INTRODUCTION TO THE FORMULATION OF SOME UNITARY THEORIES AND THE GANGE THEORY OF THE GROUP U(3,1)

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1 - INITIAL MATHEMATICAL CONCEPTS

In this section we will consider four dimensional spacetimes with a symmetric (and real) metric tensor. Several mathematical properties of such spaces are listed in sequence.

1.A - VARIATION IN THE LENGHT OF VECTORS UNDER PARALLEL TRANSPORT

The lenght of a vector A is given by the expression

$$\ell^2 = g_{\mu\nu} A^\mu A^\nu$$

The absolute differential form is defined by

$$DA^\mu = dA^\mu + \Gamma^\mu_{\sigma\rho} A^\sigma dx^\rho = A^\mu_{;\alpha} dx^\alpha$$

here $A^\mu_{;\alpha}$ is the covariant derivative of A^μ. Similarly

$$DA_\mu = dA_\mu - \Gamma^\sigma_{\mu\rho} A_\sigma dx^\rho = A_{\mu;\alpha} dx^\alpha$$

Accordingly, we have

$$Dg_{\mu\nu} = dg_{\mu\nu} - (\Gamma^\sigma_{\mu\lambda} g_{\sigma\nu} + \Gamma^\sigma_{\nu\lambda} g_{\mu\sigma}) dx^\lambda$$

Suppose that the vector A describes by parallel transport an infinitesimal closed contour. Between two points P and P + dP the lenght of this vector suffers the variation

$$2\ell d\ell = dg_{\mu\nu} A^\mu A^\nu + g_{\mu\nu} dA^\mu_{||} A^\nu + g_{\mu\nu} A^\mu dA^\nu_{||}$$

where $dA^\mu_{||}$ is given by

$$dA^\mu_{||} = DA^\mu - dA^\mu .$$

Then

$$2\ell d\ell = (Dg_{\mu\nu}) A^\mu A^\nu \qquad (1.a.1)$$

Thus, we have

(1) If $Dg_{\mu\nu} = 0$ holds, the lenght of the vector does not vary under parallel transport. In this circumstance it is possible to define an absolute scale of lenght which holds on all points of the manifold. The lenght defined in this way is independent on the trajectory chosen between two points close to each other.

(2) If $Dg_{\mu\nu} \neq 0$ it will not exist an absolute scale of lenght on the manifold.

1.B - VARIATION IN THE COMPONENTS OF A VECTOR UNDER PARALLEL DISPLACEMENT AROUND AN INFINITESIMAL CLOSED CONTOUR

This variation clearly takes the form

$$\oint_c \delta A_\mu^{||} = \oint \Gamma^\nu_{\mu\rho} A_\nu \, dx^\rho$$

From Stoke's theorem we get

$$\oint_c \delta A_\mu^{||} = \frac{1}{2} \int_s \left[\frac{\partial(\Gamma^\nu_{\mu\rho} A_\nu)}{\partial x^\lambda} - \frac{\partial(\Gamma^\nu_{\mu\lambda} A_\nu)}{\partial x^\rho} \right] d\sigma^{\lambda\rho}$$

Since the change in the components A_μ along the contour is due to parallel transport, we have

$$\delta A_\mu = \Gamma^\lambda_{\mu\tau} A_\lambda \, dx^\tau$$

which gives

$$\frac{\partial A_\mu}{\partial x^\tau} = \Gamma^\lambda_{\mu\tau} A_\lambda$$

Substitution of this result in the previous formula gives the final result

$$\oint_c \delta A_\mu^{||} = \frac{1}{2} \int_s R^\alpha_{\mu\lambda\rho} \quad A_\alpha \quad d\sigma^{\lambda\rho}$$

where

$$R^\alpha{}_{\mu\lambda\rho} = \frac{\partial \Gamma^\alpha_{\mu\rho}}{\partial x^\lambda} - \frac{\partial \Gamma^\alpha_{\mu\lambda}}{\partial x^\rho} + \Gamma^\alpha_{\beta\lambda} \Gamma^\beta_{\mu\rho} - \Gamma^\alpha_{\beta\rho} \Gamma^\beta_{\mu\lambda}$$

is the affine curvature tensor. We will call the quantity

$$\Omega^\alpha{}_\mu = R^\alpha{}_{\mu\lambda\rho} \, d\sigma^{\lambda\rho} \qquad (1.b.1)$$

as the rotation curvature. Then,

$$\oint_c \delta A_\mu^{||} = \frac{1}{2} \int_s \Omega^\alpha{}_\mu A_\alpha \qquad (1.b.2)$$

1.c - <u>CURVATURE OF SEGMENTATION</u>

Given the conditions $D_\rho g_{\mu\nu} \equiv g_{\mu\nu;\rho} = K_{\mu\nu,\rho}$, for a symmetric $g_{\mu\nu}$, the general solution for the spacetime connection is of the form

$$\Gamma^\mu_{\nu\rho} = \{^{\ \mu}_{\nu\rho}\} + g^{\mu\sigma}(\underset{V}{\Gamma_{\nu\sigma,\rho}} + \underset{V}{\Gamma_{\rho\sigma,\nu}} + \underset{V}{\Gamma_{\nu\rho,\sigma}}) - \frac{1}{2} g^{\mu\sigma}(K_{\nu\sigma,\rho} + K_{\rho\sigma,\nu} - K_{\nu\rho,\sigma}) \qquad (1.c.1)$$

where

$$\underset{V}{\Gamma_{\nu\sigma,\rho}} = \frac{1}{2} (\Gamma_{\nu\sigma,\rho} - \Gamma_{\sigma\nu,\rho})$$

$$\Gamma_{\mu\rho,\sigma} = g_{\sigma\nu} \Gamma^\nu_{\mu\rho}$$

For a Riemannian geometry both $\underset{V}{\Gamma_{\nu\sigma,\rho}}$ and $K_{\nu\sigma,\rho}$ vanish. Consequently, the spacetime connection is given by the familiar expression $\{^{\ \mu}_{\nu\rho}\}$, the Christoffel symbols. For a Riemann - Cartan geometry only the $K_{\nu\sigma,\rho}$ vanish. For both situations the connection is said to be metrical. We will see later examples of geometries which have $\underset{V}{\Gamma_{\nu\sigma,\rho}}=0$ but $K_{\nu\sigma,\rho}$ is different from zero. These geometries have connections which are said to be semi-metrical.

The contraction $\Omega = \Omega^\alpha_\alpha$ is given by

$$\Omega = R^{\mu}_{\mu\lambda\rho}\, d\sigma^{\lambda\rho} = (\partial_\rho\, \Gamma^{\mu}_{\mu\lambda} - \partial_\lambda\, \Gamma^{\mu}_{\mu\rho})\, d\sigma^{\lambda\rho} \qquad (1.c.2)$$

This expression vanishes if $\Gamma^{\mu}_{\alpha\beta} = \{^{\mu}_{\alpha\beta}\}$ or equivalently if $R^{\mu}_{\nu\lambda\rho}$ is the Riemann-Christoffel tensor. It may be shown that

$$\Omega = \frac{1}{2}\, g^{\mu\nu}\, K_{\mu\nu\lambda\rho}\, d\sigma^{\lambda\rho} \qquad (1.c.3)$$

where

$$K_{\mu\nu\lambda\rho} = \partial_\rho\, K_{\mu\nu,\lambda} - \partial_\lambda\, K_{\mu\nu,\rho} + 2\, \Gamma^{\tau}_{\mu\lambda}\, K_{\tau\nu,\rho} - 2\, \Gamma^{\tau}_{\mu\rho}\, K_{\tau\nu,\lambda}$$

The invariant Ω of (1.c.2) or (1.c.3), is called as the curvature of segmentation (1). If $\Omega^{\alpha}{}_{\mu} \to 0$ then $\Omega \to 0$. Clearly, the reciprocal is not valid. As an example, for the Riemannian geometry $\Omega = 0$ but $\Omega^{\alpha}{}_{\mu} \neq 0$.

1.c - NON-HOLONOMIC SYSTEMS AND THE TORSION

On each point of the fourdimensional spacetime one may define four linearly independent vector fields which are denoted by $h^{a}_{\mu}(x)$, $a = 1...4$. The functions $h^{a}_{\mu}(x)$ are regular and of the class C^2.

The vectors $h^{a}_{\mu}(x)$ may be, or may be not, the basis vectors of a coordinate system. The conditions for these vectors to belong to a coordinate system are

$$\partial_{[\lambda} h^{a}_{\mu]} = 0 \quad , \quad \text{or} \quad h^{a}_{\mu} = \frac{\partial \xi^{a}}{\partial \xi^{\mu}} \qquad (1.d.1)$$

Since the matrix $(h^{a}_{\mu}(x))$ is non-singular at each fixed point x, there exists the inverse which is denoted by $(h^{\mu}_{a}(x))$:

$$h^{\mu}_{a}\, h^{a}_{\nu} = \delta^{\mu}_{\nu} \quad , \quad h^{\mu}_{a}\, h^{b}_{\mu} = \delta^{b}_{a}$$

A field of vectors $h^{a}_{\mu}(x)$ which do not satisfy the conditions (1.d.1) is said to be Non-Holonomic. Any tensor $T^{\alpha}{}_{\beta\gamma}$ may be projected on a given Non-Holonomic system according to $T^{a}{}_{\ell m} = h^{a}_{\alpha} h^{\beta}_{\ell} h^{\gamma}_{m} T^{\alpha}$

The connection on the spacetime manifold is defined as the set of functions $\Gamma^{\nu}_{\mu\lambda}$ which transform as

$$\Gamma'^{\lambda}_{\mu\nu} = A^{\lambda'}_{\alpha}\, A^{\beta}_{\mu'}\, A^{\rho}_{\nu'}\, \Gamma^{\alpha}_{\beta\rho} - A^{\alpha}_{\mu'}\, A^{\beta}_{\nu'}\, \partial_\alpha\, A^{\lambda'}_{\beta} \qquad (1.d.2)$$

under the change of Holonomic systems $(x) \to (x')$:

$$A^{\lambda'}_{\alpha} = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \quad , \quad A^{\lambda}_{\alpha'} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}}$$

Accordingly, in Non-Holonomic systems (K) the connection is the aggregate of functions $\Gamma^{a}_{bc}(x)$ given by

$$\Gamma^{a}_{mn} = h^{a}_{\lambda}\, h^{\mu}_{m}\, h^{\nu}_{n}\, \Gamma^{\lambda}_{\mu\nu} - h^{\mu}_{m}\, h^{\nu}_{n}\, \partial_\mu\, h^{a}_{\nu} \qquad (1.d.3)$$

obtained from (1.d.2) for the transition $(x) \to (k)$ given by $A^{\lambda'}_{\alpha} \to h^{a}_{\alpha}$. Defining the Pfaffian derivative

$$\partial_a = h^{\mu}_{a}\, \partial_\mu$$

(this derivative satisfies the Leibniz rule), one has

$$\Gamma^{a}_{mn} = h^{a}_{\lambda}\, h^{\mu}_{m}\, h^{\nu}_{n}\, \Gamma^{\lambda}_{\mu\nu} + h^{a}_{\nu}\, \partial_m\, h^{\nu}_{n} \qquad (1.d.4)$$

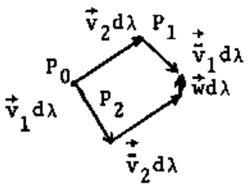
In what follows we give a geometrical interpretation of $\Gamma^{\lambda}_{[\mu\nu]}$. In order to have a general situation we use Non-Holonomic reference systems. On a given a point P_0 we construct two linearly independent vectors $v_1^a\, d\lambda$ and $v_2^a\, d\lambda$. Making the parallel transport of $v_1^a\, d\lambda$ along the vector $v_2^a\, d\lambda$ we arrive at a point P_1

$$\bar{v}_1^a\, d\lambda = v_1^a\, d\lambda - \Gamma^a_{\ell m}\, v_1^m\, v_2^\ell\, (d\lambda)^2 \qquad (1.d.5)$$

Parallel transport of $v_2^a\, d\lambda$ along the vector $v_1^a\, d\lambda$ defines another end point P_2

$$\bar{v}_2^a\, d\lambda = v_2^a\, d\lambda - \Gamma^a_{\ell m}\, v_2^m\, v_1^\ell\, (d\lambda)^2 \qquad (1.d.6)$$

The difference between these vectors determines the vector $w^a\, d\lambda$ which closes the parallelogram



$$w^a = v_2^a + \bar{v}_1^a - v_1^a - \bar{v}_2^a$$

Substituting $\bar{v}_1^a$ and $\bar{v}_2^a$ according to (1.d.5) and (1.d.6) one finds

$$w^a = 2\, \Gamma^a_{[\ell m]}\, v_2^{[m} v_1^{\ell]}\, d\lambda$$

Define

$$\Omega^a = \frac{1}{2}\, W^a\, d\lambda$$

Then

$$\Omega^a = \Gamma^a_{[\ell m]}\, v_2^{[m} v_1^{\ell]}\, (d\lambda)^2 = \Gamma^a_{[\ell m]}\, d\sigma^{m\ell} \qquad (1.d.7)$$

where $d\sigma^{m\ell}$ is the area of the parallelogram. The formula (1.d.7) in Holonomic reference systems takes the form

$$\Omega^\mu = \Gamma^\mu_{[\rho\sigma]}\, d\sigma^{\rho\sigma} \qquad (1.d.8)$$

which is called as the translation vector associated to the torsion $\Gamma^\mu_{[\rho\sigma]}$. Consequently, for manifolds where exists a torsion field it is not possible to draw a closed infinitesimal parallelogram. This property characterizes the existence of a torsion.

