) \qquad W_k = \omega_k \{1 + \frac{1}{T^2} [- \frac{1}{2} (\xi - \frac{1}{2}) \frac{R}{\omega_k^2} + \frac{1}{6} (H^2 + \frac{R}{6}) \frac{m^2}{\omega_k^4} + \frac{5}{8} \frac{H^2 m^4}{\omega_k^6}] + O(\frac{1}{T^4})\} \quad .$$

Thus a factor T^{-2} appears for each iteration in the W.K.B. procedure (a term with a T^{-n} factor will be called a term of adiabatic order n). When $T \to \infty$, $W_k \to \omega_k$, and $u_k \sim e^{-i\omega_k t}$ thus to the proper positive frequency solution of Minkowski space-time, while $\overline{u}_k$ of eq. (3.5.9) becomes in the limit $T \to \infty$ a mixture of positive and negative frequency solutions of Minkowski space-time. Therefore, the adiabatic vacuum is privileged, it is the only one with the correct Minkowski limit.

It seems that we have solved the vacuum problem but it is not so[8]. In fact,(3.5.8) is a power expansion and therefore an analytical function of its variables at the point where the expansion is made e.g. on analytical function of T^{-1} at $T^{-1} = 0$, or an analytical function of k^{-1} at $k^{-1} = 0$ etc. Eventual non-analytical solutions of equation (3.5.5), in these variables, cannot be computed by this method, therefore solutions (3.5.8) is not completely reliable: it certainly does not contain non-analytical term that there might exist. In fact, these non analytical terms exist and they can be found in the examples where we have on exact solution, they are precisely the responsible of the particle creation which is measured by $|\beta_k|^2$ which in fact it turns out to be a now analytical function of k^{-1}, as we can see in the two examples of paragraph 3.3 (cfr. eq. (3.3.24) and (3.3.40)) where $|\beta_k|^2 \sim e^{-k}$.

Thus the adiabatic vacuum is only a hint, not a solution. What is exactly the adiabatic vacuum ?

From the two examples of paragraph 3.3 we know that $|\beta_k|^2_{k\to\infty} \to 0$. This fact has a very simple physical meaning: it is much easier to create particles of low energy (or mass) then particles of high energy. In the limit, no physical force can create particles with infinite energy so logically $|\beta_k|^2 \to 0$ when $k \to \infty$. Thus in the limit of very high energies we must have a unique positive frequency solutions (or a unique negative one) and a unique hight energy vacuum, this is precisely the adiabatic vacuum.

All positive frequency solutions must go asymptotically

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(8) In the second example of paragraph 3.3, we have two trivial vacua , which one is the adiabatic vacuum ?

to the adiabatic positive frequency solution eq. (3.5.4) with eq. (3.5.3) where $k \to \infty$, this is so because they all are the adiabatic solution, plus non-analytical terms in k^{-1} that $\to 0$ when $k \to \infty$, faster than any power of k^{-1}, this can be seen in all the examples in the literature(9).

Thus the adiabatic vacuum solves the problem completely in the high energy limit, but the real problem is not there, because particle creation occurs more frequently at low energies. Anohow, the adiabatic vacuum prescribes a common behaviour for all positive energy solution in high energy regions, being therefore very useful.

In the next paragraph we shall generalize this idea to more general geomtries.

3.6 DE-WITT-SCHWINGER GREEN FUNCTION

Particle creation for high energies is very weak. This can also be explained in a geometric way. The universe expansion creates particles. Hight energy particles have short wave lenght and therefore in such a range the universe seems almost flat, or better it seems much flatter than in the case of a low energy particles with big wage lenght,i.e. high energy particles "see" the universe almost flat and are insensitive to the curvature of space--time, low energy particles see the universe curved and are sensitive to the geometry. Then for high energy the universe "is flat" and,

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(9) Really in almost all, as we shall see in paragraph 5.

ı fact, flat universe do not creates particles, thus there is weak high energy particle creation.

Then high energies correspond to short distances, and we shall try to find a Green Function for short distance i.e. in the neighbourhood of a point (Bunch & Parker (1979)). Let x' be a fix point, that we take as an origin of Riemannian normal coordinates (Schouten (1951)). Then a generic point x will have Riemannian normal coordinate y^μ and the metric tensor can be expand as:

(3.6.1)
$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta - \frac{1}{6} R_{\mu\alpha\nu\beta;\gamma} y^\alpha y^\beta y^\gamma + \left[\frac{1}{20} R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45} R_{\alpha\mu\beta\lambda} R^\lambda{}_{\gamma\nu\delta}\right] y^\alpha y^\beta y^\gamma y^\delta + \ldots$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor and the coefficients are all evaluated in x', i.e. $y^\mu = 0$. Now let us define:

(3.6.2)
$$\mathcal{G}_F(x,x') = (-g)^{1/4}(x) G_F(x,x') \quad ,$$

and make the Fourier transformation of this kernel:

(3.6.3)
$$\mathcal{G}_F(x,x') = (2\pi)^{-4} \int d^4k\, e^{-iky} \mathcal{G}_F(k) \quad ,$$

where $ky = \eta^{\alpha\beta} k_\alpha y_\beta$. Then the k-space can be considered as a kind of momentum space localized at x'. G_F must satisfy eq. (2.4.26), thus we can deduce the equation that must be satisfied by $\mathcal{G}_F(k)$ and we can solve it by iteration up to any order in k. Up to order four we have:

$$(3.6.4) \qquad \mathcal{G}_F(k) = (k^2-m^2)^{-1} - (\tfrac{1}{6} - \xi)R(k^2-m^2)^{-2} + + \tfrac{1}{2}\, i\, (\tfrac{1}{6} - \xi)R_{;\alpha}\partial^\alpha(k^2-m^2)^{-2} - - \tfrac{1}{3}\, a_{\alpha\beta}\partial^\alpha\partial^\beta(k^2-m^2)^{-2} + + [(\tfrac{1}{6} - \xi)^2 R^2 + \tfrac{2}{3}\, a^\lambda{}_\lambda](k^2-m^2)^{-3} + \dots ,$$

where $\partial_\alpha = \partial/\partial k_\alpha$ and:

$$(3.6.5) \qquad a_{\alpha\beta} = \tfrac{1}{2}\,(\xi - \tfrac{1}{6})R_{;\alpha\beta} + \tfrac{1}{120}\, R_{;\alpha\beta} - - \tfrac{1}{40}\, R_{\alpha\beta;\lambda}{}^\lambda - \tfrac{1}{30}\, R_\alpha{}^\lambda R_{\lambda\beta} + + \tfrac{1}{60}\, R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda} + \tfrac{1}{60}\, R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta} .$$

We must be sure that G_F is a time ordered product, thus we perform the k^0 integration algong the corresponding contour as in flat space-time. By the way we can also expand G_1 or any other Green Function simply changing to the correspon-ding flat space-time contour and using an homogeneous field equation if it is the case. In the case of G_F we can use also the method of replacing m^2 by m^2-$i\varepsilon$ and analogously, with the other Green Functions, we can use the corresponding tricks.

Substituying (3.6.4) into (3.6.3) we get:

$$(3.6.6) \qquad G_F(x,x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-iky} \times \times [a_0(x,x') + a_1(x,x')(-\frac{\partial}{\partial m^2}) + a_2(x,x')(\frac{\partial}{\partial m^2})^2] \times (k^2-m^2)^{-1}$$

where:

(3.6.7) $$a_0(x,x') = 1 \quad ,$$

and up to order 4:

(3.6.8) $$a_1(x,x') = (\tfrac{1}{6}-\xi)R - \tfrac{1}{2}(\tfrac{1}{6}-\xi)R_{;\alpha}y^\alpha - \tfrac{1}{3}\, a_{\alpha\beta}y^\alpha y^\beta{}_\lambda \quad ,$$

(3.6.9) $$a_2(x,x') = \tfrac{1}{2}\,(\tfrac{1}{6}-\xi)^2R^2 + \tfrac{1}{8}\, a^\lambda{}_\lambda$$

with all the quantities on the r.h.s. evaluated in x', i.e. $y^\mu = 0$ [10].

Now we can use the integral representation:

(3.6.10) $$(k^2-m^2+i\varepsilon)^{-1} = -i\int_0^\infty ds\; e^{is(k^2-m^2+i\varepsilon)} \quad ,$$

and use this formula in eq. (3.6.6). We can then interchange the order of the integrations, and performe the k integration, to obtain (we neglect $i\varepsilon$):

(3.6.11) $$\mathcal{G}_F(x,x') = -i(4\pi)^{-2}\int_0^\infty i\, ds(is)^{-2} \times$$

$$\times \exp[-im^2s + (\tfrac{\sigma}{2is})]\, F(x,x';is) \quad ,$$

where:

(3.6.12) $$\sigma(x,x') = \tfrac{1}{2}\, y_\alpha y^\alpha \quad ,$$

is one-half of the square of the space-time distance between x and x' and the function F has the following expansion:

---

(10) Note that a_i is the term of adiabatic order 2i.

(3.6.13) $$F(x,x';is) = a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2 + \dots$$

Finally, we can write G_F in normal coordinates if we use all these equations. In generic coordinates the factor $(-g)^{-1/2}$ (of equation (3.6.2)) becomes the Van Vleck determinant (Van Vleck, 1928):

(3.6.14) $$\Delta(x,x') = -[g(x)g(x')]^{-1/2} \times \det[\partial_\mu \partial_\nu \sigma(x,x')] \quad .$$

Thus, in generic coordinates, we have:

(3.6.15) $$G_F^{DS}(x,x') = -i\, \Delta^{1/2}(x,x')(4\pi)^{-2} \int_0^\infty ids(is)^{-2} \exp[-im^2 s + \frac{\sigma}{2is}]F(x,x';is) \; ,$$

known as the De-Witt-Schwinger-Green Function because it was derived by De Witt (1965,1975), following the work of Schwinger (1951a,b) for flat space-time.

Eq. (3.6.13) could be written:

(3.6.16) $$F(x,x';is) = \sum_{j=0}^{\infty} a_j(x,x')(is)^j \quad ,$$

and the a_j may also be found by a recurrent relation (De Witt (1965), Christensen (1976)). If we substitute (3.6.16) in (3.6.15) and perform the integral we have:

(3.6.17) $$G_F^{DS}(x,x') = \frac{-i\pi\Delta^{1/2}(x,x')}{(4\pi i)^2} \sum_{j=0}^{\infty} a_j(x,x') \times$$

$$\times \left(-\frac{\partial}{\partial m^2}\right)^j \left[\left(\frac{2m^2}{-\sigma}\right)^{1/2} H_1^{(2)}(2m^2\sigma)\right] \quad ,$$

where, if we remind we have neglect the $i\varepsilon$, really we must substract $i\epsilon$ from σ. Using the well known representation of the $\Delta_F(x,x')$ of flat space time as a Hankel function we get:

$$(3.6.18) \qquad G_F^{DS}(x,x') = \Delta^{1/2}(x,x') \times$$

$$\times \sum_j a_j(x,x')\left(-\frac{\partial}{\partial m^2}\right)^j \Delta_F(\sigma)$$

Analogously:

$$(3.6.19) \qquad G_1^{DS}(x,x') = \Delta^{1/2}(x,x') \times$$

$$\times \sum_j a_j(x,x')\left(-\frac{\partial}{\partial m^2}\right)^j \Delta_1(\sigma) \quad ,$$

where $\Delta_1(x,x')$ is the corresponding kernel of flat space time, and in the same way we can obtain all the curved space-time Green Functions from the flat space-time ones.

But again, if we compare (3.6.19) with (3.4.23) or (3.6.15) with (3.4.25) we could imagine that something is wrong. In fact (3.4.23) or (3.4.25) depend in the vacua we use (G_1 is a one vacuum function, G_F a two vacuum function) while (3.6.16) on (3.6.15) are really geometric constructions where we do not see any trace of the vacua. The answer is the same that the one we give in paragraph 3.5 for the same kind of problem: the integrand of the De-Witt-Schwinger-Green function in its integral representation (3.6.15) is a expansion in powers of σ, an analytical function at $\sigma = 0$. We could add non-analytical terms, that "cannot be seen" with an expansion at $\sigma = 0$, i.e.

they $\to 0$ when $\sigma \to 0$ faster than any power of σ[11]. These non--analytical terms do not change the local behaviour of G_F in the coincidente limit $\sigma \to 0$ or $x \to x'$, but they do change the boundary condition for large σ. Thus we can considered the different vacua that we can use in eqs. (3.4.25) or (3.4.23) as different boundary conducting for large σ while when $\sigma \to 0$ we always use the same local structure, the one of G_F^{DS} or G_1^{DS}.

Now, using the expansion (3.4.23) or (3.4.27) we can compare the results of paragraph 3.5 and 2.6. In fact, if we work in Robertson-Walker universe the adiabatic G_F and G_1 obtained by the eqs. (3.4.23) or (3.4.27) with the adiabatic base u_k, u_k^* coincide order by order with G_F^{DS} and G_1^{DS} (Birrell (1978), Bunch, Christensen and Fulling (1978), Bunch and Parker (1979)) we shall prove this statement in paragraph 5.6. The same thing happens in Bianchi I type universes (Castagnino & Nuñez (1984)). The above reasoning completes our understanding of the adiabatic vacuum: G_F^{DS} or G_1^{DS} also define the adiabatic vacuum and the expansion in powers of T^{-1} or k^{-1} coincide with the expansion in powers of σ, and we could call the order of the terms in either expansion, an adiabatic order.

Therefore the adiabatic vacuum is the unique high energy vacuum that we need to explain why there is a decreasing production of high energy particles and it is also the unique vacuum with the proper short distance structure. Other vacua can be obtained changing the low energy behaviour or the long distance behaviour.

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(11). There are also non analytical terms in s when $s \to 0$.

But, which is the short distance structure and what is the reason to have a unique short distance structure ? If we expand the flat space-time $\Delta_1(x,x')$ in powers of the distance squared $(x-x')^2$ we get (De Witt (1965)):

$$\acute{\Delta}_1(x,x') = \frac{m^2}{2}\{\frac{1}{m^2(x-x')^2} + \tag{3.6.20}$$

$$+ [\gamma - \log 2 + \log n + \frac{1}{2}\log|(x-x')^2|][\frac{1}{2} + \frac{m^2(x-x')^2}{2^2 4} + \ldots] -$$

$$- \frac{1}{4} - \frac{m^2(x-x')^2}{2^2\ 4}(1+\frac{1}{4}) - \frac{m^4(x-x')^4}{2^2.4^2.6}(1 + \frac{1}{2} + \frac{1}{6}) - \ldots\}\ .$$

Thus the Δ_1 expansion shows a quadratic divergence plus a logarithmic divergence plus regular functions terms when $x \to x'$, $\frac{\partial\Delta^1}{\partial m^2}$ starts with a logarithmic divergences and then has regular functions terms; $\frac{\partial^2\Delta^1}{(\partial m^2)^2}$ starts with a regular terms, etc. Also $\frac{\partial^2\Delta^1}{(\partial m^2)^2}$ is a zero and the following derivatives are zeros of increasing order when $x \to x'$. It can be shown (Castagnino, Harari & Nunes (1983)) that the only curved space-time Green Functions with the same short distence structure when $x \to x'$ (i.e., G_1^{DS} , $\partial G_1^{DS}/\partial m^2$; $\partial G_1^{DS}/(\partial m^2)^2$;... have the same divergencies, regular terms and zeros than Δ_1, $\partial\Delta_1/\gamma m^2$; $\partial^2\Delta^1/(\partial m^2)^2$; ...) are the De-Witt-Schwinger Green Functions. So these Green Functions are singled out as the unique with the following properties.

1 - They have the correct adiabatic behaviour for high energies.

2 - They have the same behaviour when $x \to x'$ than the correspon - ding flat space-time Green Functions.

