Semiclassical propagation of Gaussian wavepackets

Raúl O. Vallejos
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro
www.cbpf.br/~vallejos

Instituto do Milênio Informação Quântica

CBPF

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Title: WKB Propagation of Gaussian Wavepackets

Authors: Raphael N. P. Maia, Fernando Nicacio, Raul O. Vallejos, Fabricio Toscano
(Submitted on 17 Jul 2007)

Abstract: We analyze the semiclassical evolution of a Gaussian wavepacket in a chaotic system using standard time-dependent WKB theory (no complex trajectories). We show that the Wigner function develops the structure of a classical filament plus quantum oscillations, with phase and amplitude being determined by geometric properties of an evolving classical manifold.

Comments: 4 pages, 3 figures
Subjects: Chaotic Dynamics (nlin.CD)
Cite as: arXiv:0707.2423v1 [nlin.CD]



Outline

- Motivation
- Review of TDWKB
- Main result
- Inclusion of decoherence

Gaussian wavepackets in semiclassical regimes

Quantum-to-classical transition

Environment induced decoherence
Zurek, Paz, Dalvit, Cucchietti ...

Habib, Bhattacharya, …

Continuous quantum measurements
Sundaram, Jacobs, ...

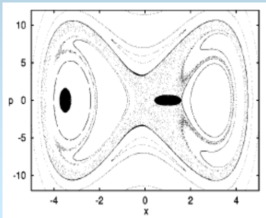
One example from EID

Monteoliva-Paz
PRL00, PRE01

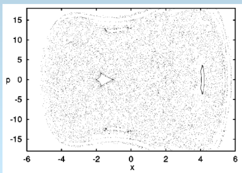
Duffing oscillator

$$H_0(x,p,t) = \frac{p^2}{2m} - bx^2 + \frac{x^4}{64a} + sx\cos(\omega t)$$ + diffusive reservoir

(or + weak continuous measurement)



mixed



mostly chaotic

Decoherence rate

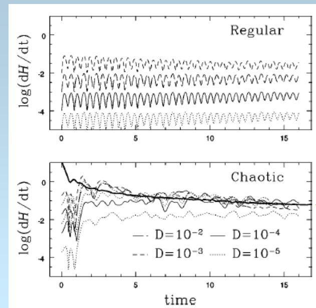


FIG. 8. Entropy production rate (in logarithmic scale) vs time (in units of the driving period). The bold curve is the (time dependent) Lyapunov exponent. The linear dependence of the rate on D

Monteoliva-Paz
PRL00, PRE01

Lyapunov exponent

Wavefunction structure – Closed system

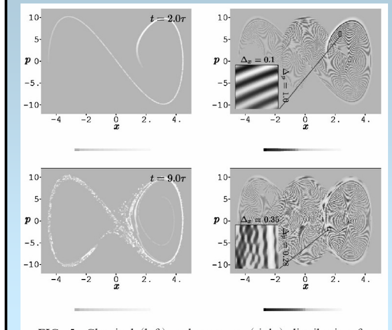


FIG. 5. Classical (left) and quantum (right) distribution functions for four different times. Already, for early times, interference

Monteoliva-Paz
PRE01

Wigner functions
D=0

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Diffusive reservoir

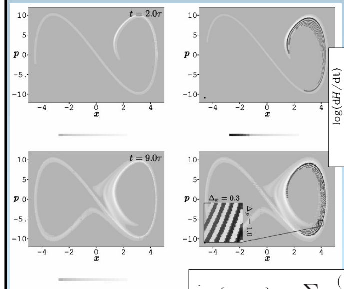
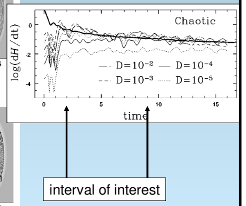


FIG. 7. Classical (left) and qua[ntum] ... tions for the same initial condition ... $=9\tau$, when the system is opened to ... ($D=0.01$).



$$\dot{W} = \{H_0, W\}_{PB} + \sum_{n\geq 1} \frac{(-1)^n \hbar^{2n}}{2^{2n}(2n+1)!} \partial_x^{(2n+1)} V \partial_p^{(2n+1)} W + \cancel{2\gamma\partial_p(pW)} + D\partial_{pp}^2 W.$$

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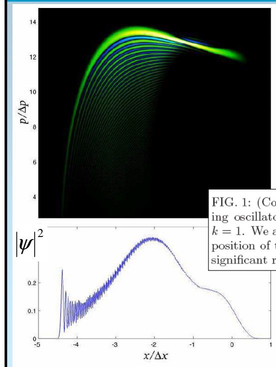
Continuous measurement



Greenbaum-Jacobs-Sundaram
quant-ph 07

Wigner function

FIG. 1: (Color online) A typical wigner function for the Duffing oscillator when $\hbar = 0.1$ and the measurement strength $k = 1$. We also plot the associated probability density for the position of the oscillator. The wave-function is spread over a significant region of the phase-space.

