Statistical bounds on the dynamical generation of entanglement

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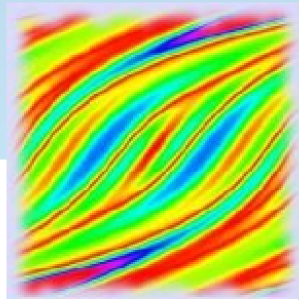

Quantum Chaos: theory and applications

12-03-07 to 16-03-07 - Tandar Laboratory - CNEA - Buenos Aires- Argentina

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Quantum Physics, abstract
quant-ph/0703057

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Statistical bounds on the dynamical production of entanglement

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We present a random-matrix analysis of the entangling power of a unitary operator as a function of the number of times it is iterated. We consider unitaries belonging to the circular ensembles of random matrices (CUE or COE) applied to random (real or complex) non-entangled states. We verify numerically that the average entangling power is a monotonic decreasing function of time. The same behavior is observed for the "operator entanglement" --an alternative measure of the entangling strength of a unitary. On the analytical side we calculate the CUE operator entanglement and asymptotic values for the entangling power. We also provide a theoretical explanation of the time dependence in the CUE cases.

Full-text: PostScript, PDF, or Other formats

Entanglement

A **pure state** of a **bipartite** system, A + B, is said ***entangled*** if it is not ***separable***.

$$\underbrace{\left|\Psi_{A+B}\right\rangle}_{H} = \underbrace{\left|\phi_A\right\rangle}_{H_A} \otimes \underbrace{\left|\psi_B\right\rangle}_{H_B}$$

separable

The Hilbert space of the composite system is the tensor product of the Hilbert spaces of both subsystems:

$$H = H_A \otimes H_B$$

Entanglement measures for bipartite pure states

$$|\Psi\rangle \in H_A \otimes H_B$$ ↔ pure state

$$\rho = |\Psi\rangle\langle\Psi|$$ ↔ density matrix

$$\rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$$ ↔ reduced density matrix

$$S_{\mathrm{vN}} = -\mathrm{tr}_A\, \rho_A \log \rho_A$$ ↔ von Neumann entropy

$$S_{\mathrm{L}} = 1 - \mathrm{tr}_A\, \rho_A^2$$ ↔ linear entropy (1 – purity)

Entangling power of unitary transformations

Zanardi, Zalka & Faoro, PRA 00

[...] how much entanglement is produced by **U** on the average, acting on a given distribution of non-entangled quantum states.

$$ep(U) = \left\langle S_{\ldots}\left(U \left|\psi_A\right\rangle \otimes \left|\psi_B\right\rangle \right) \right\rangle_{\psi_A, \psi_B}$$

entanglement measure, e.g., linear entropy

average over ensemble of product states

Alternative measure: Operator entanglement (Schmidt strength)

Zanardi, PRA 01
Wang & Zanardi, PRA 02
Nielsen et al, PRA 03
Zyczkowski & Bengtsson, OSID 04
Bengtsson & Zyczkowski´s book