The second propertie has also a physical base: the Strong Equivalence Principle, that states that in every space-time point there exists a system of coordinates where inertial-gravitational forces vanish. In this system space-time behaves locally as if it were flat. Then it is reasonable that G_1^{DS} and G_F^{DS} (and in fact all the good G_1 and G_F) must behave like Δ_1 and Δ_F when $x \to x'$ (this is the best version of what we have called perhaps too presumptously, the Quantum Equivalence Principle (Castagnino & Weder (1981), Castagnino, Laura, Foussats & Zandrón (1980)).

High energy bahaviour and short distance structure are two different features of the same physical phenomenon and the De-Witt-Schwinger Adiabatic Vacuum Green Functions: G_F^{DS} or G_1^{DS} are the only ones with both of these properties; for these reason they are so impotent. All other Green Function[(12)], have the same properties, but they contain non-analytical terms, reflecting the vacuum ambiguity explained in paragraph 3.4.

4 STRESS TENSOR RENORMALIZATION

4.1 DIVERGENCIES IN THE ACTION

In the preceeding paragraph we have seen that only the trivial or stationary vacuum and its particle model has, more or less, the same intuitive physical value than the flat space-time vacuum and its particle model: the plane-waves. When there are not Killing vectors we have neither a reasonable vacuum nor a particle model. The intuitive reason is also obvious, vacuum and particle model are global concepts, and we have

(12) That will define the "Strong Vacua" as we shall see in paragraph 5.3.

radically change the global structure of space-time and reference systems: from flat to curved and from inertial to accelerated. But we know that, while global structure changes radically in General Relativity local structure does not. In fact, the tangent space--time of the curved manyfold is always a Minkowski space-time, local equation remains the same, with the substitution of covariant derivatives in place of ordinary derivatives, etc. Thus we can imagine that it would be better to build Semiclassical Quantum Gravity using local concepts instead of global ones.

A local objects of great interest in General Relativity is the Energy-Momentum or Stress-Tensor $T_{\mu\nu}(x)$. It is a point function thus a local object, it is a the density of flux of momentum and energy, and it is the source of Einstein Equation. At the quantum level its expectation value $\langle T_{\mu\nu}\rangle = \langle\psi|T_{\mu\nu}|\phi\rangle$ (where, $|\psi\rangle$ is the quantum state of the universe) is the r.h.s. of the field equation (3.4.1) that we must solve in a real cosmological problem.

For all these reasons we shall study $\langle T_{\mu\nu}\rangle$ in this chapter. The problem is that it is a divergent quantity. In fact, if we make the corresponding computation we shall find the same divergency that we find when we calculate the energy of a free field in flat space-time. In that case we can eliminate the divergence introducing normal ordering or, which is the same thing, taking a infinite energy origin, because we only measure energy differences. But now we are searching the total value of the energy density, because it is one of the coordinates of $\langle T_{\mu\nu}\rangle$ and $\langle T_{\mu\nu}\rangle$ is the source of the Einstein equations, and we cannot take on arbitrary zero point energy, as in flat space-time where all the points are alike,because in curve space-time a different zero point energy could exist in every point of it. Thus we must relay in the generic renormalization methods of Quantum Field Theory adapted to the case. Of course we must try to obtain a method with some of the characteristics of General Relativity: it must be invariant under a general change of coordinates, $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ etc.

We shall follow the method of paragraph 2.3 and quantize a non-self interacting scalar field on a curved background. The geometric action must be a little more general then the one of eq. (2.3.1) if we want the theory to be renormalizable. Besides the usual terms -2Λ and R we must add the quadratic terms like:

(4.1.1)
$$H^{(1)} = R^2 \ , \ H^{(2)} = R^{\mu\nu} R_{\mu\nu} \ ,$$
$$H \quad = R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} \ .$$

But, the generalized Gauss-Bonnet theorem (Chern (1955),(1962)) states that:

(4.1.2)
$$\int d^4x [-g(x)]^{1/2} (R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + R^2 - 4R_{\mu\nu} R^{\mu\nu}) \quad ,$$

is a topological invariant, called the Euler number, so its metric variation will vanish identically, thus we can drop H and use only $H^{(1)}$ and $H^{(2)}$. Then, the action is:

(4.1.3)
$$S = S_g + S_m$$
$$S_g = \frac{1}{2} \int (-g)^{1/2} \{ \frac{1}{8\pi G_B} (R - 2\Lambda_B + \alpha_B H^{(1)} + \beta_B H^{(2)}) \} d^4x \ ,$$
$$S_m = \frac{1}{2} \int (-g)^{1/2} [-g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (m^2 + \xi R) \phi^2 \ ] \ d^4x \quad ,$$

where the B stands for "bere" coupling constant and S_g is the geometric action and S_m the matter action. Now we use the background field method and the $g^c{}_{\mu\nu}$ as classical field for $g_{\mu\nu}$ on zero for ϕ and use eq. (2.3.2) where we do not consider the G_g i.e. the graviton loop, that eventually can be taking into account when the spin-2 field will be studied, thus we have for the effective action[13]:

$$(4.1.4) \qquad \Gamma = S(g_{\mu\nu} = g^c{}_{\mu\nu}, \phi = 0) + W \quad ,$$

$$W = -\frac{1}{2} \mathrm{Tr}\, [\ln(-G_\phi)] \quad ,$$

where W is the matter effective action and $G_\phi = G_F$ is the Feyman propagator computed in the universe quantum state $|\phi>$:

$$(4.1.5) \qquad G_F(x,x') = <\psi|T(\phi(x),\phi(x'))|\psi> \quad .$$

Now we must compute the last term of eq. (4.1.4) . We must remember that in De-Witt notation G_F is G^{ij} where i and j are indeces that label the coordinates x and a generic set of indices if we are working with a set of fields or fields with spin $\neq 0$. But now there is only one scalar field, then to make more evident the matrix nature of G_F we can write it as

$$(4.1.6) \qquad G_F(x,x') = (x|G_F|x') \quad ,$$

where we interpret G_F as on operator which acts on a space of vectors (x) normalized by:

---
(13) In fact, you can verified that $<\phi> = <0_{in}|\phi|0_{in}> = 0$.

$$(4.1.7) \qquad (x|x') = \delta^{(4)}(x-x')[-g(x)]^{-1/2} \quad .$$

The trace of any operator M written in this formalism is:

$$(4.1.8) \qquad \mathrm{tr}\, M = \int d^4x \; [-g(x)]^{1/2} \, M_{xx} = \int d^4x \; [-g(x)]^{1/2} (x|M|x) \quad .$$

Now we know how to calculate the trace, but we must also know how to manage the logarithm in eq. (4.1.4). For that purpose we write $G_F = -K^{-1}$ and compute the inverse using an integral representation:

$$(4.1.9) \qquad -G_F = K^{-1} = -i \int_0^\infty e^{-iks}\, ds \quad .$$

From eq. (3.6.12) we know that G_F^{DS} can be written in such way if:

$$(4.1.10) \qquad \exp(-isK^{DS}(x,x')) = i(4\pi)^{-2}\Delta^{1/2}(x,x')\exp(-im^2 s + \tfrac{\sigma}{2}) \times F^{DS}(x,x';is)(is)^{-2}$$

where we have explicifically put K^{DS} and F^{DS}, and F^{DS} is given by eq. (3.6.13) and we can add non analytical terms if we want to.

Now, returning to equation (4.1.9) and assuming K has a small imaginary part (that could come from the imaginary part of m^2) we can use the integral representation:

$$(4.1.11) \qquad \int_{\Lambda}^{\infty} \exp(-iKs)(is)^{-1}\, ids = -E_i(-i\Lambda K) \quad ,$$

Where E_i is the exponential integral function, that for small values of its argument has the following expansion:

$$(4.1.12) \qquad E_i(x) = \gamma + \ln(-x) + O(x) \quad ,$$

where γ is the Euler's constant. Thus:

$$(4.3.13) \qquad e^{-iKs}(is)^{-1}ds = -\gamma - \ln(i\Lambda K) + O(i\Lambda K) = -\gamma - \ln(i\Lambda) - \ln(K) + O(i\Lambda K) \quad .$$

Letting $\Lambda \to 0$ we have:

$$(4.3.14) \qquad \ln(-G_F) = -\ln(K) = \int_0^{\infty} e^{-iKs}\,(is)^{-1}\, ids \quad ,$$

plus an infinite constant independent of the metric or the field ϕ that obviously can be ignored. The De-Witt-Schwinger Green Functions is the one obtained by the power expansion (3.6.13), thus a general Feynman Green Function would have non-analytical terms in that expansion, i.e. it would have the same integral representation (3.6.15) but with non-analytical terms, thus we must change $K^{DS} \to K$ and $F^{DS} \to F$ where the generic F is $F = F^{DS} +$ non analytical terms in (is).

Then the matrix form of eq. (4.1.14) would be:

$$(4.1.15) \qquad (x|\ln(-G_F)|x') = -\int_{m^2}^{\infty} G_F(x,x')dm^2 \quad ,$$

where the integration with respect to m^2 produces the factor $(is)^{-1}$ we need in eq. (4.1.11). Using now eq. (4.1.8) to compute the trace we get:

$$(4.1.16) \qquad W = -\frac{i}{2}\,\mathrm{Tr}\,[\ln(-G_\phi)] = = \frac{i}{2}\int d^4x\,[-g(x)]^{1/2} \lim_{x'\to x}\int_{m^2}^{\infty} dm^2\, G_F(x,x') = = \frac{i}{2}\int_{m^2}^{\infty} dm^2 \int d^4x\,[-g(x)]^{1/2}\, G_F(x,x') \quad .$$

Thus the contribution of the matter loop to the action *(4.1.4)* is a new piece for the Lagrangian density[14]:

$$(4.1.17) \qquad \mathcal{L}_{eff}(x) = [-g(x)]^{1/2}\, L_{eff} = = [-g(x)]^{1/2}\,\frac{i}{2}\lim_{x\to x'}\int_{m^2}^{\infty} dm^2\, G_F(x,x') \ ,$$

and:

$$(4.1.18) \qquad W = \int \mathcal{L}_{eff}\, d^4x \equiv \int [-g(x)]^{1/2}\, L_{eff}\, d^4x \quad .$$

Now, the new piece is divergent as we suspected. If we return to eqs. (3.6.13) and (3.6.15) we can see that L_{eff} diverges at the lower end of the integral because the damping factor $\sigma/2S$

(14) Really L_{eff} should be called L^{mat}_{eff} because it is the matter effective Lagrangian density.

vanishes when $x \to x'$, while the convergence in the upper limit is guaranteed by the term $-i\varepsilon$, which is always added to m^2. This divergence in the lower limit happens even if we have new non analytical term on F, in fact, this terms vanishes faster than any power of is and it causes no trouble when $s \to 0$.

Therefore the divergent terms are:

(4.1.19)
$$L_{div} = - \lim_{x' \to x} \frac{\Delta^{1/2}(x,x')}{32\,\pi^4} \times$$
$$\times \int_0^\infty \frac{ds}{s^3} \exp - i(m^2 s - \frac{\sigma}{2s})[a_0(x,x') +$$
$$a_1(x,x')is + a_n(x,x')(is)^2] \quad ,$$

where coefficients a_0, a_1 and a_2 are given by eqs. (3.6.7) and (3.6.9).

The remaining analytical terms a_3, a_4, ..., and the non-analytical ones, are finite when $x \to x'$. Thus only the three first analytical terms cause divergencies and this divergencies are of a pure geometrical local nature. In fact the quantum state of the universe $|\psi\rangle$, or what is the same, the large scale structure of G^F or its boundary condutions are embodied in the non analytical terms.

These divergencies are the same that the one we found in $\langle T_{\mu\nu}\rangle$ as we shall see in paragraph 4.3. Thus the renormalization of the action produces the renormalization of $\langle T_{\mu\nu}\rangle$.

4.2 Action Renormalization

We have demonstrated that the action is divergent, now we shall see how the divergencies can be absorbed in the bare coupling constant obtaining a renormalized action. We must work with infinite quantities, and this can be done in different ways. One of them is dimentional renormalization, i.e. to write the theory in n variables and see the divergencies that appear when $n \to 4$. Repeating the computations in n variables we reach to:

$$L_{eff} = \lim_{x \to x'} \frac{\Delta^{1/2}(x,x')}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x,x') \times \int_0^\infty (is)^{j-1-\frac{n}{2}} e^{-i(m^2 s - \frac{\sigma}{2s})} ids \quad , \tag{4.2.1}$$

plus the non analytical terms, and it turns out that the first $\frac{1}{2}n+1$ analytical terms are divergent as $\sigma \to 0$. Treating n as a variable that can be analytically continued throughout the complex plane and taking the limit $x \to x'$ we have:

$$L_{eff} = \frac{1}{2} (4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) \times \int_0^\infty (is)^{j-1-\frac{n}{2}} e^{-in^2 s} ids = \frac{1}{2} (4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) (m^2)^{\frac{n}{2}-j} \Gamma(j - \tfrac{n}{2}) \quad , \tag{4.2.2}$$

where $a_j(x) = a_j(x,x')$, plus the non analytical terms. But this L_{eff} has not the correct dimension $(\text{length})^{-4}$ when $n \neq 4$, then we must introduce on arbitrary mass scale μ and write (4.2.2) as:

$$L_{eff} = \frac{1}{2} (4\pi)^{-n/2} (\tfrac{m}{\mu})^{n-4} \cdot \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma(j - \tfrac{n}{2}) \quad , \tag{4.2.3}$$

$\Gamma(Z)$ has a pole in $Z = 0,-1,-2,\ldots$ thus when $n \to 4$ we shall have three divergent terms for $j = 0,1,2$:

$$(4.2.4) \qquad \Gamma(-\tfrac{n}{2}) = \frac{4}{n(n-2)}\left(\frac{2}{4-n} + \gamma\right) + O(n-4) \quad ,$$

$$\Gamma(1 - \tfrac{n}{2}) = \frac{2}{2-n}\left(\frac{2}{2-n} + \gamma\right) + O(n-4) \quad ,$$

$$\Gamma(2 - \tfrac{n}{2}) = \frac{2}{2-n} - \gamma + O(n-4) \quad .$$

Thus, as the non analytical terms are convergent we have (Bunch 1979):

$$(4.2.5) \qquad L_{div} = -(4\pi)^{-n/2}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\} \times \left(\frac{4m^4 a_0}{n(n-2)} - \frac{2m^2 a_1}{n-2} + a_2\right) \quad ,^{(15)}$$

where we have used the expansion:

$$(4.2.6) \qquad \left(\frac{m}{\mu}\right)^{n-4} = 1 + \frac{1}{2}(n-4)\,\ln\left(\frac{m^2}{\mu^2}\right) + O((n-4)^2) \quad ,$$

and we have neglected terms that vanish when $n \to 4$.

Taking the coincidence limit of eqs. (3.6.7), (3.6.8) and (3.6.9) we have:

$$(4.2.7) \qquad a_0(x) = 1 \quad ,$$

$$(4.2.8) \qquad a_1(x) = \left(\frac{1}{6} - \xi\right)R \quad ,$$

---

(15) From this equation we see that L_{div} has a convergent term.

