# Revisiting the Contact Process

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World Meeting for Women in Mathematics - July 31, 2018

## The classical contact process

- $G = (\mathbb{V}, \mathbb{E})$ graph, locally finite. Most classical example $G = \mathbb{Z}^d$.

- A Markov process $\{\xi_t\}_{t \geq 0}$ with values on $\{0, 1\}^{\mathbb{V}}$:

  $\xi_t(x) = 1$ means $x$ is infected at time $t$

  $\xi_t(x) = 0$ means $x$ is healthy at time $t$

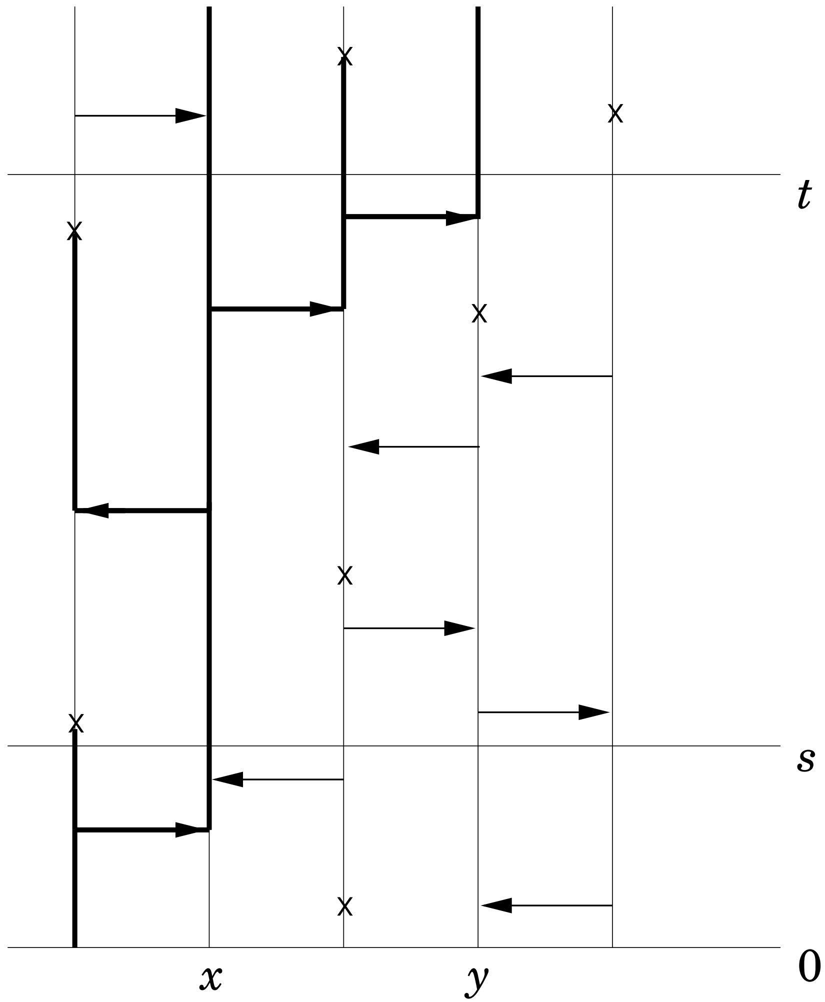
- Time evolution:

An infected individual transmits the infection with rate $\lambda > 0$ to each of its healthy neighbors, and heals with rate 1.

Identify $\xi_t$ with $\{x : \xi_t(x) = 1\}$ (set of infected individuals at time $t$)

Model introduced by T. Harris in 1974.

# The classical contact process



## Dynamical phase transition

There exists $\lambda_c \in (0, +\infty)$ so that

- If $\lambda < \lambda_c$ then $P(\xi_t^{\{0\}} = \underline{0}$ for some $t) = 1$ (subcritical)
- If $\lambda > \lambda_c$ then $P(\xi_t^{\{0\}} \neq \underline{0}$ for all $t) > 0$ (supercritical)
- $\lambda > \lambda_c \quad \Rightarrow$ positive probability that the infection remains forever

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For more general graphs than $\mathbb{Z}^d$ the supercritical regime splits into at least two:
$0 < \lambda_{1,c} < \lambda_{2,c} < \infty$ (Pemantle (1992), homogeneous tree)

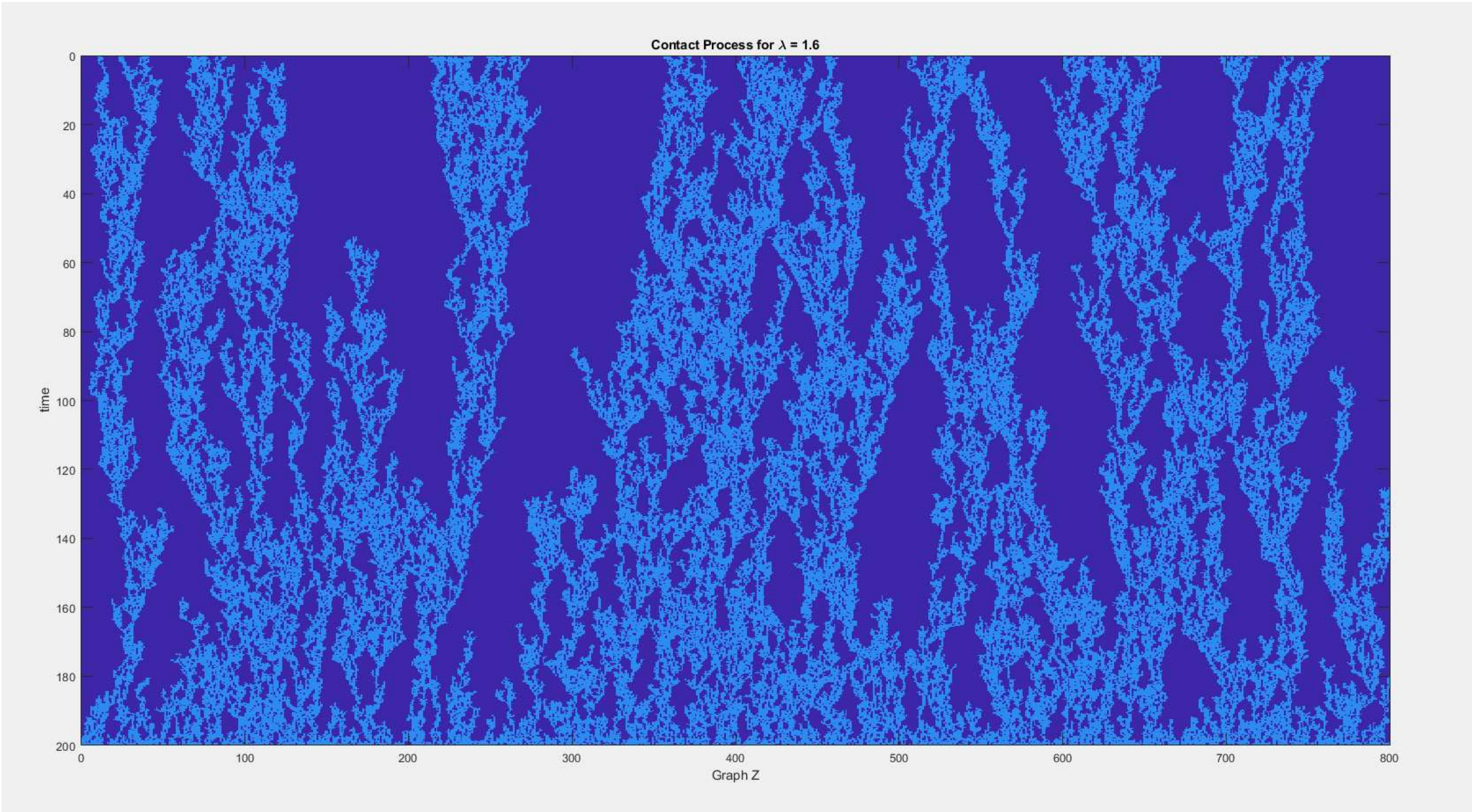
- Weak survival $\quad \lambda \in (\lambda_{1,c}, \lambda_{2,c})$
- Strong survival $\quad \lambda > \lambda_{2,c}$
- For $G = \mathbb{Z}^d$ these two critical values coincide.