$$\left|U\right\rangle \in H^{HS}_{N_A^2} \otimes H^{HS}_{N_B^2}$$

$$\left\langle A\middle|B\right\rangle \equiv \mathrm{tr}\left(A^{+}B\right)$$

Hilbert-Schmidt scalar product

$$\rho^U = \left|U\right\rangle\left\langle U\right|$$

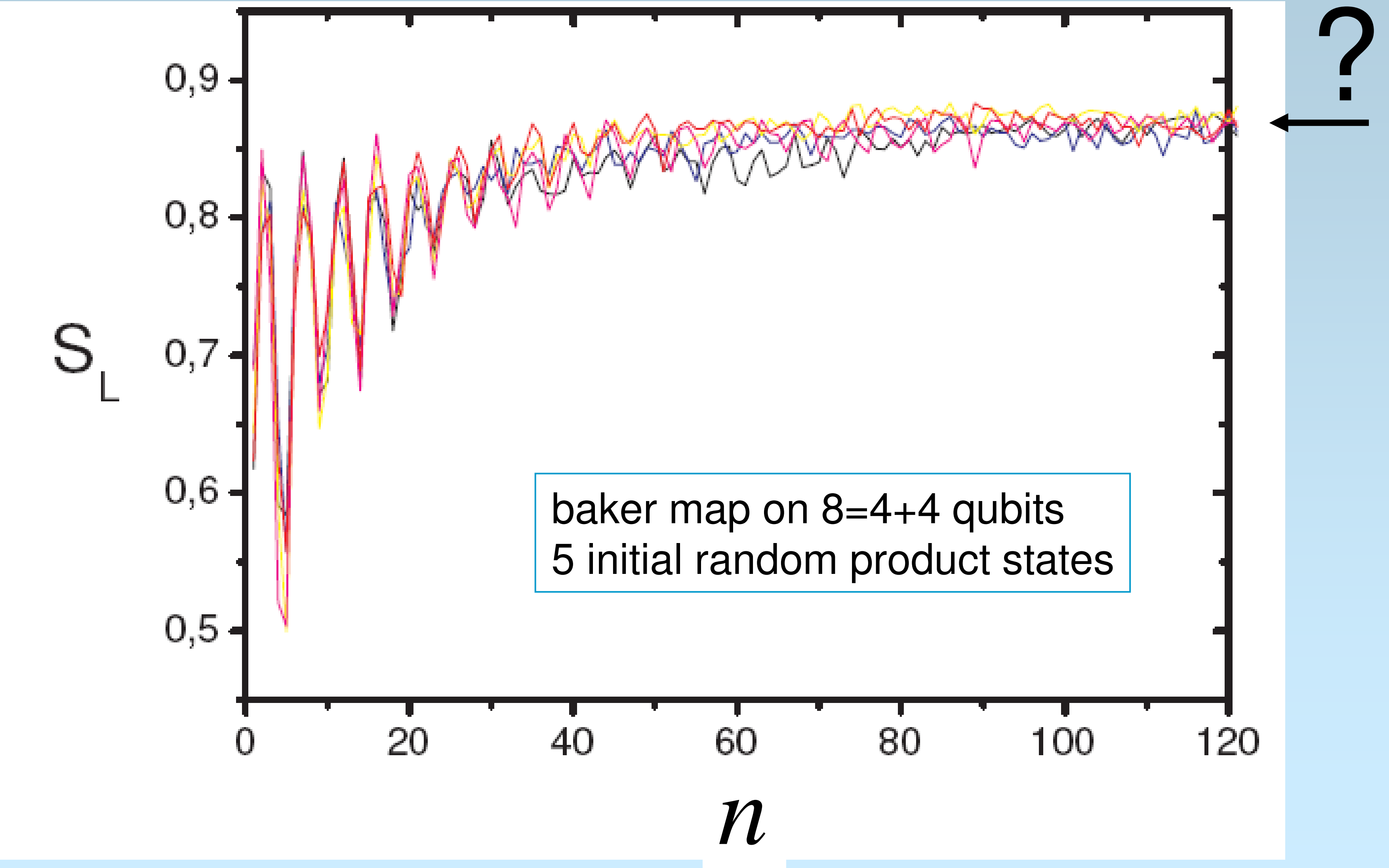
$$\rho^U_A = \mathrm{tr}_B\left|U\right\rangle\left\langle U\right|$$

$$oe\left(U\right) = 1 - \mathrm{tr}_A\left(\rho^U_A\right)^2$$

Entangling power of the baker´s map

bipartite entanglement of pure states

Scott & Caves, JPA 03



“Random State Theory”

... for the asymptotic entangling power of a unitary operator says:

If the quantum dynamics is “chaotic”, then initial nonentangled states evolve asymptotically into random states, only restricted by the normalization condition.

Then, the asymptotic entangling power should be equal to the average entropy of random states.

Canonical Random States

$$|\psi\rangle \rightarrow \left(x_1 + i\,y_1, \quad \ldots \quad , x_N + i\,y_N\right)$$

uniform measure

$$p\left(x_1, y_1, \ldots, x_N, y_N\right) \propto \delta\left(1 - \sum_{i=1}^{N} x_i^2 + y_i^2\right)$$

Random Matrix Theory

If the quantum dynamics is “chaotic”, then it can be modeled by a random unitary map.