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Semiclassical propagation: TDWKB theory

TDWKB theory

$$H(\hat{p}, q, t)\,\varphi(q,t) = -i\hbar \frac{\partial}{\partial t} \varphi(q,t)$$

WKB
Van Vleck, PNAS28
Dirac's book
Maslov, 60's-70's
Berry-Balazs, JPA79
Littlejohn, JSP92

Initial wavefunction

$$\varphi(q, t=0) \cong A_0(q)\, e^{iS_0(q)/\hbar}$$

WKB form
A(q), S(q) real
vary slowly

At a later (short) time …

$$\varphi(q,t) \cong A_t(q)\, e^{iS_t(q)/\hbar}$$

primitive WKB

Problem: find $A_t(q)$, $S_t(q)$

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Solution

Littlejohn, JSP92

$$H\left(\frac{\partial S}{\partial q}, q, t\right)\varphi(q,t) + \frac{\partial}{\partial t} S(q,t) = 0$$

Hamilton-Jacobi

$$\rho(q,t) = |A(q,t)|^2$$

$$\frac{\partial}{\partial t}\rho(q,t) + \frac{\partial}{\partial q}\left[\rho(q,t)\,\mathrm{v}(q,t)\right] = 0$$

$$\mathrm{v}(q,t) \equiv \frac{\partial}{\partial p} H(p,q,t)\Big|_{p=\frac{\partial S}{\partial q}}$$

transport equation

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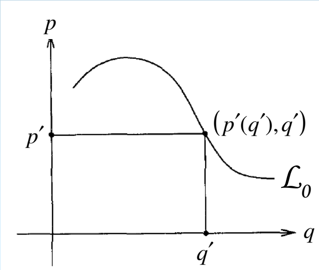
Geometrical Interpretation

$\varphi(q,t=0) \cong A_0(q)\, e^{iS_0(q)/\hbar}$ Initial wavefunction



$$p = p_0(q) = \frac{\partial S_0(q)}{\partial q}$$

Lagrangian manifold

$$S_0(q) = \int_{\dots}^{q} p\,dq$$

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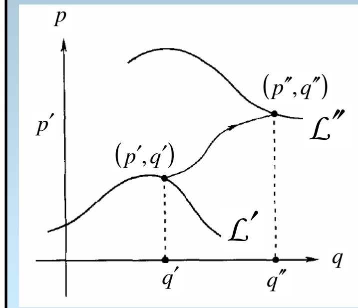
Solving the Hamilton-Jacobi equation



new phase

$$S(q'',t'') = \int_{\dots}^{q''} p\,dq$$

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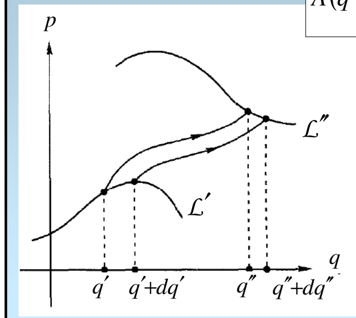
Continuity equation

$$A(q'',t) = A(q',t)\left|\frac{dq'}{dq''}\right|^{1/2}$$



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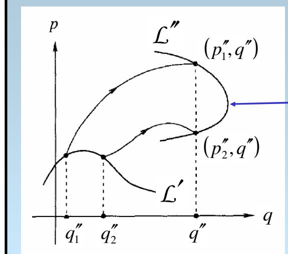
Caustics



$$A(q'',t) = A(q',t)\left|\frac{dq'}{dq''}\right|^{1/2}$$

multiple branch WKB

$$\varphi(q'',t'') \cong \sum_b A_b(q'',t'')\, e^{iS_b(q'',t'')/\hbar - i\kappa_b\pi/2}$$

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Applying TDWKB to Gaussian wavepackets

Counterexample: ground state of the harmonic oscillator

$\varphi_0(q) \propto e^{-q^2/4\sigma^2}$ $\equiv A_0(q)\, e^{iS_0(q)/\hbar}$ amplitude not smooth!

$if \quad S_0(q) = 0$

$\Rightarrow p_0(q) = 0$

$\Rightarrow p_t(q) = \kappa_t q$

$\Rightarrow S_t(q) = \kappa_t q^2/2$ wrong !!