(4.2.9) $$a_2(x) = \frac{1}{180} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180} R_{\alpha\beta}R^{\alpha\beta} - - \frac{1}{6} (\frac{1}{5} - \xi) \Box R + \frac{1}{2} (\frac{1}{6} - \xi)^2 R^2 ,$$

thus L_{div} is a purely geometrical expresion. Now writing:

(4.2.9A) $$A = \frac{4m^4}{(4\pi)^{n/2}n(n-2)} \{ \frac{1}{n-4} + \frac{1}{2} [\gamma + \ln(\frac{m^2}{\mu^2})]\} ,$$

(4.2.10) $$B = \frac{2m^2(\frac{1}{6} - \xi)}{(4\pi)^{n/2}(n-2)} \{ \frac{1}{n-4} + \frac{1}{2} [\gamma + \ln(\frac{m^2}{\mu^2})]\} ,$$

(4.2.11) $$C = \frac{1}{(4\pi)^{n/2}} \{ \frac{1}{n-4} + \frac{1}{2} [\gamma + \ln(\frac{m^2}{2})]\} ,$$

using eqs. (4.1.1) we have:

(4.2.12) $$L_{div} = - A + B (\frac{1}{6} - \xi)R - C \times$$

$$\times \frac{1}{180} H - \frac{1}{180} H^{(2)} - \frac{1}{6} (\frac{1}{5} - \xi) \Box R + \frac{1}{2}(\frac{1}{6} - \xi)^2 H^{(1)} .$$

Now we can drop the term $\Box R$ because it is a total divergence and substitute H by:

(4.2.13) $$H = -H^{(1)} + 4H^{(2)} ,$$

using eq. (4.1.2) thus:

(4.2.14) $$L_{div} = - A + B (\frac{1}{6} - \xi)R - C \times$$

$$\times [\frac{1}{60} H^{(2)} + \{ \frac{1}{2}(\frac{1}{6} - \xi)^2 - \frac{1}{180} \} H^{(1)}] .$$

Thus adding and substracting L_{div} from eq. (4.1.4) we have:

$$(4.2.15) \qquad \Gamma = \int (d^n x)(-g)^{1/2} \times$$

$$\times \{-(A + \frac{\Lambda_B}{8\pi G_B}) + [\frac{1}{16\pi G_B} + B(\frac{1}{6} - \xi)]R +$$

$$+ [\frac{\alpha_B}{16\pi G_B} + \{\frac{1}{2}(\frac{1}{6} - \xi)^2 - \frac{1}{180}\} C] H^{(1)} +$$

$$+ [\frac{\beta_B}{16\pi G_B} + \frac{C}{60}] H^{(2)} + L_{eff} - L_{div} \} .$$

From eqs. (4.2.9A), (4.2.10) and (4.2.11) we can see that A,B,C diverge when $n \to 4$, but now we can define the new renormalize physical coupling constants:

$$(4.2.16) \qquad -(A + \frac{\Lambda_B}{8\pi G_B}) = -\frac{\Lambda}{3\pi G} ,$$

$$(4.2.17) \qquad \frac{1}{16\pi G_B} + B(\frac{1}{6} - \xi) = \frac{1}{16\pi G} ,$$

$$(4.2.18) \qquad \frac{\alpha_B}{16\pi G_B} + \{\frac{1}{2}(\frac{1}{6} - \xi)^2 - \frac{1}{180}\} C = \frac{\alpha}{16\pi G} ,$$

$$(4.2.19) \qquad \frac{\beta_B}{16\pi G_B} + \frac{C}{60} = \frac{\beta}{16\pi G} ,$$

and we obtain the renormalize effective action (when $n \to 4$):

$$(4.2.20) \qquad \Gamma_{ren} = \int d^4x \, (-g)^{1/2} \times$$

$$\times \{\frac{1}{8\pi G} [-2\Lambda + R + \alpha H^{(1)} + \beta H^{(2)}] + L_{eff} - L_{div}\} ,$$

which is a completely finite expression because all the coupling constants are physical finite constants G, Λ, α, β and because we have taken away all the divergencies of L_{eff} substracting .

Now it is evident why we need to have the terms $H^{(1)}$ and $H^{(2)}$ in the Einstein action. They are necessary to remove the divergencies from the term $a_2(x)$. Any how the experimental coefficients α and β could be very small or even $\alpha = \beta = 0$, then the Einstein theory could be recovered.

The remainder renormalized matter Lagrangian shall be called:

$$(4.2.21) \qquad L_{ren} \equiv L_{eff} - L_{div} \quad ,$$

and it turns out to be:

$$(4.2.22) \qquad L_{ren} = \frac{1}{32\pi^2} \int_0^\infty \sum_{j=3}^\infty a_j(x)(is)^j \quad \times$$

$$\times \; e^{im^2 s} \; ids + \text{non analytic terms.}$$

This expression could be integrated by parts three times and using eq. (3.6.13) we obtain:

$$(4.2.23) \qquad L_{eff} = -\frac{1}{64\pi^2} \int_0^\infty \ln(is) \frac{\partial^3}{\partial(is)^3} \times$$

$$\times \; [F(x,x';is)\, e^{ism^2}]\, d(is) \; +$$

$$+ \frac{1}{64\pi^2} \int_0^\infty \ln(is) \frac{\partial^3}{\partial(is)^3} \{[a_0 + a_1(is) + a_2(is)^2]$$

$$e^{-ism^2}\} \; ids \quad .$$

The second integral in the r.h.s. is finite thus its terms could be added to the constants Λ, G, α and β and the function F of the first integral is $F = F^{DS}$ + non analytic terms. Thus we have:

$$(4.2.24) \qquad L_{ren} = -\frac{1}{64\pi^2}\int_0^\infty \ln(is) \times$$

$$\times \frac{\partial^3}{\partial(is)^3}[F(x,x;is)e^{-ism^2}]\, ids \quad .$$

About constant μ we can say the usual things, if we choose a value of μ we could measure constants Λ , G, α and β . If we change μ, these constants also change, and the analysis of the rescaling of μ leads us to renormalization group equations.

We could use other renormalization techniques:

- The generalized ζ-function (Dowker & Critchley 1976 a,b, 1977a).
- The point splitting method (Christensen 1978, Adler, Liberman & Ng 1977) etc.

All these techniques yield the same results[16].

4.3 Renormalization of the Stress-Tensor

Let us now compute the field equation. The classical field equation ($S_{,i} = 0$) could be obtained from action (4.1.3) derivating with respect to variable $g_{\mu\nu}$:

$$(4.3.1) \qquad \frac{2}{(-g)^{1/2}}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad .$$

Calling, as usual,

$$(4.3.2) \qquad \frac{2}{(-g)^{1/2}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu} \quad ,$$

(16) At least for spin zero fields.

it leads to the classical field equation:

$$(4.3.3)\qquad R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} + \Lambda_B g_{\mu\nu} + \alpha_B H^{(1)}_{\mu\nu} + \beta_B H^{(2)}_{\mu\nu} =$$

$$= -8\pi G_B\, T_{\mu\nu} \quad ,$$

where:

$$(4.3.4)\qquad H^{(1)}_{\mu\nu} = \frac{1}{(-g)^{1/2}} \frac{\delta}{\delta g^{\mu\nu}} \int (-g)^{1/2}\, H^{(1)} d^4x =$$

$$= 2R_{;\mu\nu} - 2g_{\mu\nu} \Box R - \frac{1}{2} g_{\mu\nu} R^2 + 2RR_{\mu\nu} \quad ,$$

and

$$(4.3.5)\qquad H^{(2)}_{\mu\nu} = \frac{1}{(-g)^{1/2}} \frac{\delta}{\delta g^{\mu\nu}} \int (-g)^{1/2}\, H^{(2)}\, d^4x =$$

$$= R_{;\mu\nu} - \frac{1}{2} g_{\mu\nu} \Box R - \Box R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} +$$

$$+ 2R^{\alpha\beta} R_{\alpha\beta\mu\nu} =$$

$$= 2R_{\mu\ \ ;\nu\alpha}^{\ \alpha} - \Box R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Box R + 2R_{\mu}^{\ \alpha} R_{\alpha\nu} -$$

$$- \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} \quad .$$

The quantum version ($\Gamma_{,i} = \langle S_{,i} \rangle = 0$) could be obtained through eq. (2.1.4) using the effective action (4.1.4):

$$(4.3.6)\qquad R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda_B g_{\mu\nu} + \alpha_B\, H^{(1)}_{\mu\nu} + \beta_B\, H^{(2)}_{\mu\nu} =$$

$$= -8\pi G_B \frac{2}{(-g)^{1/2}} \frac{\delta W}{\delta g^{\mu\nu}} \quad ,$$

and using eq. (2.1.4) we have:

(4.3.7) $$\frac{2}{(-g)^{1/2}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle \quad .$$

Thus, in fact, the divergencies that aflict W are the same of $\langle T_{\mu\nu} \rangle$ and the quantum equation is:

(4.3.8) $$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} + \alpha_B H^{(1)} + \beta_B H^{(2)} =$$
$$= -8\pi G_B \langle T_{\mu\nu} \rangle \quad ,$$

i.e. eq. (2.4.1). The renormalized equation is obtained substituting L_{eff} for L_{ren} in (4.1.18) and calling:

(4.3.9) $$W_{ren} = \int [-g(x)]^{1/2} L_{ren} \, d^4x \quad ,$$

(4.3.10) $$\langle T_{\mu\nu} \rangle ren = \frac{2}{(-g)^{1/2}} \frac{\delta W_{ren}}{\delta g^{\mu\nu}} \quad ,$$

and then obtaining:

(4.3.11) $$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H^{(1)} + \beta H^{(2)} =$$
$$= -8\pi G \langle T_{\mu\nu} \rangle ren \quad .$$

This equations could also be obtained directly from action (4.2.20) i.e. $\Gamma_{ren,i} = 0$. Really this equation is the one we need to know to solve the cosmological problem; thus a short-cut to obtain $\langle T_{\mu\nu} \rangle_{ren}$ directly will be welcome. This procedure can be deduced from all the just developped theory. We can write eq. (4.1.19) as:

(4.3.12) $$L_{div} = \frac{i}{2} \lim_{x \to x'} \int_{m^2}^{\infty} dm^2 \; G_F^{div}(x,x') \quad ,$$

where $G_F^{div}(x,x')$ is only the divergent part of $G_F^{DS}(x,x')$ i.e. the terms a_0, a_1, a_2. Thus equation (4.2.21) could be written:

(4.3.13) $$L_{ren} = \frac{i}{2} \lim_{x \to x'} \int_{m^2}^{\infty} dm^2 \; [G_F(x,x') - G_F^{div}(x,x')] \quad ,$$

and also eq. (4.3.10) becomes:

(4.3.14) $$\langle T_{\mu\nu} \rangle_{ren} = \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle_{div} \quad ,$$

where:

(4.3.15) $$W_{div} = \int [-g(x)]^{1/2} \, L_{div} \, d^4x \quad ,$$

(4.3.16) $$\langle T_{\mu\nu} \rangle_{div} = \frac{2}{(-g)^{1/2}} \frac{\delta W_{div}}{\delta g^{\mu\nu}} \quad .$$

Thus we obtain the following recipe "construct $\langle T_{\mu\nu} \rangle$ from $G_F(x,x')$ and $\langle T_{\mu\nu} \rangle_{div}$ from $G_F^{div}(x,x')$ and sustract". But the procedure is almost impossible because to perform derivatives like (4.3.10) or (4.3.16) W or W_{div} must be known for all possible geometries, an impossible task.

It is much easier to construct $\langle T_{\mu\nu} \rangle$ from $G^{(1)}(x,x')$ directly, in fact it can be proved (Christensen (1976)) that:

(4.3.17) $$\langle T_{\mu\nu} \rangle = \lim_{x' \to x} D_{\mu\nu}(xx') G^{(1)}(x,x') \quad ,$$

where $D_{\mu\nu}$ is a differential operator precisely:

$$(4.3.18) \qquad \langle T_{\mu\nu}\rangle = \lim_{x\to x'} g^{1/2}\{\tfrac{1}{2}(\tfrac{1}{2}-\xi) \times$$
$$\times (G^{(1)}_{;\mu'\nu} + G^{(1)}_{;\mu\nu'}) + (\xi - \tfrac{1}{4}) G^{(1)\ \rho'}_{;\ \rho} g_{\mu\nu} -$$
$$- \tfrac{1}{2}\xi (G^{(1)}_{;\mu\nu} + G^{(1)}_{;\mu'\nu'}) + \tfrac{1}{8}\xi g_{\mu\nu} \times$$
$$\times (G^{(1)}{}_{;\rho}{}^{\rho} + G^{(1)}{}_{;\rho'}{}^{\rho'}) +$$
$$+ \tfrac{3}{4}\xi(\xi R+m^2)G^{(1)} g_{\mu\nu} + \tfrac{1}{2}\xi (R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}) \times$$
$$\times G^{(1)} - \tfrac{1}{4} m^2 g_{\mu\nu} G^{(1)} \} .$$

Then we can compute:

$$(4.3.19) \qquad \langle T_{\mu\nu}\rangle_{div} = \lim_{x'\to x} D_{\mu\nu}(x,x')G^{div}_{(1)}(x,x') \quad ,$$

and use (4.1.14) to obtain $\langle T_{\mu\nu}\rangle_{ren}$. Details of this method could be obtained in Christensen (1976,1978) Davies, Fulling, Christensen & Bunch (1977), and Adler, Lieberman & Ng (1977, 1978).

Perhaps it is northwhile to give a heuristic explanation of the normalization procedure. Let us consider again Einstein unrenormalized equation (4.3.8), its r.h.s. is $-8\pi G_B\langle\psi|T_{\mu\nu}(x)|\psi\rangle$ where $|\psi\rangle$ is the quantum state of the universe and it is infinite. Then one can say that if we would know a "local vacuum" $|0\rangle_x$ at x we can compute $_x\langle 0|T_{\mu\nu}(x)|0\rangle_x$ and if this quantity is also divergent we can say that in fact, these divergencies are unreal because the vacuum expactation value of $T_{\mu\nu}$ obviously must vanish . But we know a very convincing local vacuum; if is the adiabatic vacuum related with $G_F^{DS}(x,x')$, thus $|0\rangle_x$ is the adiabatic vacuum and we know that:

$$(4.3.20) \qquad {}_x\langle 0|T_{\mu\nu}|0\rangle_x = \langle T_{\mu\nu}\rangle_{div} + \langle T_{\mu\nu}\rangle_{conv} \quad ,$$

where $\langle T_{\mu\nu}\rangle_{div}$ is produced by the terms a_0, a_1, a_2 and $\langle T_{\mu\nu}\rangle_{conv}$ by the remaining term a_3, a_4,.... Now if we substract (4.3.20) from (4.3.9) we have:

$$(4.3.21) \qquad - \frac{1}{8\pi G_B} \{R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} + \Lambda_B g_{\mu\nu} + \alpha_B H^{(1)} + \beta_B H^{(2)}\} - \langle T_{\mu\nu}\rangle_{div} - \langle T_{\mu\nu}\rangle_{conv} = \langle\psi|T_{\mu\nu}|\psi\rangle - {}_x\langle 0|T_{\mu\nu}|0\rangle_x \quad ,$$

and taking into account the structure of $\langle T_{\mu\nu}\rangle_{div}$ we get:

$$(4.3.22) \qquad - \frac{1}{8\pi G} \{R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H^{(1)} + \beta H^{(2)}\} - \langle T_{\mu\nu}\rangle_{conv} = \langle\psi|T_{\mu\nu}|\psi\rangle - {}_x\langle 0|T_{\mu\nu}|0\rangle_x \quad .$$

This is a completely logical equation:

- in its r.h.s. we have a convergent stress tensor with a reasonable property: it vanishes when $|\psi\rangle = |0\rangle_x$ i.e. the local vacuum
- in its l.h.s. we have a generalized finite Einstein Equation where we add an infinite set of geometrical terms originated by a_3, a_4 ,..., that in fact could exist in Einstein Equation and could be derived from a generalized Einstein Lagrangian with terms higher than R^2.

But as (4.3.22) is difficult to manage because it has infinite terms, it is easier to substract $<T_{\mu\nu}>_{conv}$ from both sides. Then we arrive to eq. (4.3.11) a field equation with exactly the same physic consequences than (4.3.22). In this way the origin of the renormalized Einstein Equation (4.3.11) could be easily explained.