1.E - GEOMETRICAL OBJECTS ASSOCIATED TO THE NEIGHBORHOOD OF A POINT ON THE MANIFOLD

Given some point P on the centre of an infinitesimal parallelogram we can define at this region three fundamental objects:

(i) A translation vector: $\Omega^\mu = \Gamma^\mu_{[\rho\sigma]}\, d\sigma^{\rho\sigma}$

(ii) A rotation curvature: $\Omega^\mu_\nu = R^\mu{}_{\nu\rho\sigma}\, d\sigma^{\rho\sigma}$

(iii) A curvature of segmentation: $\Omega = R^\mu{}_{\mu\sigma\rho}\, d\sigma^{\sigma\rho}$

This mathematical structure is sufficiently general for describing theories of gravitation and electromagnetism as geometrical properties of the fourdimensional manifold. However, this situation is so general that each one of such theories will be a particular case of this general geometry. Among the several possibilities we may list the following options:

(1) $\Omega^{\mu} = \Omega = 0$

the manifold differs from an Euclidean spacetime by the presence of the curvature Ω^{μ}_{ν}. This is a Riemannian geometry. Since the spacetime dimension is 4 only gravitation can be described in this case. In order to interpret the electromagnetic phenomena in this case it is necessary to increase the dimensionality of the space (at least 5 dimensions). As examples of such theories we have (2):

(1.1) Kaluza-Klein (5 dimensions): γ_{55} = constant.

(1.2) P.Jordan-G.Ludwig - E.Schmutzer (5 dimensions): γ_{55} is an arbitrary function.

(1.3) Y.Thirry: the metric has 15 independent components.

(1.4) Einstein-Mayer (5 dimensions). the fifth dimension is used as a mathematical tool.

(1.5) Einstein-Bergmann-Bargmann (5 dimensions): the five-dimensional space has cilindrical symmetry along the fifth-direction.

(1.6) J.Podolanski - uses a manifold with six dimensions.

None of such theories will be considered here, since we work with four-dimensional manifolds.

(2) $\Omega^{\mu} = 0$

there is no torsion on the manifold. If the dimension is four, this corresponds to the Weyl geometry, or to the Eddington geometry. If it corresponds to Weyl's geometry the connection is determined by the knowledge of:

(2.1) the field of a symmetric metric tensor.

(2.2) the field of a vector $\phi_{\mu}(x)$, which will be associated to the change on the lenght-scale.

This case will be treated in more detail at the section (2).

(3) $\Omega = 0$

The unity of lenght is constant along the paths on the manifold. If the dimension is four, this case corresponds to the unitary field theory of Infeld. In this theory the basic field variables are the $g_{(\mu\nu)}(x)$ and a skew-symmetric object $\Psi_{[\bar{\mu}\bar{\nu}]}(x)$.

(4) $\Omega^{\mu}_{\nu} = 0$

By contraction this implies in $\Omega = 0$. Thus, there is no curvature in this case.The torsion takes the role of the basic geometrical quantity. A manifold with such property is called as a Weitzenböck space. In this space one can determine parallelism at large distances (tele-parallelism). This geometry was used by Einstein (1928) in his second attempt for constructing an unitary field theory of gravitation and electromagnetism. Recently, this geometry was again used by K.Hayashi (3) with the intention of obtaining a theory of gravitation at microscopical level.

(5) All the previous cases work with symmetric metric fields. There are also examples of of theories which consider asymmetric metric fields. Since the symmetry presented by the metric is independent of the symmetry associated to the connection, in this case we can also have a torsion. Such theories are of two types:

(4.1) real asymmetric metric.

(4.2) complex asymmetric metric.

The first case was treated by Schrödinger and by Einstein. The second one was considered by Einstein. A short review of this second tentative will be written at the section (5).

2 - THE THEORY OF WEYL

In this theory (4), similarly to what happens in general relativity, there exists at each point of the space a geodetic coordinate system. This assumption implies in the existence of a symmetric affine connection. Thus, in this theory the Ω^{μ} vanish Weyl's theory is distinguished from general relativity due to the property that the unity of lenght is no longer an absolute quantity. As it will be seen this implies in the existence of a curvature of segmentation.

The local structure of the manifold is determined from the two differential forms

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \quad , \; d\phi = \phi_{\mu}\, dx^{\mu}$$

ds^2 is a bilinear form (symmetric metric). The metric $g_{\mu\nu}$ is subjected to the semi-metrical conditions

$$K_{\mu\nu,\rho} \equiv g_{\mu\nu,\rho} = - g_{\mu\nu}\, \phi_{\rho} \tag{2.1}$$

These equations determine the expression of the connection: (use directly the equations (1.c.1) for this value of the $K_{\mu\nu,\rho}$)

$$\Gamma^{\mu}_{\nu\rho} = \left\{{}^{\mu}_{\nu\rho}\right\} + \frac{1}{2}\left(\delta^{\rho}_{\mu}\, \phi_{\nu} + \delta^{\rho}_{\nu}\, \phi_{\mu} - g_{\mu\nu}\, \phi^{\rho}\right) \tag{2.2}$$

where we have used the assumption that $\Gamma^{\mu}_{[\nu\rho]} = 0$.

It should be noted that the equations (2.1) do not uniquely determine the set of quantities $g_{\mu\nu}$, ϕ_{ρ}. Indeed, any pair of quantities given by

$$g'_{\mu\nu} = \lambda\, g_{\mu\nu} \quad , \qquad \phi'_{\rho} = \phi_{\rho} - \partial_{\rho}\, \ell n\, \lambda \tag{2.3}$$

satisfies the same conditions (2.1). However, the $\Gamma^{\mu}_{\nu\rho}$ are uniquely determined (the $\Gamma^{\mu}_{\nu\rho}$ have the same expression for any choice of variables $g_{\mu\nu}$, ϕ_{ρ}). Due to the form of the variation $\delta\phi_{\rho}$ we call the transformation (2.3) as the Weyl gauge transformation. Thus

$$ds'^2 = \lambda\, ds^2$$

$$d\phi' = d\phi - d\, \ell n\, \lambda$$

The curvature tensor has the form

$$R^{\rho}{}_{\mu\nu\sigma} = G^{\rho}{}_{\mu\nu\sigma} + \frac{1}{2}\, \delta^{\rho}_{\mu}\phi_{\nu\sigma} + \delta^{\rho}_{[\nu}\nabla_{\sigma]}\phi_{\mu} + g_{\mu[\sigma}\nabla_{\nu]}\phi^{\rho} + \frac{1}{2}\, g_{\mu[\nu}\phi_{\sigma]}\phi^{\rho} + \frac{1}{2}\delta^{\rho}_{[\nu}\, g_{\sigma]\mu}\phi_{\lambda}\phi^{\lambda} + \frac{1}{2}\delta^{\rho}_{[\sigma}\phi_{\nu]}\phi_{\mu} \tag{2.4}$$

where $G^{\rho}_{\mu\nu\sigma}$ is the Riemann-Christoffel tensor, $\phi_{\nu\sigma}$ is the field intensity associated to the ϕ_{ν}, namely

$$\phi_{\nu\sigma} = \phi_{\nu,\sigma} - \phi_{\sigma,\nu} \quad .$$

In the formula (2.4) the symbol ∇_{ν} indicates the covariant derivative associated to the Christoffel symbols. From (2.4) it is easy to show that the curvature of segmentation and the Ricci tensor have the expressions

$$\Omega = 2\, \phi_{\nu\sigma}\, d\sigma^{\nu\sigma}$$

$$R_{\mu\nu} = R^{\rho}_{\mu\nu\rho} = G_{\mu\nu} + \frac{1}{2}\, \phi_{\mu\nu} - \nabla_{\nu}\phi_{\mu} - \frac{1}{2}\, g_{\mu\nu}\nabla_{\lambda}\phi^{\lambda} - \frac{1}{2}\, g_{\mu\nu}\phi_{\rho}\phi^{\rho} + \frac{1}{2}\, \phi_{\mu}\phi_{\nu}$$

Therefore the $R_{\mu\nu}$ is asymmetric.

3 - THE ACTION PRINCIPLE FOR WEYL'S THEORY

The quantities $g_{\mu\nu}$,ϕ_ρ are the potentials of the theory. Thus, the Action principle is of the form

$$I = \int L \, d_4 x$$

$$L = L\,(g_{\mu\nu},\ g_{\mu\nu,\alpha},\ g_{\mu\nu,\alpha\beta},\ \phi_\alpha,\ \phi_{\alpha,\beta})$$

The eventual presence of second order derivatives of the $g_{\alpha\beta}$ is due to the fact that the gravitational part of the Lagrangian density will be different of the Einstein's Lagrangian (namely, the quantity $g^{\mu\nu}G_{\mu\nu}$). In order to have an explicit form for L one imposes the following invariance principles:

(a) The Action function is invariant under arbitrary coordinate transformations.

(b) The Action function is invariant under gauge transformations of the potentials ϕ_ρ.

Then, L is dependent only on quantities which are scalar densities of weight (+ 1) with respect to coordinate transformations and which are gauge invariant. We shall call such objects as "in-invariants". By extension, "in-tensors" means tensors which are gange-invariant. The affinity $\Gamma^\mu_{\nu\rho}$ is gauge-invariant, consequently the curvature tensor of Weyl's theory is an "in-tensor". We shall use the notation

$$R^\mu{}_{\nu\rho\sigma} = {}^*G^\mu{}_{\nu\rho\sigma}$$

Accordingly, the first contraction generates another "in-tensor"

$$R_{\nu\rho} = R^\mu_{\nu\rho\mu} = {}^*G_{\nu\rho}$$

The next contraction which generates an"in-scalar-density" is of the form

$$\alpha_1 \equiv ({}^*G)^2 \sqrt{-g} \ , \tag{3.1}$$

since ${}^*G' = \frac{1}{\lambda}\,{}^*G$, and $\sqrt{-g'} = \lambda^2 \sqrt{-g}$.

The expression of α_1 given by (3.1) is the simplest expression which involves the curvature and satisfies conditions (a) and (b). It is possible to construct more complicated quantities with these properties, as for example:

$$\alpha_2 = {}^*G_{\mu\nu}\,{}^*G^{\mu\nu} \sqrt{-g} \quad , \quad \alpha_3 = {}^*G^\rho_{\mu\nu\sigma}\,{}^*G^{\mu\nu\sigma}_{\rho\ldots} \sqrt{-g}$$

However, we will not consider these possibilities (the situation here is similar to what happens in general relativity, where the Einstein's Lagrangian is the simplest expression of a possible Lagrangian).

To the expression of α_1 one adds a Maxwell Lagrangian $\phi_{\mu\nu}\,\phi^{\mu\nu}\sqrt{-g}$ and obtain the expression of Weyl's Lagrangian:

$$L = ({}^*G^2 - \alpha\,\phi_{\mu\nu}\,\phi^{\mu\nu}) \sqrt{-g} \tag{3.2}$$

The field equations obtained from this Lagrangian density have the form

$$G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} G = x\, T^{\mu\nu}$$

$$\phi^{\mu\nu}_{;\nu} + \frac{3}{4\beta} J^\mu = 0$$

where $G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} G$ corresponds to the Einstein's tensor, and $T^{\mu\nu}$ is the analog of a canonical energy-momentum tensor. Its explicit expression being

$$\chi T^{\mu\nu} = 8\ \beta\ \tau^{\mu\nu} - g^{\mu\nu}\lambda + \frac{3}{2}\ (\phi^{\mu}\phi^{\nu} - \frac{1}{2}\ g^{\mu\nu}\ \phi_{\rho}\phi^{\rho})$$

The tensor $\tau^{\mu\nu}$ is the Einstein-Maxwell tensor, with the well known properties: $\tau^{\mu\nu} = \tau^{\nu\mu}$, $\tau^{\mu}_{\mu} = 0$. The constants λ, α and β appear due to the following choice of a gauge condition

$$*G = 4\ \lambda \quad , \ \beta = \frac{d}{8\lambda}$$

In equation (3.5) J^{μ} represents the coefficient of the variation of the "gravitational part" of L with respect to the potentials ϕ_{μ}:

$$J^{\mu} = \frac{\delta(*G^2\ \sqrt{-g})}{\delta\phi_{\mu}}$$

The trace of the tensor $T^{\mu\nu}$ has the form

$$T \approx \mu c^2 = -\ \frac{4\lambda}{x} + \frac{3}{2x}\ (\phi_{\rho}\phi^{\rho} - 2\ \phi_{\rho}\phi^{\rho}) = -\ \frac{4\lambda}{x} - \frac{3}{2x}\ \phi_{\rho}\phi^{\rho} \tag{3.6}$$

From the expression of *G (which is identical to the scalar of curvature of Weyl's curvature) one gets

$$J^{\mu} = -\ 3\ \phi^{\mu}\ \sqrt{-g}\quad *G$$

in the Lorentz gauge $\nabla_{\mu}\phi^{\mu} = 0$. Then, (note that here $*G = 4\lambda$)

$$T \approx \mu c^2 = -\ \frac{4\lambda}{X} - \frac{3}{2x}\ J_{\rho}J^{\rho}\ \left(\frac{1}{3\ \sqrt{-g}\quad *G}\right)^2$$

Thus, in Weyl's theory matter cannot exist without the presence of charges or currents. Since it is well known that indeed matter can be detected without the presence of electric charges, this is a negative result of this theory.

From the mathematical stand-point Weyl's theory seems to be the simplest formulation in a four-dimensional spacetime without torsion, which may, in principle, describe gravitation and electromagnetism. However, this formulation has some serious negative points from the physical point of view:

(a) The property referred above, regarding the structure of matter.