$\lambda > \lambda_c \Rightarrow$ two extremal invariant measures: $\nu_\lambda, \delta_{\underline{0}}$.

**Remark:** There is a huge literature. Not all detailed credits in this too quick review. (See the related monograph by T. Liggett)

## Dynamical phase transition

For $G = \mathbb{Z}^d$, the process dies out at criticality (Bezuidenhout and Grimmett (1990)).

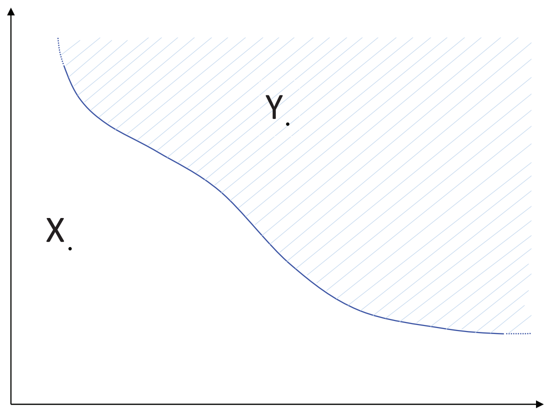



Simulation by Stefanos van Dijk $d = 1$, $\lambda$ close to $\lambda_c$.

## Metastability

If $G = \mathbb{Z}^d$ and $\lambda > \lambda_c$ the model exhibits metastability.
(Valid also for more general graphs if $\lambda$ large enough).

Metaestability: frequent phenomenon in thermodynamic systems close of a first order phase transition.



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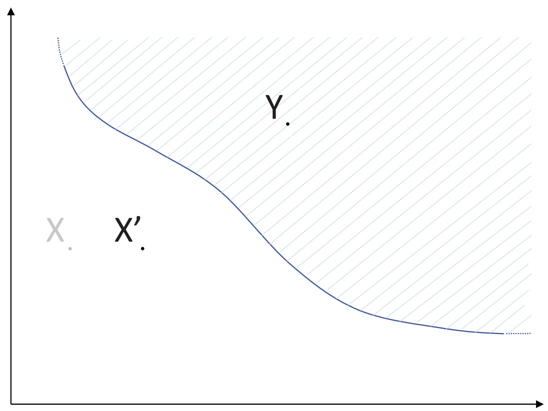



Plenty of examples in nature, in physical systems.

- supercooled liquids, super-saturated vapors;
- ferromagnets, finds many applications.

## Metastability for the contact process

Process restricted to a large finite box $\Lambda_N = \{x \colon \|x\|_\infty \leq N\}$ in $\mathbb{Z}^d$.

$\lambda > \lambda_c$ and large initial configuration.

• Extinction time $\tau_N$ is finite but exponentially large in $|\Lambda_N|$, and loses memory as $N \to \infty$.

$$\frac{\tau_N}{E(\tau_N)} \to \text{EXP }(1)$$

• Process behaves as if in equilibrium with the largest invariant measure before collapsing.

