V Workshop Challenges of New Physics in Space

A Kolmogorov-Zakharov Spectrum in AdS Gravitational Collapse

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Outline

I- Introduction.

II – Field equations: AdS_4/AdS_5.

III – Numerical scheme: using the Galerkin-Collocation method.

IV – Numerical results. Turbulence in AdS collapse.

V – Final remarks.

I – Introduction

- Minkowski (Λ=0) and de Sitter ($\Lambda > 0$) spacetimes are stable. Is AdS spacetime ($\Lambda < 0$) stable (Bizon et al, 2011)?

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 0$$

- Interest in studying the dynamics in AdS spacetimes: AdS/CFT conjecture (Maldacena, 1998, 1999).

- Collapse of scalar fields in AdS: see Choptuik and Pretorius (2000), Hussain et al (1999,2000). Models in 2+1 and higher dimensions.

- Critical collapse in gravitational collapse was found in asymptotically flat spacetimes (Λ=0) (Choptuik, 1993), and also in AdS spacetimes (Choptuik and Pretorius, 2000; Bizon, 2011). Other studies in connection with AdS/DFT conjecture (Pando Zayas et al, 2011).

- Nonlinear Instability of the AdS spacetimes (Bizon et al, 2011):

  - Any small amount of scalar field will eventually form a black hole.
  - Evidence of turbulence after a weakly nonlinear analysis.

- Our approach: to provide a more detailed picture by pursuing the full dynamics of a scalar field in AdS spacetimes.

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II – Field equations: AdS$_N$

- Fixing the notation: AdS spacetimes in d+1 dimensions (equivalent to Minkowski solution in asymptotically flat spacetimes).

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2$$

$$\Lambda = -d(d-1)/(2l^2)$$

Choosing a new coordinate system:

$$ds^2 = \sec^2\left(\frac{x}{\ell}\right)\left[-dt^2 + dx^2 + \ell^2 \sin^2\left(\frac{x}{\ell}\right) d\Omega_{d-1}^2\right]$$

Here $r/l = \tan(x)$, then $0 \leq r < \infty$ is equivalent to $0 \leq x < \pi/2$.

- Einstein's field equations with cosmological constant:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

- Matter content: massless scalar field.

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2} g_{\alpha\beta}\varphi_{,\mu}\varphi^{,\mu}$$

- Metric: spherically symmetric spacetimes (Bizon et al, 2011).

$$ds^2 = \sec^2\left(\frac{x}{\ell}\right)\left[-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \ell^2\sin^2\left(\frac{x}{\ell}\right)\, d\Omega^2_{d-1}\right] \qquad (2.3)$$

where $d=3,4$. $A=A(t,x)$ and $\delta=\delta(t,x)$; $A=1$, $\delta=0$ and $\varphi=0$ recovers the AdS solution. Also:

$$\ell^2 = -\frac{d(d-1)}{2\Lambda} \qquad (2.4)$$

- Field equations:

$$
\begin{aligned}
\delta' &= -\frac{8\pi G\ell}{d-1}\cos\left(\frac{x}{\ell}\right)\sin\left(\frac{x}{\ell}\right)(\Pi^2+\Phi^2),\\
A' &= -\frac{8\pi GA\ell}{d-1}\cos\left(\frac{x}{\ell}\right)\sin\left(\frac{x}{\ell}\right)(\Pi^2+\Phi^2),\\
&\quad +\frac{1-A}{\ell\cos\left(\frac{x}{\ell}\right)\sin\left(\frac{x}{\ell}\right)}\left[d-2+2\sin^2\left(\frac{x}{\ell}\right)\right]\\
\dot{\Phi} &= (A\mathrm{e}^{-\delta}\Pi)',\\
\dot{\Pi} &= \frac{1}{\tan^{d-1}\left(\frac{x}{\ell}\right)}\left[\tan^{d-1}\left(\frac{x}{\ell}\right)A\mathrm{e}^{-\delta}\Phi\right]'.
\end{aligned}
$$

The auxiliary functions Π and Φ are defined by:

$$
\Pi = A^{-1}e^{\delta}\dot{\varphi}, \qquad \Phi = \varphi'.
$$

Also, $8\pi G = d\text{-}1$.

