Entangling power of the baker map

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Como, 12 ott 05

Plan of the seminar

1. Introduction

2. Entangling power of the baker's map revisited,
   with Rômulo F. Abreu

Introduction

Motivation

Why study entangling power of the baker map?

Entanglement is an essential resource for quantum computation

What kind of quantum operations produce high levels of entanglement?

Or, what is the property that makes a quantum operation a good entangler?

Baker's map: The Quantum Chaos generation

selected papers

Balazs & Voros, EL 86
birth (quantization)

Saraceno, AP 90
striking scar phenomenon

Saraceno & Ozorio de Almeida, AOP 91
semiclassical theory

Heller, Tomsovic, Kaplan, O'Connor, ... 90's
semiclassical theory beyond the log-time

Dittes, Doron, Smilansky, PRE 94
computation of long-time semiclassical traces

Baker's map in Quantum Information, Computation, Quantum Open Systems

Schack & Caves, PRL 92, PRL 93, PRE 96
information-theoretical characterization of quantum chaos

Schack, PRA 98
efficient realization in terms of quantum gates

Brun & Schack, PRA 99
proposal of 3-bit NMR experiment

Weinstein, Lloyd, Emerson & Cory, PRL 02
NMR experiment

Schack & Caves, AAECC 00
quantum binary shift, family of quantizations

Baker's map in QC, QOSystems, etc

Tracy & Scott, JPA 02
limiting cases of Schack-Caves bakers

Soklakov & Schack, PRE 00; PRE 02
classical limit; decoherence studies

Lozinski & Pakonski, PRE 02
irreversible quantum baker

Bianucci, Paz & Saraceno, PRE 02
decoherence studies

Scott & Caves, JPA 03
entangling power

Meenakshisundaram & Lakshminarayan, PRE 05;
Lakshminarayan, JPA 05
multifractal eigenstates, Hadamard

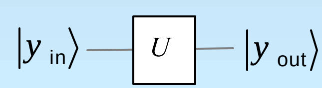
New language: qubits, gates, circuits, ...

Barenco, Ekert, Suominen & Törmä, PRA 96

A ***qubit*** is a two-state quantum system.
It has a chosen **computational basis** $|0\rangle, |1\rangle$.

A collection of qubits is called a ***register***.

A ***quantum logic gate*** is an elementary quantum computation device which performs a fixed **unitary operation** on selected qubits in a fixed period of time.

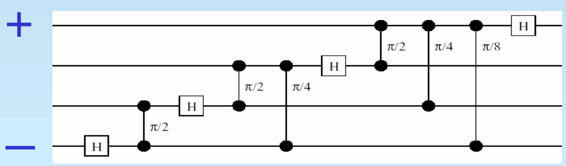


Quantum networks

A ***quantum network*** is a quantum computing device consisting of quantum logic gates whose computational steps are synchronized in time.

The outputs of some of the gates are connected by wires to the inputs of others.

The **size** of the network is its number of gates.

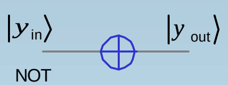


Quantum computer

A ***quantum computer*** is a quantum network (or a family of quantum networks).

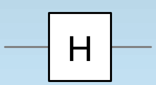
Quantum computation is defined as a unitary evolution of the network which takes its initial state ***input*** into some final state ***output***.

One qubit gates



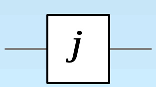
NOT

$$|0\rangle \rightarrow |1\rangle \qquad |1\rangle \rightarrow |0\rangle \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Hadamard

$$|0\rangle \rightarrow |0\rangle + |1\rangle \qquad |1\rangle \rightarrow |0\rangle - |1\rangle \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \div \sqrt{2}$$

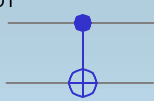


phase gate

$$|0\rangle \rightarrow |0\rangle \qquad |1\rangle \rightarrow e^{ij}|1\rangle \qquad \begin{pmatrix} 1 & 0 \\ 0 & e^{ij} \end{pmatrix}$$