4.4 UNIQUENESS AND CONSISTENCY OF THE RENORMALIZATION METHOD

As several manipulations of the preceeding paragraph are not completely rigorous, from the mathematical point of view, some one could question their consistency. Thus we can check these results using a different technique: we must simply try to see what $<T_{\mu\nu}>_{ren}$ make sense in the r.h.s. of Einstein field equation (Christensen (1975) Wald (1977,1978 a,b), Castagnino & Harari (1984)). Precisely we could state a set of reasonable axioms, that $<T_{\mu\nu}>_{ren}$ must satisfy, and see if we find one, and only one, $<T_{\mu\nu}>_{ren}$ that satisfies the set.

These axioms are:

(1) Covariant conservation

(2) Causality

(3) Standard results for "off-diagonal" elements

(4) Standard results for Minkowski space.

Let us see the meaning of the axioms. (1) is simply:

$$\nabla_{\mu} <T^{\mu\nu}>_{ren} = 0 \quad , \tag{4.4.1}$$

and it is imposed because the l.h.s. of Einstein Equation is divergentless; (2) the causality axiom says that for a fixed point x, $<T^{\mu\nu}(x)>_{ren}$ depends only in the causal past of x i.e. $E^-(x)$. In our case the causal past of x is the geometry of space-time inside the past null cone of x and eventually the quantum state of the universe, if this state is defined in the far past. $<T^{\mu\nu}(x)>_{ren}$ depends only in those elements, so we can change all other features of the problem, e.g. the geometry out side the x-past null case, and $<T^{\mu\nu}(x)>_{ren}$ must remain fixed. As all the equations of the theory are invariant by a time inversion the same statement must be true with past change by future; (3) is the condition that as $<\phi|T_{\mu\nu}|\psi>$ is finite for $<\phi|\psi> = 0$, thus $|\phi> \neq |\psi>$, for this quantity we must obtain the usual value; (4) means that when we particularize the geometry in the one of Minkowski space-time the normal ordering procedure should be valid.

Now we can prove that if $<T_{\mu\nu}>_{ren}$ satisfies condition (1), (2), (3) it is unique, within a local conversed tensor, i.e. a divergentless tensor that depends on the geometry at x only.

In fact, if $<T_{\mu\nu}>$ and $<\tilde{T}_{\mu\nu}>$ are two renormalized energy-momentum operator whose expectation values satisfy axioms (1), (2) and (3), then we must show that:

$$U_{\mu\nu} = T_{\mu\nu} - \tilde{T}_{\mu\nu} \quad , \tag{4.4.2}$$

is a local, c-number, divergentless tensor. From axiom (3) the matrix elements of $U_{\mu\nu}$ between orthogonal states must vanish because:

(4.4.3) $$\langle\phi|T_{\mu\nu}|\psi\rangle = \langle\phi|\tilde{T}_{\mu\nu}|\psi\rangle$$

Besides calling $|\pi_\pm\rangle = 2^{-1/2}(|\psi\rangle \pm |\phi\rangle)$ as:

(4.4.4) $$\langle\pi_+|U_{\mu\nu}|\pi_-\rangle = 0 \qquad \forall\ \pi_+, \pi_- \quad ,$$

we have:

(4.4.5) $$\langle\psi|U_{\mu\nu}|\psi\rangle = \langle\phi|U_{\mu\nu}|\phi\rangle \qquad \forall\psi,\phi$$

From these equations we obtain that:

(4.4.6) $$U_{\mu\nu} = u_{\mu\nu}\ I \quad ,$$

where $u_{\mu\nu}$ is a c-number tensor field and I the unit operator of the states Hilbert space.

Now $u_{\mu\nu}$ must be a local tensor because from eq. (4.4.6) we have:

(4.4.7) $$\langle 0_{in}|U_{\mu\nu}(x)|0_{in}\rangle = u_{\mu\nu}(x) \quad ,$$

(4.4.8) $$\langle 0_{out}|U_{\mu\nu}(x)|0_{out}\rangle = u_{\mu\nu}(x) \quad .$$

But from axiom (2) and eq. (4.4.7) $u_{\mu\nu}(x)$ can only depend on the past of x and from the same axiom and eq. (4.4.8) it can only depend on the future of x, thus $u_{\mu\nu}(x)$ is a function of the past and the future of x i.e. only the point x.

Finally from axiom (1):

(4.4.9) $$\nabla_\nu u^{\mu\nu} = 0 \qquad \text{q.e.d.}$$

Thus $\langle T_{\mu\nu}\rangle_{ren}$, satisfying axioms (1) to (4), is unique, witl a local conserved tensor that more likely belongs to the l.h of the Einstein Field Equation accordingly to the spirit of paragraph 4.3.

Of course one would like to know wether or not the renormalization prescriptions of the preceeding paragraph satisfy the axioms (1)–(4) and see if we can give an explicit expression of the local tensor $u_{\mu\nu}$. We would like also to know if the Lagrangian term that gives rise to $u_{\mu\nu}$ are contained in the primitive Lagrangian (4.1.3) i.e. if the theory that yields the $u_{\mu\nu}$ term is a renormalizable one. We will answer these questions in the next paragraph.

4.5 HADAMARD RENORMALIZATION

Let us start with eq. (4.3.19) and let us introduce the symbol:

$$(4.5.1) \qquad [G^{(1)}] = \lim_{x\to x'} G^{(1)}(x,x') \quad .$$

From Christensen's generalization of the Synge's theorem (Christensen (1976)) which can be applied to symmetric biscalars, we know that:

$$(4.5.2) \qquad \begin{aligned} [G^{(1)}{}_{;\mu}] &= \tfrac{1}{2}\, [G^{(1)}]_{;\mu} \quad , \\ [G^{(1)}{}_{;\mu\nu'}] &= -[G^{(1)}{}_{,\mu'\nu}] + \tfrac{1}{2}\, [G_1]_{;\mu\nu} \quad , \\ [G^{(1)}{}_{;\mu'\nu'}] &= [G^{(1)}{}_{;\mu\nu}] \quad . \end{aligned}$$

Therefore, with these formulas eq. (4.2.19) reads:

$$(4.5.3) \qquad \langle T_{\mu\nu}\rangle = -\frac{1}{2}[G_{1;\mu\nu}] -$$

$$-\frac{3}{4}(\xi-\frac{1}{3})[\Box G_1]g_{\mu\nu} + \frac{1}{2}(\frac{1}{2}-\xi)[G_1]_{;\mu\nu} +$$

$$+\frac{1}{2}(\xi-\frac{1}{4})\Box[G_1]g_{\mu\nu} + [G_1]\times$$

$$\times\{\frac{3}{4}(\xi-\frac{1}{3})(m^2+\xi R)g_{\mu\nu} + \frac{1}{2}\xi R_{\mu\nu}\} \quad .$$

Now we can use the technique of eq. (4.3.15), but now we realize that perhaps the choice of G_1^{DS} is too particular. Can we find different renormalizations if we change G_1^{DS} ? We can suppose that there exists a class of different "local" G_1 that can be used in the renormalization. Of course they must satisfy a minimal set of requirements e.g. conditions (3.4.15) and a reasonable divergent behaviours, a behaviour similar to the one of $\Delta_1(x,x')$ of flat space-time when $x \to x'$ i.e.:

$$(4.5.4) \qquad \Delta_1(x,x') \sim \frac{1}{\sigma} + \ln\sigma + R(\sigma) \quad ,$$

where σ is half the square of the geodesic distance and $R(\sigma)$ is a regular function when $\sigma \to 0$. Elementary solutions of the field equation with these properties are call Hadamard solutions (cfr. Hadamard (1952), Garabedian (1964)) that can be written as:

$$(4.5.5) \qquad G_1(x,x') = \frac{\Delta^{1/2}(x,x')}{8\pi^2}\{\frac{2}{\sigma} + v\ln\mu^2\sigma + w\} \quad ,$$

where $\Delta(x,x')$ is the Van Vleck determinant (cfr. eq. (3.6.11)), v and w are regular functions of x and x' when $x \to x'$, and μ is a mass scale to make the product $\mu^2\sigma$ dimensionless. If we expand $v(x,x')$ and $w(x,x')$ as:

$$(4.5.6) \qquad v(x,x') = \sum_{n=0}^{\infty} v_n(x,x')\,\sigma^n \quad ,$$

$$w(x,x') = \sum_{n=0}^{\infty} w_n(x,x')\,\sigma^n \quad ,$$

from the field equation (3.4.15III) we obtain:

$$(4.5.7) \qquad v_0 + v_0^{;\mu}\,\sigma_{;\mu} = V - \Delta^{1/2}\,\Box\,(\Delta^{1/2}) \quad ,$$

where $V = m^2 + \xi R$,

$$(4.5.8) \qquad v_n + \frac{1}{n+1}\,v_n^{;\mu}\sigma_{;\mu} = \frac{1}{2n(n+1)}\{ V v_{n-1} - \Delta^{-1/2}\,\Box\,(\Delta^{1/2} v_{n-1})\} \qquad (n \geq 1)$$

$$(4.5.9) \qquad w_n + \frac{1}{n+1}\,w_n^{;\mu}\sigma_{;\mu} = \frac{1}{2n(n+1)}\{ V\, w_{n-1} - \Delta^{-1/2}\,\Box\,(\Delta^{1/2}\, w_{n-1})\} - \frac{2n+1}{n(n+1)}\,v_n - \frac{1}{n(n+1)}\,v_n^{;\mu}\sigma_{;\mu} \qquad (n \geq 1)$$

Thus in eq. (4.5.5) $\Delta(x,x')$ and $v(x,x')$ are univocally determinated by the background geometry, the only arbitrary function is $w_0(x,x')$. Every choice of that function determinates the complet $w(x,x')$ through eq. (4.5.9)[17]. We shall suppose that

[17] Thus the infinite part of G_1 is fixed and we have the temptation to use only this infinite part to built L_{div} or G_1^{div}(eqs.(4.2.5) or (4.3.12)) but it does not work, we need finite terms to make: $\langle T_{\mu\nu}\rangle_{ren;}{}^{\nu} = 0$ (cfr.eq.(4.2.9)to(4.2.12)).

the indetermination in $<T_{\mu\nu}>_{ren}$ is originated in this indetermination. For instance, the flat space-time kernel $\Delta_1(x,x')$:

$$(4.5.10) \qquad \Delta_1(x,x') = \frac{m^2}{4\pi} \ \ln \left(\frac{H_1^{(1)} \sqrt{2m^2\sigma}}{\sqrt{2m^2\sigma}} \right) \quad ,$$

can be shown to be the Hadamard solution characterized by:

$$(4.5.11) \qquad w_0^M = m^2 \ (2\gamma-1) \qquad , \qquad \mu^2 = m^2 \ \text{[18]}$$

where we used a superscript "M" to denote a "Minkowskian" value and γ is the Euler constant. The v_n^M on w_n^M, satisfying the recurrence relations (4.5.8) and (4.5.9), are:

$$(4.5.12) \qquad v_n^M = 2 \ (\frac{m^2}{2})^{n+1} \ / \ n!(n+1)! \quad ,$$

$$w_n^M = -2 \ (\frac{m^2}{2})^{n+1} \{ \log 2 + \psi(n+2) + \psi(n+1)\}/n! \quad ,$$

where $\psi(n)$ is the derivative of the logarithm of the Γ function.

All the arbitrariness in any Hadamard solution is completely contained in $w_0(x,x')$ (except for changes in the mass scale μ). In other words, changing $w_0(x,x')$ we can find different local $G_1(x,x')$ and obtain different renormalization that we shall call Hadamard renormalizations.

But, even so, $w_0(x,x')$ is not completely arbitrary because to satisfy eq. (3.4.15 II) $w(x,x')$ must be symmetric. Also, as it was proved by Wald (1978), $G_1(x,x')$ and thus $w(x,x')$ must be symmetric if the vacuum expectation value of the

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(18) We shall use this last equation in all massive cases.

energy-momentum, constructed after them, is expected to be covariantly conserved. However covariant conservation does not strictly recquire a completely symmetric w. Indeed, taking into account that in the construction of $\langle T_{\mu\nu}\rangle_{ren}$ from G_1, at most second derivatives of G_1 appear, it would be enough for $\langle T_{\mu\nu}\rangle_{ren}$ to be conserved that the following properties would be satisfied:

(4.5.13) $$[w_{;\mu}(x,x')] = [w_{;\mu}(x',x)] \quad ,$$
$$[w_{;\mu\nu}(x,x')] = [w_{;\mu\nu}(x',x)] \quad ,$$
$$[w_{;\mu\nu}{}^{\nu}(x,x')] = [w_{;\mu\nu}{}^{\nu}(x',x)] \quad .$$

The quoted Christensen generalization of Synge theorem reads:

(4.5.14) $$[T_{\mu_0\ldots\mu_n\nu'_1\ldots\nu'_m;\mu'}] =$$
$$= -[T_{\mu_0\ldots\mu_n\nu'_1\ldots\nu'_m;\mu}] +$$
$$+[T_{\mu_1\ldots\mu_n\nu'_1\ldots\nu'_m}]_{;\mu} \quad .$$

Repeated use of this equation and the anticommutation rule for covariant derivatives allow us to transform conditions (4.5.13) into the following relations between $[w_0]$, $[w_{0;\mu}]$ and $[w_{0;\mu\nu}]$:

(4.5.15) $$[w_{0;\alpha}] = \frac{1}{2}\,[w_0]_{;\alpha}$$

(4.5.16) $$[w_{0;\alpha\beta}]_{;}{}^{\beta} - \frac{1}{4}\,[\Box\, w_0]_{;\alpha} =$$

$$= \frac{1}{4} \Box [w_0]_{;\alpha} + \frac{1}{12} R_{\alpha\beta} [w_0]_;{}^{\beta} +$$

$$+ \frac{1}{4} (\xi - \frac{1}{6}) R_{;\alpha}[w_0] - \frac{1}{4} [m^2 + (\xi - \frac{1}{6})R] \times$$

$$\times [w_0]_{;\alpha} + \frac{1}{2} [v_1]_{;\alpha} \quad .$$

Now, as we are looking for all possible local solutions at x, $[w_0]$ and $[w_{0,\mu\nu}]$ an only be functions of the geometric quantities defined at x. The dimension must be that of m^2 for $[w_0]$ and m^4 for $[w_{0,\mu\nu}]$ and also the Minkowskian value (4.5.11) must be obtained in the Minkowski limit.

Thus it can be shown that the more general expression is:

(4.5.17) $$[w_0] = w_0^M + AR + \frac{1}{m^2} \{T + (3C_1 - 2C_2) \Box R \} \quad ,$$

(4.5.18) $$[w_{0;\mu\nu}] = -m^2 AR_{\mu\nu} + (w_0^M + m^2)(\xi - \frac{1}{6})R_{\mu\nu} +$$

$$+ C_1 RR_{\mu\nu} + (\frac{1}{3} A + C_2 - C_1)R_{;\mu\nu} - 2C_2(R^{\theta\rho}R_{\mu\theta\nu\rho} + \frac{1}{2} \Box R_{\mu\nu}) \quad ,$$

where:

(4.5.19) $$T = \frac{1}{180} (R_{\theta\rho\tau\epsilon}R^{\theta\rho\tau\epsilon} - R_{\theta\rho}R^{\theta\rho}) + \frac{1}{2}(\xi - \frac{1}{6})^2 R^2 - \frac{1}{6}(\xi - \frac{1}{5}) \Box R,$$

and A, C_1 and C_2 are arbitrary coefficients.