- Mass function *m(t,x)*:

$$g^{\mu\nu} r_{,\mu} r_{,\nu} = 1 - \frac{2m(t,x)}{r^{d-2}} + \frac{r^2}{\ell^2} \;\Rightarrow\; m(t,x) = \frac{\ell^{d-2}}{2}(1-A)\sec^2\left(\frac{x}{\ell}\right)\tan^{d-2}\left(\frac{x}{\ell}\right)$$

- ADM mass:

$$M_{ADM} = \lim_{x\to\frac{\pi}{2}} m(t,x) = \frac{\ell^{d-1}}{2}\int_0^{\pi/2} A(\Pi^2+\Phi^2)\tan^{d-1}\left(\frac{x}{\ell}\right)dx$$

- It can be shown that the ADM mass is constant. This property is used as a test to our numerical code.

- Before describing the numerical scheme the following regularity conditions must be satisfied. Near *x=0* (origin):

$$\begin{aligned} \varphi(t,x) &= \varphi_0(t) + \mathcal{O}(x^2), \qquad \delta(t,x) = \mathcal{O}(x^2) \\ A(t,x) &= 1 + \mathcal{O}(x^2). \end{aligned}$$

Near the spatial infinity *$x=\pi/2$* (we use *$\rho=\pi/2-x$*):

$$
\begin{aligned}
\varphi(t,x) &= \varphi_\infty(t)\rho^d, \qquad \delta(t,x) = \delta_\infty(t) + \mathcal{O}(x^{2d}) \\
A(t,x) &= 1 + \mathcal{O}(x^d)
\end{aligned}
$$

Notice the dependence on *d*.

III – Numerical scheme: using the Galerkin-Collocation method

- Spectral methods: approximating functions by series of suitable functions known as basis functions. For instance:

$$\Pi_a(t,y) = \sum_{k=0}^{N} a_k(t)\psi_k(y),$$

where $a_k(t)$ are the unknown modes, N is the truncation order and $\psi_k(y)$ are the basis functions that satisfy the boundary conditions (Galerkin method).

- New spatial variable:

$$y = \frac{4x}{\pi} - 1$$

$x=0,\ \pi/2$ corresponds to $y=-1,+1$.

$$\Phi_a(t,y) = \sum_{k=0}^{N-1} f_k(t)\chi_k(y)$$

$$A_a(t,y) = 1 + \sum_{k=0}^{M} c_k(t)\psi_k^{(A)}(y)$$

$$\delta_a(t,y) = \sum_{k=0}^{M} b_k(t)\psi_k^{(\delta)}(y)$$

- Here $f_k(t)$, $c_k(t)$ and $b_k(t)$ are the other unkowns modes .
- The basis functions are expressed as suitable linear combination of the Chebyshev polynomials.

- Numerical procedure:

  - Forming the residual equations: substitution of the spectral expansions of $\Pi(t,x)$, $\Phi(t,x)$, $A(t,x)$ and $\delta(t,x)$ into the field equations. For instance:

$$\mathrm{Res}_\delta(t,y) = \frac{4}{\pi}\frac{\partial \delta_a}{\partial y} + \sin\left[\frac{\pi}{4}(1+y)\right]\cos\left[\frac{\pi}{4}(1+y)\right](\Pi_a^2 + \Phi_a^2)$$

  - The residual equations are forced to vanish at the collocation points (as in Collocation method).

$$\mathrm{Res}_\delta(t,y_k) = \frac{4}{\pi}\left(\frac{\partial \delta_a}{\partial y}\right)_{y_k} + \sin\left[\frac{\pi}{4}(1+y_k)\right]\cos\left[\frac{\pi}{4}(1+y_k)\right](\Pi_a^2 + \Phi_a^2)_{y_k} = 0$$

  - The field equations are reduced to sets of ODEs and algebraic equations.

IV – Numerical results.