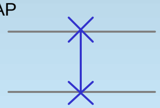
Two qubit gates

CNOT



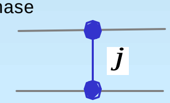
$$|00\rangle \rightarrow |00\rangle \quad |01\rangle \rightarrow |01\rangle \quad |10\rangle \rightarrow |11\rangle \quad |11\rangle \rightarrow |10\rangle \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

SWAP



$$|00\rangle \rightarrow |00\rangle \quad |01\rangle \rightarrow |10\rangle \quad |10\rangle \rightarrow |01\rangle \quad |11\rangle \rightarrow |11\rangle \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c-phase



$$|00\rangle \rightarrow |00\rangle \quad |01\rangle \rightarrow |01\rangle \quad |10\rangle \rightarrow |10\rangle \quad |11\rangle \rightarrow e^{if}|11\rangle \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{if} \end{pmatrix}$$

Quantum Fourier Transform

$$|j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_n\rangle \equiv |j\rangle$$

computational basis (position basis)

$$j = j_1 2^{n-1} + \ldots + j_n 2^0$$

Periodic Fourier transform

$$F|j\rangle = 2^{-n/2} \sum_{k=0}^{2^n-1} e^{2\pi i\, j\, k/2^n} |k\rangle$$

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Factorization of QFT

$$\begin{aligned} F|j\rangle = 2^{-n/2}\ & \left(|0\rangle + e^{2\pi i 0.j_n}\ |1\rangle\right) \otimes \\ & \left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}\ |1\rangle\right) \otimes \\ & \ldots \\ & \left(|0\rangle + e^{2\pi i 0.j_2 \ldots j_n}|1\rangle\right) \otimes \\ & \left(|0\rangle + e^{2\pi i 0.j_1 \ldots j_n}\ |1\rangle\right) \end{aligned}$$

Fourier transform does not entangle states of the computational basis

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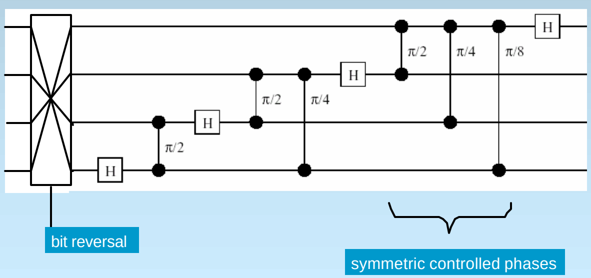
Quantum Fourier Transform (periodic)

$$F_4 =$$



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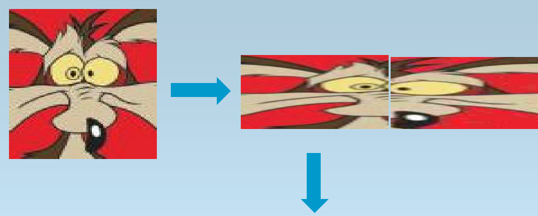
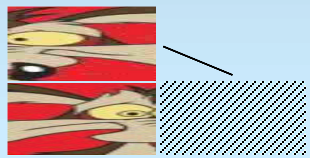
The baker's map

Classical baker



fully chaotic dynamics
(binary shift)
unit square
➔ phase space



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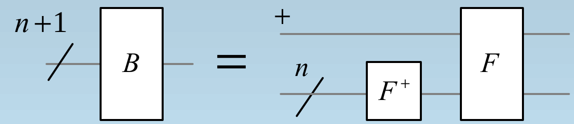
Quantum baker's map



$$N = 2^{n+1}$$

$$B|j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_{n+1}\rangle = F_{n+1}\left[\,|j_1\rangle \otimes F_n^{+}\left(|j_2\rangle \otimes \ldots \otimes |j_{n+1}\rangle\right)\right]$$

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Baker matrix in the computational basis

$$B = F_N^{*}\begin{pmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{pmatrix}$$

N is the dimension of Hilbert space, e.g., $N=2^{n}$,
but may take any even value.