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End of Introduction

Proposition

1. In closed chaotic systems Gaussian wavepackets eventually evolve into WKB states:

$$\varphi(q'',t'') \cong \sum_b A_b(q'',t'')\, e^{iS_b(q'',t'')/\hbar - i\kappa_b \pi/2}$$

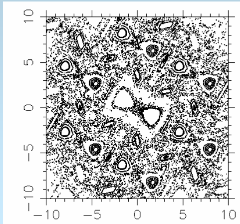
2. Even if TDWKB fails to propagate Gaussian wavepackets !

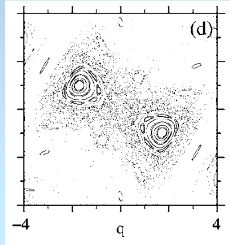
3. Construction (exact)

Kicked Harmonic Oscillator

$$H(p,q,t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + K\cos(kq)\sum_{n=0}^{\infty}\delta(t - n\tau)$$

Berman-Zaslavsky

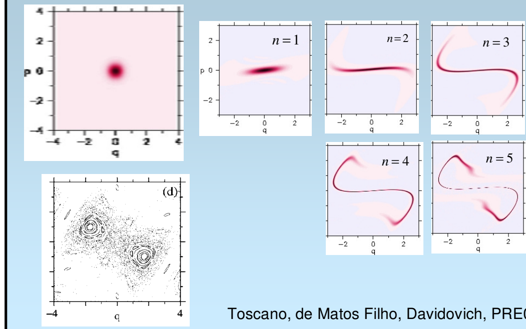




$m = \omega = k = 1,\ K = 2$

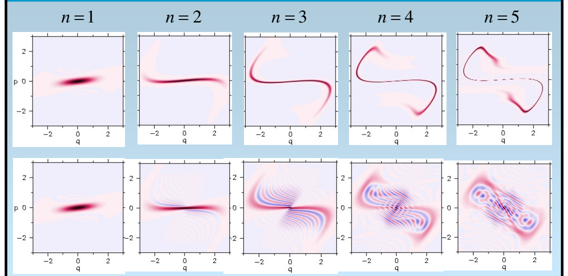
$\tau = T/6$

Classical Dynamics (Liouville)

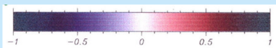


Toscano, de Matos Filho, Davidovich, PRE05

Classical vs. Quantum



Wigner functions



Toscano, de Matos Filho, Davidovich, PRE05

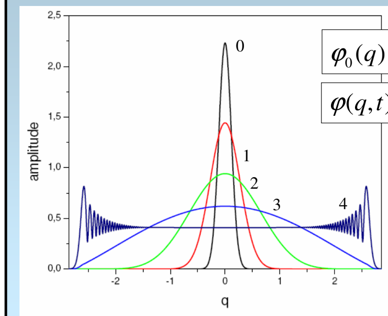
Observation

Chaotic dynamics stretches wavepackets, first linearly, then nonlinearly.

After a certain time (log $\hbar$) a wavepacket becomes a smooth primitive WKB state.

From then on it can be propagated with TDWKB.

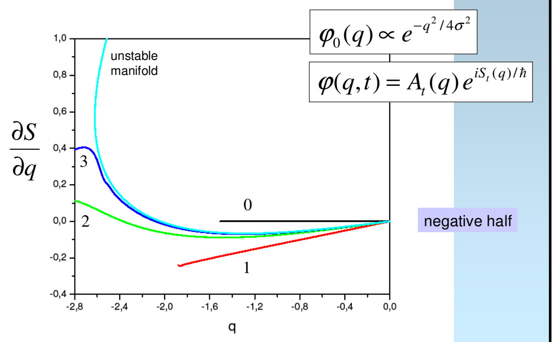
Amplitude: exact propagation



$$\varphi_0(q) \propto e^{-q^2/4\sigma^2}$$

$$\varphi(q,t) = A_t(q)\, e^{iS_t(q)/\hbar}$$

Phase: exact propagation



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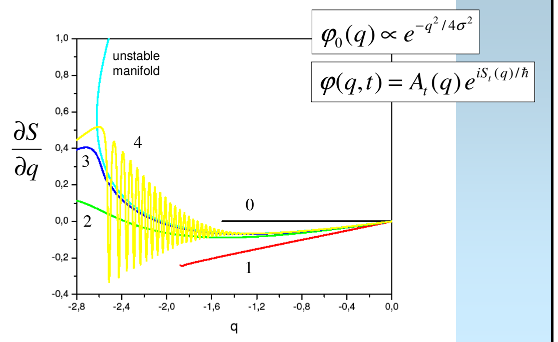
Phase: exact propagation