Asymptotic states are the result of the repeated application of a random map to nonentangled states:

$$\left|\psi(n)\right\rangle = U^{n}\left|\phi_A\right\rangle \otimes \left|\phi_B\right\rangle$$

$$n \to \infty$$

Initial Objective

Compare the predictions of both theories, i.e.,
Random State Theory *vs* Random Matrix Theory,
for the asymptotic entanglement of "typical" maps.

typical = describable by any of the circular ensembles of random unitary matrices (CUE or COE)

Observation

RMT $\left|\psi(n)\right\rangle = U^{n}\left|\phi_A\right\rangle \otimes \left|\phi_B\right\rangle$

RST $\left|\psi(n)\right\rangle = U^{1}\left|\phi_A\right\rangle \otimes \left|\phi_B\right\rangle$

$n \to \infty$

If U is a random unitary operator belonging to CUE, then a canonical random vector can be constructed as

$$U\left|\phi_A\right\rangle \otimes \left|\phi_B\right\rangle$$

Extended Objective

1. Entangling power of U^n as a function of *n*

2. Operator entanglement of U^n as a function of *n*

Choosing the ensembles

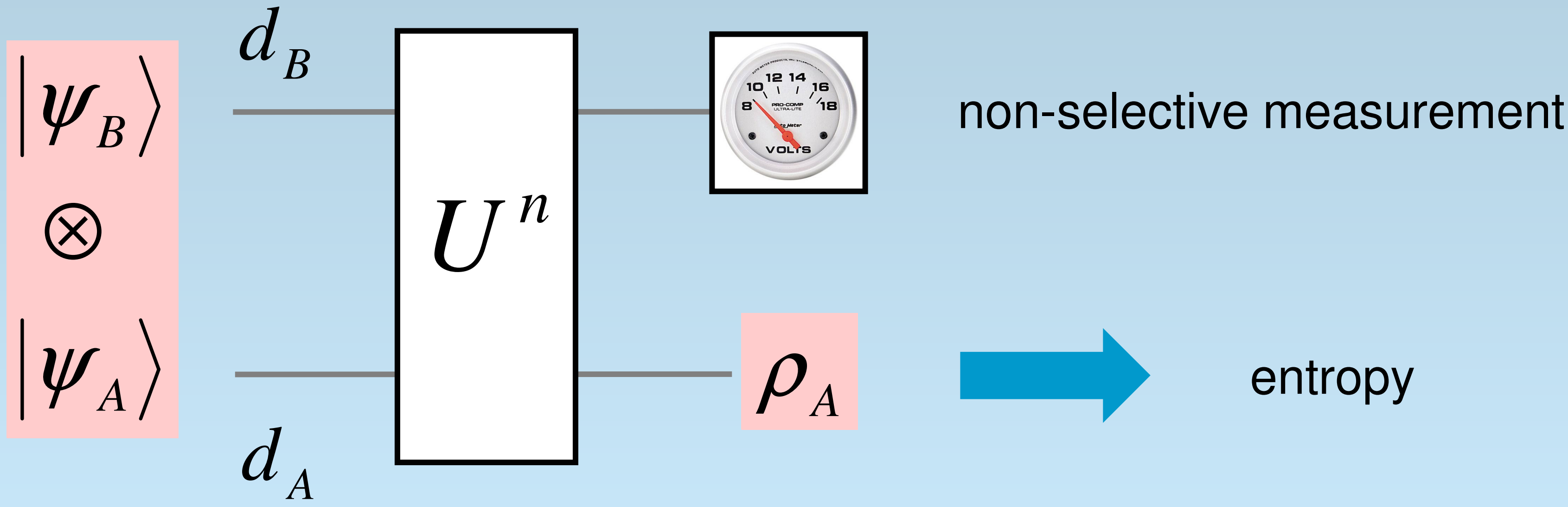
states: $$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

