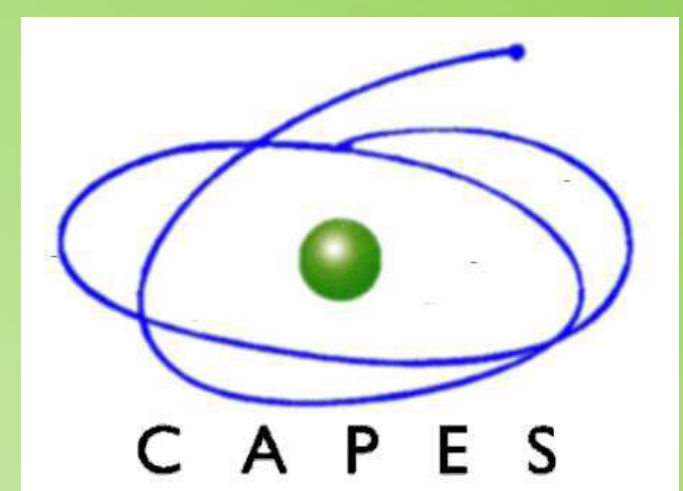# THE CONFORMAL AND PROJECTIVE STRUCTURES OF PALATINI f(R) THEORIES OF GRAVITY

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In the early 70s Ehlers-Pirani-Schild (EPS) proposed an axiomatic approach to gravitational physics whereby the geometry of spacetime **M** is described by two structures: (i) a conformal structure, a class of Lorentzian metrics given by $[g] = \{\tilde{g} = \Omega^2 g\}$ (Ω is a positive function on **M)**; and (ii) a projective strucuture, a class of connections given by $[\Gamma] = \{\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + A_{(\mu}\delta^\alpha_{\nu)}\}$ (A_μ is a 1-form). In the EPS approach, chrono-geometry is associated with the Lorentzian metrics, while inertio-gravitational effects are associated with the torsion-free affine connections; and it is the compatibility conditions between the two structures that connect them. From a physical point of view, this decoupling means that the chronological structure of spacetime (timelike, spacelike directions as well as light cones) is determined by the conformal class [g], while geodesics (auto-parallel paths) of particles are governed by the projective structure [Γ].

As is well known, in the Palatini formalism of $f(R)$ gravity the equations of motion are derived from a variational principle treating the metric g and the connection Γ as completely independent fields. Interestingly, this is naturally translated, by means of the Palatini field equations, into a bi-metric structure of spacetime compatible with the EPS axiomatic approach. Beside the metric g of the spacetime manifold, another metric g̃ is involved, which is related to the connection Γ̃. In this panel, following the EPS framework, we find that this new metric g̃ is directly linked to conformal Killing symmetries preexistent in the spacetime manifold. We relate the conformal Killing vectors in Friedmann-Lemaitre-Robertson-Walker (FLRW) flat $(k = 0)$ spacetime to the derivative of $f(R)$ and show that, in principle, there are 9 classes of Palatini $f(R)$ theories of gravity associated to the conformal symmetries in FLRW.

Ehlers-Pirani-Schild Compatibility Conditions

EPS analysis [1] is based on physical and mathematical axioms and turned into the understanding that gravity and causality require on spacetime both an affine and a metric structure a priori independent, but related by suitable compatibility conditions. The necessary condition for a connection Γ to be EPS-compatible with the metric g (or any other metric h conformal to g) is that

$$\nabla_\mu(\sqrt{g}\, g^{\alpha\beta}) = A_\mu \sqrt{g}\, g^{\alpha\beta} \qquad \text{Eq.(1)}$$

Where ∇ is the Levi-Civita connection of g and A_μ is a 1-form. In a recent paper [Di Mauro et. al.[2] showed that Palatini f(R) theories of gravity are in fact EPS-compatible with the conformal structure identified by g (see below).

Palatini Approach to Modified f(R) Gravity

The action that defines an $f(R)$ gravity is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{f(R)}{2\kappa^2} + \mathcal{L}_m \right] , \qquad \text{Eq.(2)}$$

Where $\kappa^2 = 8\pi G$, g is the determinant of the metric tensor, $\mathcal{L}_m$ is the Lagrangian density for the matter fields.