All the terms of eqs. (4.5.17) and (4.5.18) are independent, as can be shown using Bianchi identities and Gauss-Bonnet theorem (in the sence used in eqs. (4.1.1) and (4.1.2)). Now if we introduce these expression in (4.5.3) with adequate arrangements of the terms we have that, up to the fourth order:

(4.5.20) $$16\pi^2 \langle T_{\mu\nu}\rangle_{div} =$$

$$= -m^2 (\xi - \tfrac{1}{6})(R_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu}R) +$$

$$+ 9(\xi - \tfrac{1}{6}) [v_1] g_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} T +$$

$$+ m^2 A G_{\mu\nu} + \tfrac{1}{2} [C_1 - (\xi - \tfrac{1}{6})A] H^{(1)}_{\mu\nu}$$

$$- C_2 H^{(2)}_{\mu\nu} + \text{divergent components.}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R$.

This is the generic $\langle T_{\mu\nu}\rangle_{div}$ that must be substracted from $\langle T_{\mu\nu}\rangle$ (cfr. eq. (4.3.15)) if we want to perform a generic Hadamard renormalization. The particular case of a De-Witt--Schwinger renormalization is obtained with the choice:

(4.5.21) $$A = (\xi - \tfrac{1}{6})M \ ; \ C_1 = \tfrac{2}{3} C_2 = - \tfrac{M}{90} \quad ,$$

where $M = 2\gamma - \ln 2$.

Then we obtain:

(4.5.22) $$16\pi^2 \langle T_{\mu\nu}\rangle^{DS}_{div} =$$

$$= -m^2(\xi - \tfrac{1}{6})(R_{\mu\nu} - \tfrac{1}{4} R g_{\mu\nu}) +$$

$$+9(\xi - \tfrac{1}{6})[v_1] g_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} T +$$

$$+ (\xi - \tfrac{1}{6})(w^M_0 + m^2)G_{\mu\nu} - \tfrac{1}{2} M(\xi - \tfrac{1}{6})^2 \times$$

$$\times H^{(1)}_{\mu\nu} - \tfrac{M}{180} (H^{(1)}_{\mu\nu} - 3 H^{(2)}_{\mu\nu}) +$$

$$+ \text{divergent components.}$$

Thus the difference between two different Hadamard renormalized tensor (e.g. the difference between eqs. (4.5.20) and (4.5.22)) taking into account that the divergent component that comes from terms $2/\sigma$ and $v \ln \mu^2\sigma$ of eq. (4.5.5) are the same it results:

$$u_{\mu\nu} = am^2 G_{\mu\nu} + b H^{(1)}_{\mu\nu} + c H^{(2)}_{\mu\nu} \quad , \tag{4.5.23}$$

where a, b, c are arbitrary constant.

Therefore we have actually find the local, c-number, divergentless tensor $u_{\mu\nu}$ of paragraph 4.4. It is, in fact, divergentless because $\langle T_{\mu\nu}\rangle_{div}$ of eq. (4.5.20) (and also $\langle T_{\mu\nu}\rangle^{DS}_{div}$ of eq. (4.5.21)) is divergentless because it is obtained through eq. (4.3.13) from a symmetric kernel $G_1(x,x')$.

Now if we move $u_{\mu\nu}$ from the r.h.s. of Einstein Field Equation (4.3.11) to the l.h.s., constant a,b,c must be added to the coupling constants G, α, β, thus they remain undeterminated being only then final coupling constants the ones that must be determinated by a set of experiments. Therefore all the Hadamard renormalizations yield the same physical result and we obtain the same conclusion as in paragraph 4.3 with the G_1^{DS} Green function.

In the massive case that we have studied we have taken the scale $\mu = m$, thus we can ask what the scale would be in the massless case. Of course in this case the scale remains arbitrary but there is no problem with a change of scale $\mu \to \mu'$ either. In fact the change in the kernel is given by:

$$G_1' = G - 2\,\frac{\Delta^{1/2}}{8\pi^2}\, v \ln\left(\frac{\mu'}{\mu}\right) \quad . \tag{4.5.24}$$

We can deduce the change in $<T_{\mu\nu}>$ using eq. (4.5.3) and compute the difference $G_1' - G_1$. The result is:

$$16\pi^2 <T_{\mu\nu}>' = 16\pi^2 <T_{\mu\nu}> + \left[\frac{1}{180}\left(3H^{(2)}_{\mu\nu} - H^{(1)}_{\mu\nu}\right) - \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 H^{(2)}_{\mu\nu} + \left(\xi - \frac{1}{6}\right) m^2 G_{\mu\nu}\right] \ln \frac{\mu'}{\mu} \quad . \tag{4.5.25}$$

Again the ambiguity is proportional to $G_{\mu\nu}$, $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$ and it is absorbed by the coupling coefficients. So if we added to axioms of paragraph 4.4 the following one: (5) "The renormalization must be an Hadamard one", the theory turns out to be renormalizable because $u_{\mu\nu}$ is given by eq. (4.5.22).

Finally we may verify that the renormalized $<T_{\mu\nu}>_{ren}$ computed using Hadamard renormalization satisfies axioms (1) to (4) of paragraph 4.4. From eq. (4.3.15) we know that $<T_{\mu\nu}>_{ren}$ is defined using eq. (4.3.18) by a kernel:

$$G_{1\ ren} = G_1 - G_{1\ div} \quad , \tag{4.5.26}$$

where G_1 is the primitive and divergent kernel, $G_{1\ div}$ the Hadamard kernel eq. (4.5.5) up to the four order, and $G_{1\ ren}$ the renormalized kernel. Then:

- It clearly results that $G_{1\ ren}$ satisfies axiom (2) because the two pieces G_1 and $G_{1\ div}$ are both causals G_1 because is a computed from $\phi(x)$ using eq. (3.4.23) and $\phi(x)$ propagates causally and because $G_{1\ div}$ is a pure local object when $x \to x'$.

- Axiom (3) can be proved using a similar proceeding than the one used in eqs. (4.4.3) to (4.4.8). Let us compute:

(4.5.27)
$$\langle\pi_+|T_{\mu\nu}|\pi_-\rangle_{ren} =$$
$$= \frac{1}{2}\langle\psi|T_{\mu\nu}|\psi\rangle_{ren} - \frac{1}{2}\langle\phi|T_{\mu\nu}|\phi\rangle_{ren} =$$
$$= \frac{1}{2}\lim_{x\to x'} D_{\mu\nu}(G_{\psi ren} - G_{\phi ren}) =$$
$$= \frac{1}{2}\lim_{x\to x'} D_{\mu\nu}(G_{\psi} - G_{\phi}) =$$
$$= \frac{1}{2}\langle\psi|T_{\mu\nu}|\psi\rangle - \frac{1}{2}\langle\phi|T_{\mu\nu}|\phi\rangle =$$
$$= \langle\pi_+|T_{\mu\nu}|\pi_-\rangle \qquad \text{q.e.d.}$$

Thus we have obtained the standard results for the off-diagonal elements simply using the definition (4.5.26).

- In Minkowski space G_1 becomes Δ_1 (cfr. eq. (4.5.17)) thus we obtain the standard results of Minkowski space-time and axiom (4) is satisfied.

- Finally both terms of the r.h.s. of eq. (4.5.26) yields divergentless tensor, because both satisfy the field equations and they are symmetric. Thus our understanding of the renormalization of $T_{\mu\nu}$ is complete.

5. Vacuum Definition and Stress Tensor Renormalization in Robertson Walker Universe

5.1 Introduction

From the preceeding paragraph we can see that nowadays we have a satisfactory and acceptable renormalized version of the stress-tensor. Thus problem 1 of paragraph 2.4 is completely solved. On the other hand we only have a very limitated set of reliable vacua, the trivial ones (cf. paragraph 3.3) that appears when we have a Killing vector field (or at least a field that behaves like a Killing vector on a Cauchy surface).

For problem 2 we only have a beginning of a general solutions. Anyhow almost all the book of Birrell and Davies, that is a sample of the state of the art up to 1982, is based in these two ingredients, and in fact a lot of things could be done . But it seems necessary to try to enlarge the set of possible vacua. We shall see that we can only do it based on the satisfactory solution of problem 1.

In this paragraph we shall study the problem in Robertson-Walker universes and in its comoving frame, because it is the most important model for cosmological evolution, and we shall take into account two features of the renormalized stress tensor.

1 - Ultraviolet renormalization eliminates the divergences that appear when $k \to \infty$ by the susstraction (4.3.14). Thus we are forced to state that the divergent parts of $G_F(x,x')$, or $G_1(x,x')$, that is defined by the vacuum, must be the same that

the one of $G_F^{DS}(x,x')$ or $G_1^{DS}(x,x')$, if we want a finite diference.[19] A stronger statement would be that the local structure of $G_F(x,x')$ or $G_1(x,x')$, i.e. its analitical terms, would be equal to $G_F^{DS}(x,x')$ or $G_1^{DS}(x,x')$, that as we know are the most accurate local copy of $\Delta_F(x,x')$) or $\Delta_1(x,x')$ of flat space-time[20]. This would be a reasonable interpretation of a "Quantum Equivalence Principle"; the local behaviour of the $G_1(x,x')$ defined by the vacuum must be the same that the one of $G_1^{DS}(x,x')$, the best curved space-time version of flat space $\Delta_1(x,x')$. Thus we have a weak and a strong criterium that we *can impose to the vacuum.*

2 - But there is another, less studied feature. If we want to define a vacuum at a time t_0 (i.e. at a Cauchy surface Σ_{t_0}; $t = t_0$ of Robertson-Walker universe) and in coordinates adapted to the comoving frame of reference and we know how to renormalize the stress-tensor for every quantum states, we can compute the energy vacuum expectation value for every candidate vacuum $|0_{t_0}>$ (from a set of vacuum related with the comoving frame):

$$(5.1.1) \qquad <0_{t_0}|H|0_{t_0}> = \int_\Sigma <0_{t_0}|T^0{}_0|0_{t_0}>_{ren}\, d\sigma$$

Thus we can use the standard notion of vacuum as the state that minimizes the energy. Only in this way we interpret other states as linear combination of many particle states and we construct a reasonable Fock space and we reach a logical decom-

(19) In this case all terms that do no yield to divergencies can change.

(20) In this case only the non analytical terms can change.

position of the energy momentum tensor in a Casimir term and a term due to the particle creation (cfr. paragraph 5.7).

We shall see that these two criteria are not compatible in general, and this incompatibility originates the well known vacuum ambiguity in curved space-time.

We shall find a method to bound the degree of ambiguity and to single out the good vacua.

5.2 ENERGY MINIMIZATION

Let us go back to paragraph 3.5 and work in a spatially flat Robertson-Walker universe with metric (3.5.1), with a few mirror changes we could also consider a non-flat spatial geometry but as this change has not a physical relevance we studied only the flat case. We shall work in the inertial comoving reference fluid only. The space-time paths of this reference fluid are the lines x,y,z = const, t = variable. This fluid is irrotational and geodesic and coordinates t, x, y, z are completely adapted, the vector u^μ is giving by eq. (3.1.24) but as the spatial metric is:

$$\gamma_{\mu\nu} = a^2(t)\, d_{\mu\nu} \qquad , \tag{5.2.1}$$

with $d_{\mu\nu} = 0$ if μ or $\nu = 0$ and $d_{\mu\nu} = \delta_{\mu\nu}$ if $\mu,\nu = 1,2,3$ we have:

$$\tilde{K}_{\mu\nu} = 2a\dot{a}\, d_{\mu\nu} \neq 0 \qquad , \tag{5.2.2}$$

thus , the fluid defines a Killing vector field only if $\dot{a} = 0$ i.e. in the static case, in this case only we shall have a trivial vacuum (like in the in and out case of the second example of paragraph 3.3). Now from eqs. (3.3.1), (3.1.24) and (5.2.2) we can obtain the field equation in this universe, make a variable separation as in eq. (3.5.2) and reach to eq. (3.5.3) for the time factor f(t).

The classical energy-momentum tensor is:

$$T_{\mu\nu} = \underset{\mu \leftrightarrow \nu}{\mathrm{sym}} \, [(1-2\xi)\, \partial_\mu\phi\partial_\nu\phi + (2\xi - \tfrac{1}{2}) g_{\mu\nu} g^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi - 2\xi\phi\nabla_\mu\partial_\nu\phi - 2\xi\, g_{\mu\nu}\phi\Delta\phi - \xi\, G_{\mu\nu}\phi^2 + \tfrac{1}{2} m^2 g_{\mu\nu}\phi^2] \tag{5.2.3}$$

where $G_{\mu\nu}$ is the Einstein tensor. Its T_{00} component is:

$$T_{00} = \frac{1}{2}(\partial_t\phi)^2 + 6\xi H\phi\partial_t\phi + \frac{1}{2a^2}\sum_{j=1}^{3}(\partial_j\phi)^2 + \frac{1}{2} m^2\phi^2 + 3\xi H^2\phi^2 - \frac{2\xi}{a^2}\sum_{j=1}^{3}\partial_j(\phi\partial_j\phi) \quad , \tag{5.2.4}$$

where $j = 1,2,3$ and $H = \dot{a}/a$ is the Hubble coefficient.

Now we can use eq. (5.1.1) to define the vacuum espectation value of the energy and try to minimize it at an arbitrary time. Precisely, using spacial simmetry of the Robertson-Walker metric we guess that we can define a vacuum at every time i.e. at every Cauchy surface t = const. The vacuum would be the quantum state that minimizes the energy at that time. But first we must realize that we can minimize either the classical energy or the renormalized energy with the same result. In fact, from paragraph 4 we know that we renormalize substracting a geometrical local quantity that it is independent of the quantum state of the system.

Then let us write the field equation solution like eqs. (3.5.2) and (3.5.4) with an arbitrary $W_k(t) = \Lambda_k(t)$ to obtain an arbitrary orthogonal basis. The Cauchy data at time t would then be:

$$(5.2.5) \qquad u_k = (2\pi a)^{-3/2} \exp(-i\overline{k}.\overline{x}) \times \times (2W_k)^{-1/2} \exp(-i \int_0^t W_k \, dt)$$

$$(5.2.6) \qquad \dot{u}_k = -(2\pi a)^{-3/2} \exp(-i\overline{k}.\overline{x}) \times \times (2W_k)^{-1/2} \exp(- i \int_0^t W_t \, dt) \times \times [\tfrac{3}{2} H + \tfrac{1}{2} \frac{\dot{W}_k}{W_k} + i W_k] \quad ,$$

where W_k and $\dot{W}_k$ are arbitrary. The vacuum expectation value for a vacuum $|0>$ that corresponds to the negative and positive frequency solutions from these Cauchy data and for a volume $(2\pi a)^3$ is:

$$(5.2.7) \qquad <0|H|0> = \int_{\Sigma;(2\pi a)^3} <0|T^0{}_0|0> d\sigma = = \frac{1}{2} \sum_k \dot{u}_k\dot{u}_k^* + \frac{1}{a^2} \sum_{j=1}^{3} \partial_j u_k \partial_j u_k^* + + m^2 u_k u_k^* + \xi \{6H(u_k\dot{u}_k^* + u_k^*\dot{u}_k) + 6H^2 u_k u_k^* - \frac{2}{a^2} \sum_{j=1}^{3} [\partial_j(u_k\partial_j u_k^*) + \partial_j(\overset{*}{u}_k \partial_j u_k)\} \quad .$$

Now we can write this quantity as a function of W_k and $\dot{W}_k$ using eqs. (5.2.5) and (5.2.6) and minimize (5.2.7) taking the W_k and $\dot{W}_k$ derivations to be zero, then we obtain:

$$(5.2.8) \qquad W_k^{ME} = + [w_k^2 - 6\xi H^2(6\xi - 1)]^{1/2} \quad ,$$

where $w_k = (m^2 + \frac{k^2}{a})^{1/2}$ and:

$$(5.2.9) \qquad \dot{W}_k^{ME} = 3H(4\xi - 1)W_k \quad ^{(21)} \quad ,$$

and the energy vacuum expectation value at this minimum reads:

$$(5.2.10) \qquad \langle 0|H|0\rangle = \frac{1}{2}\sum_k W_k^{ME} \quad .$$

We can see immediately that the method works in flat space-time, because for $H = 0$ we have $W_k = w_k$ and $\dot{W}_k = 0$ and the positive and negative frequency solutions turn out to be the usual ones

$$\exp \pm i(\underline{k}.\underline{x} + w_k t)$$

We can also compute the energy for a many particle state $|n\rangle$ such that:

$$(5.2.11) \qquad n_k = \langle n|N_k|n\rangle = \langle n|a_k^+ a_k|n\rangle \quad ,$$

in this case we obtain:

$$(5.2.12) \qquad \langle n|H|n\rangle = \sum_k n_k W_k^{ME} + \frac{1}{2}\sum_k W_k^{ME} \quad .$$

---

(21) ME = Minimal Energy.

