Gödel-type Universes and Chronology Protection in Hořava-Lifshitz Gravity

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HL GRAVITY AND CAUSAL ANOMALIES

We are investigating the following questions:

- HL quantum gravity permits:
  - Gödel-type solutions?
  - Gödel-type space-time regions with closed time-like curves?
  - Gödel-type space-times with physically well motivated matter content?
- Or HL quantum gravity somehow
  - incorporates a chronology protection for Gödel-type space-times?

HOŘAVA-LIFSHITZ GRAVITY

- **Quantum Formulation:**

  Power-counting renormalizable;

  Ultra-high energy scales (trans-Planckian).

- **Causal Structure of Space-Time:**

  ADM formulation (space and time splitting);

  Lifshitz anisotropic scaling (no local Lorentz invariance).

- **Compatibility with General Relativity (GR)**

  GR to be recovered at low and medium energy scales (sub-Planckian).

P. Hořava, Phys. Rev. D **79**, 084008 (2009)

VIOLATION OF CAUSALITY IN GENERAL RELATIVY

- **Godel-type Space-Times**

$$ds^2 = -[dt + H(r)\,d\phi]^2 + dr^2 + D(r)^2 d\phi^2 + dz^2$$

- **ST-Homogeneity**

$$\frac{1}{D}\frac{dH}{dr} = 2\omega \qquad (m, \omega) \text{ are constants such that } \omega^2 > 0 \text{ and } -\infty \leq m^2 \leq \infty$$

$$\frac{1}{D}\frac{d^2 D}{dr^2} = m^2 \qquad \text{Identical pairs } (m^2, \omega^2) \text{ determine isometric Gödel-type space-times.}$$

$m^2 = 2\omega^2 \Rightarrow$ **Gödel geometry!!**

- **Closed Time-like curves**

$$G(r) = D^2(r) - H^2(r)$$

$$ds^2 = -dt^2 - 2\,H(r)\,dt\,d\phi + dr^2 + G(r)\,d\phi^2 + dz^2$$

Gödel circles $t, z, r = const$, when $ds^2 = G(r)\,d\phi^2 < 0$

$$G(r) = D^2(r) - H^2(r) < 0$$

M.J. Rebouças and J. Tiomno, Phys. Rev. D **28**, 1251(1983)

VIOLATION OF CAUSALITY IN GENERAL RELATIVY

- **Classes of Godel-type Space-Times**

$$ds^2 = -[dt + H(r)\, d\phi]^2 + dr^2 + D(r)^2 d\phi^2 + dz^2$$

- **Hyperbolic Class**

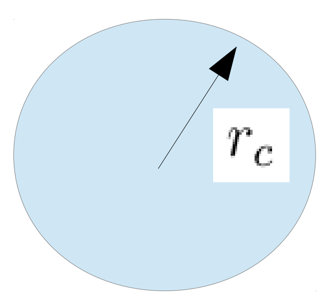
$$m^2 > 0 \qquad H(r) = \frac{4\omega}{m^2} sinh^2(\frac{m\, r}{2}) \, , \; D(r) = \frac{1}{m}\sinh(m\, r)$$

- **Critical radius**

$$t, z, r = \text{const}, \quad G(r) < 0 \quad \text{for} \quad r > r_c$$

$$\sinh^2 \frac{m r_c}{2} = \left[\frac{4\omega^2}{m^2} - 1\right]^{-1} \quad \text{for} \quad 0 < m^2 \leq 4\omega^2$$

- **Causal circle** $G(r) > 0$



$r < r_c$

M.J. Rebouças and J. Tiomno, Phys. Rev. D **28**, 1251(1983)

VIOLATION OF CAUSALITY IN GENERAL RELATIVY

- **Trigonometric Class**

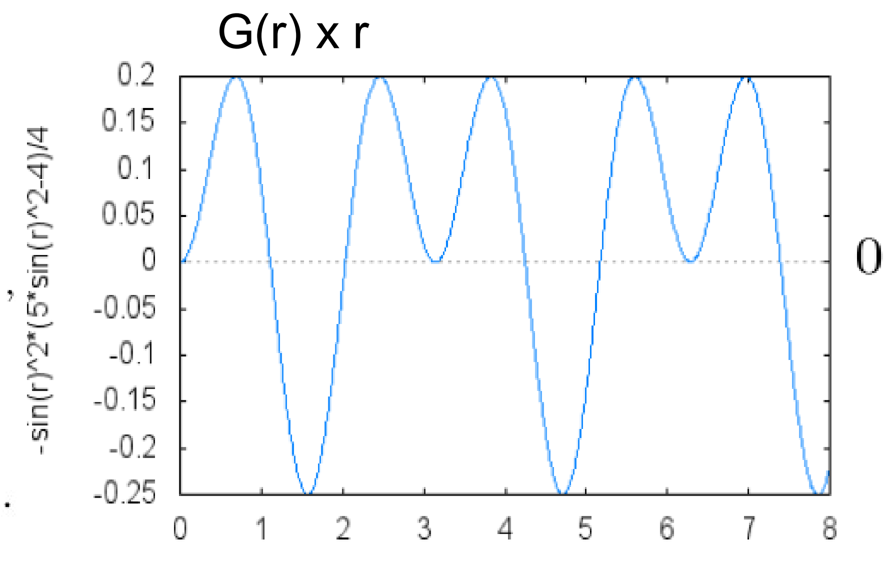
$$m^2 = -\mu^2 < 0 \qquad H(r) = \frac{4\omega}{\mu^2} sin^2(\frac{\mu r}{2}), \; D(r) = \frac{1}{\mu}\sin(\mu r)$$