(b) The electromagnetic gauge has a direct interference with the measure of lenghts: $\ell'^2 = \lambda\ell^2$. Since $\lambda = \lambda(x)$, it follows that the observation of the lenght of a vector will depend on the location of this vector. For the propagation of light in vacuum, this would imply in a variation of the frequency ($c = \nu\lambda$ = constant). This in turn would imply that atoms radiate different spectral lines at different locations, and this is not observable.

In order to avoid these negative results one has to consider only the light-cone at each event in spacetime, since these regions are not influenced by conformal changes. However, a formulation involving tensor fields is, in principle, a theory which may be described in any region, and so it is not adequate for such restrictions. In principle, it may be possible to reformulate Weyl's theory in terms of two-component spinor fields. Since such fields are associated to null-vectors, the above mentioned problems may be avoided. Due to these problems, Weyl's theory, as originally formulated, is not considered as a correct possible formulation of an unitary field theory.

4 - EINSTEIN'S THEORY WITH ABSOLUTE PARALLELISM

The second unitary field theory proposed by Einstein in 1928 (5) is characterized by the two conditions: $\Omega^{\mu}_{\nu} = 0$, $\Omega = 0$. The only quantity which distinguishes this manifold from an Euclidean spacetime is the torsion. In this situation it is possible to determine a field of parallel vectors over the manifold. In principle, one of the properties of this theory is the substitution of the description of gravitational phenomena from the language of the curvature to the language of the torsion. Accordingly, a correspondence principle with general relativity is lost, and this fact is also present in almost all the unitary theories proposed by this time. These theories proposed to formulate a new theory of gravitation and electromagnetism completely dissociated from general relativity. This later theory was interpreted as provisional formulation of the gravitational phenomena.

Due to the possibility of the existence of a tele-parallelism, the natural basic geometric quantities are the vierbeins $h^{\mu}_{(\alpha)}$.

$$g_{\mu\nu} = h^{(\alpha)}_{\mu}\ h_{\nu(\alpha)}$$

The $h^{\mu}_{(\alpha)}$ are subjected to the conditions

$$h^{\mu}_{(\alpha);\nu} = \partial_{\nu}\ h^{\mu}_{(\alpha)} + \Gamma^{\mu}_{\lambda\nu}\ h^{\lambda}_{(\alpha)} = 0 \tag{4.1}$$

Accordingly, the curvature associated to the connection Γ vanishes. We have,

$$\Gamma^{\mu}_{\nu\lambda} = h^{\mu}_{(\alpha)}\ \partial_{\nu}\ h^{(\alpha)}_{\lambda} \tag{4.2}$$

As in general relativity, we also have that $g_{\mu\nu;\alpha} = 0$. Since the general solution of these equations has the form given by (1.c.1), it follows that $R^{\mu}_{\nu\alpha\beta}(\{\ \})$ is in general different from zero. Thus, it isstill possible to define the objects $\{^{\alpha}_{\beta\gamma}\}$ and the "Riemann-Christoffel" tensor. However, such quantities are directly written in function of the $h^{\mu}_{(\alpha)}$, which are different from the vierbeins of a Riemannian spacetime (these vierbeins do not satysfy(4.1)).

From the metricity condition $g_{\mu\nu;\alpha} = 0$ we also have that the full covariant derivatives of the vierbeins vanish:

$$h^{\mu}_{(\alpha)|\nu} = h^{\mu}_{(\alpha);\nu} - \Delta^{(\lambda)}_{\nu\ (\alpha)}\ h^{\mu}_{(\lambda)} = 0$$

Therefore

$$\Delta^{(\lambda)}_{\nu(\alpha)} = h^{\mu}_{(\alpha);\nu}\ h^{(\lambda)}_{\mu} = 0$$

Thus, the internal connection associated to Γ vanishes over all spacetime. Consequently the internal space of this theory is an Euclidian four-space.

The general expression for the commutator of covariant derivatives in a space with curvature and torsion has the form (2)

$$\left[D_{\rho},D_{\sigma}\right]A^{\nu\ \cdots}_{\cdot\mu\cdots} = -R^{\nu}_{\cdot\mu\rho\sigma}(\Gamma)A^{\tau\ \cdots}_{\cdot\mu\cdots} + R^{\tau}_{\cdot\mu\rho\sigma}(\Gamma)A^{\nu\ \cdots}_{\cdot\tau\cdots} + 2\ \underset{V}{\Gamma^{\tau}_{\rho\sigma}}\ D_{\tau}A^{\nu\cdots}_{\cdot\mu\cdots}$$

Presently it assumes the form

$$\left[D_{\rho},D_{\sigma}\right]A^{\nu\ \cdots}_{\cdot\mu\cdots} = 2\ \underset{V}{\Gamma^{\tau}_{\rho\sigma}}\ D_{\tau}A^{\ \nu\cdots}_{\cdot\mu\cdots}$$

here D_{ρ} indicates covariant derivative with respect to the connection Γ.

4.1 - THE TENSOR OF TORSION

The torsion $\Gamma^{\rho}_{[\mu\nu]}$ has 24 independent components. Among such components there will be a certain number of identities. The differential Bianchi identities in a space with curvature and torsion assume the form (here, covariant derivatives indicated by ; refer to the connection Γ)

$$R^{\tau}_{\mu\sigma\nu;\rho} + R^{\tau}_{\mu\rho\sigma;\nu} + R^{\tau}_{\mu\nu\rho;\sigma} - 2R^{\tau}_{\mu\lambda\rho}\Gamma^{\lambda}_{[\sigma\nu]} - 2R^{\tau}_{\mu\lambda\nu}\Gamma^{\lambda}_{[\rho\sigma]} - 2R^{\tau}_{\mu\lambda\sigma}\Gamma^{\lambda}_{[\nu\rho]} \equiv 0$$

Thus, if $R^{\tau}{}_{\mu\sigma\nu} \to 0$ over all space it will be identically satisfied. From the identity

$$R^{\tau}{}_{\nu\rho\sigma}+R^{\tau}{}_{\sigma\nu\rho}+R^{\tau}{}_{\rho\sigma\nu}+2(\Gamma^{\tau}_{[\rho\nu]};_{\sigma}+\Gamma^{\tau}_{[\sigma\rho]};_{\nu}+\Gamma^{\tau}_{[\nu\sigma]};_{\rho})-4\Gamma^{\lambda}_{[\rho\nu]}\Gamma^{\tau}_{[\lambda\sigma]}-4\Gamma^{\lambda}_{[\sigma\rho]}\Gamma^{\tau}_{[\lambda\nu]}-4\Gamma^{\lambda}_{[\nu\sigma]}\Gamma^{\tau}_{[\lambda\rho]} \equiv 0$$

in the case of vanishing curvature, one gets

$$2(\Gamma^{\tau}_{[\rho\nu]};_{\sigma}+\Gamma^{\tau}_{[\sigma\rho]};_{\nu}+\Gamma^{\tau}_{[\nu\sigma]};_{\rho})-4\Gamma^{\lambda}_{[\rho\nu]}\Gamma^{\tau}_{[\lambda\sigma]}-4\Gamma^{\lambda}_{[\sigma\rho]}\Gamma^{\tau}_{[\lambda\nu]}-4\Gamma^{\lambda}_{[\nu\sigma]}\Gamma^{\tau}_{[\lambda\rho]} \equiv 0 \qquad (4.1.1)$$

Presently, the covariant derivative with respect to Γ has to be defined in a proper way since this connection is asymmetric. We shall use the following definition of this derivative:

$$\Gamma^{\tau}_{[\nu\sigma]};_{\rho}=\partial_{\rho}\Gamma^{\tau}_{[\nu\sigma]}+\Gamma^{\tau}_{\lambda\rho}\Gamma^{\lambda}_{[\nu\sigma]}-\Gamma^{\lambda}_{\nu\rho}\Gamma^{\tau}_{[\lambda\sigma]}-\Gamma^{\lambda}_{\sigma\rho}\Gamma^{\tau}_{[\nu\lambda]}$$

Contracting the indices (τ,σ), one finds

$$\Gamma_{\nu;\rho} = \partial_{\rho}\,\Gamma_{\nu} - \Gamma^{\lambda}_{\nu\rho}\,\Gamma_{\lambda} \qquad (4.1.2)$$

where Γ_{ν} is the vector of torsion:

$$\Gamma_{\nu} = \Gamma^{\sigma}_{[\nu\sigma]}$$

Contracting the indices (τ,σ) in (4.1.1) and using (4.1.2) one obtains

$$\Gamma^{\sigma}_{[\rho\nu]};_{\sigma}+\partial_{\rho}\Gamma_{\nu}-\partial_{\nu}\Gamma_{\rho}+2(\Gamma^{\lambda}_{[\nu\rho]}\Gamma_{\lambda}+\Gamma^{\lambda}_{[\rho\sigma]}\Gamma^{\sigma}_{[\lambda\nu]}+\Gamma^{\lambda}_{[\sigma\nu]}\Gamma^{\sigma}_{[\lambda\rho]}-\Gamma^{\lambda}_{[\nu\rho]}\Gamma_{\lambda}) \equiv 0$$

In the parenthesis the first term cancels with the fourth one, the same happening with the second and third terms, giving

$$\Gamma^{\sigma}_{[\rho\nu]};_{\sigma} = -(\partial_{\rho}\,\Gamma_{\nu} - \partial_{\nu}\,\Gamma_{\rho}) \qquad (4.1.3)$$

Defining

$$F_{\rho\nu} = \Gamma^{\sigma}_{[\rho\nu]};_{\sigma}$$

we have

$$F_{\rho\nu} = \partial_{\nu}\,\Gamma_{\rho} - \partial_{\rho}\,\Gamma_{\nu} \qquad (4.1.4)$$

Similarly,

$$F^{\rho\nu} = \Gamma^{\sigma[\rho\nu]}{}_{;\sigma}$$

It may be shown that

$$G^{\mu\sigma}{}_{;\sigma} - 2F^{\mu\sigma}{}_{;\sigma} - 4\Gamma^{\rho[\mu\nu]}\,F_{\nu\rho} = 0 \qquad (4.1.5)$$

where

$$\frac{1}{2}\,G^{\mu\sigma} = \Gamma^{\sigma[\mu\nu]}{}_{;\nu} + 2\,\Gamma^{\rho[\mu\nu]}\,\Gamma^{\sigma}_{[\nu\rho]} \qquad (4.1.6)$$

4.2 - THE FIELD EQUATIONS

The field variables are the sixteen components of the vierbeins. The torsion is given in terms of the first derivatives of these quantities

$$\Gamma^{\mu}_{[\lambda\nu]} = h^{\mu}_{(\alpha)}\, \partial_{[\nu} h^{(\alpha)}_{\lambda]}$$

Thus, in order to generate second order differential equations one must consider as field equations, expressions which contain the first derivatives of the torsion. It is natural to select the objects $G^{\mu\nu}$ and $F^{\mu\nu}$ which contain respectively 16 and 6 components, since they have the correct differential order. Having this in mind, we postulate the field equations as:

$$\begin{cases} G^{\mu\nu} = 0 & (4.2.1) \\ F^{\mu\nu} = 0 & (4.2.2) \end{cases}$$

However, in this form they represent 22 equations for the sixteen unknowns $h^{\mu}_{(\alpha)}$. From(4.1.4) and (4.2.2) it follows that

$$\Gamma_{\mu} = \partial_{\mu} \log \Psi \qquad (4.2.3)$$

$\Psi(x)$ is an arbitrary scalar field. Accordingly, one may use instead of the (4.2.1)and(4.2.2) the set of equations

$$G^{\mu\nu} = 0 \quad , \quad \Gamma_{\mu} = \partial_{\mu} \log \Psi$$

which are 20 differential equations for the determination of 17 quantities: $h^{\mu}_{(\alpha)}$ and Ψ. These equations are highly non-linear in the $h^{\mu}_{(\alpha)}$. The tensor $G^{\mu\nu}$ given by (4.1.6) has terms up to the order h^{8} in the vierbeins. Due to this mathematical complexity it is not hoped that exact solutions can be found for such equations. Thus, one proceeds to find out particular solutions, and from such solutions one looks for a physical interpretation of the theory.

4.3 - WEAK FIELD APPROXIMATION FOR THE FIELD EQUATIONS

Consider the linear approximation

$$h^{\mu}_{(\alpha)} = \delta^{\mu}_{\alpha} + r^{\mu}_{\alpha}$$

where r^{μ}_{α} are first order infinitesimals. The symmetry transformations of the theory are:

a - arbitrary coordinate transformations;
b - global Lorentz transformations.

The transformations (a) appear due to the fact that the manifold has a connection.The global Lorentz rotations are associated to the property that the internal space is globally flat. In the linear approximation both transformations coincide up to gauge transformations, thus, presently we will have no distinction between these two types of transformations.