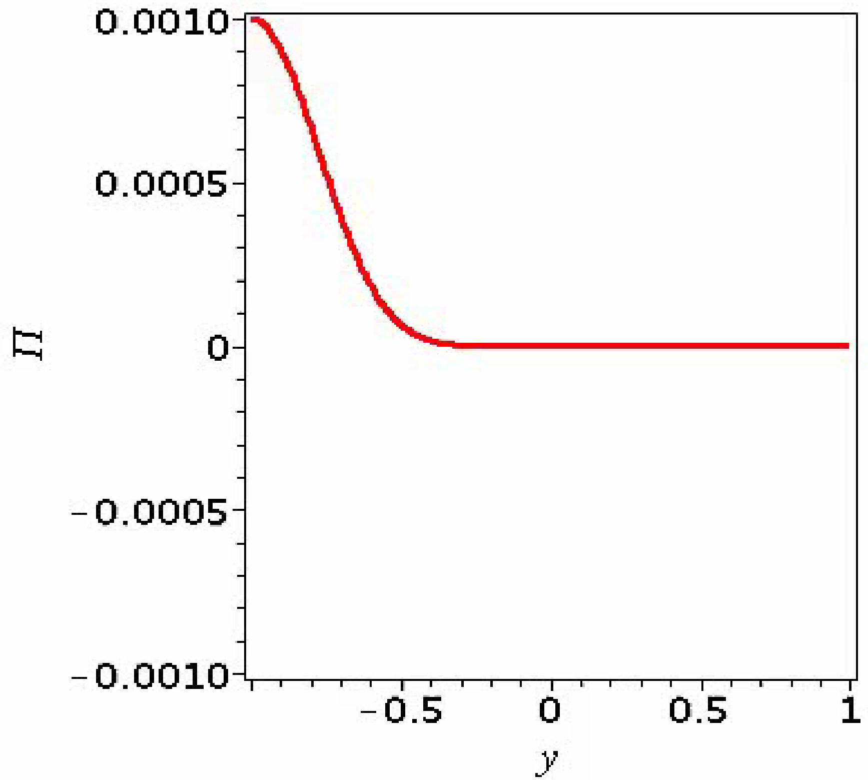
- Initial data (Bizon et al, 2011):

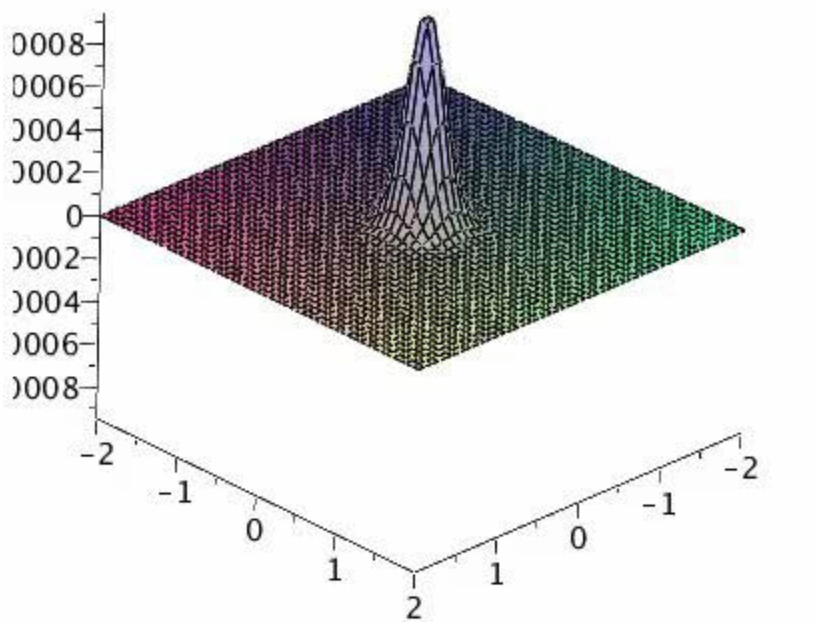
$$\Phi(0,y) = 0, \qquad \Pi(0,y) = \epsilon_0 \exp\left[-\frac{\tan^2(\pi/4(1+y))}{\sigma^2}\right]$$

ε_0 is the initial amplitude and $\sigma=1/4$.