The second term of the r.h.s. is divergent equal to (5.2.10), and independent of the quantum state e.g. we can see that *it is independent* of the n_k. If we renormalize this term substracting a local quantity we shall find a finite and local component of the renormalize energy, known as the "Casimir" component. The *first term is a finite component* for a quantum state with a finite number of particles and can be considered the energy of these particles. In the important cases $\xi = 0$ and $\xi = 1/6$ it becomes:

(5.2.13) $$\sum_k n_k w_k \quad .$$

Thus, we can see that the Energy Minimization coincid with the criterion commonly known as Hamiltonian diagonalizatio that states that the renormalized energy must have the form (5.2.13) and so it can be interpretated as the energy of the particles of the state $|n\rangle$.

Hamiltonian Diagonalization is strongly criticied in the literature. In fact, there are possible different definit of Hamiltonians (Fulling (1979)), the created particles could be infinite (Castagnino, Verbeure & Weder (1975)) and the uncertainty relation prevents an instantaneous definition of the energy (Parker (1969)). But we shall only use the Energy deduced from the Stress-tensor integrated on the Cauchy surface where we want to minimize this energy and define a vacuum, therefore there will be no Hamiltonian definition ambiguity. Also, as we shall see, we shall only consider the vacua as reliable when there is a finite number of particles created among these

Finally, the energy is, of course, determinated up to some uncertainty because the uncertainty principle states that:

$$\Delta E \Delta t \sim 1 \quad . \tag{5.2.14}$$

We can make the following heuristic reasoning. Δt cannot be greater than $R^{-1/2}$, the radious of curvature, because only in a neighbourhood smaller than $R^{-1/2}$ space-time can be considered more or less flat (more precisely Δt must be smaller than $(R_{max})^{-1/2} = (R_{\mu\nu\lambda\rho\ max})^{-1/2}$ being $R_{\mu\nu\lambda\rho\ max}$ the maximum component of $R_{\mu\nu\lambda\rho}$ at the considered point). In fact, we can measure the energy in a more or less flat patch where we can neglect the energy of the created particles. In our units $\Delta\omega \sim \Delta E$, thus we need that $\Delta\omega < \omega$ to have a reliable definition of positive and negative frequency solution of frequency ω, thus we have:

$$\omega > R^{1/2} \tag{5.2.15}$$

Thus Vacuum Minimization gives reliable high frequency definitions and we immediately see that the range of good definition is complete in flat space because $R = 0$, or in asyntotic regions, where $R \to 0$. In fact, we can only use this principle for all frequencies if $R = 0$ but this will be the case, not only in flat space-time but in some special kind of singularities, as we shall see.

Therefore all the objections to the Vacuum Minimization method have been analized.

5.3 THE STRONG VACUUM

Now let us consider the criterion 1 of paragraph 5.1 in its stronger version: the local analytical structure of $G_1(x,x')$ must be the same structure of kernel $G_1^{DS}(x,x')$ or the adiabatic Green function. From eq. (3.5.8) we know that the W_k and the $\dot{W}_k$ that correspond to that the kernel are:

$$W_k^{DS} = w_k[1 - \frac{1}{2}(\xi - \frac{1}{6})\frac{R}{w_k^2} + \frac{1}{6}(H^2 + \frac{R}{6}) \cdot \frac{m^2}{w_k^4} + \frac{5}{8}\frac{H^2m^4}{w_k^6} + \dots] \tag{5.3.1}$$

$$\dot{W}_k^{DS} = -Hw_k + [m^2H - \frac{1}{2}(\xi - \frac{1}{6})\dot{R} - \frac{1}{2}(\xi - \frac{1}{6})RH]\frac{1}{w_k} + [\frac{1}{6}(2H\dot{H} + \frac{\dot{R}}{6})m^2 + \frac{1}{2}(\xi - \frac{1}{6})Rm^2H + \frac{1}{6}(H^2 + \frac{R}{6})]\frac{1}{w_k^3} + [\frac{5}{4}H\dot{H}m^4 - \frac{1}{2}(H^2 + \frac{R}{6})m^2H + \frac{25}{8}H^3m^4]\frac{1}{w_k^5} + \dots \text{ (*)} \tag{5.3.2}$$

Now, Cauchy data from eqs. (5.2.8), (5.2.9) are different than the Cauchy data from eqs. (5.3.1), (5.3.2). If we consider that criteria 1 and 2 of paragraph 5.1 are both physical reasonable criteria we must arrive to the conclusion that we have a realiable vacuum only when both criteria coincide, we shall call such a vacuum on <u>Strong Vacuum</u>.

Let us clarified the issue with some figures, that are

(*) DS = De Witt, Schwinger.

merely qualitative. In figure 2 the behaviour of both W_k^{DS} and W_k^{ME} are shown in the important cases $\xi = 0$ and $\xi = 1/6$. We see that both curves $\to k/a$ when $k \to \infty$ and near $k = 0$ W_k^{DS} is not defined, because we do not know if De Witt-Schwinger expansion converges for low k. The shaded area reflects the ambiguity in the vacuum definition because in a strong vacuum $W_k^{DS} = W_k^{ME}$ and the shaded area must desappear.

In figure 3 we have plotted eq. (5.2.9)

$$\dot{W}_k - 3H\,(4\xi-1)\,W_k = 0 \qquad (5.3.3)$$

that must be fulfilled to obtain a minimal vacuum in the case $\xi = 0$ for W_k^{DS} and $\dot{W}_k^{DS}$. It can be seen that these functions do not satisfy eq. (5.3.3) in general, that $\dot{W}_k^{DS} + 3HW_k^{DS} \to 2Hk/a$ when $k \to \infty$, that also here, the curve is undefined for $k \to 0$ and that we have also a shaded area.

In figure 4 we have the same picture but in the case $\xi = 1/6$ where the only but important difference is that, condition (5.5.3) that it is now $\dot{W}_k^{DS} + HW_k^{DS}$, vanish when $k \to \infty$. Therefore the coupling $\xi = 1/6$ shows a better behaviour than coupling $\xi = 0$ because in fact, we have that at least both curves converge in figure 3 and 4 when $k \to \infty$, giving a reasonable vacuum with the correct ultraviolet behaviour for the energies. Going back to the reasoning of the last part of paragraph 3.5: "the vacuum must be well defined for high energies and must coincide with the adiabatic vacuum", we conclude that most likelly, coupling $\xi = 1/6$ is the good coupling. There are other reasons that justifie this statement, therefore we shall only study the case $\xi = 1/6$ from now on.

Anyhow we shall have an strong vacuum only in the cases where the shaded areas vanish (or it is only reduced t non analytical terms that of course we can add to W^{DS}) e.g.

1 - When $a = $ const., then $H = 0$, $R = 0$, etc. and we have the static or Minkowski vacuum, a trivial vacuum, in fact the most trivial one.

2 - But we could have $a = $ const only between t_1 and t_2, then we have a Killing vector there and a trivial vacuum, the only one that we can consider in this paragraph, this trivial vacuum is also a strong vacuum.

Thus, probably all trivial vacua are strong vacua. Is the set of strong vacua bigger than the set of trivial vacua ? In fact it is, because we can add the following examples.

3 - When $t \to \infty$ and $H \to 0$, $R \to 0, \ldots$, etc. This happens in an adiabatic flat region, it is the adiabatic out vacuum for the far future, that appears in most universe evolutions.

4 - We can imagine a universe such that at a time $t = t_0$ we have $H = 0$, $R = 0$, ..., etc, we have a strong vacuum there (e.g. in evolution $a = t^\alpha$, at $t = 0$ for $\alpha > 1$).

In fact in both examples 3 and 4 we have a local Killing field i.e. the field u^μ tangent to the comoving fluid behaves as a Killing field at the far future in example 3 and at time t_0 in example 4. Thus we have a strong vacuum every time we have a global Killing field or when the field u^μ behaves like a Killing field at a Cauchy surface. We must remark that in all of these cases as $R = 0$ eq. (5.2.15) allows to measure all frequencies.

But, are there strong vacua unrelated with Killing vectors ? In fact there is a very important example:

5 - When $m = 0$, $\xi = \frac{1}{6}$ for every evolution and for all times we have a strong vacuum because shaded areas vanish. This is the conformal vacuum that we shall study in the next paragraph.

Finally we must remark that there are other reasons to believe that the strong vacuum is a reasonable concept. In fact, it seems that the only method to define a vacuum in curved space-time is to find the quantum state that has as many properties of flat space-time vacuum as possible. Calzetta & Castagnino (1983,1984) analized a set of properties of flat space-time trying to generalize them to curved space-time to single a good vacuum, them Castagnino and Mazzitelli (1984) find two properties that can be generalized to curved space-time. The first one is based in the Wick trick; we pass from a Lorentz space to an euclidean space making the change $t \to it$. In euclidean space these is a unique Green Function that must correspond to a unique Feynman propagator in curved space-time. The second one is based in the addition of a term $i\varepsilon$ to the squared mass to single out the right Feynman propagator.

It turns out that the strong vacua are endowed with these two properties (the proof is done in Robertsen-Walker universe and in Bianchi Type I universe) this fact improves the base of the strong vacuum concept.

Also as the Cauchy data of a strong vacuum coincides with the ones of the adiabatic vacuum all the analytical terms are the same in all the strong vacuum that may exist in a Robertson--Walker universe. Thus between two strong vacua the $|\beta|^2$ of the created particles is always non analytical, like the thermal spectra of the examples of paragraph 3.3. Thus the total number of created particles is convergent and also the created energy etc, i.e. criterion 1 of paragraph 5.1 fixes a common analytical part for both vacua while criterion 2 defines the non analytical part that produces the particle creation. In this way the divergent particles creation that aflicts Hamiltonian Diagonalization desappears completely.

5.4 THE CONFORMAL VACUUM

We shall call a conformal transformation to a map from one Riemannian manifold of metric $g_{\mu\nu}(x)$ to another Riemannia manifold of metric $\bar{g}_{\mu\nu}(x)$ such that:

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \quad , \tag{5.4.1}$$

where $\Omega(x)$ is a real, non-vanishing, continuous function. From such a transformation we can find that:

$$\Gamma^{\rho}_{\mu\nu} \to \bar{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \Omega^{-1} \left(\delta^{\rho}_{\mu}\Omega_{;\nu} + \delta^{\rho}_{\nu}\,\Omega_{;\mu} - g_{\mu\nu}g^{\rho\alpha}\Omega_{\alpha}\right) \quad , \tag{5.4.2}$$

$$R^{\nu}{}_{\mu} \to \bar{R}^{\nu}{}_{\mu} = \Omega^{-2}R^{\nu}{}_{\mu} - 2\Omega^{-1}(\Omega^{-1})_{;\mu\rho}g^{\rho\nu} + \frac{1}{2}\,\Omega^{-1}(\Omega^{2})_{;\rho\sigma}\, g^{\rho\sigma}\delta^{\nu}{}_{\mu} \quad , \tag{5.4.3}$$

$$R \to \bar{R} = \Omega^{-2}R + 6\Omega^{-3}\Omega_{,\mu\nu}g^{\mu\nu} + 6\Omega^{-4}\Omega_{;\mu}\Omega_{;\nu}\, g^{\mu\nu} \quad , \tag{5.4.4}$$

If we define a conformal change of the field like:

(5.4.5) $$\phi(x) \to \bar{\phi}(x) = \Omega(x)\phi(x) \quad ,$$

the massless field equation with conformal coupling $\xi = 1/6$ turns out to be invariant because:

(5.4.6) $$(\Box + \frac{1}{6} R)\phi \to (\bar{\Box} + \frac{1}{6} \bar{R})\ \bar{\phi} =$$

$$= \Omega^{-3}\ (\Box + \frac{1}{6} R)\phi \quad .$$

Examples of conformally flat space-time are all two dimensional space-time and the spatially flat Robertson-Walker universe because a conformal transformation exists that changes these space-time in Minkowsky space-time. In fact their metric may always be cast in the form:

(5.5.7) $$g_{\mu\nu}(x) = \Omega^2(x)\ \eta_{\mu\nu} \quad ,$$

as it can be proved in the latest case generalizing eqs. (3.3.26), (3.3.27) and (3.3.28) to the four dimensional case.

Thus if we take the conformal invariant scalar field equation ($m = 0$; $\xi = 1/6$):

(5.5.8) $$(\Box + \frac{1}{6} R)\phi = 0 \quad ,$$

under the conformal transformation:

(5.5.9) $$g_{\mu\nu} \to \eta_{\mu\nu} = \Omega^{-2}\ g_{\mu\nu} \quad ,$$

we obtain:

(5.5.10) $$\bar{\Box}\ \bar{\phi} \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu(\Omega^{-1}\phi) = 0 \quad ,$$

as $\overline{R} = 0$ in Minkowski space. Now in Minkowski space we can use the familiar positive and negative frequency solution:

$$\overline{u}_{\underline{k}}(x) = [2\omega(2\pi)^3]^{-1/2} e^{-ik_\mu x^\mu} \quad , \tag{5.5.11}$$

where $k^0 = \omega$. These modes satisfy eq. (3.3.10) with respect to the time like Killing vector ∂_{x^0} of the Minkowski space:

$$L_{x^0} \overline{u}_{\underline{k}}(x) = -i \, \overline{u}_{\underline{k}}(x) \quad . \tag{5.5.12}$$

Now as $\tilde{\phi} = \Omega\overline{\phi}$, we have a mode decomposition of the Robertson--Walker field ϕ as:

$$\phi(x) = \Omega(t) \sum_{\underline{k}} [a_{\underline{k}}\overline{u}_{\underline{k}}(x) + a^+_{\underline{k}}\overline{u}^*_{\underline{k}}(x)] \quad , \tag{5.5.13}$$

and it is natural to define as positive and negativa frequency basis, of the Robertson-Walker theory to $\{\Omega\overline{u}_{\underline{k}}(x)\} \cup \{\Omega u_{\underline{k}}(x)\}$, the vacuum associated with this base is the <u>conformal vacuum</u> $|0\rangle$ defined by $a_k|0\rangle = 0$. This is so because in this case, both the geometry and the field equation are obtain by conformal transformation from the Minkowski case.

Precisely: the metric of Robertson-Walker universe can be written as:

$$dS^2 = a^2(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{5.5.14}$$

where η is the "conformal time" eq. (3.3.27), thus the time factor of the positive and negative frequency solution is

$$a \; e^{\omega \int \frac{dt}{a}} \quad . \tag{5.5.15}$$

If we put $m = 0$, $\xi = 1/6$ in eqs. (5.2.8), (5.2.9) or (5.3.1), (5.3.2) we obtain:

$$W_k = \omega_k = \frac{k}{a} \quad , \tag{5.5.16}$$

$$\dot{W}_k = -H\omega_k = -\frac{\dot{a}}{a^2}\,k \quad . \tag{5.5.17}$$

and we can verify that k/a is a solutions of eq. (3.5.5) in this case and that f_k is precisely proportional to (5.5.16). Thus in this case criteria 1 and 2 of paragraph 5.1 coincide and give a strong vacuum that it is the same one that we obtain using a conformal transformation i.e. the conformal vacuum.