The Vector of torsion takes the form

$$\Gamma_{\lambda} = \partial_{[\mu} r^{\mu}_{\lambda]}$$

Accordingly, the equations (4.2.3) assume the form

$$\partial_{[\mu} r^{\mu}_{\lambda]} = \partial_{\lambda} \log \Psi \qquad (4.3.1)$$

We have (here $\eta^{\mu\nu}$ indicates the Minkowski tensor)

$$\Gamma^{\sigma[\mu\nu]} \simeq \eta^{\mu\lambda}\eta^{\nu\alpha}\Gamma^{\sigma}_{\lambda\alpha} = \frac{1}{2}(\partial_{\alpha} r^{\sigma}_{\lambda} - \partial_{\lambda} r^{\sigma}_{\alpha})\eta^{\mu\lambda}\eta^{\nu\alpha} = \frac{1}{2}(\partial_{\alpha} r^{\sigma\mu}\eta^{\nu\alpha} - \eta^{\mu\lambda}\partial_{\lambda} r^{\sigma\mu})$$

Therefore, the expression $\Gamma^{\rho[\mu\nu]}\Gamma^{\sigma}_{[\nu\rho]}$ is of the second order and may be neglected. Then,

$$\Gamma^{\sigma[\mu\nu]}{}_{\nu} = \Gamma^{\sigma[\mu\nu]}{}_{,\nu} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha[\mu\nu]} + \Gamma^{\mu}_{\alpha\nu}\Gamma^{\sigma[\alpha\nu]} + \Gamma^{\nu}_{\alpha\nu}\Gamma^{\sigma[\mu\alpha]} \simeq \Gamma^{\sigma[\mu\nu]}{}_{,\nu} = \frac{1}{2}\left(\Box\, r^{\sigma\mu} - \eta^{\mu\lambda}\frac{\partial^{2} r^{\sigma\nu}}{\partial x^{\lambda}\partial x^{\nu}}\right)$$

From (4.1.6) the field equations (4.2.1) are written as

$$G^{\mu\sigma} \approx \Box\, r^{\sigma\mu} - \partial^{\mu}\, \partial_{\nu}\, r^{\sigma\nu} = 0 \qquad (4.3.2)$$

Presently, the field equations ar the (4.3.1) and (4.3.2). The symmetry transformations are the Lorentz rotations and the gauge transformations which act on the vierbeins. Under these later transformations we have for the variation of the $r^{\mu\nu}(x)$:

$$r'^{\mu\nu}(x) = r^{\mu\nu}(x) + \xi^{\mu}_{,\alpha}\, r^{\alpha\nu} + \xi^{\nu}_{,\beta}\, r^{\mu\beta} - \xi^{\alpha}\, r^{\mu\nu}_{,\alpha}$$

Thus,

$$r'^{\mu\nu}_{,\nu}(x) = r^{\mu\nu}_{,\nu}(x) + \xi^{\mu}_{,\alpha} r^{\alpha\nu}_{\nu} + \xi^{\mu}_{,\alpha\nu} r^{\alpha\nu} + \xi^{\nu}_{,\beta}\, r^{\mu\beta}_{\nu} + \xi^{\nu}_{,\beta\nu}\, r^{\mu\beta} - \xi^{\alpha}\, r^{\mu\nu}_{\alpha\nu} - \xi^{\alpha}_{,\nu}\, r^{\mu\nu}_{\alpha}$$

Imposing the gauge condition

$$r^{\mu\nu}_{,\nu} = 0 \quad , \qquad (4.3.3)$$

the ga ge function remains conditioned by the equations

$$\xi^{\mu}_{\alpha\nu}\, r^{\alpha\nu} + \xi^{\nu}_{,\beta}\, r^{\mu\beta}_{\nu} + \xi^{\nu}_{,\beta\nu}\, r^{\mu\beta} - \xi^{\alpha}_{,\nu}\, r^{\mu\nu}_{\alpha} = 0$$

Since both ξ^{μ} and $r^{\alpha\nu}$ are first order quantities these conditions are trivially satisfied in the first order approximation. From (4.3.3) we have:

$$r^{(\mu\nu)}_{,\nu} = 0 \quad , \quad r^{[\mu\nu]}_{,\nu} = 0 \qquad (4.3.4)$$

Note that under the linear approximation for the $h^{\mu}_{(\alpha)}$ written at the beginning of this section we have

$$g^{\mu\nu} = h^{\mu}_{(\alpha)} h^{\nu}_{(\beta)} \eta^{\alpha\beta} \approx \eta^{\mu\nu} + r^{\mu\nu} + r^{\nu\mu} = \eta^{\mu\nu} + 2r^{(\mu\nu)}$$

Then, the symmetric part of the $r^{\mu\nu}$ is expected to describe gravitation in this approximation. Since this is a field of spin 2 we still need another condition on the $r^{(\mu\nu)}$ (from the ten components of $r^{(\mu\nu)}$ we need five elements for the description of spin 2). As it is usually done in field theory, for the definition of a "TT" potential, we impose that

$$r = \eta_{\mu\nu}\, r^{(\mu\nu)} = 0 \qquad (4.3.5)$$

From the conditions (4.3.3) and (4.3.5) one has in equation (4.3.1)

$$\log \Psi = \text{constant}$$

Thus, for the gange conditions (4.3.3) and (4.3.5) the scalar Ψ becomes a constant. For the equation (4.3.2) one has

$$\Box\, r^{\sigma\mu} = 0 \qquad (4.3.6)$$

which means that both $r^{(\sigma\mu)}$ and $r^{[\sigma\mu]}$ satisfy a wave equation. Therefore, the field equations for these variables are wave equations supplemented by the conditions (4.3.4) and (4.3.5). This set of equations for the $r^{[\sigma\mu]}$ have the form of the free Maxwell's equations in the field intensity $F^{\sigma\mu} \equiv r^{[\sigma\mu]}$. The equations for the $r^{(\sigma\mu)}$ have the form of the equations for gravitational radiation in empty spaces (a "TT" field). In this approximation both fields are uncoupled. Indeed, in this formulation the coupling between the fields $r^{(\mu\nu)}$ and $r^{[\mu\nu]}$ is of the second-order, since the Maxwell's stress energy tensor $T^{\mu\nu}$ is quadratic in $r^{[\mu\nu]}$. By the other hand, in the exact theory the gravitational and electromagnetic fields are described by the set of equations (4.2.1) and (4.2.3), and in these equations we cannot separate the components which describe one of these fields from the other components, since it is not clear how one can impose gange conditions in the exact theory without breaking the symmetry of the group of general coordinate transformations.

However, the theory presents some problems. As was said its linear approximation describes free fields, and usually in field theory for a spin 2 field there exist interactions with electromagnetic fields via the Maxwell's tensor $T_{\mu\nu}$. For the exact theory exact solutions are not known. Thus, it is not clear if this formulation can indeed describe gravitational and electromagnetic phenomena at a classical level.

It should be mentioned that L.Civita (6) has shown that is possible to obtain all results of this theory working in a Riemannian space. Thus, it is possible that the change in language from the curvature to the torsion is merely a mathematical procedure.

5 - THE ASYMMETRIC UNITARY THEORY OF EINSTEIN-SCHRÖDINGER

In this section we will briefly review the unitary field theory proposed by Einstein (7) and by Schrödinger (8). In this theory use is made of a non-symmetric metric and of a non-symmetric connection. Two versions of the theory are known, one which use asymmetric real metric and connection, due both to Einstein and to Schrödinger, and the other which is based on a complex Hermitian Metric tensor. This later version is mainly due to Einstein. The mathematical results for these two versions are similar. Presently we consider only the case for complex metrics (9).

5.1 - INITIAL CONCEPTS

In general relativity, or in general for the unitary theories we have to determine the expression of the connection in terms of the metric tensor $g_{\mu\nu}$ through the conditions

$$D_\alpha g_{\mu\nu} = K_{\mu\nu,\alpha}$$

for given $K_{\mu\nu,\alpha}$. Thus, in general relativity $K_{\mu\nu,\alpha} = 0$, and imposing the conditions of symmetry for the connection, one finds $\Gamma^\lambda_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}$. In Weyl's theory $K_{\mu\nu,\alpha} = -\phi_\alpha \; g_{\mu\nu}$, and with the same symmetry requirement one determines Weyl's connection. In order to complete the structure of the theory we need the expression for the curvature, Weyl tensors, etc, and the field equations. These equations are of the second differential order, and will represent conditions on the curvature tensor. As example, we may have equations of the form

$$R_{\mu\nu} = 0 \quad \text{or} \quad G_{\mu\nu} = 0$$

which hold outside of the matter distributions. The definition of the energy-momentum tensor is phenomenalogical, and is distinguished from the definition of its action, namely, the field $g_{\mu\nu}$ which is by itself essentially geometric. The introduction of the tensor $T_{\mu\nu}$ on the right-hand-side of the field equations leads to the equations in presence of matter and energy.

The mathematical structure of these equations is, in general, required to be obtained from a variational principle. From such principle one constructs a Lagrangian density which is the sum of a free term L_0 associated to the field, and of an interacting term L_i which describes the coupling of the field with the external sources. This coupling is usually of the form $T_{\mu\nu} g^{\mu\nu}$ and is called as the "minimal interaction" (as an extension of the concept of minimal electromagnetic interaction). The field equations follow by variations on the potentials $g_{\mu\nu}$ (Hilbert's variational principle), or on the $g_{\mu\nu}$ and the $\Gamma^\mu_{\nu\rho}$ (Palatini's variational principle).

One of the basic properties of the unified theories is that no phenomenological term $T_{\mu\nu}$ is introduced in the Lagrangian density (or in the postulated field equations). The structure of the interactions of the field with the sources are expected to be described by the generalized field equations.

In these equations the field variables are not only the $g_{\mu\nu}$ but also the variables which describe the sources (as for instance, the electromagnetic potentials). Accordingly, we can only derive field equations with the structure of equations in a region exterior to matter. Thus, in such theories all factors are reduced to the geometrical properties of the space under conside

In 1945 Einstein proposed his last attempt for a consistent unitary theory, and he has worked in this theory up to 1955. Schrödinger has also worked in this theory and has made substantial mathematical improvements in the foundations of the formalism. The basic principles of the theory may be outlined as follows:

a - The metric $g_{\mu\nu}$ and the connection $\Gamma^{\rho}_{\mu\nu}$ are asymmetrical.

b - The skew symmetric part of $g_{\mu\nu}$ is proportional to an object which will be interpreted as the electromagnetic field strength.

c - The curvature has the only symmetry $R^{\mu}_{\ \nu\alpha\beta}(\Gamma) = -R^{\mu}_{\ \nu\beta\alpha}(\Gamma)$.

In order to determine the expression for the field equations one has to work out some mathematical properties of geometries with Hermitian metrics.

5.2 - SOME MATHEMATICAL RESULTS ASSOCIATED TO EINSTEIN'S THEORY

The bilinear form defining the inner product of vectors in spaces with real, symmetric metrics is presently replaced by a sesquilinear form which determines the inner product of two arbitrary complex vectors.

$$g(A,B) = g_{\mu\nu} A^{\mu} B^{*\nu} = A_{\nu} B^{*\nu} \tag{5.2.1}$$

with the property

$$g(A,B) = [g(B,A)]^{*}$$

Accordingly, the $g_{\mu\nu}$ is a Hermitian second-rank tensor

$$g_{\mu\nu} = g^{*}_{\mu\nu} \tag{5.2.2}$$

The coordinates (x^{α}) are still real quantities. Thus, under coordinate transformations we have a formula similar to that used in general relativity.

The affine connection is also asymmetric, and satisfies a condition similar to (5.2.2):

$$\Gamma^{\mu}_{\nu\alpha} = \Gamma^{*\mu}_{\alpha\nu} \tag{5.2.3}$$

Objects satisfying (5.2.2) and (5.2.3) are of the form

$$g_{\mu\nu} = g_{(\mu\nu)} + i\, f_{[\mu\nu]} \qquad , \qquad \Gamma^{\mu}_{\nu\alpha} = \Gamma^{\mu}_{(\nu\alpha)} + i\, B^{\mu}_{[\nu\alpha]}$$

Due to the asymmetry of the connection one may define two types of variation under parallel displacement. These expression are denoted by

$$\delta \overset{\mu}{A_{+}} = - \Gamma^{\mu}_{\sigma\rho} A^{\sigma} dx^{\rho}$$

$$\delta \overset{\mu}{A_{-}} = - \Gamma^{\mu}_{\rho\sigma} A^{\sigma} dx^{\rho}$$

Accordingly, one has two kinds of covariant derivatives

$$\overset{\mu}{A}_{\overset{+}{;}\rho} = \partial_{\rho} A^{\mu} + \Gamma^{\mu}_{\sigma\rho} A^{\sigma}$$

$$\overset{\mu}{A}_{\overset{-}{;}\rho} = \partial_{\rho} A^{\mu} + \Gamma^{\mu}_{\rho\sigma} A^{\sigma}$$

and for a covariant vector

$$B_{\underset{+}{\mu};\rho} = \partial_{\rho} B_{\mu} - \Gamma^{\sigma}_{\mu\rho} A_{\sigma}$$

$$B_{\underset{-}{\mu};\rho} = \partial_{\rho} B_{\mu} - \Gamma^{\sigma}_{\rho\mu} A_{\sigma}$$

The process of raising (lowering) indices is defined as

$$A^{\mu} = A_{\nu}\, g^{\nu\mu} \qquad A^{*\mu} = g^{\mu\nu} A^{*}{}_{\nu}$$

$$A_{\mu} = A^{\nu}\, g_{\nu\mu} \qquad A^{*}{}_{\mu} = g_{\mu\nu} A^{*\nu}$$

$$g^{\lambda\nu} g_{\lambda\mu} = g^{\nu\lambda} g_{\mu\nu} = \delta^{\lambda}_{\mu}$$

A curvature of rotation is introduced according to the formula derived previously

$$\oint_C \underset{+}{\delta} A_{\mu} = \frac{1}{2}\int_S R^{\alpha}{}_{\mu\rho\sigma}\; A_{\nu}\; ds^{\rho\sigma}$$

$$R^{\nu}{}_{\mu\rho\sigma} = \partial_{\sigma}\Gamma^{\nu}_{\mu\rho} - \partial_{\rho}\Gamma^{\nu}_{\mu\sigma} + \Gamma^{\nu}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\rho} - \Gamma^{\nu}_{\lambda\rho}\Gamma^{\lambda}_{\mu\sigma} = R^{\nu}{}_{\mu\rho\sigma}(\Gamma)$$

This affine curvature tensor also follows from the commutator

$$\overset{\sigma}{A}{}_{\underset{+}{+};\mu;\nu} - \overset{\sigma}{A}{}_{\underset{+}{+};\nu;\mu} = R^{\sigma}{}_{\lambda\mu\nu} + 2\,\Gamma^{\lambda}_{[\mu\nu]}\,\overset{\sigma}{A}{}_{+;\lambda}$$

In general, the curvature is a complex set of quantities with the form

$$R^{\sigma}{}_{\mu\rho\lambda}(\Gamma) = T^{\sigma}{}_{\cdot\mu\rho\lambda} + i\, V^{\sigma}{}_{\cdot\mu\rho\lambda}$$

where

$$T^{\sigma}{}_{\mu\rho\lambda}\, G^{\sigma}{}_{\mu\rho\sigma} + \Gamma^{\sigma}_{[\alpha\lambda]}\Gamma^{\alpha}_{[\mu\rho]} - \Gamma^{\sigma}_{[\alpha\rho]}\Gamma^{\alpha}_{[\mu\lambda]} + 2\Gamma^{\alpha}_{[\lambda\rho]}\Gamma^{\sigma}_{[\mu\alpha]}$$

$$V^{\sigma}{}_{\mu\rho\lambda} = \overset{\sigma}{\Gamma}{}^{+}_{\underset{++}{[\mu\rho]};\nu} - \overset{\sigma}{\Gamma}{}^{+}_{\underset{++}{[\mu\nu]};\rho}$$

$G^{\sigma}_{\mu\rho\lambda}$ being the affine curvature tensor constructed with the symmetric part of the connection $\Gamma^{\lambda}_{\mu\nu}$. We mention that here $\Gamma^{\lambda}_{(\mu\nu)}$ is not necessarily equal to the Christoffel symbols. Indeed up to now we have not derived a metricity condition.