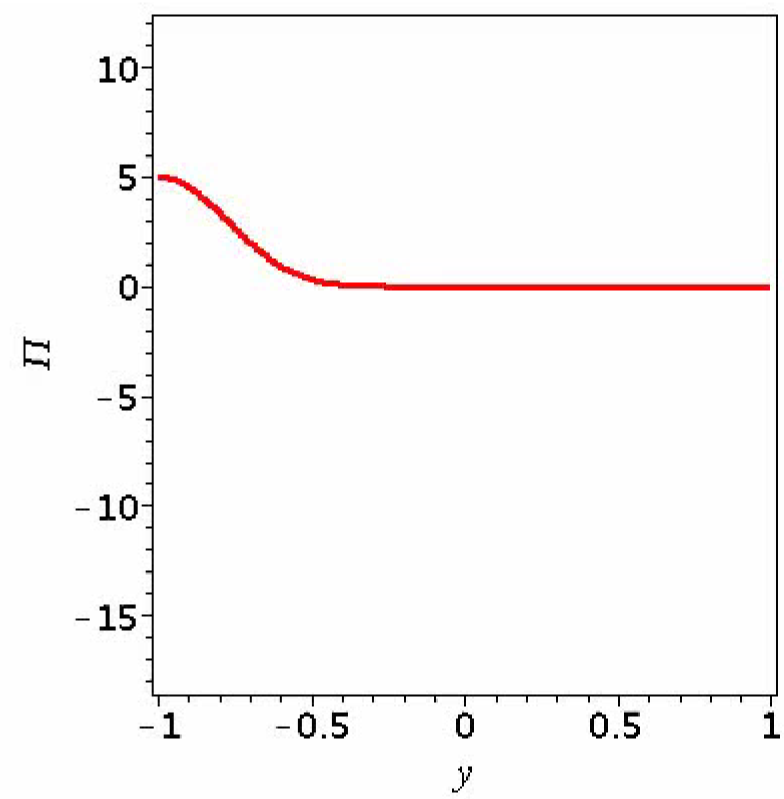
- Evolving initially very weak scalar field: ε_0 =0.001. Observing the bounces of the scalar field