canonical random states, complex or real

maps: CUE or COE

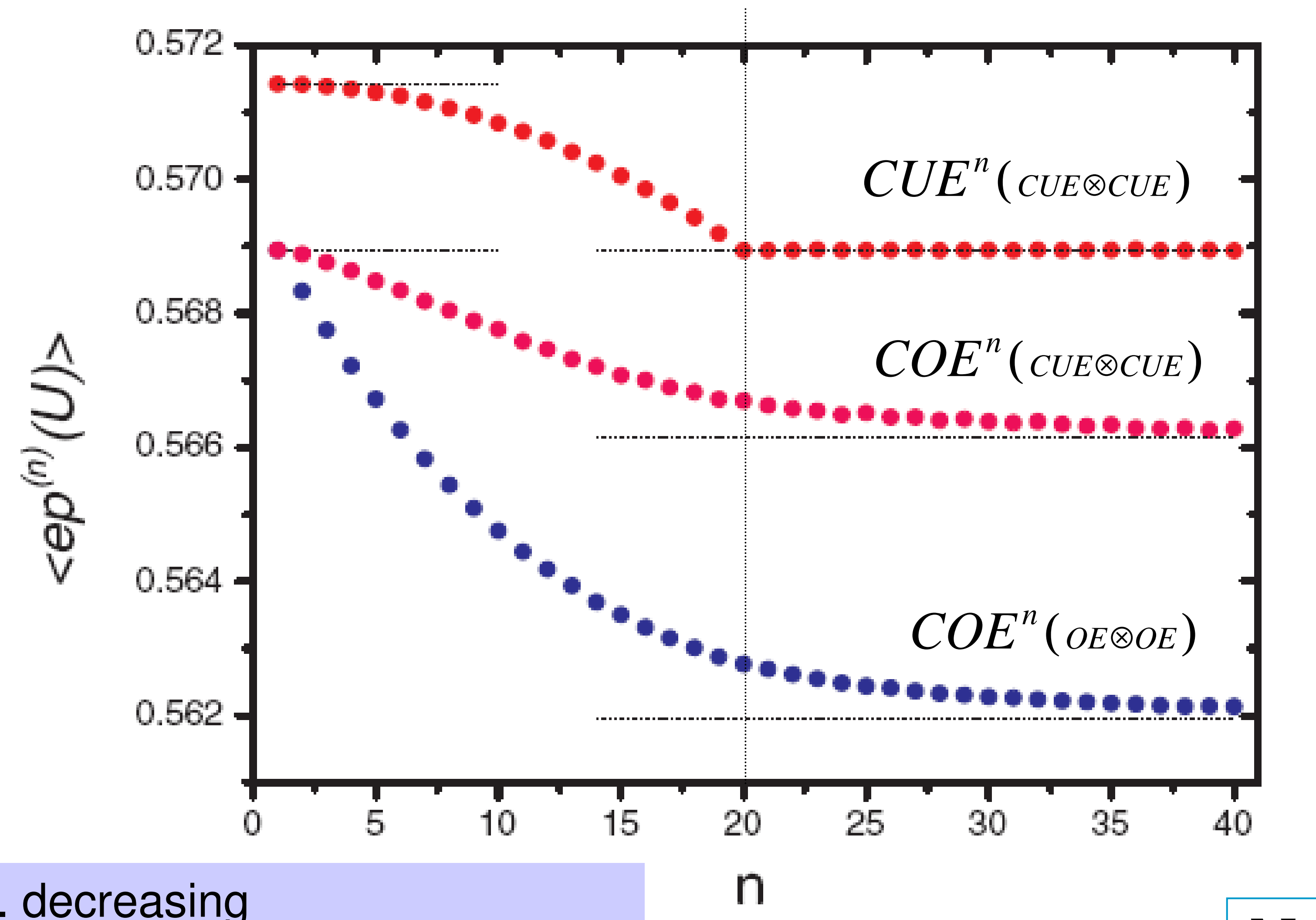
Entangling Power: Simulations

<u>*Experimental setup*</u>



- Entanglement (linear entropy) *<u>as a function of time</u>*
- Double average over input states and maps