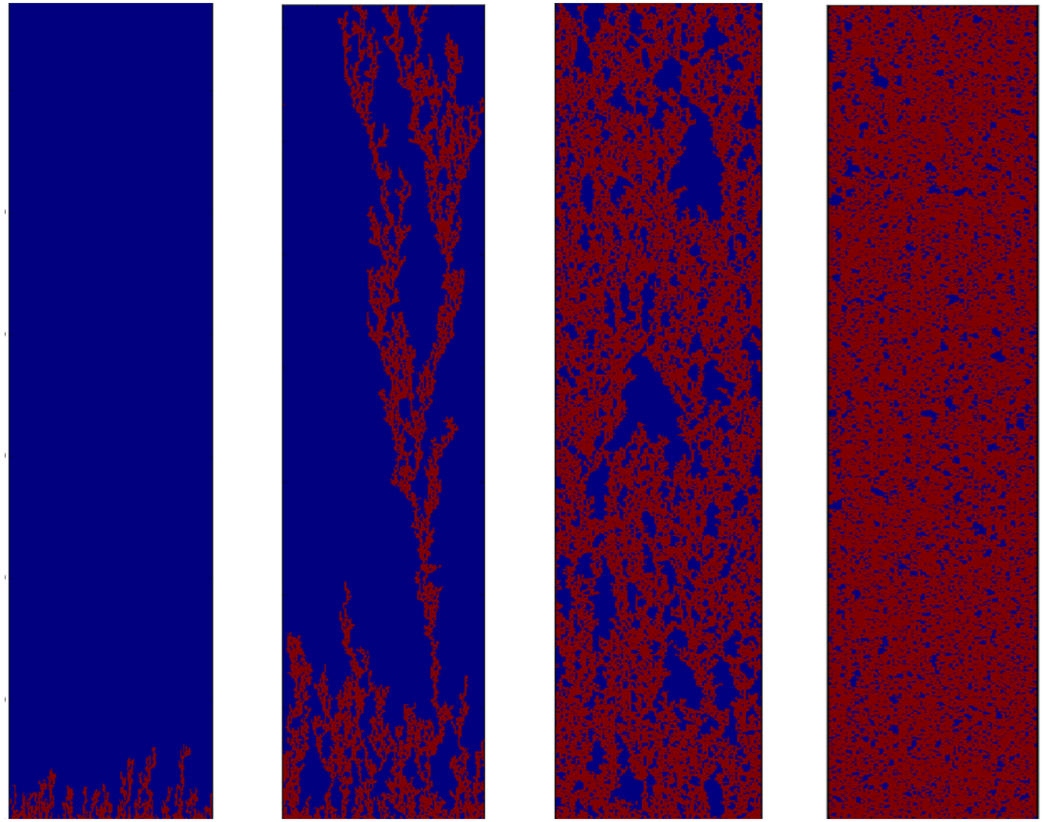
$d = 1$ Cassandro, Galves, Olivieri, V. (1984) Schonmann (1985)

$d \geq 2$ Mountford (1993, 1999)

Trees and more general graphs Mountford, Mourrat, Schapira, Valesin, Yao (2016)

A contact process with two species. Mariela P. Machado (2018) - preprint

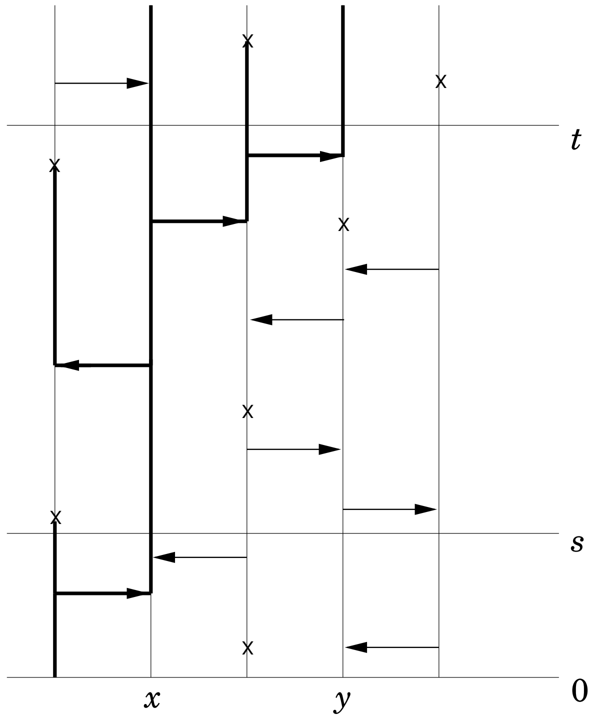
# Metastability for the contact process



Simulations by Daniel Valesin

## The renewal contact process

Same construction except that the recovery times are not anymore given by Poisson processes.




• For each ordered pair $(x, y)$ of neighbouring points in $\mathbb{Z}^d$ a Poisson process $N_{x,y}$ of rate $\lambda$. (The arrows)

• Take independent renewal processes $\mathcal{R}_x$ for $x \in \mathbb{Z}^d$. (The crosses)

• Parameters: $\lambda$ and $\mu$ (the law of the times between two consecutive crosses, assumed i.i.d.)

## The renewal contact process

Our process is then constructed via *paths* as before.

The contact renewal process starting at $A \subseteq \mathbb{Z}^d$, $\xi_t^A$

$$\xi_t^A = \{y\colon\ \exists \text{ a path from } (x,0) \text{ to } (y,t) \text{ for some } x \in A\}.$$

- We no longer have a Markov process on $\{0,1\}^{\mathbb{Z}^d}$.
- The questions regarding percolation continue to make sense:

$$\lambda_c = \inf\{\lambda\colon P(\xi_t^{\{0\}} \neq \varnothing\, \forall t) > 0\}$$

- May we have $\lambda_c = 0$?

**Theorem 1.** (Fontes, Marchetti, Mountford, V)
If $\mu(t, +\infty) \geq t^{-\alpha}$ for some $\alpha < 1$ (all $t$ large) plus some regularity conditions, then $\lambda_c = 0$.

**Theorem 2.** (Fontes, Mountford, V, 2018)
If $\int t^2 \mu(dt) < \infty$ then $\lambda_c > 0$ for any $d \geq 1$.

(Robust argument; branching)

**How to improve this?**

**Hypothesis A:** $\mu$ has a density $f$ and the hazard rate $h(t) = \frac{f(t)}{\mu(t,+\infty)}$ is decreasing in $t$.

**Theorem 3.(Fontes, Mountford, V, 2018)**
Let $d = 1$. If $\mu$ satisfies Hypothesis A and $\int t^{\alpha}\mu(dt) < \infty$ for some $\alpha > 1$, then $\lambda_c > 0$.

Our arguments rely on putting together distinct crossing paths. They require $d = 1$.

## Hypothesis A and FKG inequalities

A very convenient construction:

• $h$ the hazard rate function

$\eta$ be a P.p.p. on $\mathbb{R} \times (0, +\infty)$ with rate 1.

To construct a renewal process starting at some point $t_0 \in \mathbb{R}$, consider all points of $\eta$ in $(t_0, +\infty) \times (0, +\infty)$ that are under the graph of the function $t \mapsto h(t - t_0)$.

• Take the point with the smallest first coordinate, say $(t_1, u_1)$;

$P(t_1 - t_0 > s) = e^{-\int_0^s h(v)dv} = \mu(s, \infty)$ i.e. $t_1 - t_0$ distributed according to $\mu$.