ε_0=0.001

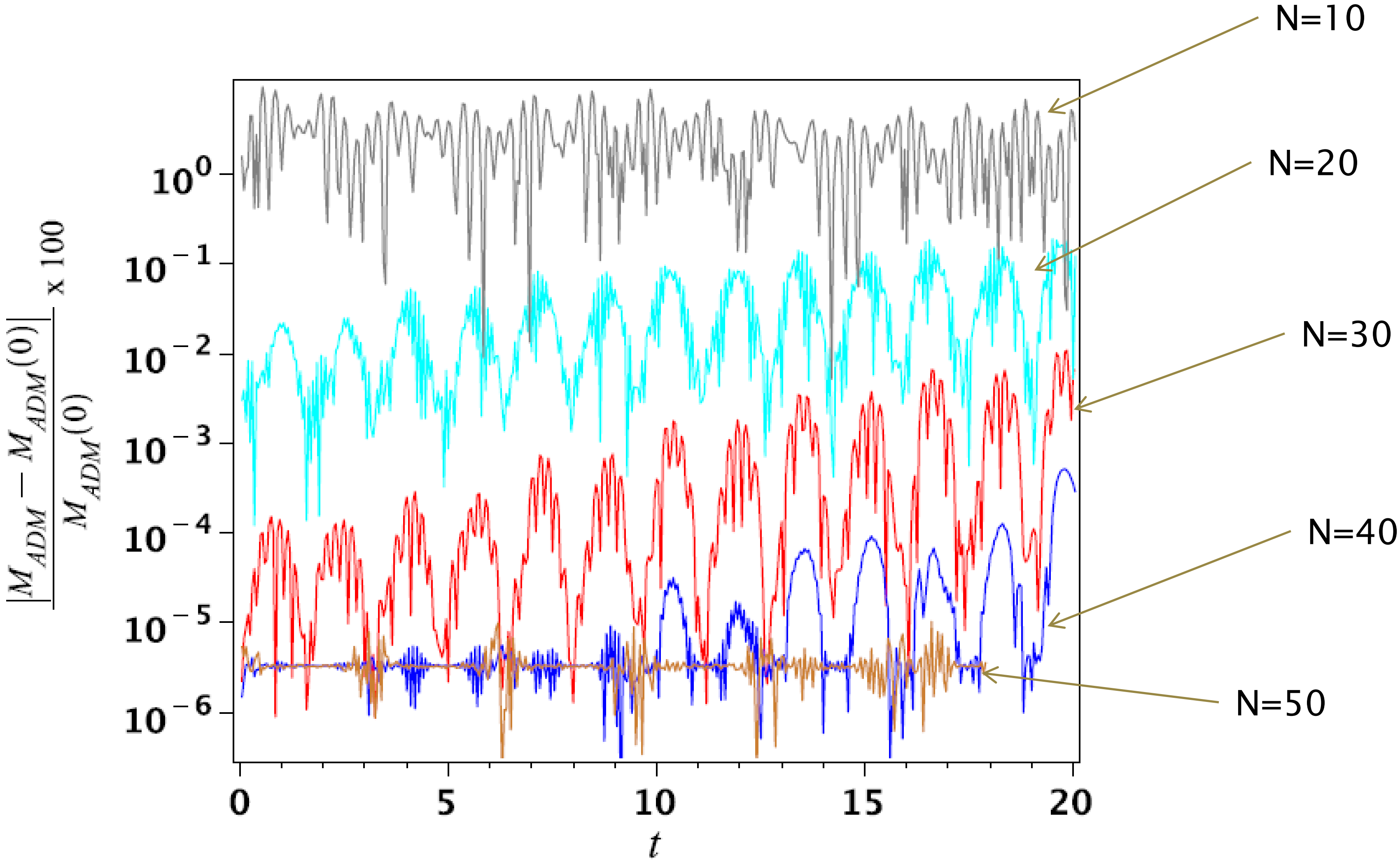




ε_0=5.0



Case ε_0=3.7, AdS_5 for truncation orders N=10, 20,30,40 and 50.



- A dynamical system approach: from the weakly non-linear regime to turbulence.
  - Let us consider the weak field regime: δ,1-A <<1. The Klein-Gordon equation in the spectral representation can be read as:

$$\ddot{a}_k(t) - \frac{1}{\tan^{d-1}(x(y))}[\tan^{d-1}(x(y))]' a_k(t) + \mathcal{F}(\delta, 1-A, a_k, \dot{a}_k, \ddot{a}_k, ..) = 0$$

linearized evolution

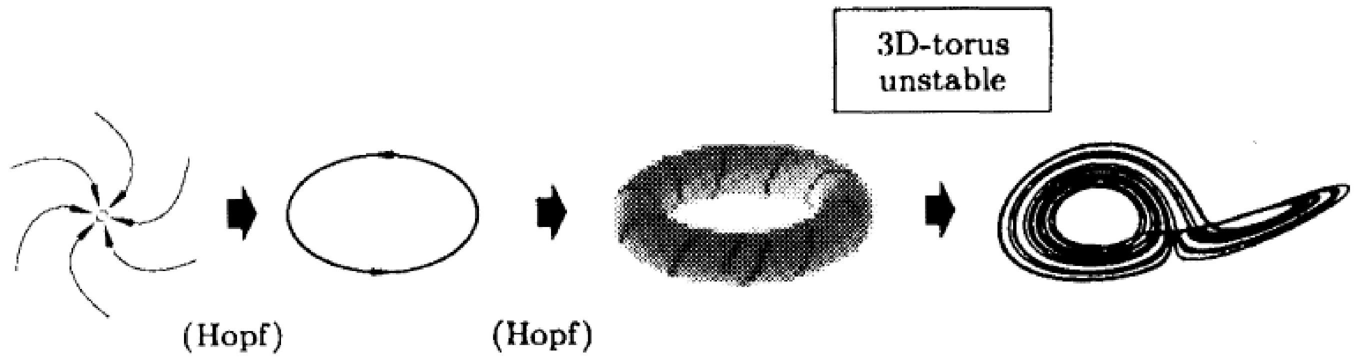
$\omega^2_j = (d+2j)^2$

(Ishibashi and Wald, 2004)