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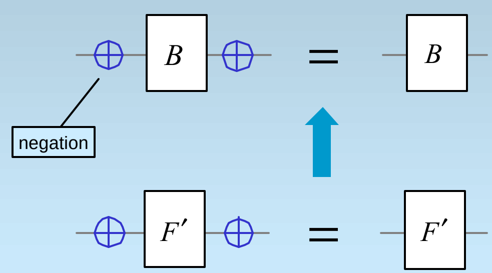
NOT Symmetry

Saraceno, AP 90

If antiperiodic Fourier is used, then



Corresponds to phase-space reflection symmetry

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Symmetry reduced baker

Because of reflection symmetry the baker can be block diagonalized

$$B = U\begin{pmatrix} B_- & 0 \\ 0 & B_+ \end{pmatrix} U^+ = U\begin{pmatrix} D_0 & 0 \\ 0 & R\ D_p\ R \end{pmatrix} U^+$$

reflection operator

$D_{0,p}$ desymmetrized baker

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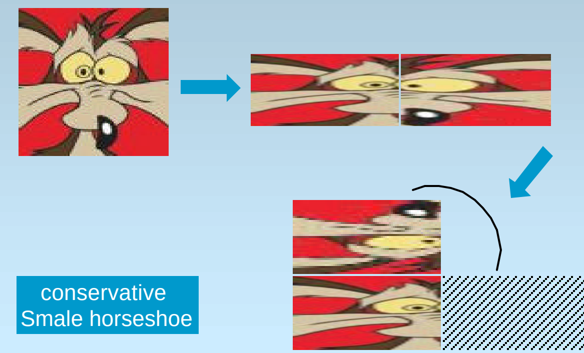
The classical D-baker (D-map)

Cvitanovic, Gunaratne & Procaccia PRA 88



conservative Smale horseshoe

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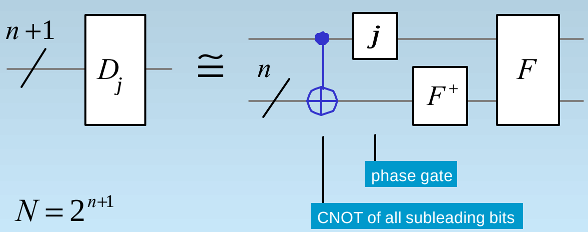
The quantum D-baker

Saraceno & Vallejos, CHAOS 95



$N = 2^{n+1}$

Does not have reflection symmetry

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D-baker matrix in the computational basis

$$D_j = F_N^* \begin{pmatrix} F_{N/2} & 0 \\ 0 & e^{ij}\ F_{N/2}^* \end{pmatrix}$$

$F_{N/2}$ only difference from baker's map

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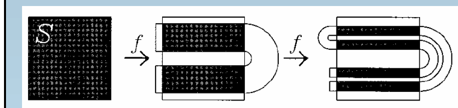
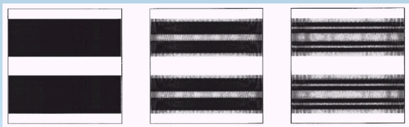
Digression: Quantum Smale horseshoe



Saraceno & Vallejos, CHAOS 95



Kirkwood phase space functions of iterations of the identity

$$U^k \, 1 U^{+k}$$

N=108

$$U = F_N^* \begin{pmatrix} 0_{N/12} & 0 & 0 & 0 & 0 \\ 0 & F_{N/3} & 0 & 0 & 0 \\ 0 & 0 & 0_{N/6} & 0 & 0 \\ 0 & 0 & 0 & F_{N/3}^* & 0 \\ 0 & 0 & 0 & 0 & 0_{N/12} \end{pmatrix}$$

Nonnenmacher & Zworski, nlin.CD 04-05
fractal Weyl laws, resonance distribution, etc

Entangling power of the baker's map

Entanglement

A **pure state** of a **pair** of quantum systems, A and B, is said ***entangled*** if it is not ***separable***.