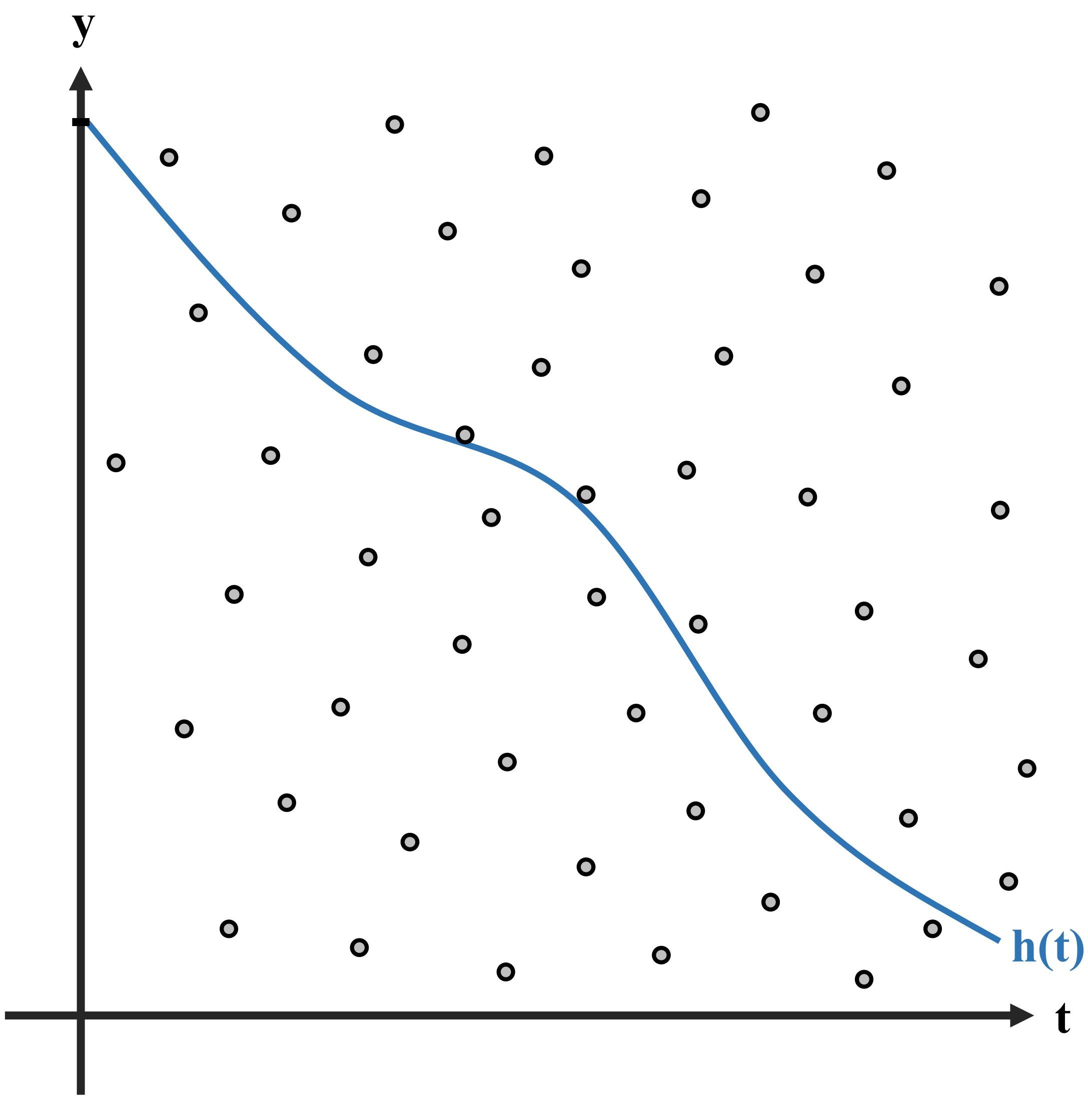
• Having obtained $t_1$ we repeat the procedure replacing $t_0$ by $t_1$.

• The properties of the P.p.p. $\Rightarrow t_1 < t_2 < \ldots$ so that $t_i - t_{i-1}, i \geq 1$ are i.i.d. with density $f$.

**Useful consequence:**

• If $h$ is decreasing, the renewal process is an increasing function of the points in the P.p.p.

# Hypothesis A and FKG inequalities



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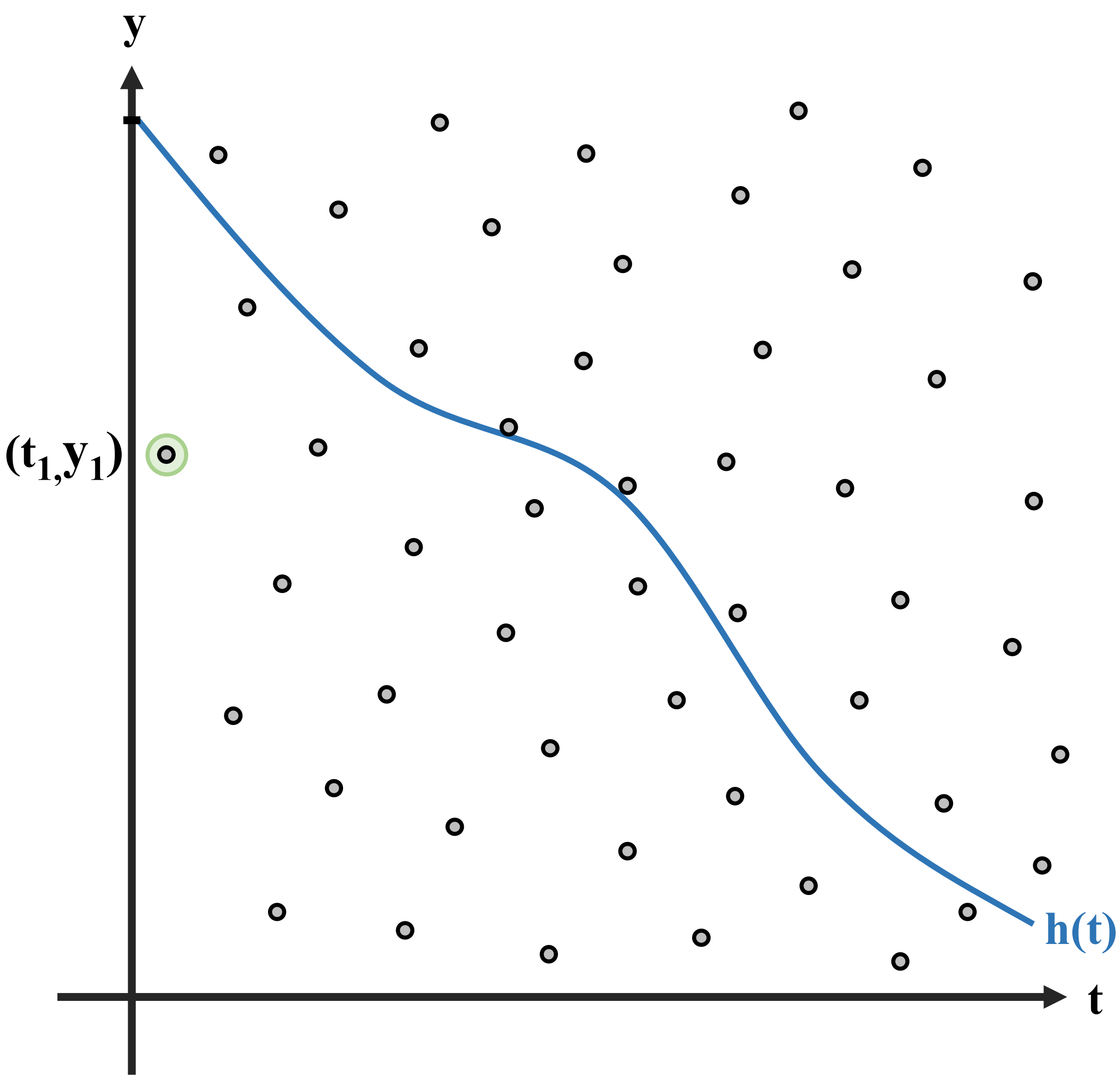