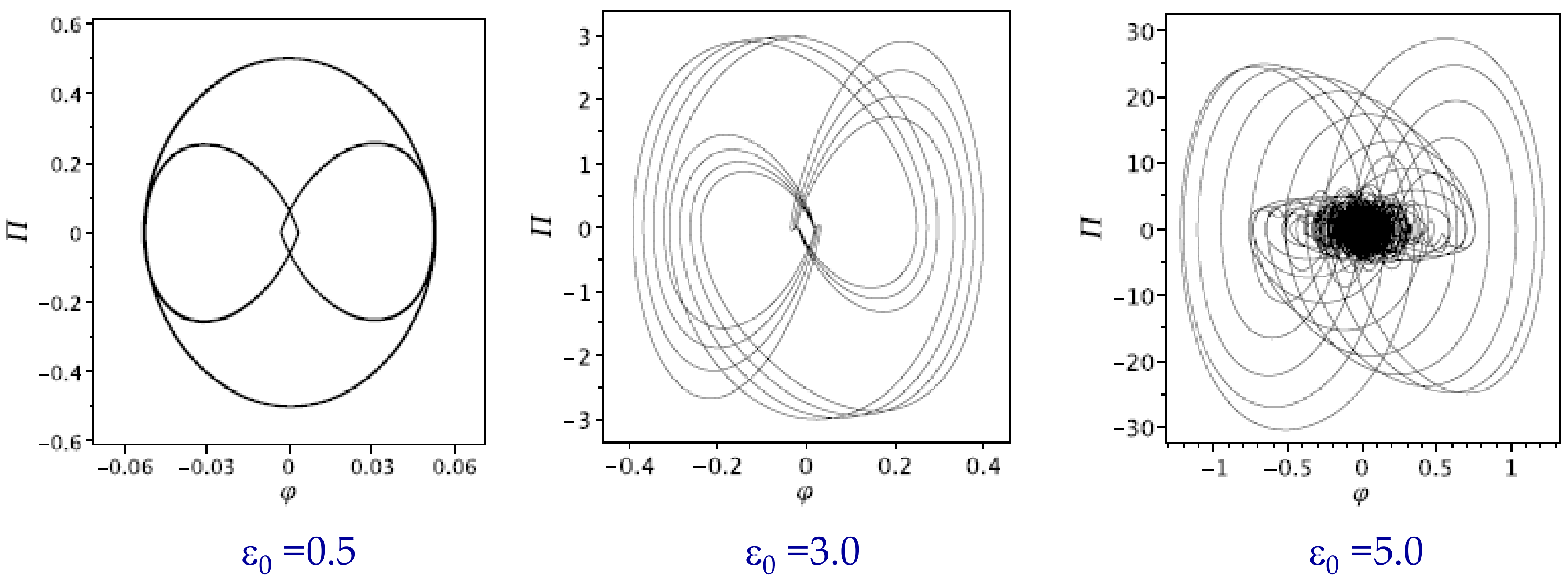
effect of the gravitational sector: responsible for the instability!

  - Evolution of the scalar field in AdS: set of nonlinearly interacting oscillators. Periodic motion: motion on a torus (very small *F*).
  - Action of *F*: alter the natural frequencies producing quasi-periodic motion. Later, the torus is destroyed producing chaotic (turbulent) motion.

- Ruelle-Takens scenario for turbulence.