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Recipe

Propagate during a short time either
- numerically,
- using the linear dynamics (if satisfactory),
- complex TDWKB,
- etc

Resume propagation with (real) TDWKB

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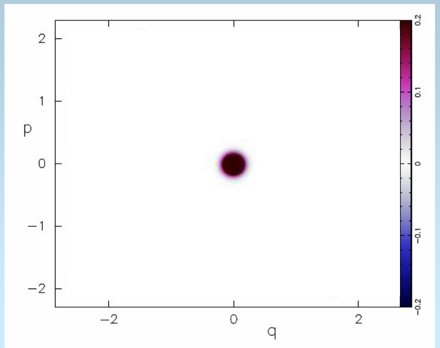
Wigner function $\pi\hbar W(p,q)$



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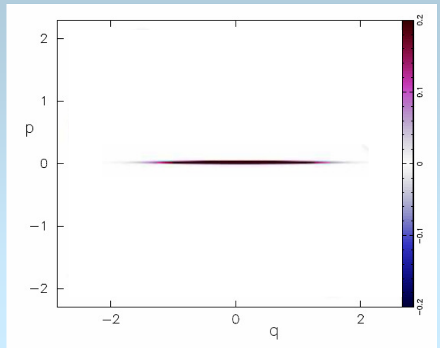
Wigner function $\pi\hbar W(p,q)$



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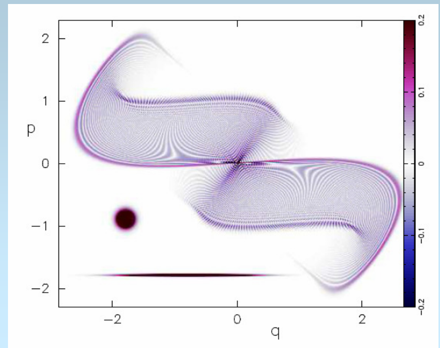
Wigner function $\pi\hbar W(p,q)$



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Semiclassical (TDWKB) Wigner function

$$\varphi(q'',t'') \cong \sum_b A_b(q'',t'')\, e^{iS_b(q'',t'')/\hbar - i\kappa_b \pi/2}$$

$$W(p,q) = \frac{1}{\pi\hbar}\int d\xi\, \varphi^*(q-\xi/2)\,\varphi(q+\xi/2)\, e^{-ip\xi/\hbar}$$

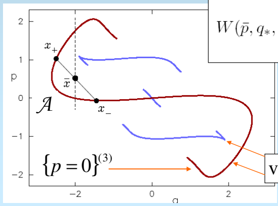
stationary phase

Semiclassical (TDWKB) Wigner function

$$W(\bar p,\bar q,t) = \frac{2\sqrt{2}}{\sqrt{\pi\hbar}} A_0(q_+)A_0(q_-)\frac{\cos(\mathcal{A}/\hbar - \pi/4)}{\sqrt{|v_+ \wedge v_-|}} \,.$$

one chord

$\bar x = (\bar p, \bar q)$

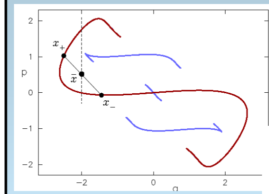


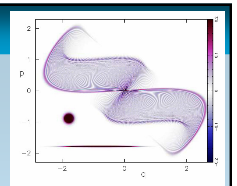
$$W(\bar p, q_*, t) \approx \frac{2A_t^2(q_*)}{(\hbar^2 p_*'')^{1/3}}\mathrm{Ai}\left[-\frac{2(\bar p - p_*)}{(\hbar^2 p_*'')^{1/3}}\right]$$

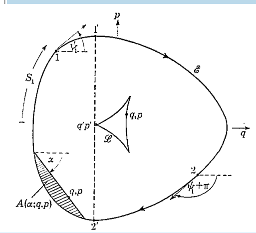
close to the manifold

$v_+ \wedge v_- = 0$

Caustics

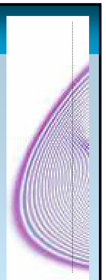




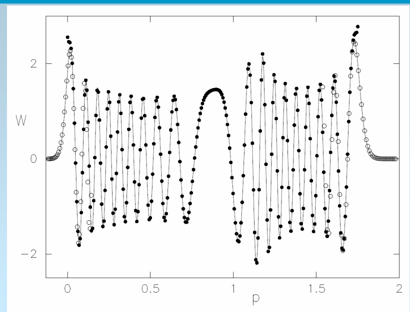


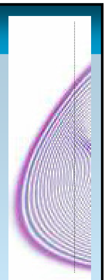
Berry 77

Quantum vs. WKB – Wigner section



Quantum vs. WKB – Wigner section





full line = exact;
black circles = primitive WKB
open circles = Airy transitional approximation