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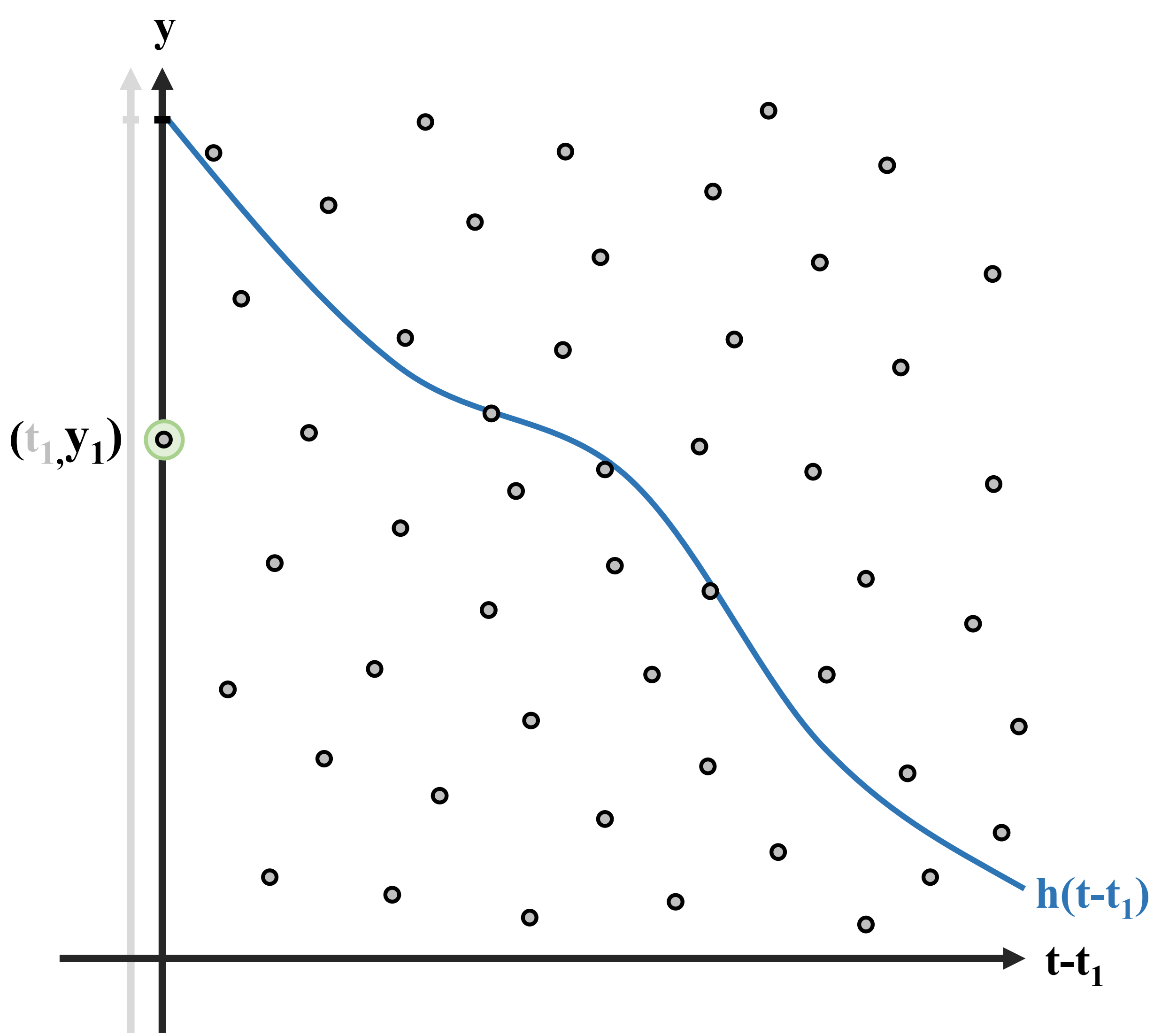