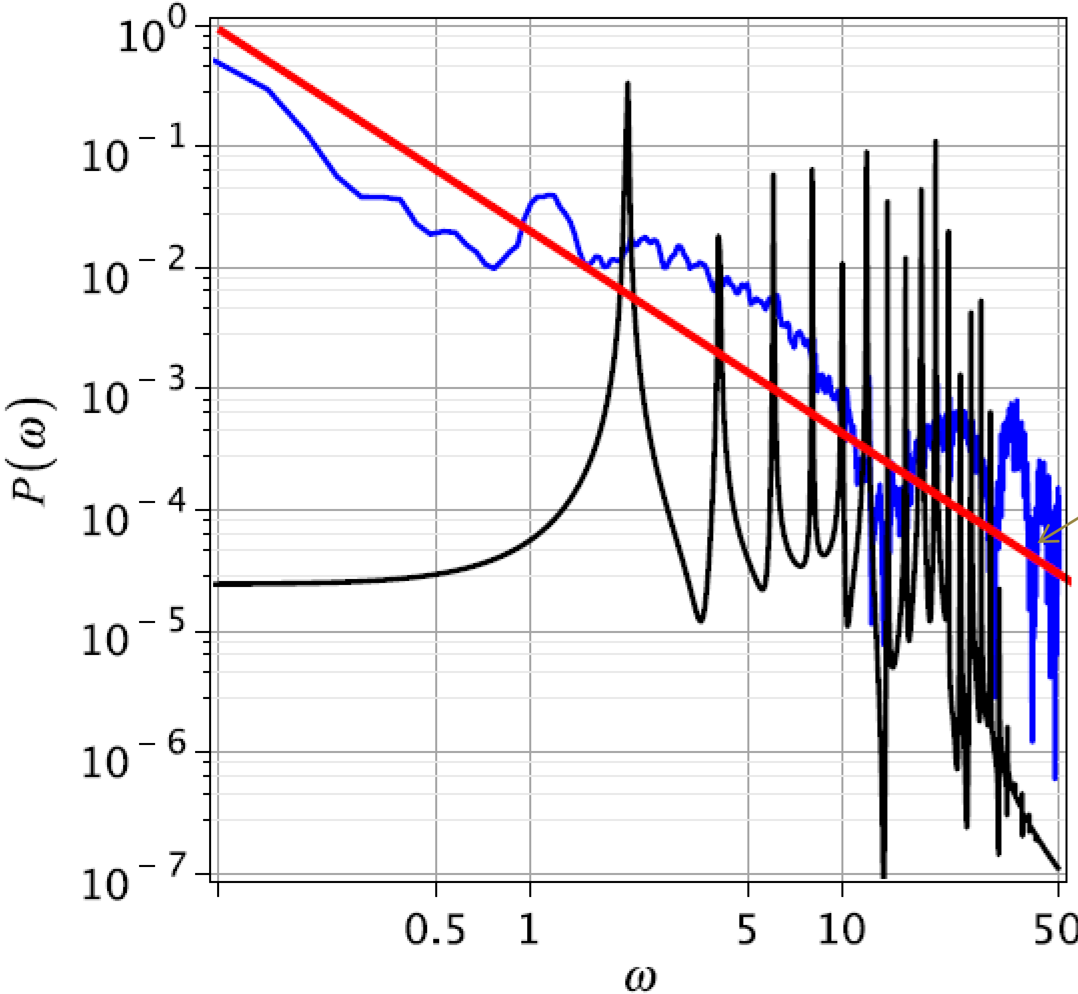


- A suggestive numerical experiment. Strange attractor?



- Nonlinear regime: a Kolmogorov-Zakharov spectrum.
  - Power spectrum of the scalar of curvature R (evaluated at y=0): AdS_5.

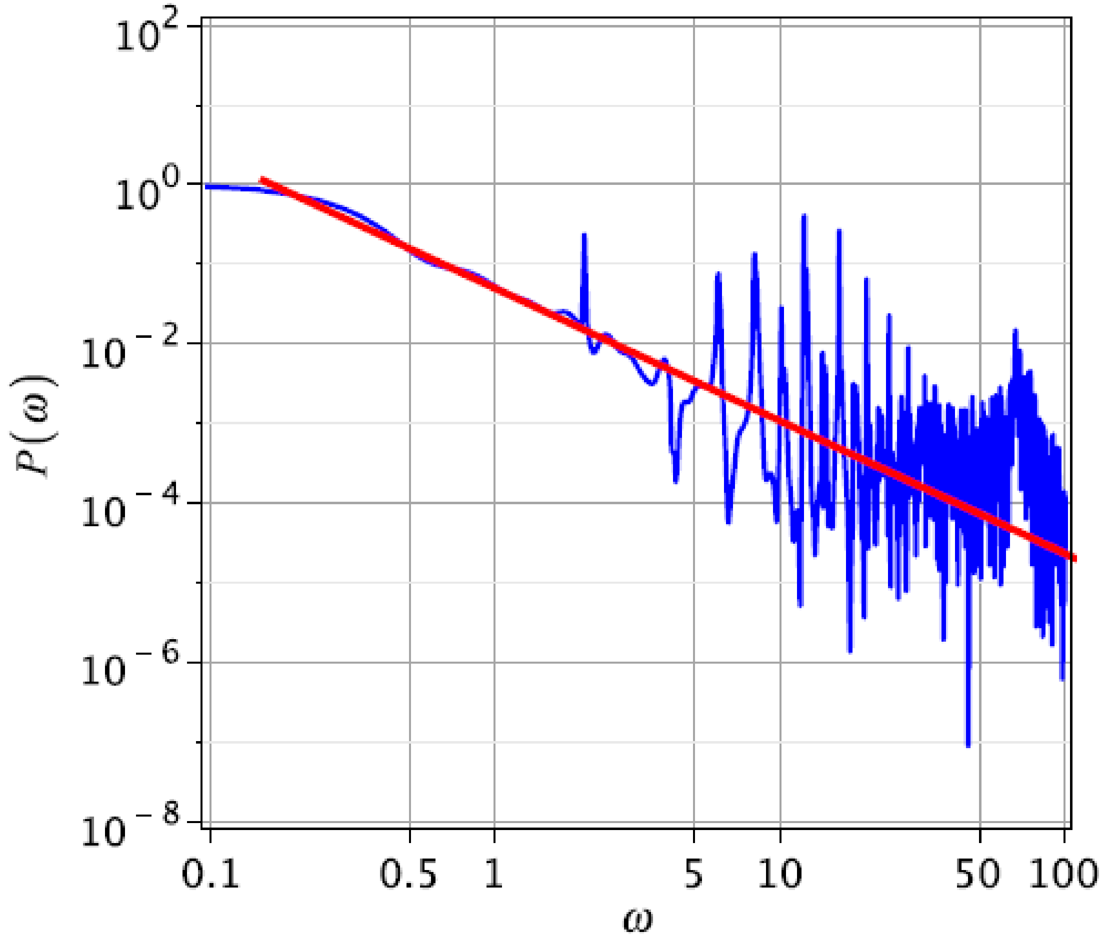
$$R = 1/l^2[A\cos(x)^2(-\Pi^2 + \Phi^2) - d(d-1)]$$