$$\underbrace{|\Psi_{A+B}\rangle}_{H} = \underbrace{|f_A\rangle}_{H_A} \otimes \underbrace{|y_B\rangle}_{H_B}$$

separable

The Hilbert space of the composite system is the tensor product of the Hilbert spaces of both subsystems:

$$H = H_A \otimes H_B$$

Entangling measures for bipartite pure states

$$|\Psi\rangle \in H_A \otimes H_B$$ — pure state

$$r = |\Psi\rangle\langle\Psi|$$ — density matrix

$$r_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$$ — reduced density matrix

$$S_{\mathrm{vN}} = -\mathrm{tr}_A \, r_A \log r_A$$ — von Neumann entropy

$$S_{\mathrm{L}} = 1 - \mathrm{tr}_A \, r_A^2$$ — linear entropy (1 minus purity)

Entangling power of unitary transformations

Zanardi, Zalka & Faoro, PRA 00

[...] how much entanglement is produced by **U** on the average, acting on a given distribution of non-entangled quantum states.

$$e_{\mathrm{p}} = \left\langle E\left(U\left|y_A\right\rangle \otimes \left|y_B\right\rangle\right)\right\rangle_{y_A, y_B}$$

entanglement measure, i.e., von Neumann or linear entropy

average over ensemble of product states, with probability **p**

Properties

Zanardi, Zalka & Faoro, PRA 00

$$e_{\mathrm{p}} = \left\langle E\left(U\left|y_A\right\rangle \otimes \left|y_B\right\rangle\right)\right\rangle_{y_A, y_B}$$

If *E* is von Neumann (or linear) entropy, the measure above satisfies the properties required for an entanglement measure:

$$e_{\mathrm{p}}(U_A \otimes U_B\, U) = e_{\mathrm{p}}(U)$$

$$e_{\mathrm{p}}(T\,U) = e_{\mathrm{p}}(U)$$

T=SWAP (if dimA= dimB)

$$e_{\mathrm{p}}(1) = 0$$

Ensembles of states

$$\left|y\right\rangle \rightarrow (x_1 + i\,y_1, \quad \ldots \quad, x_N + i\,y_N)$$

uniform measure (CUE, Haar)

$$p\,(x_1, y_1, \ldots, x_N, y_N) \propto d\left(1 - \sum_{i=1}^{N} x_i^2 + y_i^2\right)$$

Invariant with respect to unitary transformations

States generated in this way are called *random states*

Ensembles of product states

$$\left|y\right\rangle = \left|y_A\right\rangle \otimes \left|y_B\right\rangle$$

$CUE\,(\mathrm{N_A})$ $CUE\,(\mathrm{N_B})$

$$CUE(\mathrm{N_A}) \otimes CUE(\mathrm{N_B})$$

Invariant with respect to *local* unitary transformations

Minimum information ensemble

Ensembles of unitary operators

$$U \rightarrow \begin{pmatrix} x_{11} + i\,y_{11} & & \\ & \dots & \\ & & x_{NN} + i\,y_{NN} \end{pmatrix}$$

Measure in the group of unitary matrices $p\,(U)$

Invariance with respect to left or right group actions

$$p\,(U) = p\,(V_1\,U\,V_2)$$

➔ leads to CUE ensemble

Ensemble average in ...

Zanardi, Zalka & Faoro, PRA 00

$$e_{\mathrm{p}} = \left\langle E\,\left(U\,|y_A\rangle \otimes |y_B\rangle \right) \right\rangle_{y_A, y_B}$$

The “natural” measure in the set of product states is CUE X CUE, to be denoted p_0 .

Proposition

$$\langle e_{\mathrm{p}_0}(U) \rangle_{U \in CUE} = \frac{(N_A - 1)(N_B - 1)}{N_A N_B + 1}$$

Comment

$$\langle e_{\mathrm{p}_0}(U) \rangle_{U \in CUE} = \left\langle E\,\left(U\,|y\rangle \right) \right\rangle_{U \in CUE}$$

CUE ensemble of product states, U in CUE

any input state, U in CUE

Other measures: 2) Operator entanglement (Schmidt strength)