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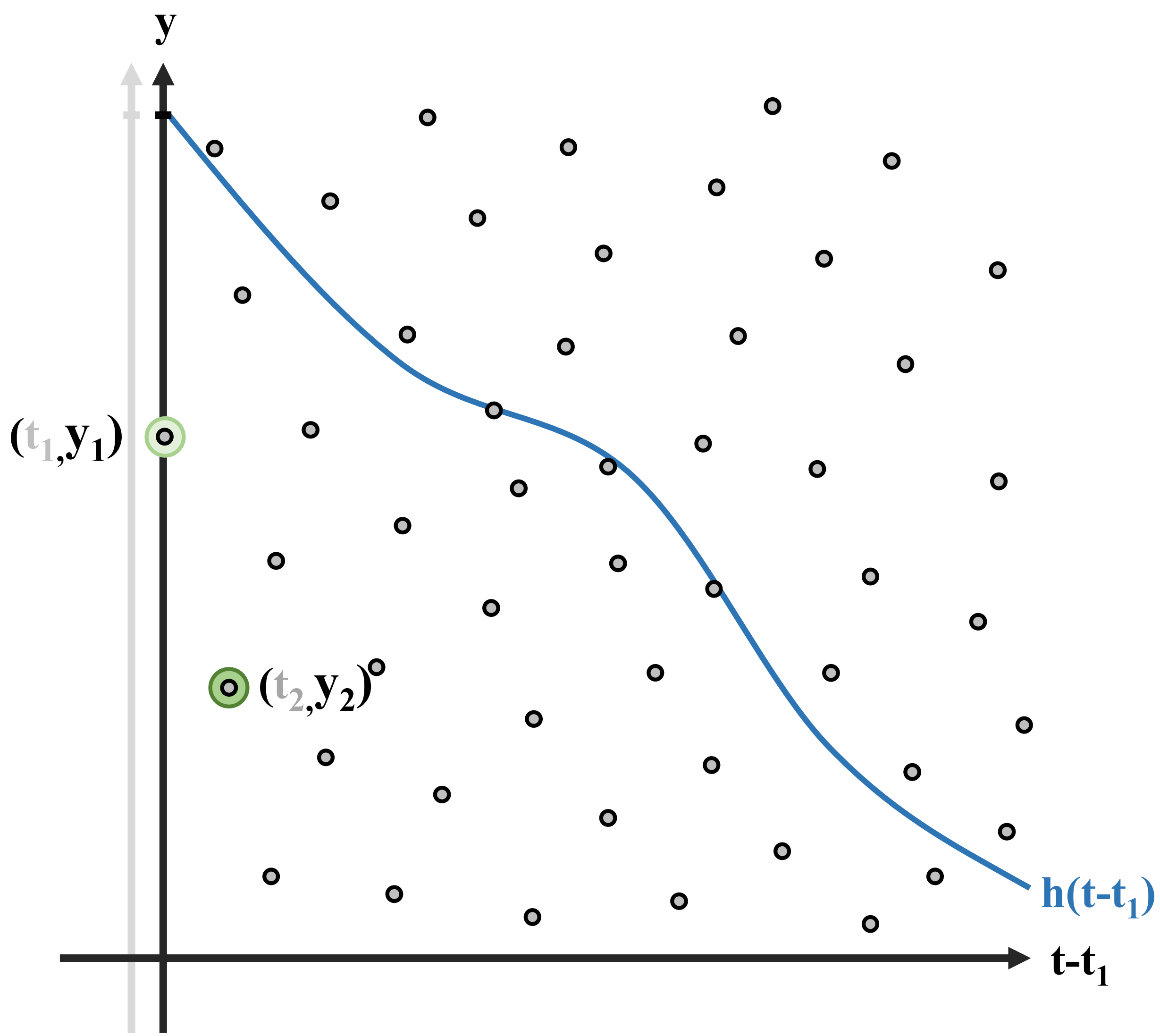

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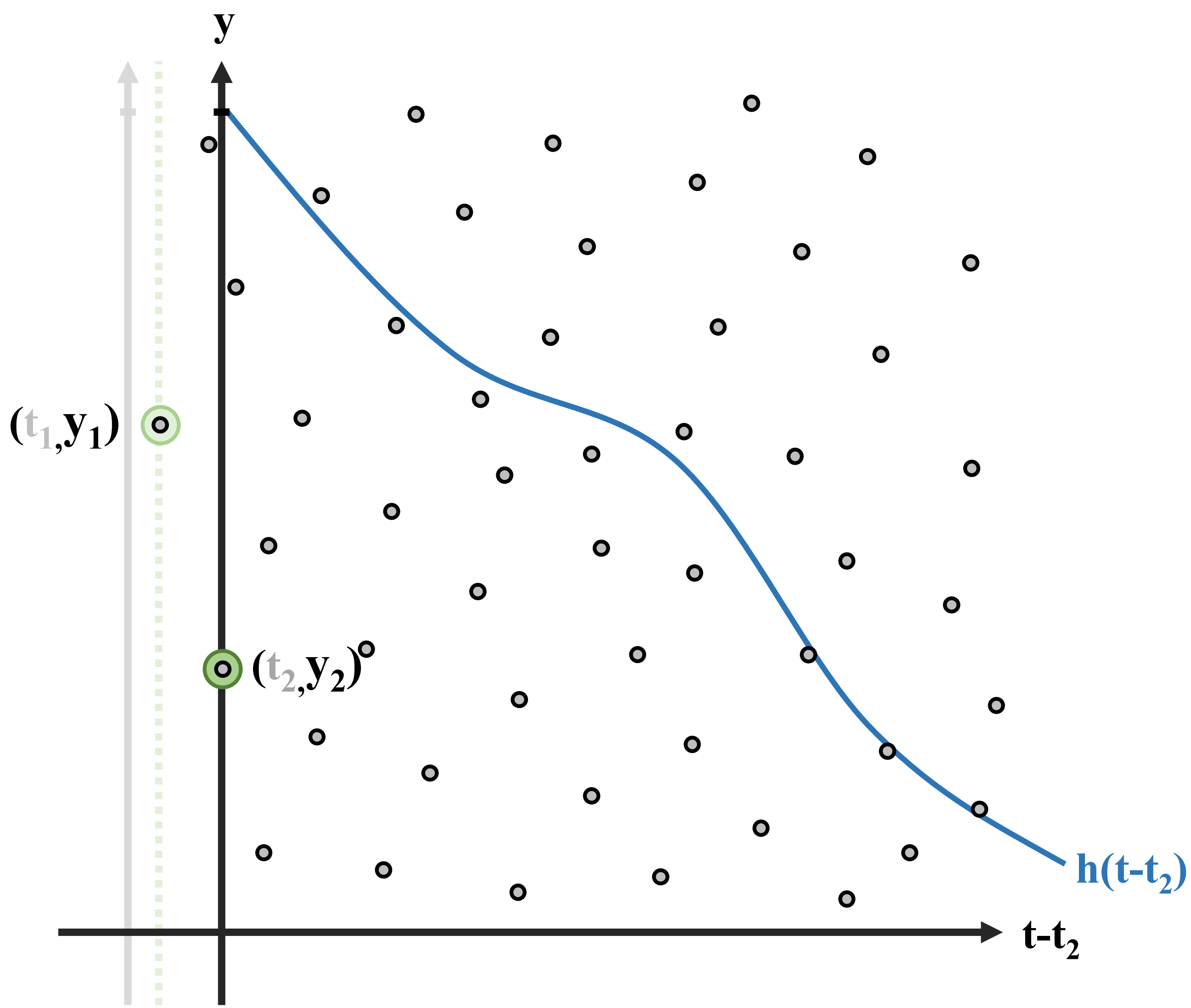