Average entangling power versus time

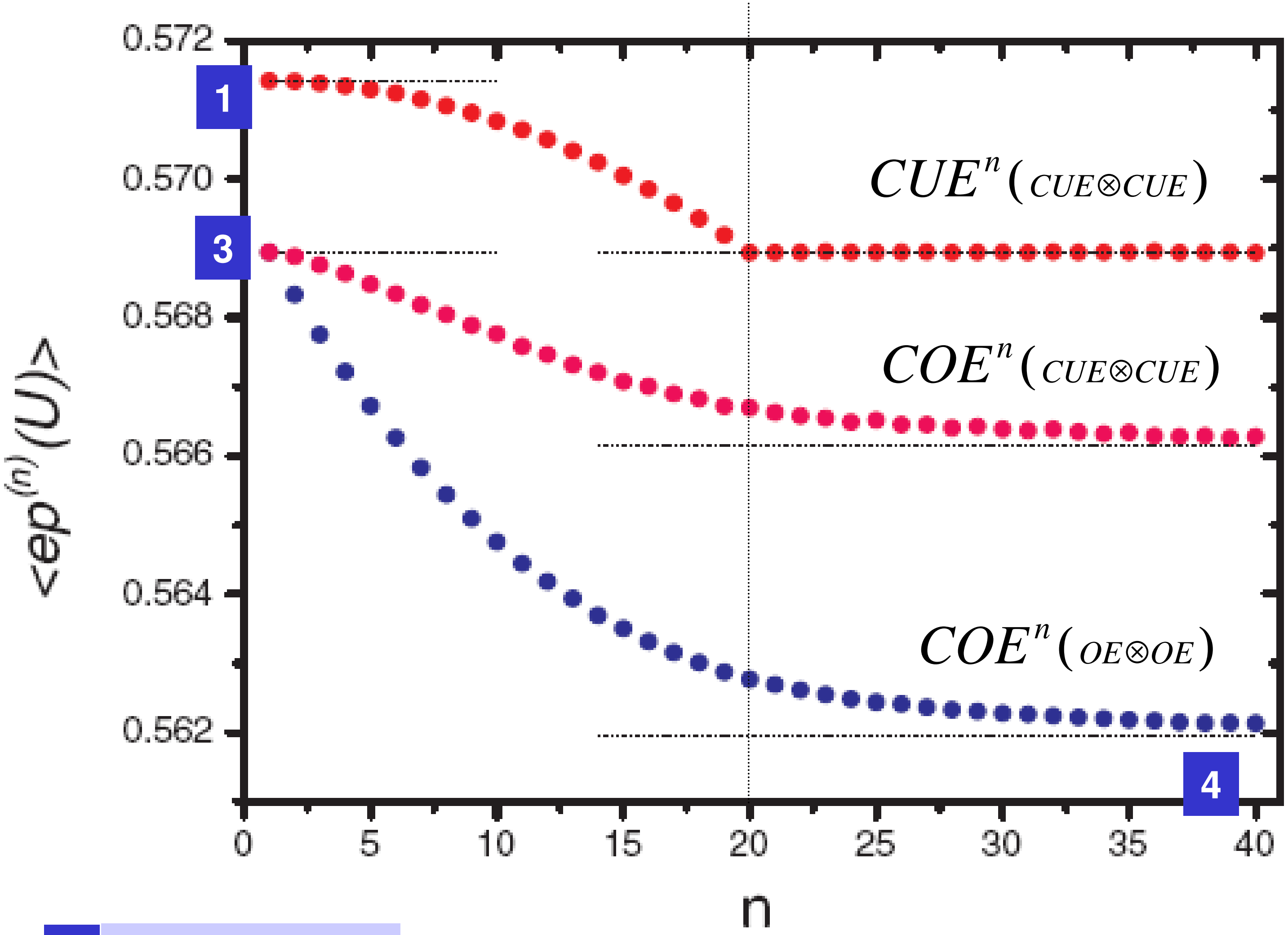


1. decreasing
2. saturation at Heisenberg time
3. coincidence?

$\tau_H = 20$

$N = 4 \otimes 5$

Analytical results



1 random vectors

3 4 Gorin-Seligman

τ_H

$N = 4 \otimes 5$

Two-vector averages of monomials of order 8

$$\left\langle ep\left(U^{\infty}\right)\right\rangle_{CUE} = 1 - \Big\langle \sum_{\alpha} \sum_{r,r',s,s'} |U_{11,\alpha}|^4 U_{rs,\alpha} U^*_{r's,\alpha} U_{r's',\alpha} U^*_{rs',\alpha} -$$

$$\sum_{\alpha\neq\beta} \sum_{r,r',s,s'} |U_{11,\alpha}|^2 |U_{11,\beta}|^2 U_{rs,\alpha} U^*_{r's,\alpha} U_{r's',\beta} U^*_{rs',\beta} -$$

$$\sum_{\alpha\neq\beta} \sum_{r,r',s,s'} |U_{11,\alpha}|^2 |U_{11,\beta}|^2 U_{rs,\alpha} U^*_{rs',\alpha} U_{r's',\beta} U^*_{r's,\beta} \Big\rangle .$$