- **Critical radius**

$$r_1^{(n)} = \frac{2\pi n}{\mu}, \; n = 0, 1, 2, \ldots$$

$$r_2^{(n)} = -\frac{2\left[\arcsin\left(\frac{\mu}{\sqrt{4\omega^2+\mu^2}}\right) - \pi n\right]}{\mu}, \; n = 1, 2, \ldots ,$$

$$r_3^{(n)} = \frac{2\left[\arcsin\left(\frac{\mu}{\sqrt{4\omega^2+\mu^2}}\right) + \pi n\right]}{\mu}, \; n = 0, 1, 2, \ldots$$



- **Causal circles (altenating with non-causal circles)** $G(r) > 0$

$$R_1 = \left\{ r \mid (r_1^{(0)} = 0) \le r \le r_3^{(0)} \right\}, \; R_n = \left\{ r \mid r_2^{(n-1)} \le r \le r_3^{(n-1)} \right\}, \; n = 2, 3, ..$$

VIOLATION OF CAUSALITY IN GENERAL RELATIVY

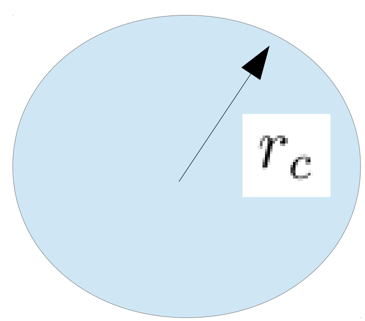
- **Linear Class**

$$m^2 = 0 \qquad H(r) = \omega r^2 \,,\; D(r) = r$$

- **Critical radius**

$$r_c = 1/\omega$$

- **Causal circle** $G(r) > 0$



$$r < r_c$$

CAUSAL STRUCTURE OF SPACE-TIME

Two interconnected physically signicant ingredients:

1 - the **space-time geometry**, which may include **non-causal regions**;

2 - the **gravity theory**, which involves the **field equations** and the **matter source**.

We investigate Gödel-type models in HL gravity in two steps:

1 - **Gödel-type geometries** in the **ADM framework** of HL gravity;

2 - **Solutions of HL gravity** with **Gödel-type** spacetime metrics and their **matter contents**.

DYNAMICAL VARIABLES OF HL GRAVITY

Framework of HL gravity:

ADM splitting of space-time in space and time

Space-time metric recast in the ADM framework:

$$ds^2 = g_{ij} dx^i dx^j = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

.

$$(i, j = 1, 2, 3)$$

ADM dynamical variables:

The lapse function: $N(t, \vec{x})$;
The shift vector: $N^i(t, \vec{x})$;
The spatial metric: $g_{ij}(t, \vec{x})$.

GÖDEL-TYPE METRICS IN HL GRAVITY

ADM restrictions on the Gödel-type metric:

- The lapse is a real function: $N = \frac{D(r)}{\sqrt{G(r)}} \in \mathbb{R}$;
- The spatial metric is positive defined: $det(g_{ij}) = \sqrt{G(r)} > 0$;
- The metric function $G(r)$ is positive defined: $G(r) > 0$.

ADM restrictions on closed time-like curves:

- Gödel circles $t, z, r = const$ where $ds^2 = G(r)\, d\phi^2 < 0$
  Not allowed.
- Non-causal regions: Not allowed.

CHRONOLOGY PROTECTION IN HL GRAVITY

- Quantum effects prevents violations of causality (Hawking, 1992).
- A quantum gravity theory:
  - Incorporates a cronology proctection;
  - Excludes closed time-like curves.

- The ADM framework of HL quantum gravity permits:
  - Gödel-type metrics only on the chronology respecting space-time regions.
- The ADM framework of HL quantum gravity excludes:
  - The causal anomalies of all Gödel-type space-time metrics.

this result is valid for any theory in the ADM framework.

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

Now we arrive at our second step::

Solutions of HL gravity with **Gödel-type** spacetime metrics and their **matter contents**.

$m^2 > 0$ Does HL gravity admit **Godel-type solutions** in **the chronology preserving regions**?