Zanardi, PRA 01
Wang & Zanardi, PRA 02
Nielsen et al, PRA 03
Zyczkowski & Bengtsson, OSID 04

$$|U\rangle \in H^{HS}_{N_A^2} \otimes H^{HS}_{N_B^2}$$

$$\langle A | B \rangle \equiv \mathrm{tr}\,(A^+ B)$$ — Hilbert-Schmidt scalar product

$$r^U = |U\rangle\langle U|$$

$$r^U_A = \mathrm{tr}_B |U\rangle\langle U|$$

$$e_{\mathrm{Sch}}(U) = -\mathrm{tr}_A\; r^U_A \log r^U_A$$

Question

Is there a measure
in the set of separable states
such that the entangling power
coincides with Schmidt strength?

$$? \exists\, p\,(y_A, y_B)\ :$$

$$e_p = \left\langle E\left(U\left|y_A\right\rangle \otimes \left|y_B\right\rangle\right)\right\rangle_{y_A, y_B} = e_{\text{Sch}}(U)$$

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Other measures: 3) Eigenvector entanglement

U unitary operator

$\{|y_k\rangle\}_{1\le k\le N}$ eigenvectors

Calculate the entanglement of each eigenvector and then average over all eigenvectors

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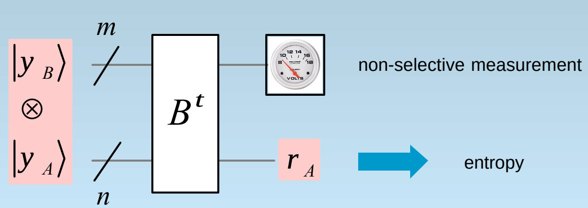
Simulations with the baker

Experimental setup



- Entanglement between most (***m***) and least (***n***) significant qubits, *as a function of time*
- Average over set of input pure states (Zanardi)

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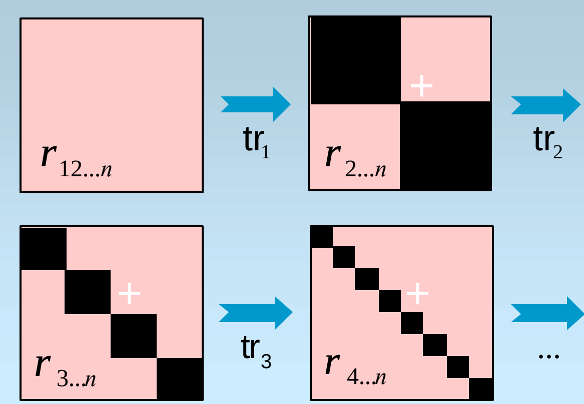
Tracing out most significant qubits



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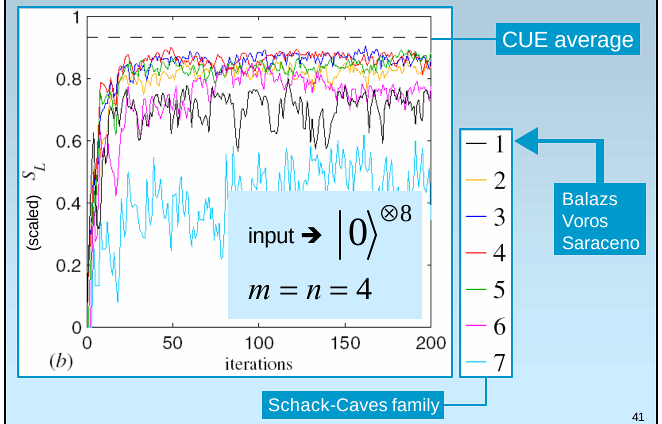
Scott & Caves, JPA 03



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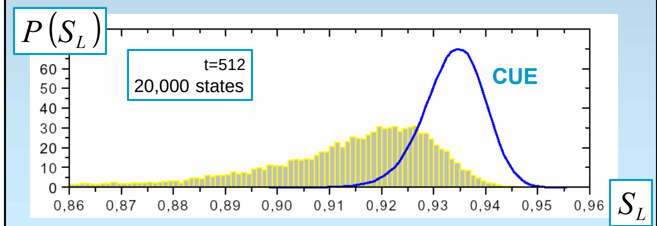
Distribution of entropies

Scott & Caves, JPA 03

input ensemble = $\mathrm{CUE}_1^{\otimes 8}$ $\quad |f_1\rangle \otimes |f_2\rangle \otimes \ldots \otimes |f_8\rangle$