Tools for averaging over the classical groups

Mello, JPA 90

Gorin, JMP 02

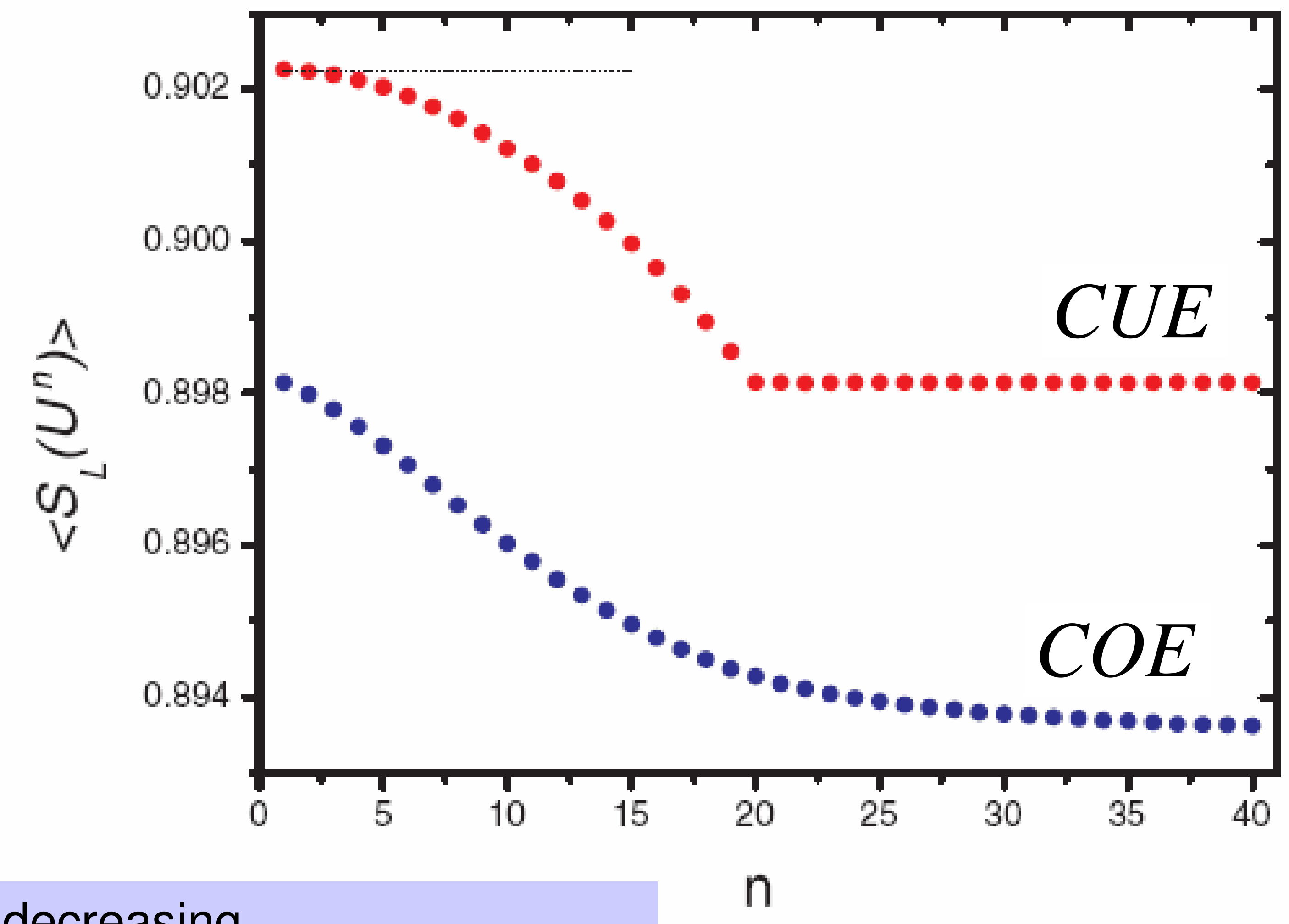
Collins, math-ph/02

Aubert & Lam, JMP 03, JMP 04

Braun, math-ph/06

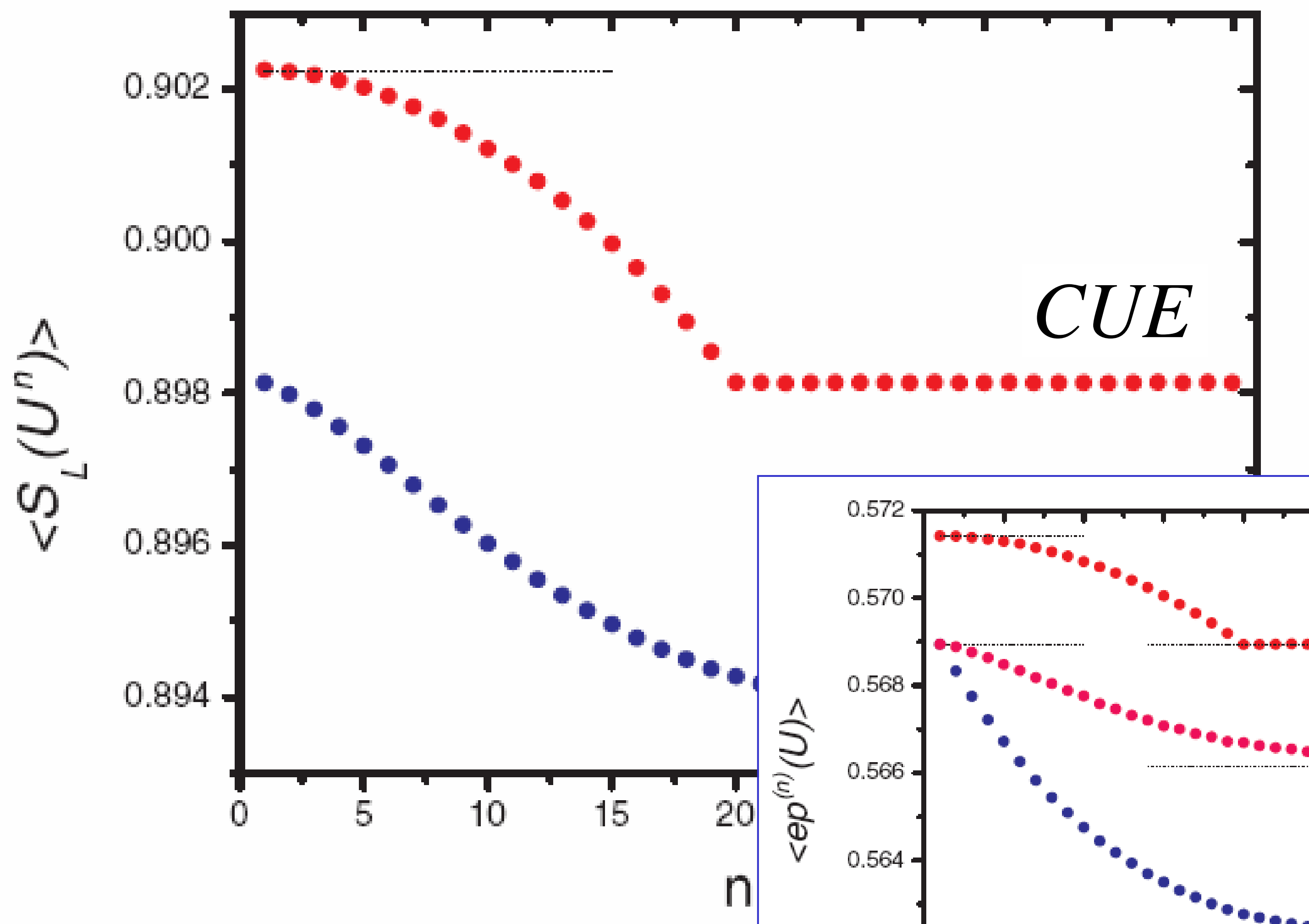
Collins & Sniady, CMP 06

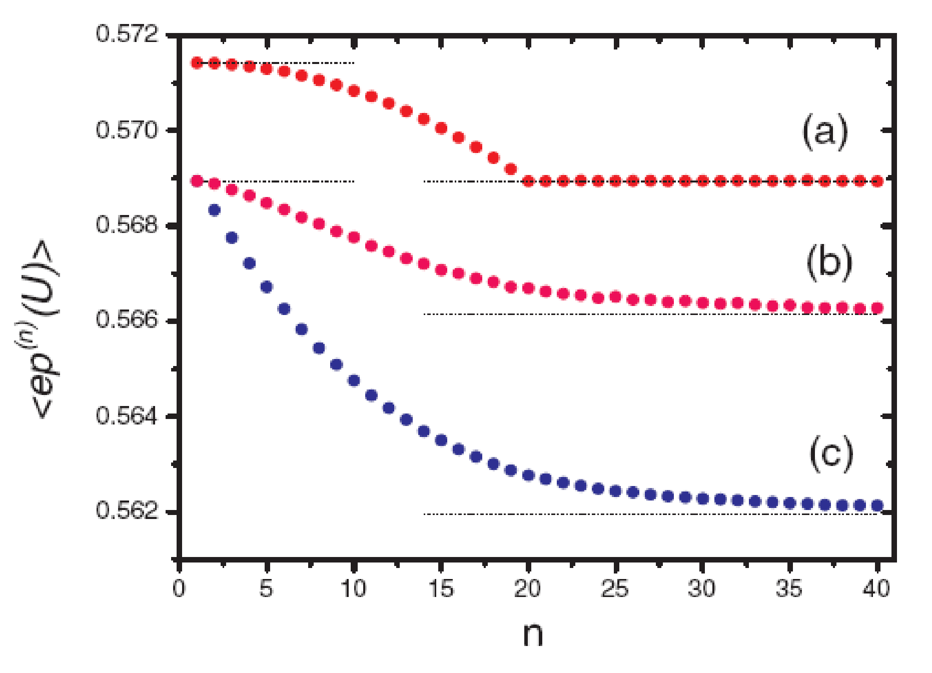
Operator entanglement



1. decreasing
2. saturation at Heisenberg time
3. coincidence !

Analytical results





1. CUE operator entanglement
2. Shape for CUE cases (also *ep*)

Conclusions

Asymptotic states generated dynamically are not canonical random states