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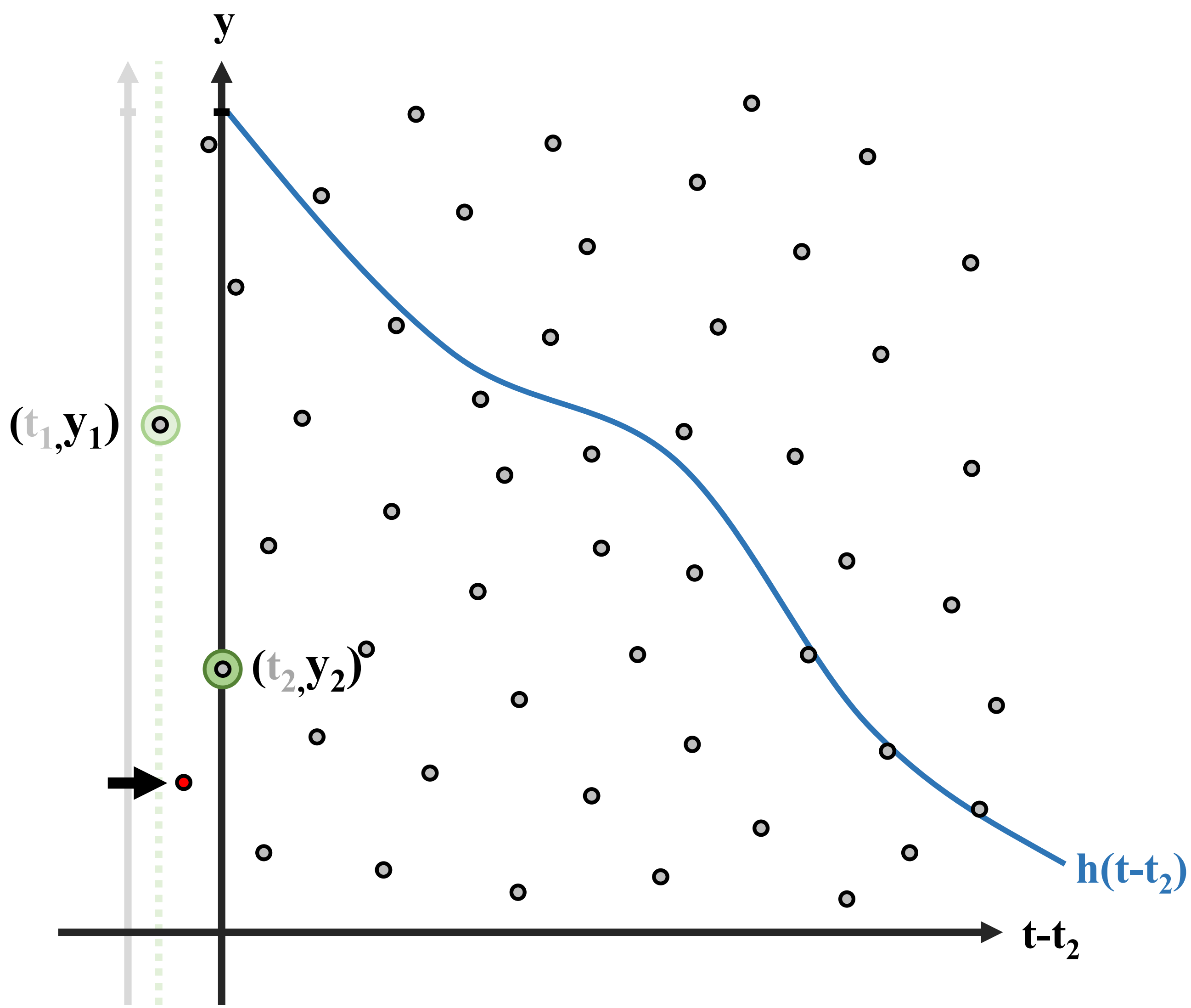