This coincidence of three reasonable criteria make the conformal vacuum a universally accepted one. It is also the most important example of non trivial strong vacuum.

5.6 G_1^{DS} AND G_1^{SD} ARE THE EQUAL IN ROBERTSON-WALKER UNIVERSE. THE MINIMAL VACUUM

To renormalize the stress-tensor in Robertson-Walker universe and to introduce the concept of Minimal Vacuum it is necessary to know G_1^{DS} in this universe. Then let us demonstrate that G_1^{Ad} constructed with adiabatic solutions, is G_1^{DS} as we promised in paragraph 3.6. Thus let us return to paragraph 3.5 and let us try to solve eq. (3.5.5) that we can write as:

$$\frac{1}{2}\frac{\ddot{W}_k}{W_k} - \frac{3}{4}\left(\frac{\dot{W}_k}{W_k}\right)^2 + W_k^2 = \omega_k^2 - \left(\xi - \frac{1}{4}\right)R + \frac{3}{4}H^2 \tag{5.6.1}$$

not with an expansion like eq. (3.5.3) in inverse powers of ω_k

but in a expansion in the metric and its derivatives, like those of paragraph 3.6. The reason is simple: we need to know G_1^{DS} up to the fourth adiabatic order to renormalize i.e. up to the fourth order in the metric derivatives.

Thus we must know all the geometrical independents variable built from a. The second order ones are:

$$\alpha_1 = H^2 \quad ; \quad \alpha_2 = \frac{1}{6} R \quad , \tag{5.6.2}$$

the fourth order are:

$$\beta_1 = H^4 \quad , \quad \beta_2 = \frac{1}{36} R^2 \quad , \quad \beta_3 = \frac{1}{6} H^2 R \tag{5.6.3}$$

$$\beta_4 = \frac{1}{6} H\dot{R} \quad , \quad \beta_5 = \frac{1}{6} \ddot{R} \quad .$$

All other fourth order variables can be written as a function of these β , e.g.:

$$\beta_6 = H^2\dot{H} = - 2\beta_1 + \beta_3 \quad , \tag{5.6.4}$$

$$\beta_7 = H\ddot{H} = 8\beta_1 - 4\beta_3 + \beta_4 \quad ,$$

$$\beta_8 = \dddot{H} = - 48\beta_1 - 4\beta_2 + 32\beta_3 - 4\beta_4 + \beta_5 \quad ,$$

$$\beta_9 = \dot{H} \frac{R}{6} = \beta_2 - 2\beta_3 \quad ,$$

$$\beta_{10} = \dot{H}^2 = 4\beta_1 + \beta_2 - 4\beta_3 \quad .$$

Therefore our expansion will be:

$$W_k(t) = \omega_k(t) \; [1 + \sum_{n=1}^{2} A_n\alpha_n + \sum_{n=1}^{5} B_n\beta_n + \ldots] \tag{5.6.5}$$

If we substitute eq. (5.6.5) in eq. (5.6.1) and solve this equation for each adiabatic order independently we find:

(5.6.6) $$A_1 = -\frac{1}{4}\frac{m^2}{\omega_k^2} + \frac{5}{8}\frac{m^4}{\omega_k^6} \quad ,$$

$$A_2 = \frac{1}{2}\frac{(6\xi-1)}{\omega_k^2} - \frac{1}{4}\frac{m^2}{\omega_k^4} \quad , \ \dots \ \text{etc.}$$

The final result is:

(5.6.7) $$W_k(t) = \omega_k(t)\,\{1 + \frac{1}{\omega_k^2}\,[\frac{1}{2}\,(6\xi-1)\alpha_2] \; -$$

$$- \frac{1}{\omega_k^4}\,[\frac{1}{4}\,m^2(\alpha_1+\alpha_2) + \frac{1}{8}\,(6\xi-1)^2\,\beta_2 + \frac{1}{4}\,(6\xi-1)\,\times$$

$$\times\,(\beta_2+\beta_3 + \frac{5}{2}\,\beta_4 + \frac{1}{2}\,\beta_5)] \; +$$

$$+ \frac{m^2}{\omega_k^4}\,[\frac{5}{8}\,m^2\alpha_1 + \frac{1}{4}\,\{9\xi\beta_2 - (2-39\xi)\beta_3 \, +$$

$$+ (15\xi + \frac{1}{4})\beta_4 + \frac{1}{4}\,\beta_5\}\,] \; -$$

$$- \frac{1}{4}\frac{m^4}{\omega_k^8}\,\{\frac{19}{8}\,(\beta_1+\beta_2) + \frac{1}{2}\,(75\xi+39)\beta_3 \, +$$

$$+ \frac{7}{2}\,\beta_4 + \frac{221}{32}\frac{m^6}{\omega_k^{10}}\,(\beta_1+\beta_3) - \frac{1105}{123}\frac{m^8}{\omega_k^{12}}\,\beta_1] + \dots$$

We shall also need the expansion of $\dot{W}_k$:

(5.6.8) $$\frac{\dot{W}_k}{W_k} = -H + \frac{1}{\omega_k^2}\,[m^2H + (6\xi-1)(H\alpha_2 + \frac{1}{2}\,\dot{\alpha}_2)] \; -$$

$$- \frac{m^2}{\omega_k^4}\,[(6\xi + \frac{1}{2})H\alpha_2 + \frac{1}{4}\,\dot{\alpha}_2] + \frac{m^2}{\omega_k^6}\,\frac{9}{4}\,H(\alpha_1+\alpha_2) \; -$$

$$- \frac{15}{4}\frac{m^6}{\omega_k^8}\,H\,\alpha_1 + \dots\dots \quad .$$

Now we can find $G_1^{Ad}(x,x')$ from eqs. (3.4.33), (3.5.2), and (5.6.7), it will be sufficient for our purpose to take $t = t'$:

$$(5.6.9) \qquad G_1^{Ad}(x,x')\Big|_{t=t'} = \frac{1}{(2\pi a)^3} \int d^3\bar{k}\, e^{i\bar{k}.(\bar{x}-\bar{x}')} \times$$

$$\times \sum_{n=0}^{6} A_n \omega_k^{-(2n+1)} \qquad ,$$

with $\qquad A_0 = 1 \quad , \quad A_1 = -\frac{1}{2}(6\xi-1)\alpha_2 \quad ,$

$$A_2 = \frac{1}{4} m^2(\alpha_1+\alpha_2) + \frac{3}{8}(6\xi-1)^2\beta_2 +$$

$$+ \frac{1}{4}(6\xi-1)(\beta_2+\beta_3+\frac{5}{2}\beta_4+\frac{1}{2}\beta_5) \quad ,$$

$$A_3 = -\frac{5}{8} m^4\alpha_1 - \frac{1}{4} m^2 \left[(15\xi-1)\beta_2 + (45\xi-3)\beta_3 +\right.$$

$$\left.+ (15\xi+\frac{1}{4})\beta_4 + \frac{1}{4}\beta_5\right] \quad ,$$

$$A_4 = \frac{1}{4} m^2 \left[\frac{21}{8}(\beta_1+\beta_2) + \frac{1}{2}(105\xi+35)\beta_3 + \frac{7}{2}\beta_4\right] \quad ,$$

$$A_5 = -\frac{231}{32} m^6 (\beta_1+\beta_3) \quad ; \quad A_6 = \frac{1155}{128} m^3\beta_1 \quad .$$

It is easy to make all the Fourier transformation of this equation, from the equation of the flat Δ_1:

$$(5.6.10) \qquad \Delta_1(\bar{\sigma}) = \frac{1}{(2\pi a)^3} \int d^3\vec{k}\; \frac{e^{i\vec{k}.\vec{r}}}{\omega_k} \quad ,$$

where $\bar{\sigma} = \frac{1}{2} a^2 r^2$. Δ_1 is an Hadamard solution in flat space-

-time and its expansion is:

$$(5.6.11) \qquad 8\pi^2\Delta_1(\bar\sigma) = \frac{2}{\bar\sigma} + (m^2 + \frac{m^4}{4}\bar\sigma + \dots)\ln\bar\sigma + \dot\omega_0^M + 0(\bar\sigma) ,$$

where $\dot\omega_0^M = m^2[\ln(m^2/2) + 2\gamma - 1]$. If we use now:

$$(5.6.12) \qquad \omega_k^{-(2n+1)} = \frac{2^n}{(2n-1)(2n-3)\times\dots\times 3\times 1}\left(-\frac{\partial}{\partial m^2}\right)^n\left(\frac{1}{\omega_k}\right) ,$$

we can write each one of the integrals of eq. (5.6.9) as derivativs of $\Delta_1(\bar\sigma)$ and obtain:

$$(5.6.13) \qquad 8\pi^2\, G_1^{Ad}(x,x')\Big|_{t=t'} =$$

$$= \frac{2}{\bar\sigma} + \ln\bar\sigma\,\{m^2 + (6\xi-1)\alpha_2 +$$

$$+ \bar\sigma\,[\frac{m^4}{4} + (6\xi-1)\frac{m^2}{2}\alpha_2 + \frac{1}{6}m^2(\alpha_1+\alpha_2) +$$

$$+ \frac{1}{4}(6\xi-1)^2\beta_2 + (\xi - \frac{1}{6})(\beta_2 + \beta_3 + \frac{5}{2}\beta_4$$

$$+ \frac{1}{2}\beta_5)] + 0(\bar\sigma^2)\} + \dot\omega_0^M + \frac{1}{3}\alpha_2 + (6\xi-1)(\ln\frac{m^2}{2} + 2\gamma)\alpha_2 +$$

$$+ \frac{1}{m^2}[\frac{1}{2}(6\xi-1)^2\beta_2 + (\xi - \frac{1}{5})(3\beta_4+\beta_5) +$$

$$+ \frac{1}{15}(\beta_1-\beta_3)] + 0(\bar\sigma) \quad .$$

Now we can write a generic Hadamard kernel in Robertson-Walker universe using all the formalism of paragraph 4.5 and eq.(3.6.9) for this universe, i.e.:

(5.6.14) $$\overline{\sigma} = -\tau^2 + \overline{\sigma} + \overline{\sigma}H\tau + \frac{1}{3}\,\overline{\sigma}(\frac{R}{6} - H^2)\tau^2 + \frac{1}{6}\,\overline{\sigma}^2H^2 + \ldots$$

Then we have:

(5.6.15) $$8\pi G_1^{Had}(x,x')\Big|_{t=t'} = \frac{2}{\overline{\sigma}} + \ln\overline{\sigma}\{m^2+(6\xi-1)\alpha_2+\overline{\sigma}[\frac{m^4}{4} + (6\xi-1)\frac{m^2}{2}\alpha_2 + \frac{1}{6}m^2(\alpha_1+\alpha_2) + \frac{1}{4}(6\xi-1)^2\beta_2 + (\xi-\frac{1}{6})(\beta_2+\beta_3+\frac{5}{2}\beta_4+\frac{1}{2}\beta_5)] + 0(\overline{\sigma}^2)\} + [\omega_0] + \frac{1}{3}\alpha_2 + 0(\overline{\sigma}) \quad .$$

If we compare eq. (5.6.13) with eq. (5.6.15) we can see that $G_1^{Ad}(x,x')$ is on Hadamard solution characterized by:

(5.6.16) $$[\omega_0]^{Ad} = \omega_0^M + (6\xi-1)\,(\ln\frac{m^2}{2} + 2\gamma)\alpha_2 + \frac{1}{m^2}\,[\frac{1}{2}\,(6\xi-1)^2\,\beta_2 + (\xi-\frac{1}{5})(3\beta_4+\beta_5) + \frac{1}{15}\,(\beta_1-\beta_3)]\;,$$

that is precisely eq. (4.5.17) with eq. (4.5.41) thus up to the order considered $[\omega_0^{Ad}] = [\omega_0^{DS}]$ and $G_1^{Ad}(x,x')$ and $G_1^{DS}(x,x')$ coincide when $t = t'$. Analogously we can make the computation of the time derivatives of the kernels, that defined the corresponding Cauchy data on the surface $t = t'$ and we shall find the same coincidente, thus $G_1^{Ad}(x,x') = G_1^{DS}(x,x')$ even if $t \neq t'$.

Alternatively we can use a theorem by Davies, Fulling, Christensen & Wald (1977) that assure that every propagator that is an Hadamard structure on a neighbourhood of a Cauchy surface it is a the same Hadamard structure everywhere. The coincidence of $G_1^{Ad}(x,x')$ and $G_1^{DS}(x,x')$ if $t = t'$ both beeing Hadamard structures means that both kernels are equal in all space-time.

Now we can use $G_1^{Ad}(x,x')$ given by eq. (5.6.9) up to the fourth adiabatic order to renormalize.

We can now compute the divergent part of the adiabatic stress-tensor:

(5.6.17)
$$\langle T_{00}\rangle_{Ad}^{div} = \int \frac{d^3\vec{k}}{(2\pi a)^3} \frac{1}{2} \omega_k \cdot$$
$$\cdot \{ 1 + \frac{1}{\omega_k^2} [- \frac{1}{2}(6\xi-1)\alpha_1] +$$
$$+ \frac{1}{\omega_k^4} [- \frac{1}{2} (6\xi-1) m^2\alpha_1 + \frac{1}{8} (6\xi-1)^2(\beta_2-2\beta_3-2\beta_4)] +$$
$$+ O(\omega_k^{-6}) \} = \frac{1}{4\pi^2} \int_0^\infty dk \frac{k^3}{a^4} \times$$
$$\times \{1 + \frac{a^2}{k^2} [\frac{m^2}{2} - \frac{1}{2} (6\xi-1)\alpha_1] + \frac{a^4}{k^4} [- \frac{1}{8} m^4 -$$
$$- \frac{1}{4} (6\xi-1) m^2\alpha_1] + \frac{1}{8}(6\xi-1)^2(\beta_2-2\beta_3-2\beta_4) + O(k^{-6})\} ,$$

(5.6.18)
$$\langle T_{ij}\rangle_{Ad}^{div} = \delta_{ij} \int \frac{d^3\vec{k}}{(2\pi a)^3} \frac{1}{2} \omega_k \times$$
$$\times \{\frac{1}{3} + \frac{1}{\omega_k^2} [- \frac{m^2}{3} - \frac{1}{2} (6\xi-1)(\alpha_1 - \frac{2}{3} \alpha_2)] +$$
$$+ \frac{1}{\omega_k^4}[- \frac{1}{2} (6\xi-1)m^2(\alpha_1 - \frac{2}{3} \alpha_2) + \frac{1}{24} (6\xi-1)^2 \times$$
$$\times (\beta_2-2\beta_3+4\beta_4+2\beta_5)] + O(\omega_k^{-6}) \} .$$

Now we can define the <u>Minimal Vacuum</u>, it is the vacuum that minimizes the energy (criterion 2 of paragraph 5.1) and produces an energy momentum tensor with the divergencies of these last equations (criterion 1 of paragraph 5.1 in its weaker version).[22] It is in fact, the minimal condition that a vacuum must have, it must be a minimum of something (Hajicek (1976)), most likely the energy that the stress-tensor defines in its Cauchy surface and it must produce a renormalizable stress tensor by the ordinary renormalization method i.e. by the substraction of $<T_{\mu\nu}>_{AC}$ of eqs. (5.6.17), (5.6.18), otherwise it would be useless.

We shall see on important example of minimal vacuum in the next paragraph.

5.7 MINIMAL VACUUM IN SCALAR FIELD THEORIES WITH CONFORMAL COUPLING

Let us consider the scalar field theories with conformal coupling $\xi = 1/6$ in Robertson-Walker universe, from paragraph 5.3 we know that this coupling has a better high energy behavior than the minimal one $\xi = 0$, thus we shall consider that the former is the real physical coupling.