Dynamics produces correlations → lower entropy

Small effect (second order in system size), but equivalent to imposing time reversal symmetry

A1. Extreme values

monomials of order 4

$$ep^{(1)}(U)_{(a)} = \frac{d - (d_A + d_B) + 1}{d + 1}$$

$$ep^{(1)}(U)_{(b),(c)} = \frac{d^3 - (d_A + d_B - 4)\, d^2 - [3(d_A + d_B) - 1]\, d + 2(d_A + d_B - 1)}{d(d+1)(d+3)}$$

$$\langle S_L(U) \rangle_{\mathrm{CUE}} = \frac{d^2 - (d_A^2 + d_B^2) + 1}{d^2 - 1}$$

monomials of order 8

$$ep^{\infty}(U)_{(\mathrm{a})} = ep^{(1)}(U)_{(b),(c)} \,,$$

$$ep^{\infty}(U)_{(\mathrm{c})} = \frac{d^4 - (d_A + d_B - 13)\, d^3 - [12(d_A + d_B) - 47]\, d^2 - 35(d_A + d_B - 1)\, d}{(d+1)(d+2)(d+4)(d+6)}$$

A2. Shapes

$$S(n) = 1 - \sum_{\alpha,\beta,\delta,\gamma=1}^{d} C_{\alpha\beta\delta\gamma} \left\langle e^{in(\phi_\alpha + \phi_\beta - \phi_\delta - \phi_\gamma)} \right\rangle$$

$$S(n) = c_1 + c_2 \left\langle |\, t_n|^2 \right\rangle^2 + c_3 \left\langle |\, t_{2n}|^2 \right\rangle + c_4 \left\langle |\, t_n|^2 \right\rangle$$

$$\left\langle |\, t_n|^2 \right\rangle_{\mathrm{CUE}} = \begin{cases} n & \text{if} \quad 1 \le n \le d \ , \\ d & \text{if} \quad n \ge d \ . \end{cases}$$