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Bakers vs CUE

Scott & Caves, JPA 03

Bakers are not as good entanglers as typical CUE maps

5. Discussion and conclusion

The numerical calculations of the previous section show that the quantum baker's maps are, in general, good at creating multipartite entanglement amongst the qubits. It was found, however, that some quantum baker's maps can, on average, entangle better than others, and that all quantum baker's maps fall somewhat short of generating the levels of entanglement expected in random states. This might be related to the fact that spatial symmetries in the baker's map allow deviations from the predictions of random matrix theory [2]. Such deviations are apparent in the statistics of the eigenvectors of $B_{N,n}$ and might also taint the randomness of our quantum-baked states. In this light, an entanglement measure might, in fact, provide a reasonable test for the randomness of states produced by a quantum map.

Symmetries ?

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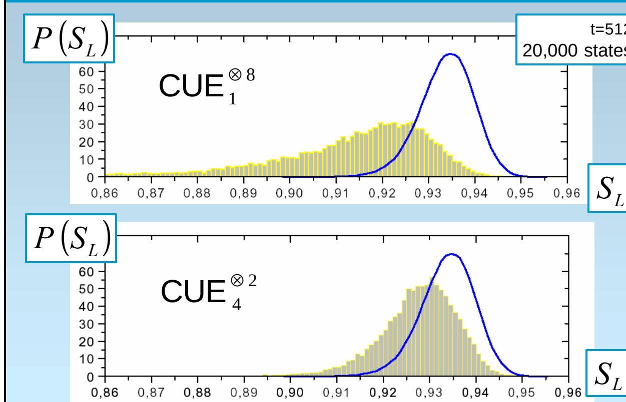
Our simulations: 1. Enlarging the ensemble



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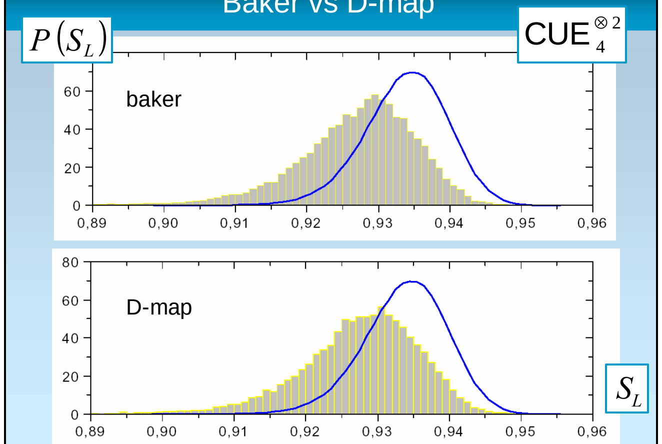
Baker vs D-map



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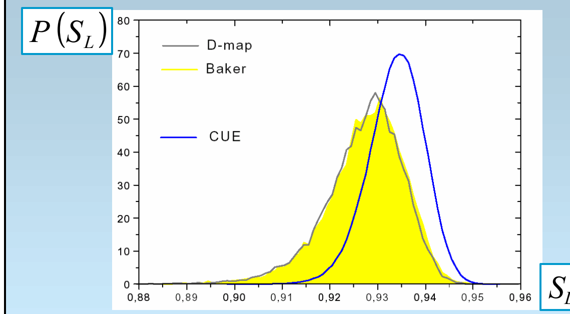
Baker vs D-map II



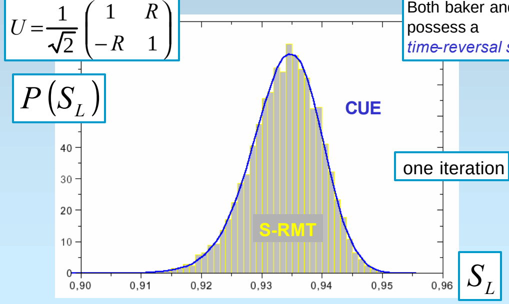
Elimination of symmetry does not increase asymptotic entangling power

Complementary test: Random Symmetric Map

$$B = U \begin{pmatrix} B_- & 0 \\ 0 & B_+ \end{pmatrix} U^+ \to U \begin{pmatrix} COE_1 & 0 \\ 0 & COE_2 \end{pmatrix} U^+$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & R \\ -R & 1 \end{pmatrix}$$

Both baker and D-map possess a *time-reversal symmetry*



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Then ...