We examine this question for **the hyperbolic class** ($m^2 > 0$), which has the most important solutions in GR:

(1) The **Godel solution**, where $m^2 = 2\omega^2$;

(2) The **only causal Godel-type solution**, where $m^2 = 4\omega^2$.

The matter content considered is a **perfect fluid.**

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

To simplify the calculations we define new (Cartesian) t', x, y, z' coordinates:

$$\tan[\,\phi/2 + (m^2/4\omega)\,(t' - t)\,] = e^{-mr} \tan(\phi/2),$$
$$e^{mx} = \cosh(mr) + \sinh(mr)\,\cos\phi\,,$$
$$m\,y\,e^{mx} = \sinh(mr)\,\sin\phi\,,$$
$$z' = z\,,$$

where the Gödel-type metric is given by

.

$$ds^2 = -[dt' + (2\omega/m)\,e^{mx}dy]^2 + e^{2mx}dy^2 + dx^2 + dz'^2$$

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

The ADM Variables for the Gödel-type metric in Cartesian coordinates

$$N = 1/v \qquad v = \sqrt{1 - \left(\frac{2\omega}{m}\right)^2}$$

$$N_i = (0, -(2\omega/m)\, e^{mx}, 0)$$

$$g_{ij} = \mathrm{diag}\,(1, G(x), 1) \qquad G(x) = v^2 e^{2mx}$$

The chronology preserving interval

$$m^2 > 4\omega^2$$

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

The Lagrangian for the HL gravity we consider is

$$L = \sqrt{g}N\Big[\frac{2}{\kappa^2}\Big(K_{ij}K^{ij} - \lambda K^2\Big) - \frac{\kappa^2}{2w^4}C_{ij}C^{ij} + \\ + \frac{\kappa^2\mu}{2w^2}\frac{\epsilon^{ijk}}{\sqrt{g}}R_{il}\nabla_j R^l_k - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} \\ + \frac{\kappa^2\mu^2}{8(1-3\lambda)}\Big(\frac{1-4\lambda}{4}R^2 + \Lambda R - 3\Lambda^2\Big) + \mathcal{L}_m\Big],$$

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \qquad C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}}\nabla_k(R^j_l - \frac{1}{4}R\delta^j_l)$$

Λ is a cosmological constant $\quad\kappa^2$ is a gravitational constant

λ, w, μ are coupling parameters of the theory

$\mathcal{L}_m$ is the matter lagrangian

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

The HL Field Equations

$$
\begin{aligned}
-4\,m^4\,\tau\,v\,\zeta + 2\,\Lambda\,m^2\,\tau\,v\,\zeta + 3\,\Lambda^2\,\tau\,v\,\zeta & \\
+2\,m^4\,v\,\zeta + 2\,p\,v^2 - 4\,\omega^2\,v - 2\,\rho - 2\,p &= 0 \\
2\,\omega\,(p\,v - 4\,m^2) &= 0 \\
-4\,m^4\,\tau\,\zeta - 3\,\Lambda^2\,\tau\,\zeta + 2\,m^4\,\zeta - p\,v + 4\,\omega^2 &= 0 \\
-4\,m^4\,\tau\,v\,\zeta - 3\,\Lambda^2\,\tau\,v\,\zeta + 2\,m^4\,v\,\zeta & \\
+\rho\,v^2 - 12\,\omega^2\,v - \rho - p &= 0 \\
4\,m^4\,\tau\,\zeta - 2\,\Lambda\,m^2\,\tau\,\zeta - 3\,\Lambda^2\,\tau\,\zeta & \\
-2\,m^4\,\zeta - p\,v - 4\,\omega^2 &= 0
\end{aligned}
$$

independent parameters $\rho, \Lambda, \tau, \zeta, \omega,$ and m^2

Perfect Fluid with energy-momentum tensor

$$T^{\mu\nu} = (\rho + p)\,u^\mu\,u^\nu + p\,g^{\mu\nu}$$

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

Solution of the HL Field Equations

$$p = \frac{4\,m^2}{v} \qquad \rho = \frac{8\,(2\,\omega^2 - m^2)}{v}$$

$$\tau = \frac{2\,m^6}{(\Lambda\,m^2 + 3\,\Lambda^2)\,\omega^2 + 4\,m^6 - \Lambda\,m^4},$$

$$\zeta = -\frac{(2\,\Lambda\,m^2 + 6\,\Lambda^2)\,\omega^2 + 8\,m^6 - 2\,\Lambda\,m^4}{\Lambda\,m^6 + 3\,\Lambda^2\,m^4}$$

$$m^2 = \frac{2}{3}\omega^2 \quad \text{and} \quad m^2 = \frac{1}{4}\omega^2$$

The solution is outside the chronology preserving interval

$$m^2 > 4\omega^2$$

GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

Conclusions

HL gravity excludes
perfect fluid solution with
Gödel-type hyperbolic metrics in the
allowed chronology preserving region.

This results holds regardless of the equation of state p/ρ