Thus, the complex Ricci tensor is of the form

$$R^{(\Gamma)}_{\mu\rho} = R^{\sigma}_{\mu\rho\sigma}(\Gamma)$$

It may be shown that $R_{\mu\rho}(\Gamma)$ is Hermitian if the vector of torsion of the Hermitian connection vanishes (10):

$$\Gamma_{\rho} = \Gamma^{\sigma}_{[\rho\sigma]} = 0 \tag{5.2.4}$$

In the definition of the connection we follow the Schrödinger's prescription of introducing another affine connection $W^{\lambda}_{\mu\nu}$ such that

$$\Gamma^{\lambda}_{\mu\nu} = W^{\lambda}_{\mu\nu} + \frac{2}{3}\,\delta^{\lambda}_{\mu}\, W_{\nu} \tag{5.2.5}$$

$$W_{\nu} = W^{\sigma}_{[\nu\sigma]}$$

Note that $W^{\lambda}_{\mu\nu}$ is not Hermitian. From (5.2.5) it follows that the vector Γ_{ρ} vanishes. Accordingly, $R_{\mu\nu}(\Gamma)$ is Hermitian.

5.3 - THE FIELD EQUATIONS

Since $g_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\alpha}$ are in principle independent quantities, the field equations have to follow from Palatini's variational principle. For the Lagrangian density Einstein chooses the same formal expressions for general relativity:

$$L = \sqrt{-g} \quad g^{\mu\nu} \quad R_{\mu\nu}(W)$$

where $R_{\mu\nu}(W)$ is the Ricci tensor for the affine connection W. The field equations have the form

$$\frac{\delta L}{\delta W^{\alpha}_{\mu\nu}} = (\sqrt{-g} \quad \overset{\mu\nu}{g{+}{-}})_{;\alpha} = 0 \tag{5.3.1}$$

$$\frac{\delta \; L}{\delta(\sqrt{-g} \quad g_{\mu\nu})} = R_{\mu\nu}(W) = 0 \tag{5.3.2}$$

where the semi-colon denotes covariant derivative for the connection Γ. From (5.3.1) it follows by contraction

$$\partial_{\rho} \; (\sqrt{-g} \quad g^{[\mu\rho]}) = 0 \tag{5.3.3}$$

Equation (5.3.2) can also be written in terms of Γ as

$$R_{(\mu\nu)}(\Gamma) = 0 \tag{5.3.4}$$

$$R_{[\mu\nu]}(\Gamma) = \frac{2}{3} \; (W_{\nu,\mu} - W_{\mu,\nu}) \tag{5.3.5}$$

(use that $R_{\mu\nu}(\Gamma) = R_{\mu\nu}(W) + \frac{2}{3} \; (W_{\nu,\mu} - W_{\mu,\nu})$).

From the structure of (5.3.3) we see that it has the form of the Maxwell equations and (5.3.5) corresponds to the remaining electromagnetic equations. Equation (5.3.4) will describe the gravitational field. However, we still have the equations (5.3.1). Such equations in general relativity determine the connection in closed form. Presently, there is no exact closed solution for these equations. This means that the connection $\Gamma^{\mu}_{\nu\rho}$ plays the role of a dynamical function, similarly to the $g_{\mu\nu}$ and the W_{μ} (the vector W_{μ} is proportional to the electromagnetic potentials). From the point of view of a comparison with general relativity, this set of equations are much more complicated than the Einstein-Maxwell equations.

5.4 - INTERPRETATION OF THE FIELD EQUATIONS

Using a comparison with general relativity, the theory has only an "exterior region" since $T_{\mu\nu}$ is not present in the field equations. The action of $T_{\mu\nu}$, that means the generation of gravitational fields, is therefore included in the geometrical factor $R_{(\mu\nu)}(\Gamma)$in eqs. (5.3.4). However, in order to obtain such result one has to know explicitly the expression for the $\Gamma^{\mu}_{\nu\rho}$. Accordingly, (5.3.4) and (5.3.1) have to be solved simultaneously. The remaining equations also have to be solved with these later two equations since they involve the metric, connection and the vector W_{μ} which is part of the connection (see (5.2.5)). As result, the question of obtaining exact solutions for these equations is complicated. In the literature the exact solution for spherical symmetry is known (11). However, this solution does not generate a correct motion for charged test particles in the field (12).

Presently, we will consider the more simple situation corresponding to the weak field approximation (13). We have the two quantities $g_{(\mu\nu)}$ and $g_{[\mu\nu]}$. In a process of linearization we may expand both objects in power series of some infinitesimal parameters. Here we will consider a more general situation where only the $\overset{*}{g}_{[\mu\nu]} = \phi_{\mu\nu}$ is subjected to such expansion. Writing

$$\phi_{\mu\nu} = \varepsilon \underset{1}{\phi}_{\mu\nu} + \varepsilon^2 \underset{2}{\phi}_{\mu\nu} + \dots$$

$$\gamma_{\mu\nu} = g_{(\mu\nu)}$$

and using the notation $B^{\mu} = \gamma^{\mu\nu} B_{\nu}$, which holds for arbitrary B_{ν}, we can write (5.3.3)in the first order approximation as

$$\partial_\rho(\sqrt{-\gamma}\ \underset{1}{\phi}^{\mu\rho}) = \nabla_\rho\ \underset{1}{\phi}^{\mu\rho} = 0 \tag{5.4.1}$$

where ∇_ρ indicates a covariant derivative with respect to a connection $\{{}^{\alpha}_{\mu\nu}\}_\gamma$ constructed with the metric $\gamma_{\mu\nu}$. Equation (5.3.5) takes the form

$$\Box\ \underset{1}{\phi}_{\mu\nu} - \frac{1}{2}\nabla^\rho\ \phi_{\mu\nu\rho} = \frac{2}{3}(\partial_\mu W_\nu - \partial_\nu W_\mu) \quad , \tag{5.4.2}$$

for,

$$\phi_{\mu\nu\rho} = \partial_\mu\ \underset{1}{\phi}_{\nu\rho} + \partial_\rho\ \underset{1}{\phi}_{\mu\nu} + \partial_\nu\ \underset{1}{\phi}_{\rho\mu} = \nabla_\mu\ \underset{1}{\phi}_{\nu\rho} + \nabla_\rho\ \underset{1}{\phi}_{\mu\nu} + \Phi_\nu\ \underset{1}{\phi}_{\rho\mu} = -\ \phi_{\nu\mu\rho}$$

The equation (5.3.4) for the gravitational field assumes the form

$$R_{\mu\nu} - \nabla^\rho(\underset{1}{\phi}_{\mu\lambda}\nabla^\lambda\underset{1}{\phi}_{\nu\rho} + \underset{1}{\phi}_{\nu\lambda}\nabla^\lambda\underset{1}{\phi}_{\mu\rho}) + \frac{1}{2}\nabla^\rho(\underset{1}{\phi}_{\mu\lambda}\phi_{\nu\rho}{}^\lambda + \underset{1}{\phi}_{\nu\lambda}\phi_{\mu\rho}{}^\lambda) - \frac{1}{2}\nabla_\mu\nabla_\nu \log\gamma - (\nabla_\sigma\underset{1}{\phi}_{\mu\rho} - \frac{1}{2}\phi_{\mu\rho\sigma})(\nabla^\rho\underset{1}{\phi}{}^\sigma_\nu - \frac{1}{2}\phi^\sigma{}_\nu{}^\rho) = 0 \tag{5.4.3}$$

writing

$$\partial_\mu W_\nu - \partial_\nu W_\mu = -\frac{3}{4}\ X\ F_{\mu\nu}$$

We can rewrite (5.4.2) as

$$\Box\ \underset{1}{\phi}_{\mu\nu} = -\ X\ F_{\mu\nu} - G^{\tau\sigma}{}_{\mu\nu}\ \underset{1}{\phi}_{\tau\sigma} \tag{5.4.4}$$

where $G^{\sigma\tau}{}_{\mu\nu} = \gamma^{\sigma\rho}\ \gamma^{\tau\lambda}\ G_{\rho\lambda\mu\nu}$ and $G_{\rho\eta\mu\nu}$ is the Riemann-Christoffel tensor, and $\Box = \nabla^\rho\nabla_\rho$.

The theory is complicated in this approximation since we have two second-order anti-symmetrical tensors: $F_{\mu\nu}$ and $\underset{1}{\phi}_{\mu\nu}$ for the description of electromagnetism. For the equations associated to gravitation, one finds after some calculations

$$R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} R = X\ (\tau_{\mu\nu} + M_{\mu\nu} + X_{\mu\nu} + Y_{\mu\nu}) \tag{5.4.5}$$

where,

$$\tau_{\mu\nu} = -\frac{1}{2X}(\underset{1}{\phi}_{\mu\tau}F^\tau_\nu + \underset{1}{\phi}_{\nu\tau}F^\tau_\mu) + \frac{1}{4X}\gamma_{\mu\nu}\ \underset{1}{\phi}_{\lambda\tau}\ F^{\lambda\tau}$$

$$M_{\mu\nu} = \frac{1}{4X}(\phi_{\tau\mu\rho}\ \phi^\tau{}_\nu{}^\rho - \frac{1}{6}\gamma_{\mu\nu}\ \phi^{\rho\sigma\lambda}\ \phi_{\rho\sigma\lambda})$$

$$X_{\mu\nu} = \frac{1}{X}\Big[(\nabla^\rho\ \underset{1}{\phi}_{\nu\lambda})\ (\nabla^\lambda\ \underset{1}{\phi}_{\mu\rho}) + \frac{1}{4}\nabla_\mu\ \nabla_\nu(\underset{1}{\phi}_{\rho\sigma}\ \underset{1}{\phi}^{\rho\sigma}) - \frac{1}{2}\gamma_{\mu\nu}(\nabla_\lambda\underset{1}{\phi}_{\rho\sigma})(\nabla^\lambda\underset{1}{\phi}^{\rho\sigma}) + \frac{1}{2}\gamma_{\mu\nu}\underset{1}{\phi}_{\rho\sigma}F^{\rho\sigma})\Big]$$

$$Y_{\mu\nu} = -\frac{1}{2X}\Big[G^{\rho\sigma}{}_{\nu\tau}\underset{1}{\phi}_{\rho\sigma}\underset{1}{\phi}_\mu{}^\tau + G^{\rho\sigma}{}_{\mu\tau}\underset{1}{\phi}_{\rho\sigma}\underset{1}{\phi}_\nu{}^\tau - \frac{2}{3}\gamma_{\mu\nu}\ G^{\rho\sigma}{}_{\lambda\tau}\underset{1}{\phi}_{\rho\sigma}\underset{1}{\phi}^{\lambda\sigma}\Big]$$

Thus, several candidates to tensors $T_{\mu\nu}$, of the sources, appear in the right-hand-side of equations (5.4.5). Of such candidates only $\tau_{\mu\nu}$ has the form of a Maxwell stress tensor if $\underset{1}{\phi}_{\mu\nu} = F_{\mu\nu}$. However, there is no possibility to get rid of the remaining tensors. Due to such difficulties, it is not easy to interpret the physical content of such approximation. The reader should consult the literature for further discussions on this subject.

A weaker approximation is obtained by taking a series expansion on the gravitational variables

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\ \underset{1}{\gamma}_{\mu\nu} + \varepsilon^2\ \underset{2}{\gamma}_{\mu\nu} + \ldots$$

The equations for the electromagnetic field take the form

$$\eta^{\rho\sigma}\,\partial_\sigma\,\underset{1}{\phi}_{\mu\rho} = 0$$

$$\overset{o}{\Box}\,\underset{1}{\phi}_{\mu\nu} = -\,X\,F_{\mu\nu} \quad , \qquad \overset{o}{\Box} = \partial^\mu\,\partial_\mu$$

with the conditions

$$\eta^{\rho\sigma}\,\partial_\sigma\,\phi_{\mu\nu\rho} = -\,x\,F_{\mu\nu}$$

$$\eta^{\rho\sigma}\,\partial_\rho\,F_{\mu\sigma} = 0$$

$$\Box\,\phi_{\mu\nu\rho} = 0$$

For gravitation one has the equations

$$\underset{1}{R}_{\mu\nu} = 0$$

As was to be expected we still have two skew symmetric fields $\underset{1}{\phi}_{\mu\nu}$ and $F_{\mu\nu}$. Besides this the gravitational and electromagnetic equations are not coupled in this approximation. This result is not consistent since it is known that even gravitational theories in special relativity explain the deviation of light vays in the presence of gravitating masses. The coupling between the two fields will appear in the second-order approximation, with the presence of three sources factors:

$$\underset{2}{R}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\underset{2}{R} - \frac{1}{2}\,\underset{1}{R}\,\underset{1}{\gamma}_{\mu\nu} = x\,(\underset{2}{\tau}_{\mu\nu} + \underset{2}{M}_{\mu\nu} + \underset{2}{X}_{\mu\nu})$$

However, explicit calculations apparently, do not conduct to results similar to those obtained in the Einstein-Maxwell formalism.