From eqs. (5.2.3), (3.4.13) and (3.5.2) we can compute the vacuum expectation value of the energy momentum tensor in a basis f_k, f_k^* that correspond to that vacuum:

(5.7.1) $$<T_{00}> = \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi a)^3} \{|f_k|^2 (\omega_k^2 + \frac{1}{4} H^2) + |\dot{f}_k|^2 - \frac{1}{2} H(\dot{f}_k f_k^* + \dot{f}_k^* f_k)\} ,$$

[22] I.e. We can choose arbitrary $a_3, a_4, \ldots$, and the non analytical terms.

$$(5.7.2)\qquad \langle T_{ij}\rangle = 3\,\delta_{ij}\,\{T_{00} + T\} \quad ,$$

where:

$$(5.7.3)\qquad T = -\,m^2 \int \frac{d^3k}{(2\pi a)^3}\,|f_k|^2 \quad .$$

We shall call $|0\rangle_\tau$ the vacuum state that minimizes the energy at time $t = \tau$ and let $f_k^{(\tau)}(t)$, $\dot{f}_k^{(\tau)}(t)$ be the time factor of the corresponding positive and negative frequency structions. We can find an easy expression for $\langle 0|T_{\mu\nu}|0\rangle_\tau$ writing $f_k^{(\tau)}(t)$ as:

$$(5.7.4)\qquad f_k^{(\tau)}(t) = \alpha_k(\tau,t)\,\frac{e^{i\int^t \omega_k(t')\,dt'}}{\sqrt{2\omega_k(t)}} + \beta_k(\tau,t)\,\frac{e^{-i\int^t \omega_k(t')dt'}}{\sqrt{2\omega_k(t)}} \quad ,$$

and its derivatives as:

$$(5.7.5)\qquad \dot{f}_k^{(\tau)}(t) = i\omega_k\left(\alpha_k(\tau,t)\,\frac{e^{i\int^t \omega_k\,dt'}}{\sqrt{2\omega_k}} - \beta_k(\tau,t)\,\frac{e^{-i\int^t \omega_k\,dt'}}{\sqrt{2\omega_k}}\right) + \frac{1}{2}\,H\,f_t^{(\tau)}(t) \quad ,$$

if we know $f_k^{(\tau)}$ and $f_{k'}^{(\tau)}$ we can compute α_k and β_k from system (5.7.4), (5.7.5). From the Cauchy data that minimize the energy at τ, eq. (5.2.8) and (5.2.9), we can obtain the values of α_1 and β_k at $t = \tau$:

$$(5.7.6) \qquad \alpha_k(\tau,\tau) = 1 \quad , \quad \beta_k(\tau,\tau) = 0 \quad .$$

Besides the normalization condition that the basis must satisfy to be orthonormal (i.e. eq. (3.2.5)) in this case is:

$$(5.7.7) \qquad \dot{f}_k^* f_k - f_k^* \dot{f}_k = i \quad ,$$

and yields:

$$(5.7.8) \qquad |\alpha_k(\tau,t)|^2 - |\beta_k(\tau,t)|^2 = 1 \quad .$$

Also from system (5.7.4), (5.7.5) and eq. (5.7.7) we can deduce that:

$$(5.7.9) \qquad \dot{\alpha}_k \; e^{i\int^t \omega_k \, dt'} + \dot{\beta}_k \; e^{-i\int^t \omega_k dt'} =$$

$$= \frac{m^2 H}{2\omega_k^2} (\alpha_k \, e^{i\int^t \omega_k dt'} + \beta_k \; e^{-i\int^t \omega_k dt'} \quad) \quad .$$

Also, if we compute $\ddot{f}_k$ from eq. (5.7.5) and we use eq. (3.5.3) for $\xi = 1/6$ we obtain:

$$(5.7.10) \qquad \dot{\alpha}_k \; e^{i\int^t \omega_k dt'} - \dot{\beta}_k \; e^{-i\int^t \omega_k dt} =$$

$$= \frac{m^2 H}{2\omega_k^2} (\alpha_k \, e^{i\int^t \omega_k \, dt'} - \beta_k \, e^{-i\int^t \omega_k dt'} \quad) \quad .$$

From these last two equations we can see that the field equation is equivalent to the system:

$$(5.7.11) \qquad \dot{\alpha}_k = \frac{m^2 H}{2\omega_k^2} \beta_k \; e^{-2i\int^t \omega_k dt}$$

$$(5.7.12) \qquad \beta_k = \frac{m^2 H}{2\omega_k^2} \alpha_k \quad e^{2i\int^t \omega_k dt}$$

The last equation yields:

$$(5.7.13) \qquad \frac{m^2}{\omega_k^2} R_e \{\alpha_k \beta_k^* \, e^{2i \int^t \omega_t dt'}\} = $$

$$= \frac{1}{H} \frac{d}{dt} |\beta_k(\tau, t)|^2 \quad .$$

Now, from eqs. (5.7.1) and (5.7.2) and eqs. (5.7.4), (5.7.5) and (5.7.13) we obtain:

$$(5.7.14) \qquad {}_\tau\langle 0|T_{00}|0\rangle_\tau = \int \frac{d^3k}{(2\pi a)^3} \frac{1}{2} \omega_k \{1 + 2|\beta_k(\tau, t)|^2\}$$

$$(5.7.15) \qquad {}_\tau\langle 0|T_{ij}|0\rangle_\tau = \frac{1}{3} \delta_{ij} \int \frac{d^3k}{(2\pi a)^3} \frac{1}{2} \omega_k \times$$

$$\times \quad \{(1 - \frac{m^2}{\omega_k^2})(1 + 2|\beta_k(\tau, t)|^2 - \frac{2}{H} \frac{d}{dt} |\beta_k(\tau, t)|^2\} \quad .$$

These expressions are, of course, divergent and must be renormalized. We shall use adiabatic renormalization, so we shall substract from these expressions the stress-tensor for the adiabatic solution (eq. (5.6.17) and (5.6.18)) computed up to the fourth adiabatic order, that reads:

$$(5.7.16) \qquad \langle T_{00}\rangle^{div}_{Ad} = \int \frac{d^3k}{(2\pi a)^3} \frac{1}{2} \omega_k - P_{00} \quad ,$$

$$(5.7.17) \qquad \langle T_{ij}\rangle^{div}_{Ad} = \frac{1}{3} \delta_{ij} \int \frac{d^3k}{(2\pi a)^3} \frac{1}{2} \omega_k \quad (1 - \frac{m^2}{\omega_k^2}) - P_{ij} \quad ,$$

where:

(5.7.18) $$P_{00} = -\frac{m^2}{288\pi^2}\,3\alpha_1 - \frac{1}{2880\pi^2} \times$$

$$\times\ (-3\beta_1 + 3\beta_2 - 6\beta_3 - 6\beta_4) \quad ,$$

(5.7.19) $$P_{ij} = \delta_{ij}\ \{-\frac{m^2}{288\pi^2}(\alpha_1 - 2\alpha_2) -$$

$$-\frac{1}{2880\pi^2}(-5\beta_1 + \beta_2 + 2\beta_3 + 4\beta_4 + 2\beta_5)\ \} \ .$$

We can immediatelly see that the divergences of eq. (5.7.16) and (5.7.17) cancel the ones of eq. (5.7.14) and (5.7.15) all being of the form $\int d^3\bar{k}\omega_k$[23], thus the minimum energy vacuum at τ produces a renormalizable stress-tensor, and therefore it is a minimal vacuum.

The renormalized stress-tensor ${}_\tau\langle 0|T_{\mu\nu}|0\rangle_\tau^{ren} = {}_\tau\langle 0|T_{\mu\nu}|0\rangle_\tau - \langle T_{\mu\nu}\rangle_{Ad}^{(4)}$ reads:

(5.7.20) $${}_\tau\langle 0|T_{\mu\nu}|0\rangle_\tau^{ren} = P_{00} + \int \frac{d^3k}{(2\pi a)^3}\,\omega_k|\beta_k(\tau,t)|^2$$

(5.7.21) $${}_\tau\langle 0|T_{ij}|0\rangle_\tau^{ren} = P_{ij} + \frac{1}{3}\,\delta_{ij}\int \frac{d^3k}{(2\pi a)^3}\ \cdot$$

$$\cdot\ \omega_k\ \{(1 - \frac{m^2}{\omega_k^2})|\beta_k(\tau,t)|^2 - \frac{1}{H}\frac{d}{dt}|\beta_k(\tau,t)|^2\}$$

Now we can study the nature of the terms that appears in the renormalized stress-tensor. Let us compute the stress-tensor at time τ where we have defined the vacuum. From eqs. (5.7.6)

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(23) In fact, all the divergent components must be contain in eqs. (5.7.16) and (5.7.17).

we have:

(5.7.22) $$ {}_{\tau}\langle 0|T_{\mu\nu}(\tau)|0\rangle_{\tau}^{ren} = P_{\mu\nu} \, . $$

At $t = \tau$ we have the vacuum state $|0\rangle_{\tau}$, nevertheless ${}_{\tau}\langle 0|T_{\mu\nu}|0\rangle_{\tau}^{ren} \neq 0$. Thus the quantum vacuum $|0\rangle_{\tau}$ in curved space-time is the residence of a non vanishing stress-tensor that it is called the polarization tensor. The presence of a tension in a quantum vacuum in a non-trivial situation is demonstrated by the Casimir effect where an atraction appears between two perfectly conducting plates because space is not unbauded, even if there is not field nor matter between the plates. Thus the polarization tensor $P_{\mu\nu}$ belongs to this kind of phenomena. From eqs. (5.7.18) and (5.7.19) we can see that it depends on the geometry of space-time at time $t = \tau$ only. Thus at every time t we can write eq. (5.3.22) as:

(5.7.23) $$ P_{\mu\nu} = {}_{t}\langle 0|T_{\mu\nu}(t)|0\rangle_{t}^{ren} \quad , $$

i.e. $P_{\mu\nu}$ is the stress tensor at time t, if we put the vacuum at t.

The second component of eqs. (5.7.20) and (5.7.21) is the stress-tensor of the particles created between τ and t. In fact, if there is no particle creating (if $H = 0$; $\alpha = 1$, $\beta = 0$ is a solution of the system (5.7.11) and (5.7.12)) as in the mass-less case this term vanishes. Thus the minimal vacuum concept allows us to find a canonical decomposition of the stress-tensor in a polarization term and a matter term, both with a physical reasonable base, the Casimir effect for the polarization term, and the classical notion that matter produces a stress-tensor, for the matter term.

At this point we have all the elements to construct a Quantum Field Theory in Robertson-Walker unvierse (and to fors a Quantum Field Theory in a General Geometry). If somewhere in space-time there is a Minimal Vacuum we can build a Fock space there. If we have a Strong Vacuum or a Trivial Vacuum we can do the same thing and we shall have a greather confidence in the physical meaning of the construction. If there is no Minimal Vacuum in the whole space-time, we cannot build any Fock space and the implementation of a standard Quantum Field Theory is impossible, because we cannot construct a complete set of observables that allow us to interpret the ket of the space of state in the usual way: the vacuum, the one particle states, etc. If we have at least one place where we can define a Fock space we can define there a physical initial condition and write and solve the semiclassical Einstein equation (4.3.23) and compute the universe evolution. If this evolution produces a second Minimal Vacuum (or Strong or Trivial Vacuum) at another time we can build a second Fock space and study the scattering problem between these two Fock spaces (as in the second example of paragraph 3.3).

Thus the two problems that we formulated in paragraph 2.4 are solved and the main lines of the picture are drawn.

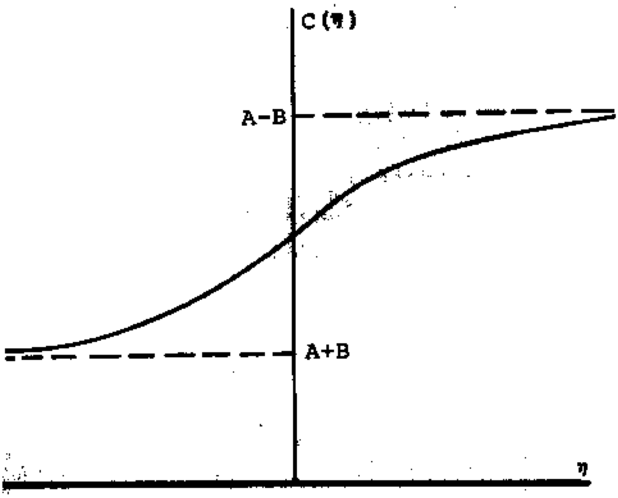


fig.1

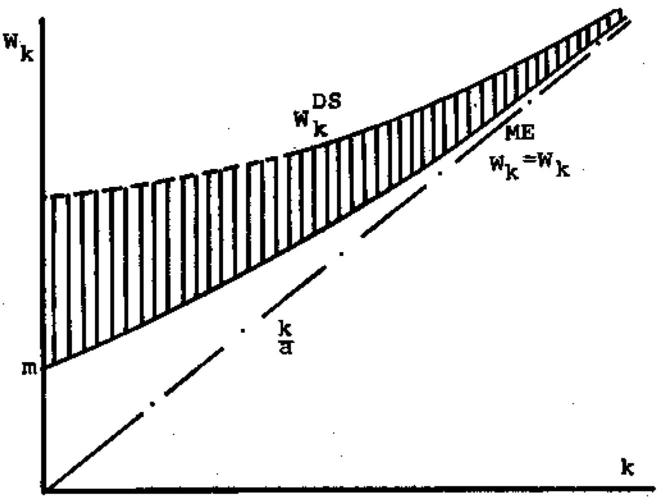
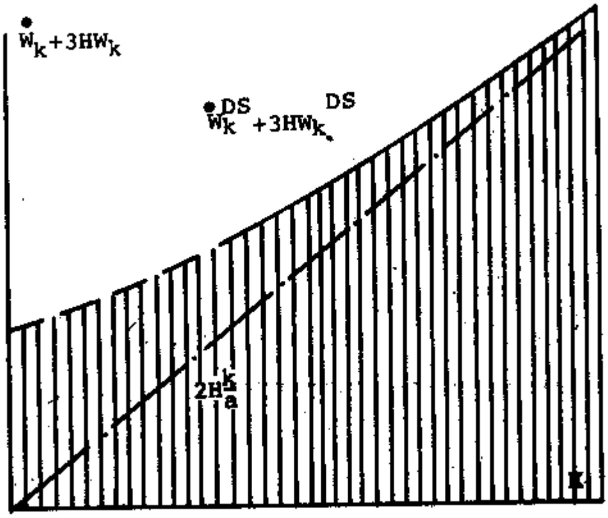
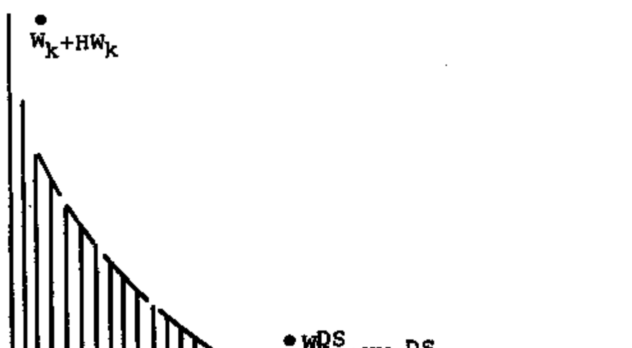


fig.2



$\dot{W}_k + 3HW_k$
$\dot{W}_k^{DS} + 3HW_k^{DS}$
$2H\frac{k}{a}$
k

fig.3



$\dot{W}_k + HW_k$
$\dot{W}_k^{DS} + HW_k^{DS}$
k

fig.4

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