The Equations of the Motion

In the Palatini variational approach the metric and the affine connections are treated as independents and the variation is taken with respect to both, giving us the *modified Einstein's Equations* :

$$f' R_{(\mu\nu)} - \frac{1}{2} f g_{\mu\nu} = 8\pi G\, T_{\mu\nu} , \qquad \text{Eq.(3)}$$

and $\quad \nabla_\alpha \left(f' \sqrt{-g}\, g^{\mu\nu} \right) = 0 , \quad$ where $\quad f' = df/dR. \qquad$ Eq.(4)

This last equation give us the connections*:*

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} h^{\alpha\sigma} \left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \right) , \qquad \text{Eq.(5)}$$

Where $h_{\mu\nu} = f' g_{\mu\nu}$ is a new conformal metric.

Next we show that this conformal metric is directly linked to *Conformal Killing Symmetries* preexistent in the manifold (M,*g*).

Conformal Transformations & Lie Derivatives

Let (M; g) be a spacetime with metric g, (**X**;ϕ_t) a smooth vector field on M with the associated diffeomorphisms ϕ_t .**X** is a Conformal Vector Field if the diffeomorphisms ϕ_t eserve the metric up to a conformal factor: $\phi_t^* g = \Omega(x)g,$ for some positive function Ω [3]. This can be cast in terms of the Lie Derivative of the metric tensor as

$$£_{X_A} g_{\mu\nu} = 2\phi_A g_{\mu\nu} \qquad \text{Eq.(6)}$$

where ϕ_A is called the conformal function of X_A .

We are interested in the Conformal Vector Fields X_A , and respective conformal functions, possibly admitted by the flat (k = 0) FLRW spacetime such that

$$£_{X_A} g_{\mu\nu} = 2f'_A g_{\mu\nu} \qquad \text{Eq.(7)}$$

where $f' = df/dR \neq 0$ are the derivatives of f (R) theories of gravity.

Conformal Vector Fields in (k = 0) FLRW

We write the metric of the flat FLRW spacetime as

$$ds^2 = a^2(\eta) \left(-d\eta^2 + dx^2 + dy^2 + dz^2 \right) \qquad \text{Eq.(8)}$$

where η is the conformal time; $d\eta = dt/a(t).$
For this metric there are 9 Conformal Vectors [4]:

$$X_0 = \partial_\eta \qquad\qquad X_i = x_i \partial_\eta + \eta\, \partial_i \qquad X_4 = x^\alpha \partial_\alpha$$
$$K_0 = -2\eta\, X_4 - (x_\alpha x^\alpha) X_0 \qquad K_i = 2x_i\, X_4$$

where $x^\alpha = (\eta, x, y, z)$ and $x_\alpha = (-\eta, x, y, z).$

The "conformal function" $f'(R)$ associated with each **Conformal Vector Field** X_A (A = 1 ...9) are

Conformal Vector	$f' = df/dR$ (associated $f(R)$ theory)
X_0	$d \ln a/d\eta$
X_i	$x_i\, d \ln a/d\eta$
X_4	$1 + \eta\, d \ln a/d\eta$
K_0	$-2\eta - (\eta^2 + x^2 + y^2 + z^2)\, d \ln a/d\eta$
K_i	$2\, x_i\, (1 + \eta\, d \ln a/d\eta)$

Conclusions

EPS compatibility of Palatini f(R) gravity enhances a physical interpretation of this theory in terms of observational quantities such as light rays and free falling mass particles. We pretend to explore the adequacy of the 9 possible (adapted) functional forms find above to cosmological tests. In closing, we call attention to the following points concerning Palatini gravity:

- In the Palatini approach to $f(R)$ gravity, beside the metric g, another metric h is involved, which gives the connection Γ. These two metrics are related by a conformal transformation such that $h_{\mu\nu} = f' g_{\mu\nu}$.
- The new metric h is directly linked to conformal Killing symmetries preexistent in the manifold (M, g).
- We relate the conformal Killing vectors in FLRW flat spacetime to $f(R)$ theories in the Palatini formalism.

Bibliography

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