6 - EXTENSION OF THE ASYMMETRIC THEORY-CORRESPONDENCE PRINCIPLE WITH THE EINSTEIN-MAXWELL FORMULATION

Bonnor (14) and Kursunoglu (15) have considered the possibility of modifying Einstein's non-symmetric theory. Based on such results Moffat (16) has developed an extension of Einstein's theory in such way that a correspondence principle with the theory of general relativity may be obtained. Presently we make a short review of Moffat's theory. The Lagrangian density is given by

$$L = \sqrt{-g}\;\;R(W) + \frac{4\pi\;G}{k^2c^4}\,\sqrt{-g}\;\;g^{[\mu\nu]}\;\;g_{[\nu\mu]} \tag{6.1}$$

k is a constant to be specified later. This Lagrangian has the form of the Einstein Lagrangian plus a "Maxwell term". Similarly as before one has to use the Palatini's variational principle. The field equations have the form

$$(\sqrt{-g}\;\;\overset{\mu\nu}{g_{+-}})_{;\alpha} = 0 \tag{6.2}$$

$$\partial_\alpha(\sqrt{-g}\;\;g^{[\mu\alpha]}) = 0 \tag{6.3}$$

$${}^*R_{(\mu\nu)}(\Gamma) = 0 \tag{6.4}$$

$$ {}^*R_{[\mu\nu]}(\Gamma) = \frac{2}{3}\,(W_{\nu,\mu} - W_{\mu,\nu}) \tag{6.5}$$

The only difference with the equations of the Einstein theory comes from the new quantities ${}^*R_{\mu\nu}$ which are given by

$${}^*R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\Gamma) + \frac{4\pi\;G}{k^2c^4}\;\;I_{\mu\nu} \tag{6.6}$$

$$I_{\mu\nu} = -\,(g_{[\mu\nu]} + g^{[\lambda\beta]}\;\;g_{\beta\nu}\;g_{\mu\lambda} + \frac{1}{2}\,g^{[\beta\lambda]}\;\;g_{\beta\lambda}\;g_{\mu\nu}) \tag{6.7}$$

Since the dimension of R(W) is L^{-2} (the metric has no dimensions), it follows that the term $4\pi G/k^2c^4$ in (6.1) has this same dimension. Thus, the constant k has the dimensions

$$\dim\ k = L^{1/2}\ M^{-1/2}\ T = \frac{L^2}{\dim\ |e|}$$

where |e| stands for the electric charge. We take k as a pure imaginary quantity. From this analysis we can write

$$g_{[\mu\nu]} = k\ F_{\mu\nu} = i\ K\ F_{\mu\nu}$$

The characteristic lenght which may be constructed with the constants ħ, G and c is the Planck lenght (17)

$$L = (\frac{\hbar G}{c^3})^{1/2} = 1.62 \times 10^{-23}\ cm$$

Using this value one obtains for the constant K the following value

$$K = \frac{\hbar\ G}{c^3 e} = 5.44 \times 10^{-57}\ g^{-1/2}\ cm^{1/2}\ seg$$

Another quantity with the dimension of a lenght and defined in function of classical variables e, G and c is

$$L' = \frac{eG^{1/2}}{c^2} = 1.38 \times 10^{-34}\ cm$$

which gives

$$K' = \frac{L'^2}{e} = 3.95 \times 10^{-59}\ g^{-1/2}\ cm^{1/2}\ seg$$

The relation between K' and K being given by

$$\frac{K'}{K} = \frac{L'^2}{L^2} = \frac{e^2}{\hbar c} = \frac{1}{137}$$

Thus, the difference between K and K' is very small. We will use the identification

$$A_\mu = \frac{kc^4}{12\pi G} W_\mu \tag{6.8}$$

In the limit $K \to 0$, $g_{[\mu\nu]} \to 0$ and so $g_{\mu\nu} \to g_{(\mu\nu)}$. It may be shown that in this limit

$$\frac{4\pi\ G}{k^2c^4}\ I_{(\mu\nu)} \to -\frac{8\pi\ G}{c^4}\ T^{Maxw.}_{\mu\nu}$$

$$\frac{4\pi\ G}{k^2c^4}\ I_{[\mu\nu]} \to -\frac{8\pi\ G}{kc^4}\ F_{\mu\nu}$$

Since in this region $g_{\mu\nu} \to g_{(\mu\nu)}$, $R_{[\mu\nu]}(\Gamma) \to 0$ and $R_{(\mu\nu)}(\Gamma) \to$ Ricci Christoffel tensor. Then, (6.4) and (6.5) assume the form

$$R_{\mu\nu} - \frac{8\pi\ G}{c^4}\ T^{Maxw.}_{\mu\nu} = 0 \tag{6.9}$$

$$-\frac{8\pi G}{kc^4}\ F_{\mu\nu} = \frac{2}{3}\ (W_{\nu,\mu} - W_{\mu,\nu}) \tag{6.10}$$

From (6.8) we have for (6.10)

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{6.11}$$

The remaining field equations, namelly equations (6.2) and (6.3) take the form,

$$g_{\mu\nu;\alpha} = 0 \tag{6.12}$$

$$\partial_\alpha(\sqrt{-g}\ \ F^{\mu\alpha}) = 0 \qquad (6.13)$$

Thus, from (6.9), (6.11), (6.12) and (6.13) we see that the field equations in this limit describe the Einstein-Maxwell theory. The effect of introducing a term quadratic in the quantities $g_{[\mu\nu]}$ in the Lagrangian density is to achieve a correspondence with the known results of general relativity.

The equations of Moffat's theory can be solved for the case of spherical symmetry and a-symmetric static pointlike fields (exact solution corresponding to Schwarzschild's solution in general relativity). In the limit $K \to 0$ the Reissner-Nosdstrom solution of the Einstein-Maxwell equations is obtained. This solution has the following properties:

(1) It has a singularity associated to the system of coordinates at $r = m + (m^2 + q^2/2)^{1/2}$ similar to that presented by the Reissner-Nosdstrom solution. Such singularity may be removed by a Kruskal transformation.

(2) The geometry associated to this solution has a sphere of radius $r_s = \sqrt{Kq}$, $K = L^2/q$ which acts as a surface of barrier for the singularity at $r = 0$. This surface is non-singular and analytic for the $g_{\mu\nu}$ solution of the field equations. The curvature tensors $R^\lambda{}_{\mu\nu\sigma}$ and $R_{\mu\nu}$ are regular at $r_s \sim L$.

(3) The component g_{oo} has the form

$$g_{oo} = \left(1 - \frac{2Gm}{c^2 r} + \frac{4Ge^2}{c^4 r^2}\right)\left(1 - \frac{k^2e^2}{r^4}\right)$$

Taking

$$K = \frac{L^2}{|e|} > 0$$

we have

(a) If $r > \sqrt{Ke} = L$, then $K^2e^2/r^4 < 1$. This means that the above value for g_{oo} follows the usual sign for the g_{oo} of the Reissner-Nordstrom's solution. In this region, for $r \to \infty$ one gets $g_{oo} \to 1$. Thus, the signature for the region $r > r_s$ is $(---+)$.

(b) If $r < \sqrt{Ke} = L$, we get $K^2e^2/r^4 > 1$ and g_{oo} changes sign. This happens inside of the sphere. Thus, the signature becomes elliptical: $(----)$. In this region the spacetime has the structure of an Euclidean fourdimensional space. Since there we cannot define a light cone, it follows that light rays cannot enter into this region.

(c) It follows that physical trajectories (represented by time-like, or by null lines) tend to be deflected from the surface of the sphere (time-like paths require indefinite metrics, and they do not exist inside of the sphere). In this way the singularity at $r = 0$ is not crossed by physical test particles.

(d) The motion of test-particles in this field may be treated and it generates trajectories which do not cross the sphere of radius $r = r_s$.

Several other problems related to practical applications of this theory have been published (18), and we refer the reader to the recent literature.

Finally, it is interesting to observe that the asymmetric theory proposed by Moffat can be generalized in order to include internal symmetries (SU(n)). In particular, for $n = 2$ we can include the standard Yang-Mills field in this theory (19). The new theory which is obtained tends to the Einstein-Maxwell-Yang Mills formalism in the limit where $K \to 0$. Mathematically, this generalization of Moffat's theory is obtained by the transition: $g_{\mu\nu}(x)$ (sixteen complex functions) $\to G_{\mu\nu}(x)$ (sixteen 2 x 2 complex matrices, given by $G_{\mu\nu}(x) = q_{\mu\nu i}(x)\sigma_i$. Here $\sigma_i = (\sigma_o, \vec{\sigma})$ are the Pauli matrices plus the 2 x 2 identity matrix).

The structure of the vierbiens associated to this extended version of the theory may be determined (20). The vierbeins being represented by complex 2 x 2 matrices.

7 - EXAMPLE OF AN UNITARY FIELD THEORY DERIVED FROM THE GAUGE FORMALISM

In this section we consider the gauge formalism, as developed in conventional field theory, as the basic structure. Following with the results of such formalism we will show how one can obtain an unitary theory based on a background space with a Hermitian metric. Since such method is an extension of the gauge theory of the group $SL_2(C)$, we first give a short review of this theory, presently written in terms of the vierbeins.

7.1 - INTERNAL STRUCTURE CORRESPONDING TO THE LOCAL GROUP SO (3,1)

Presently we consider the field of vierbeins as the basic quantities of the theory. Accordingly, the gauge theory which is obtained corresponds to the local group SO(3,1) which is isomorphical to the group of linear unimodular transformations acting in the two-dimensional complex spinor space. The vierbein associated to the Riemannian metric may be indicated by $e_a = (e^{\mu}_a)$. In this notation the world index μ denotes the four local vectors. These vectors form a column matrix e_a. In what follows we shall use matrix notation. The vierbeins may be looked as the set of four local vectors $e^1_a, \ldots e^0_a$ which transform the Riemannian metric $g_{\mu\nu}$ into the Lorentzian metric η_{ab}:

$$\eta_{ab} = e^T_a \cdot g \cdot e_b \tag{7.1.1}$$

$$g = (g_{\mu\nu}) = g^T$$

$$e^*_a = e_a$$

The gauge transformations are given by the local rotations of the vierbeins. It will be of more interest to consider the vierbeins as the column matrix

$$e_\mu = (e^a_\mu) = \begin{pmatrix} e^0 \\ \vdots \\ e^3_\mu \end{pmatrix}$$

and use the notation

$$\bar{e}_\mu = e^T_\mu \cdot \eta \qquad , \qquad \eta = (\eta_{ab}) = \eta^T$$

Then,

$$g_{\mu\nu} = \bar{e}_\mu \cdot e_\nu \tag{7.1.2}$$

are the inverse relations corresponding to (7.1.1). Thus, the local Lorentz transformations are written as

$$e'_\mu(x) = L(x)\, e_\mu(x) \quad , \quad L^T \cdot \eta \cdot L = \eta$$

which imply that

$$\bar{e}'_\mu(x) = \bar{e}_\mu(x)\, L^{-1}(x)$$

under such transformations one has $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$. From the metricity conditions $g_{\mu\nu;\alpha} = 0$ one gets from (7.1.2)

$$e_{\mu|\alpha} = e_{\mu;\alpha} + \Omega_\alpha\, e_\mu = 0$$

$$\bar{e}_{\mu|\alpha} = \bar{e}_{\mu;\alpha} - \bar{e}_\mu\, \Omega_\alpha = 0$$

These two equations are algebraically dependent if

$$\Omega^T_\mu = - \eta \cdot \Omega_\mu \cdot \eta^{-1} \quad , \tag{7.1.3}$$

and this implies that Tr Ω_μ vanishes. Equivalently, from (7.1.3) one has that $w_\mu = \eta \cdot \Omega_\mu$ is antisymmetric. In order to interpret this formalism as a gauge theory we introduce a vector $\Psi = (\Psi^a)$ in a internal space by

$$\Psi^a = e^a_\mu \Psi^\mu \tag{7.1.4}$$

Then,

$$\Psi'(x) = L(x)\ \Psi(x) \tag{7.1.5}$$

Since $\Psi'_{,\alpha}$ does not transform as Ψ under local SO(3,1) transformations one introduces a covariant derivative

$$\Psi_{|\alpha} = \partial_\alpha \Psi + \Omega_\alpha \Psi$$

such that

$$\Psi'_{|\alpha} = L\ \Psi_{|\alpha}$$

Accordingly, the connection Ω_α associated to SO(3,1) transforms as

$$\Omega'_\alpha = L \cdot \Omega_\alpha \cdot L^{-1} - L_{,\alpha} L^{-1} \tag{7.1.6}$$

In the gauge theory Ω_α is called as the gauge field. In the usual gauge theory of internal groups SU(n), the matrix L of (7.1.5) is a (n x n) unitary matrix in internal space. The simplest example is for n = 1, where we have the gauge group U(1) of phase transformations of some complex scalar field $\phi(x)$. In this case $L = e^{i\Lambda}$ and (7.1.6) gives the well known gauge transformation for the electromagnetic potentials: $\Omega_\alpha \to A_\alpha$.