Introducing symmetry does not reduce entangling power of random maps

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Comments

Symmetry does not affect entangling power, at least for the considered measure

But it might be relevant for eigenvector entanglement ...
(initial states are symmetric, symmetry is preserved by the partial trace)

Who is responsible ?

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Observation

Spectrum is not relevant for asymptotic entangling power (if incommesurate)

Demkowicz-Dobrzanski & Kus, PRE 04
Weinstein & Hellberg, quant-ph 05

$$S_L(\infty)=\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} S_L(t) \propto 1-\sum_{k_1,k_2,k_3,k_4}(\ldots)\left\langle e^{i(f_{k_1}-f_{k_2}+f_{k_3}-f_{k_4})t}\right\rangle_{\text{time}}$$

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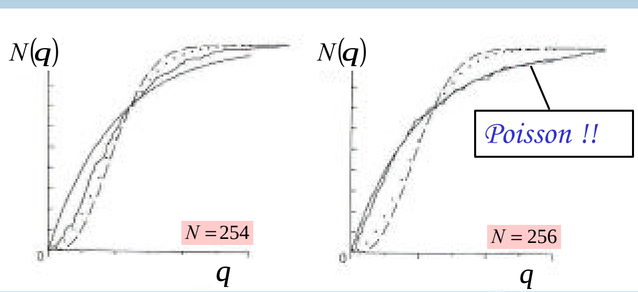
Arithmetical chaos?

It is known since Balazs-Voros' times that bakers with $N=2^n$ are anomalous:



Balazs & Voros, AOP 89

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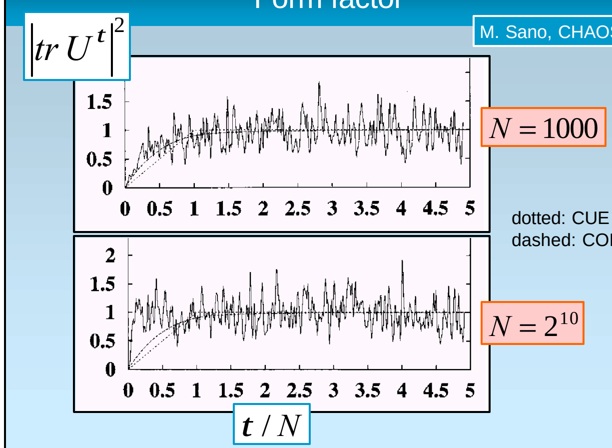
Form factor

M. Sano, CHAOS 00



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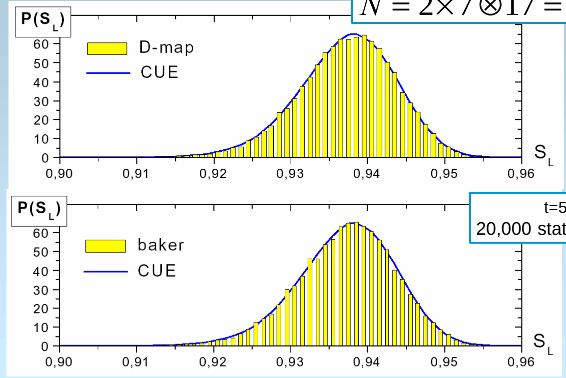
Breaking pseudo-symmetries

$$N = 2\times 7 \otimes 17 = 238$$



t=512
20,000 states

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Conclusions and perspectives

We showed in two different ways that spatial symmetry does not affect the entangling power of typical unitaries.

By avoiding power-of-two dimensions, *i.e.,* qubit systems, agreement with Random Matrix Theory is restored.

Next steps

(i) eigenvector entanglement analysis

(ii) identification of precise place where deviation from RMT occurs

(iii) analysis of pseudo-symmetries

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