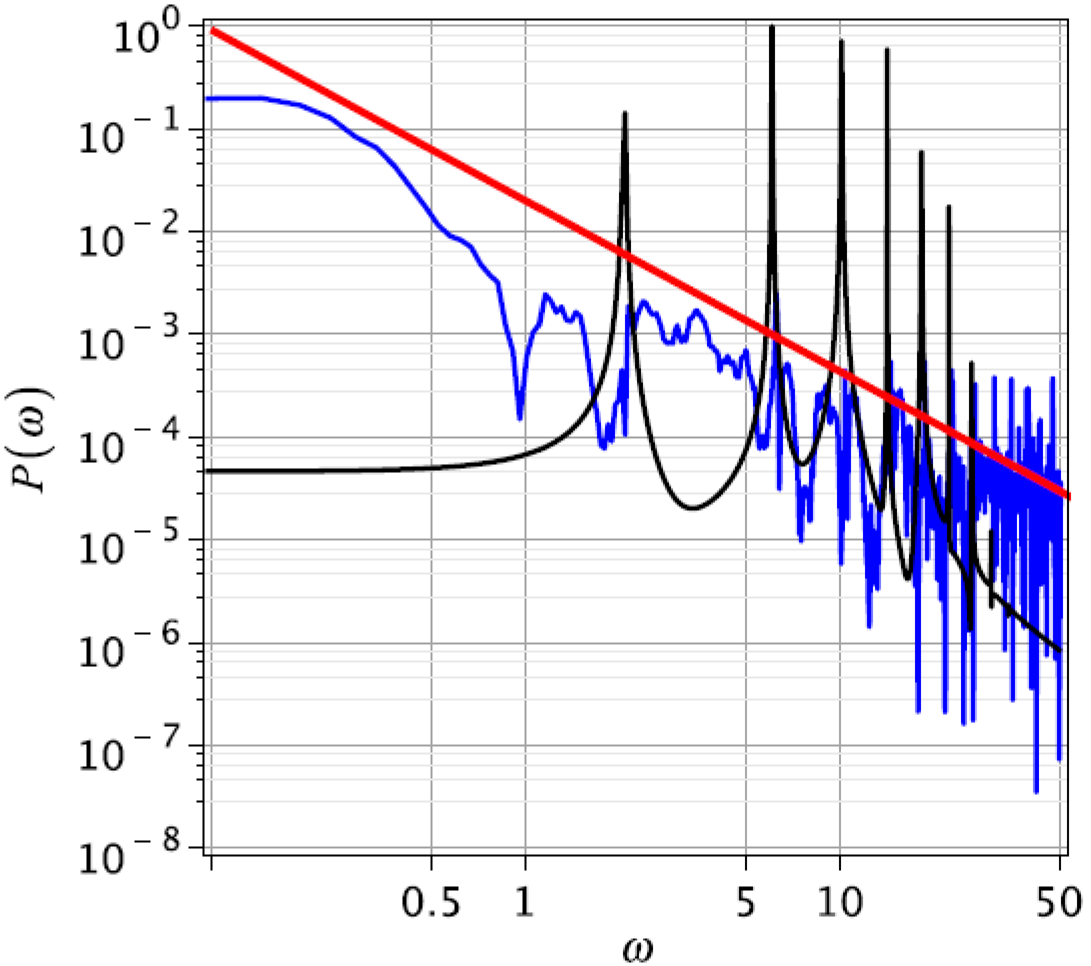
$P(\omega) \propto \omega^{-s}$

$s \approx 1.61$

Blue: ε_0 =2.5; black: ε_0 =0.001



Case AdS_5, ε_0 =3.76.



Case AdS_4, ε_0 =3.76 (blue), ε_0 =0.01(black).

V – Further perspectives.

- We have confirmed the Instability of AdS spacetimes. The formation of a black hole is a generic feature in these spacetimes.

- Evidence of turbulent behavior in the scalar field collapse in AdS spacetimes. An example of wave turbulence.

- Strange attractor in gravitational collapse?

- What happens if massive scalar field is considered?

- Consequences for the AdS/CFT conjecture.