For Riemannian spacetimes one can derive the explicit expression for the connection Ω_α from the conditions $e_{\mu|\alpha} = 0$.

$$\Omega_\alpha = - e_{\mu;\alpha} \cdot \bar{e}^\mu = e^\mu \cdot \bar{e}_{\mu;\alpha} \tag{7.1.7}$$

By this formula we again see that Ω_α, for each fixed value for α, is a matrix in internal space. The equations (7.1.7) are called as the Fock-Ivanenko connection.

The internal curvature, or field intensity associated to this gauge theory, is given by the commutator

$$(\Psi_{|\mu})_{|\nu} - (\Psi_{|\nu})_{|\mu} = P_{\nu\mu} \cdot \Psi \tag{7.1.8}$$

$$P_{\nu\mu} = \partial_\nu \Omega_\mu - \partial_\mu \Omega_\nu + [\Omega_\nu\ \Omega_\mu] \tag{7.1.9}$$

From (7.1.3) and (7.1.9) we have

$$P^T_{\mu\nu} = - \eta \cdot P_{\mu\nu} \cdot \eta^{-1} \tag{7.1.10}$$

Thus, the trace of $P_{\mu\nu}$ vanishes: Tr $P_{\mu\nu} = 0$. By (7.1.9) we see that Tr $P_{\mu\nu}$ is an Abelian field intensity, but for SO(3,1) we will not have such field. The integrability conditions for the equations $e_{\mu|\alpha} = 0$ give:

$$P_{\mu\nu} = R^\lambda{}_{\alpha\mu\nu}\ e^\alpha \cdot \bar{e}_\lambda \tag{7.1.11}$$

where $R^\lambda{}_{\alpha\mu\nu}$ is the Riemann-Christoffel tensor. Under the transformations of the SO(3,1) group one has

$$P'_{\mu\nu} = L \cdot P_{\mu\nu} \cdot L^{-1} \tag{7.1.12}$$

7,2 - EXTENSION FOR A HERMITIAN METRIC

The complex metric $g_{\mu\nu}$ satisfies the symmetry conditions $g_{\mu\nu} = g^*_{\nu\mu}$. These conditions generalize the Riemannian conditions $g_{\mu\nu} = g_{\nu\mu}$. The vierbein formalism for Hermitian metrics is known in the literature (21). Presently we use this formalism as a natural complex generalization of the results of the section (7.1). Instead of the equation (7.1.2), here we have

$$g_{\mu\nu} = \bar{e}_\mu \cdot e_\nu \tag{7.2.1}$$

$$\bar{e}_\mu = e^+_\mu \cdot \eta \tag{7.2.2}$$

where $e^+_\mu = e^{T*}_\mu$ is the Hermitian conjugate of the column matrix e_μ.

A natural extension of the equations $g_{\mu\nu;\alpha} = 0$ of the section (7.2) are given by the relations

$$g_{\underset{+-}{\mu\nu};\alpha}(\Gamma) = 0 \tag{7.2.3}$$

We recall that such relations are part of the field equations for the Einstein and for the Moffat theory. The spacetime connection which appear in these equations satisfy the symmetry properties:

$$\Gamma^\alpha_{\mu\nu} = \Gamma^{*\alpha}_{\nu\mu}$$

However, presently we will see that equations of the form (7.2.3) have to be postulated. Assuming that (7.2.3) holds we have from (7.2.1)

$$e_{\underset{\pm}{\mu}|\nu} = e_{\underset{\pm}{\mu};\alpha} + \Omega_\nu . e_\mu = 0 \tag{7.2.4}$$

$$\bar{e}_{\underset{\pm}{\mu}|\nu} = \bar{e}_{\underset{\pm}{\mu};\alpha} - \bar{e}_\mu . \Omega_\nu = 0 \tag{7.2.5}$$

As before, these two equations are algebraically dependent if

$$\Omega^+_\nu = - \eta \cdot \Omega_\nu \cdot \eta^{-1} \tag{7.2.6}$$

Thus, here Tr Ω_ν does not vanish and is a pure imaginary quantity. From (7.2.4), or equivalently from (7.2.5) one derives the explicity expression for the internal connection

$$\Omega_\nu = e^\mu \cdot \bar{e}_{\underset{\pm}{\mu;\nu}} = e^\mu . (\partial_\nu \bar{e}_\mu - \Gamma^\lambda_{\mu\nu} \bar{e}_\lambda) \tag{7.2.7}$$

The extension for complex vierbeins of the local Lorentz rotations is given by the transformations

$$e'_\mu(x) = L(x) \cdot e_\mu(x)$$

$$\bar{e}'_\mu(x) = \bar{e}_\mu(x) \cdot L^{-1}(x) \quad , L^+ \cdot \eta \cdot L = \eta \tag{7.2.8}$$

These transformations define the local group U(3,1). The connection Ω_ν is the compensating field associated to such transformations. The variation $\delta\Omega_\nu = \Omega'_\nu(x) - \Omega'_\nu(x)$ has a form similar to that obtained in the SO(3,1) gauge theory (see equations (7.1.6)).

A field intensity is here defined similarly to the definition previously made in the section (7.1).

$$P_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + \left[\Omega_\mu, \Omega_\nu\right] \tag{7.2.9}$$

Here we have

$$P^{+}_{\mu\nu} = - \eta \cdot P_{\mu\nu} \cdot \eta^{-1} \qquad (7.2.10)$$

As consequence Tr $P_{\mu\nu}$ is a pure imaginary antisymmetric second rank tensor. Note that instead of (7.2.10) one can use the equivalent conditions:

$$w_{\mu\nu} = \eta \cdot P_{\mu\nu} = - w^{+}_{\mu\nu}$$

Under the transformations (7.2.8) one gets

$$P'_{\mu\nu} = L \cdot P_{\mu\nu} \cdot L^{-1}$$

Accordingly, Tr $P_{\mu\nu}$ is gauge invariant. From (7.2.9) we see that this quantity has the form of an Abelian field intensity

$$\mathrm{Tr}\, P_{\mu\nu} = \partial_\mu \,\mathrm{Tr}\, \Omega_\nu - \partial_\nu \,\mathrm{Tr}\, \Omega_\mu \qquad (7.2.11)$$

for the gauge field Ω_ν one obtains

$$\mathrm{Tr}\, \Omega'_\mu = \mathrm{Tr}\, \Omega_\mu + \mathrm{Tr}\, (LL^{-1}_{,\mu})$$

Thus, by taking trace on the objects associated to the gauge group U(3,1) we determine its Abelian sub-group structure, in the case the group U(1).

An arbitrary element L satisfying (7.2.8) has the following infinitesimal form

$$L = 1 + \varepsilon$$

$$\varepsilon = r + iS \;,\; R = \eta \cdot r = - R^T \;,\; S = \eta \cdot s = S^T \;.$$

Such transformations depend on sixteen real parameters. We may decompose the matrix $E = \eta \cdot \varepsilon$ as

$$E = R + i\, T + \frac{i}{4}\, \eta \cdot \Lambda,$$

where,

$$T = S - \frac{i}{4} \cdot \eta \cdot \Lambda$$

$$\Lambda = \mathrm{Tr}\, S$$

T is a 4 x 4 symmetric, trace free matrix. The particular element L_o given by

$$L \equiv 1 + \frac{i}{4} \cdot 1 \cdot \Lambda = 1 \,.\, e^{i/4\, \Lambda}$$

satisfies the condition $L^{+}_{o} L_o = 1$. Thus, it belongs to the sub-group U(1). Under the action of this element we have

$$\Omega'_\mu = \Omega_\mu - 1 \cdot \frac{i}{4} \Lambda_{,\mu}$$

Then,

$$-i\, \mathrm{Tr}\, \Omega'_\nu = - i\, \mathrm{Tr}\, \Omega_\nu + \Lambda_\nu$$

this implies that the quantities $A_\nu = i\, \mathrm{Tr}\, \Omega_\nu$ satisfy the properties

(i) are real quantities
(ii) they tend to zero if $\Gamma^{\mu}_{\nu\alpha} \to \{^{\mu}_{\nu\alpha}\}$ (Tr Ω_ν vanishes in general relativity)
(iii) the field intensity for A_ν has the form of a field tensor of an Abelian gauge theory.
(iv) the sub-group formed with the set of elements $L = 1 \cdot e^{i\Lambda}$ is mathematically equivalent to U(1):

$\Psi' = L_o\, \Psi$, where Ψ has the properties:

a - is a set of four world scalars.
b - is an internal vector under the transformations of the group U(3,1).

In order to associate these properties to a real charged matter field one has to obtain the relationship between Ω_α and a spinor connection $\hat{\Omega}_\alpha$, since charged matter is usually descibed bu spinor fields. In these notes we shall not discuss what are the differences between a Dirac spinor (associated to SO(3,1)) and a spinor generated by U(3,1). The reason for avoiding such discussion is that presently we can determine $\hat{\Omega}_\alpha$ without the explicit use of four-component spinors. We use the definitions

$$\gamma^\mu(x) = e^{\mu a}(x)\ \gamma_a$$

$$\beta^\mu(x) = \bar{e}^\mu_a\ (x)\ \gamma^a$$

These formulas generalize the usual results used in the theory of Dirac spinors in general relativity. The γ_a indicate the constant Dirac matrices, and presently we have

$$\beta_\mu\ \gamma_\nu + \gamma_\nu\ \beta_\mu = 2\ g_{\mu\nu}\ \mathbb{1} \qquad (7.2.12)$$

(we use the symbol $\mathbb{1}$ for indicating the 4 x 4 identity matrix with spinor indices).

It will be of interest to determine the relation between Ω_α and $\hat{\Omega}_\alpha$ without using the metricity conditions" (7.2.3). This can be done, since we use the weaker conditions

$$\gamma^a_{\ ||\alpha} = 0 \qquad (7.2.13)$$

These conditions represent metricity conditions on the local pseudo-Euclidian space with metric η_{ab}. The notation of a double stroke indicates the full covariant derivative,given by

$$\beta_{\underset{+}{\mu}||\alpha} = \partial_\alpha\ \beta_\mu - \Gamma^\lambda_{\mu\alpha}\ \beta_\lambda + \left[\beta_\mu, \hat{\Omega}_\alpha\right]$$

From (7.2.13) one gets

$$\left[\hat{\Omega}_\alpha, \gamma^a\right] + \Omega^a_{\alpha.c}\ \gamma^c = 0$$

Solving for $\hat{\Omega}_\alpha$ one finds

$$\hat{\Omega}_\alpha = c_\alpha \cdot \mathbb{1} - \frac{i}{4}\ \Omega^a_{\alpha.b}\ \sigma_a{}^b \qquad (7.2.14)$$

The arbitrary field c_α is chosen by imposing that

$$\mathrm{Tr}\ \hat{\Omega}_\alpha = 4c_\alpha = -\ i\ k\ A_\alpha \qquad (7.2.15)$$

k is a constant with the dimension of the inverse of the electric charge(22)(or equivalently with the dimension of $e/\hbar c$).Given Ω_α we determine $\hat{\Omega}_\alpha$ through the relations (7.2.14)and(7.2.15), and use this connection for the introduction of the minimal interaction of a spinor field Ψ and the two parts of our present gange field:

$$\Psi_{,\alpha} \rightarrow \Psi_{|\alpha} = \Psi_{,\alpha} + \hat{\Omega}_\alpha\ \Psi$$

In this derivative we have a term $-ik\ s\ A_\alpha \Psi$ which has the structure of an electromagnetic interaction, besides this we also have another interaction given by $-\frac{i}{4}\ \Omega^{\ a}_{\alpha.b}\sigma_a{}^b\Psi$ which will represent a "gravitational interaction".

7.3 - APPLICATION OF THIS THEORY AS AN UNITARY FIELD THEORY

The integrability conditions associated to the equations (7.2.4) imply that the internal curvature $P_{\nu\alpha}$ is given in terms of the spacetime curvature by

$$P_{\nu\alpha} = R_{\beta\mu\nu\alpha}(\Gamma)\ e^\mu \otimes e^{-\beta} \qquad (7.3.1)$$

where $R_{\beta\mu\nu\alpha}$ is the curvature $R^{\beta}_{.\mu\nu\alpha}$ with the index β lowered with the use of the Hermitian metric. The notation ⊗ denotes a direct product in internal space.

Due to the condition (7.2.10) the curvature has to satisfy the conditions

$$R^{*}_{\beta\mu\nu\alpha}(\Gamma) = - R_{\mu\beta\nu\alpha}(\Gamma) \qquad (7.3.2)$$

We have from (7.3.1)

$$\mathrm{Tr}\ P_{\nu\alpha} = R^{\beta}_{.\beta\nu\alpha}(\Gamma) \qquad (7.3.3)$$

Using (7.3.2) one can show that

$$R^{\beta}_{.\beta\nu\alpha}(\Gamma) = \partial_{\alpha}\, \Gamma^{\beta}_{[\beta\nu]} - \partial_{\nu}\, \Gamma^{\beta}_{[\beta\alpha]} = \partial_{\alpha}\, \Gamma_{\nu} - \partial_{\nu}\, \Gamma_{\alpha}$$

Thus (7.3.3) takes the form

$$\mathrm{Tr}\ P_{\nu\alpha} = \partial_{\alpha}\, \Gamma_{\nu} - \partial_{\nu}\, \Gamma_{\alpha} \qquad (7.3.4)$$

If we work with Einstein's asymmetric theory, or with Moffat's theory, the vector of torsion of the Hermitian connection vanishes. Therefore, for the theories treated at the sections (5) and (6) there will be no Abelian field intensity coming from the local structure of the tangent space (i.e. the structure associated to the group U(3,1)). In these theories the electromagnetic potentials are given by the vector W_{ν} by means of the definition (5.2. 5) which represents a projective transformation in spacetime.