Decoherence

Decoherence

$$\dot W = \{H_0, W\}_{PB} + \sum_{n\geqslant 1} \frac{(-1)^n \hbar^{2n}}{2^{2n}(2n+1)!} \partial_x^{(2n+1)} V \partial_p^{(2n+1)} W$$
$$\cancel{+2\gamma\partial_p(pW)} + D\partial_{pp}^2 W.$$

$$if \quad \dot W = D\partial_{pp} W$$

$$then \quad \hat\rho(t) = \int d\xi\, g(\xi; D, t)\hat T_\xi \hat\rho_0 \hat T_\xi^+ \qquad \xi \in R^2$$

Gaussian channel

phase space translation (Glauber)

average over random translations

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Kicked Harmonic Oscillator

The KHO dynamics "commutes" with diffusion (and with instantaneous kicks), then …

- Kick (nonlinear shear)
- Harmonic oscillator (rotation)
- Gaussian channel
- Iterate

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Stochastic unravelling

- Kick (nonlinear shear)
- Harmonic oscillator (rotation)
- Random phase space translation
- Iterate
- Repeat for another set of random translations
- Average over translations (over ensemble of WKB manifolds)

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Continuous time

In case of continuous time dependence, use Lie-Trotter decomposition:

$$\dot W = \{H_0, W\}_{PB} + \sum_{n\geqslant 1} \frac{(-1)^n \hbar^{2n}}{2^{2n}(2n+1)!} \partial_x^{(2n+1)} V \partial_p^{(2n+1)} W$$
$$\cancel{+2\gamma\partial_p(pW)} + D\partial_{pp}^2 W.$$

$$\dot W = \{H, W\}_{MB} + D\partial_{pp} W$$

$$W(t+dt) = T\int dt\, e^{t\left[\{H,\ \}_{MB} + D\partial_{pp}\right]} W(t)$$

$$\approx e^{dt\, D\partial_{pp}}\, e^{dt\{H,\ \}_{MB}} W(t)$$

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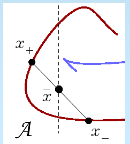
Semiclassical Wigner function with diffusion

$$W(p,q) = \left\langle \Re\, 2\sqrt{\frac{2}{\pi\hbar}} \frac{A(q_+)A(q_-)}{\sqrt{|v_+ \wedge v_-|}} \exp\left(i\frac{\mathcal{A}}{\hbar} - i\frac{\pi}{4}\right)\right\rangle_{\substack{random \\ translations}}$$

$$\approx \Re\, 2\sqrt{\frac{2}{\pi\hbar}} \frac{A(q_+)A(q_-)}{\sqrt{|v_+ \wedge v_-|}} \left\langle \exp\left(i\frac{\mathcal{A}}{\hbar} - i\frac{\pi}{4}\right)\right\rangle$$ if fast



$$\approx \Re\, 2\sqrt{\frac{2}{\pi\hbar}} \frac{A(q_+)A(q_-)}{\sqrt{|v_+ \wedge v_-|}} e^{-i\pi/4}\, e^{i\langle\mathcal{A}\rangle/\hbar}\, e^{-\langle\delta\mathcal{A}^2\rangle/2\hbar^2}$$ if Gaussian

Probably $\langle \delta\mathcal{A}^2 \rangle \propto t$

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Summary

Simple geometrical description of the evolution of a wavepacket in a closed chaotic system (or chaotic region)

Diffusion is easily included

Applications?

Long-time validity of semiclassical propagation

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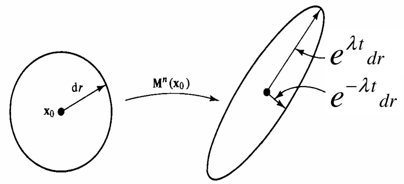
Fin

Lyapunov exponent

Chaotic regions are characterized by exponential sensitivity to initial conditions, i.e., $\lambda > 0$

Figure 4.15 Evolution of an initial infinitesimal ball after n iterations of the map.



Ott's book

Applying TDWKB to Gaussian wavepackets

Counterexample: ground state of the harmonic oscillator

$$\varphi_0(q) \propto e^{-q^2/4\sigma^2} \equiv A_0(q)\, e^{iS_0(q)/\hbar}$$

amplitude not smooth!

However …

$$A_0(q) = 1$$

$$S_0(q) = iq^2\hbar / 4\sigma^2$$

Huber-Heller 88

$$p_0(q) = iq\hbar / 2\sigma^2$$

OK !!