Therefore, if we want to exploit the possibility of existence of an Yang-Mills gauge theory for the group U(3,1) which will represent another approach for an unitary field theory (23), we have to follow a different method. In what follows we shall consider this possibility. The assumption which will be made is as follows: we take the fields Ω_{μ} and $g_{\mu\nu}$ as independent quantities, this will mean that we will have an unitary field theory involving Ω_{μ} as dynamical quantities in a background with metric $g_{\mu\nu}$. The interpretation of this theory will be made at the end of this section. As consequence of our initial assumption the two curvatures $P_{\nu\alpha}$ and $R^{\alpha}_{.\beta\mu\nu}$ become independent. We identify

$$\mathrm{Tr}\ P_{\mu\nu} = 2\ \partial_{[\mu}\ \mathrm{Tr}\ \Omega_{\nu]} = 2\ i\ k\ \partial_{[\nu}A_{\mu]} = i\ k\ F_{\mu\nu} \qquad (7.3.5)$$

The conditions $\gamma^{a}_{||\alpha} = 0$ are still valid. Besides this we have

(i) The curvature $R^{\alpha}_{.\beta\mu\nu}$ still satisfies the basic symmetry property $R^{\alpha}_{.\beta\mu\nu} = - R^{\alpha}_{.\beta\nu\mu}$.

(ii) The connection $\Gamma^{\mu} = (\Gamma^{\mu}_{\alpha\beta})$ satisfies the same symmetry condition as before: $\Gamma^{+\mu} = \Gamma^{\mu}$.

(iii) The variational principle which leads to the field equations will involve a Lagrangian density quadratic in the curvature $P_{\mu\nu}$ and is of the Palatini form.

We take as Lagrangian density the expression

$$L = \sqrt{-g}\ \ \mathrm{Tr}\ (P_{\mu\nu}\ P^{\mu\nu}) = \sqrt{-g}\ \ g^{\lambda\rho}\ g^{\beta\sigma}\ \mathrm{Tr}\ (P_{\rho\sigma}\ P_{\lambda\beta}) = L^{*}$$

The standard Yang-Mills Lagrangian has the following invariance properties:

(a) is invariant under arbitrary coordinate transformations.

(b) is invariant under transformations of the group U(3,1).

(c) is invariant for the Weyl scale transformations of the vierbeins: $e'_{\mu}(x)=\lambda(x)e_{\mu}(x)$, with λ real.

Under the transformations (c) we have $g'_{\mu\nu} = \lambda^2 g_{\mu\nu}$ but the $P_{\mu\nu}$ remain unchanged due to our basic initial assumption (24).

Using the Palatini variational principle, with Ω_μ and $g_{\mu\nu}$ as independent quantities, we get the equations

$$\frac{\delta L}{\delta \Omega_\mu} = \frac{1}{\sqrt{-g}}\left[(g^{\beta\rho}g^{\mu\sigma}+g^{\rho\beta}g^{\sigma\mu})\sqrt{-g}\; P_{\rho\sigma}\right]_{,\beta}+(g^{\beta\rho}g^{\mu\sigma}+g^{\rho\beta}g^{\sigma\mu})\left[\Omega_\beta , P_{\rho\sigma}\right] = 0 \qquad (7.3.6)$$

$$\frac{\delta L}{\delta g_{\mu\nu}} = 2g^{\nu[\rho}\mathrm{Tr}(P_{\rho\sigma}P^{\sigma]\mu}) + \frac{1}{2}\, g^{\nu\mu}\; \mathrm{Tr}\;(P_{\rho\sigma}P^{\rho\sigma}) = 0 \qquad (7.3.7)$$

If there exist external interactions one has to add at the right hand side of these equations the source factors

$I^\mu = (I^{\mu a}{}_b)$ - the complex hypermomentum tensor.

$T^{\mu\nu}$ - the Hermitian energy - momentum tensor.

Both terms appear in the minimal coupling scheme as $\mathrm{Tr}(\Omega_\mu\; I^\mu)$ and $g_{\mu\nu}\; T^{\mu\nu}$. Taking trace in (7.3.6) we obtain

$$\frac{i\;k}{\sqrt{-g}}\left[(g^{\beta\rho}g^{\mu\sigma} + g^{\rho\beta}g^{\sigma\mu})\sqrt{-g}\; F_{\rho\sigma}\right]_{,\beta} = 0 \qquad (7.3.8)$$

These equations are assumed to describe the electromagnetic field in a space with Hermitian metric. Thus, the (7.3.6) describe gravitation and electromagnetism in this theory. However, the equations describing gravitation are associated to the potentials Λ_μ, where

$$\Omega_\mu = \Lambda_\mu - \frac{ik}{4}\,\Lambda_\mu \cdot \mathbb{1}$$

$$\Lambda_\mu = L_\mu + i\;D_\mu$$

$$L^T_\mu = -\,\eta \cdot L_\mu \cdot \eta^{-1} \quad , \quad D^T_\mu = \eta \cdot D_\mu \cdot \eta^{-1} \quad , \quad \mathrm{Tr}\; D_\mu = \Theta$$

Calling by $Q_{\mu\nu}$ the curvature associated to the Λ_μ we can write the equations for the Λ_μ as

$$\frac{1}{\sqrt{-g}}\left[(g^{\beta\rho}g^{\mu\sigma} + g^{\rho\beta}g^{\sigma\mu})\sqrt{-g}\; Q_{\rho\sigma}\right]_{,\beta} + (g^{\beta\rho}g^{\mu\sigma} + g^{\rho\beta}g^{\sigma\mu})\left[\Lambda_\beta , Q_{\rho\sigma}\right] = 0 \qquad (7.3.9)$$

The two dynamical equations are (7.3.8) and (7.3.9). The equations (7.3.7) are algebraic equations for the $g_{\mu\nu}$. Thus, the $g_{\mu\nu}$ are taken as a background metric. However, the background is acted on by the dynamical quantities since (7.3.7) involves the curvature $P_{\mu\nu}$.

In the limit $g_{\mu\nu} \to g_{(\mu\nu)}$, $\Gamma^\mu_{\nu\alpha} \to \Gamma^\mu_{(\nu\alpha)}$, along with the metrical conditions $\Gamma^\mu_{(\nu\alpha)} = \{{}^{\;\mu}_{\nu\alpha}\}$ we have

$$\mathrm{Tr}\;\Omega_\nu \to 0 \quad , \qquad \mathrm{Tr}\; P_{\mu\nu} \to 0.$$

This means that the electromagnetic field becomes an external field. In this region $\Omega_\nu \to \Lambda_\nu = \Lambda^*_\nu$ and the equations (7.3.9) assume the form of the equations proposed by Ponomariev (27) for the description of the gravitational field at microscopic level. Accordingly, our present equations generalize this Yang-Mills formulation for gravitation, and propose to describe an unitary Yang-Mills formalism for gravitation and electromagnetism at microscopical level.

Some general properties may be obtained. From the expression of the Lagrangian density we have that δL is real. This implies that the matrix

$$\frac{\delta L}{\delta g} = \left(\frac{\delta L}{\delta g_{\mu\nu}}\right)$$

is Hermitian (28): $\left(\frac{\delta L}{\delta g}\right)^H = \frac{\delta L}{\delta g}$. As consequence the complex tensor $T^{\mu\nu}$ has the same symmetry

$$T^{\mu\nu} = T^{*\nu\mu}$$

For the variational derivative with respect to Ω_μ we get

$$\left(\frac{\delta L}{\delta\Omega_\mu}\right)^+ = - \eta \cdot \frac{\delta L}{\delta\Omega_\mu} \cdot \eta^{-1}$$

which implies that the complex hypermomentum tensor has the same symmetry

$$I^{\mu +} = - \eta \cdot I^\mu \cdot \eta^{-1}$$

Accordingly, $Tr\left(\frac{\delta L}{\delta\Omega_\mu}\right)$ and Tr I^μ are pure imaginary quantities. The current four-vector is given by

$$j^\mu = \frac{1}{i} Tr\ I^\mu$$

From (7.3.7) one obtains

$$\frac{\delta L}{\delta g_{\mu\nu}} g_{\mu\nu} = T^{\mu\nu} g_{\mu\nu} = T_\nu^{\ \nu} = 0$$

and T_ν^ν is a real quantity. Thus, the source term associated to the equations for $g_{\mu\nu}$ has to be traceless.

Using the previous decomposition of the curvature $P_{\mu\nu}$ (see reference (26)) we may re-write the equations (7.3.7) as

$$2g^{\nu\rho}\ Tr(Q_{\rho\sigma}Q^{\sigma\mu}) + \frac{1}{2} g^{\nu\mu}\ Tr\ (Q_{\rho\sigma}Q^{\rho\sigma}) = - 2\pi\ k^2\ T^{\mu\nu}_{(m)} + T_{\mu\nu(ext.)}$$

where $T^{\mu\nu}_{(m)}$ has the formal expression of a Maxwell tensor but is constructed with the use of the Hermitian "metric" $g^{\mu\nu}$

$$T^{\mu\nu}_{(m)} = \frac{1}{4\pi} (g^{\nu\rho}F_{\rho\sigma}F^{\mu\sigma} - \frac{1}{4} g^{\nu\mu}F_{\rho\sigma}F^{\rho\sigma})$$

$$T^{\mu}_{(m)\mu} = \overset{*}{T}{}^{\mu}_{(m)\mu} = 0 \quad , \quad \overset{*}{T}{}^{\mu\nu}_{(m)} = T^{\nu\mu}_{(m)}$$

Note that presently we do not consider a decomposition of the form $g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}$, with the intention of associating $g_{[\mu\nu]}$ to the electromagnetic field strength. Indeed, here $g_{\mu\nu}$ and Ω_μ are independent objects, and all dynamical effects are taken over by the potentials Ω_μ in the Yang-Mills approach.

It was shown in the literature (29) that exists a macroscopic region x >> L, here L is a fundamental length proportional to the Planck's length, where the present Yang-Mills theory for gravitation degenerates in the theory of general relativity. Some details are:

(i) Introduce massive terms in the Lagrangian. The Weyl invariance principle is lost.

(ii) For re-obtaining this invariance introduce a scalar field $\phi(x)$, the analogue of a Goldstone boson, with

$$L_\phi = \frac{1}{12} R(g)\ \phi^2 - \frac{1}{2} \partial_\mu\phi \cdot \partial^\mu\phi + \lambda\phi^4 \quad , \quad \lambda = \text{constant.}$$

(iii) The transformation in $g_{\mu\nu}$ due to scale variation is now equivalent to a Higgs-Kibble transformation, and may be written as $g'_{\mu\nu} = \frac{\phi(x)}{\alpha} \cdot g_{\mu\nu}$, where α = constant.

Then,

$$L_{tot-} = L_\Gamma + Z\ \{L_o + m_1^2\ Tr(B_\mu B^\mu) - m_2\ \bar{\Psi}\Psi\}$$

$$L_o = - \frac{1}{4} Tr\ (B_{\mu\nu}B^{\mu\nu}) + \frac{i}{2} \bar{\Psi}\ \gamma^\mu \overleftrightarrow{D}_\mu\ (\Gamma, B)\Psi$$

The B_μ field and the spinor field Ψ are the massive fields. Let

$$m_1^2 \rightarrow \phi^2(x)m_{10}^2$$

$$m_2^2 \rightarrow \phi^2(x)m_{20}^2$$

$$\phi'_{(x)} = \frac{1}{\lambda(x)}\,\phi(x)$$

which imply in

$$L_{tot} \rightarrow L_{tot} + \eta\, L_\phi \quad , \quad \eta = \text{constant}$$

The Weyl gange transformation is then fixed by a choice of measuring the interval. It may be shown that

$$L_{tot} = \beta\, R(g) \mp \Lambda + L_\Gamma + Z(L_o + \alpha^2 m_{10}^2\, Tr(B_\mu B^\mu) - \alpha\, m_{20}\, \bar{\Psi}\Psi)$$

with

$$\beta = \frac{\alpha^2 \eta}{12} \quad , \Lambda = \lambda\, \eta\, \alpha^4 \quad , \quad \dim \alpha = L^{-1}$$

$\dim \beta = L^{-2}$, L proportional to Planck's length

Then, for $x >> L$ we have $\beta\, R(g) >> L_\Gamma$ and in this region Einstein's equations for general relativity are recovered:

$$\beta\, R(g) + L_\Gamma = \frac{\eta}{12\, L^2}\,(R(g) + \frac{12\, L^2}{\eta}\, L_\Gamma)$$

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$$\Gamma'^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu\,\partial_\nu \log \lambda$$

This is the expression of the Einstein λ- transformation, which implies that the curvature $R^\lambda_{\mu\nu\sigma}$ is invariant. This in turn implies that $P'_{\mu\nu} = P_{\mu\nu}$. Thus, the same invariance principle may be re-obtained in the "metrical formulation". The present result is a check of consistency, since if we impose $\Omega'_\mu = \Omega_\mu$ this directly implies in $P'_{\mu\nu} = P_{\mu\nu}$.

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26- A $P_{\mu\nu}$ satisfying (7.2.10) may be decomposed as

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$$Q_{\mu\nu} = N_{\mu\nu} + i\,M_{\mu\nu}$$

$$N^T_{\mu\nu} = -\,\eta\cdot N_{\mu\nu}\,\eta^{-1} \quad , \quad M^T_{\mu\nu} = \eta\cdot M_{\mu\nu}\,\eta^{-1} \quad , \quad \operatorname{Tr} M_{\mu\nu} = 0$$

Then,

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