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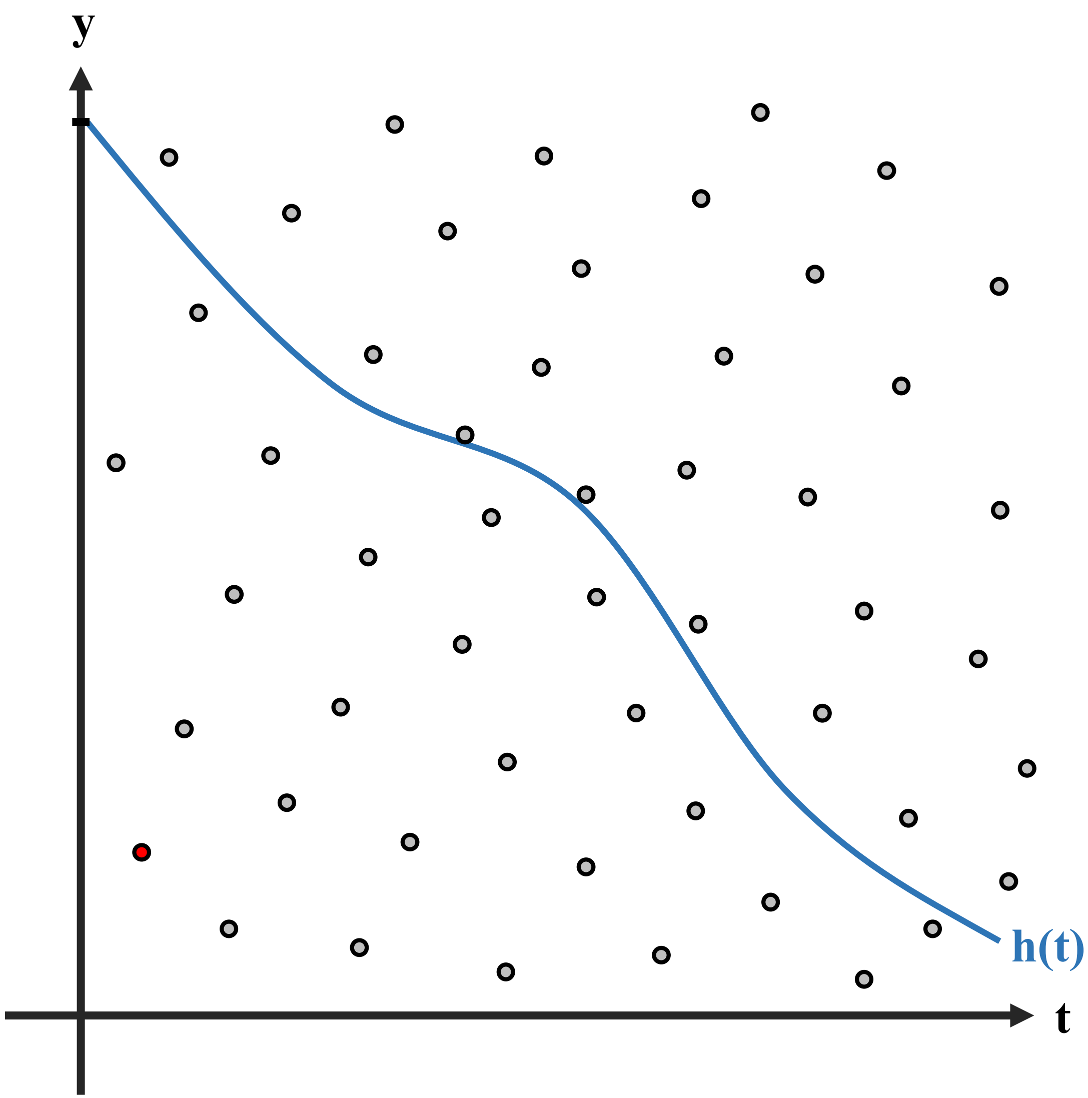

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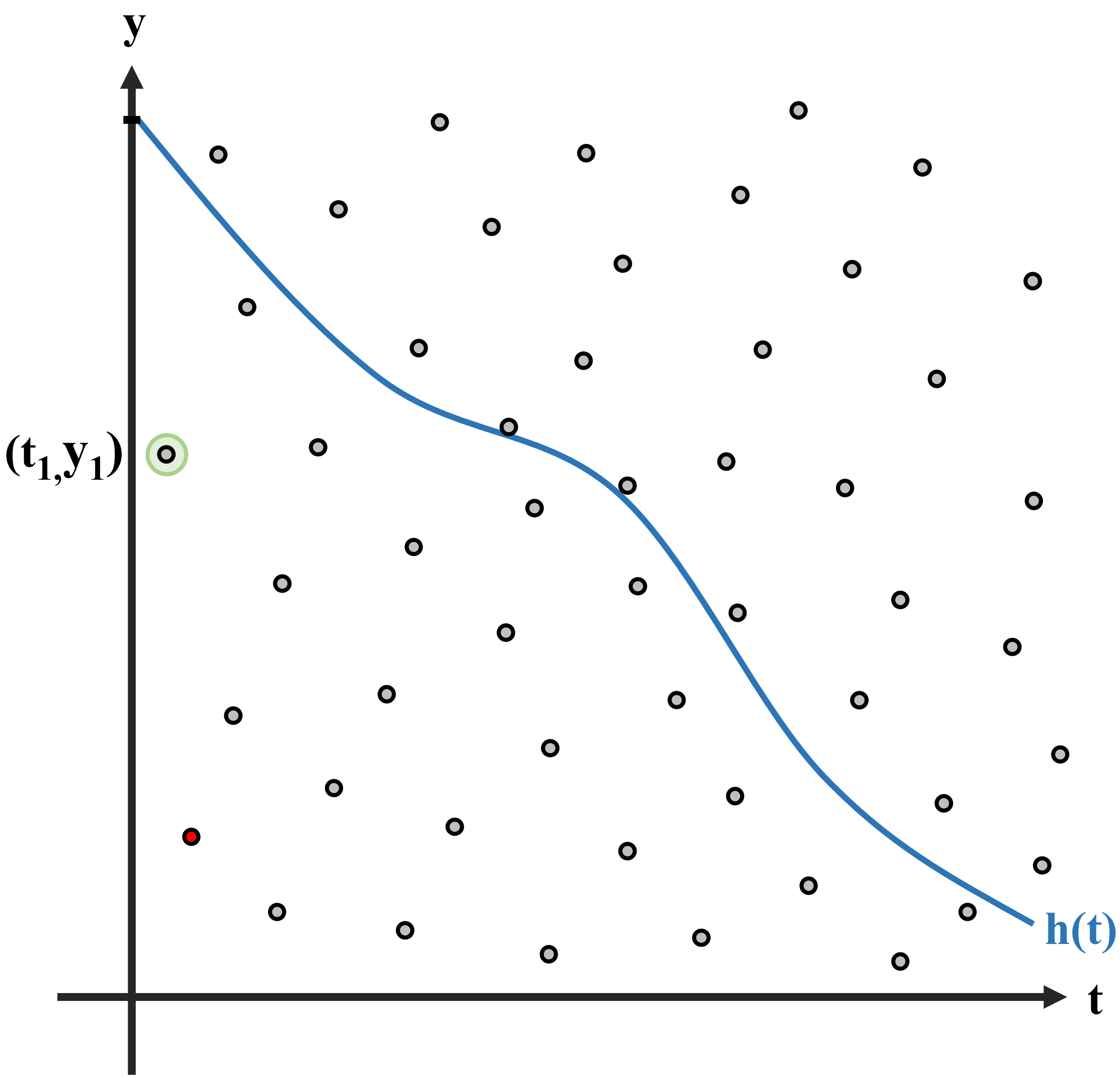

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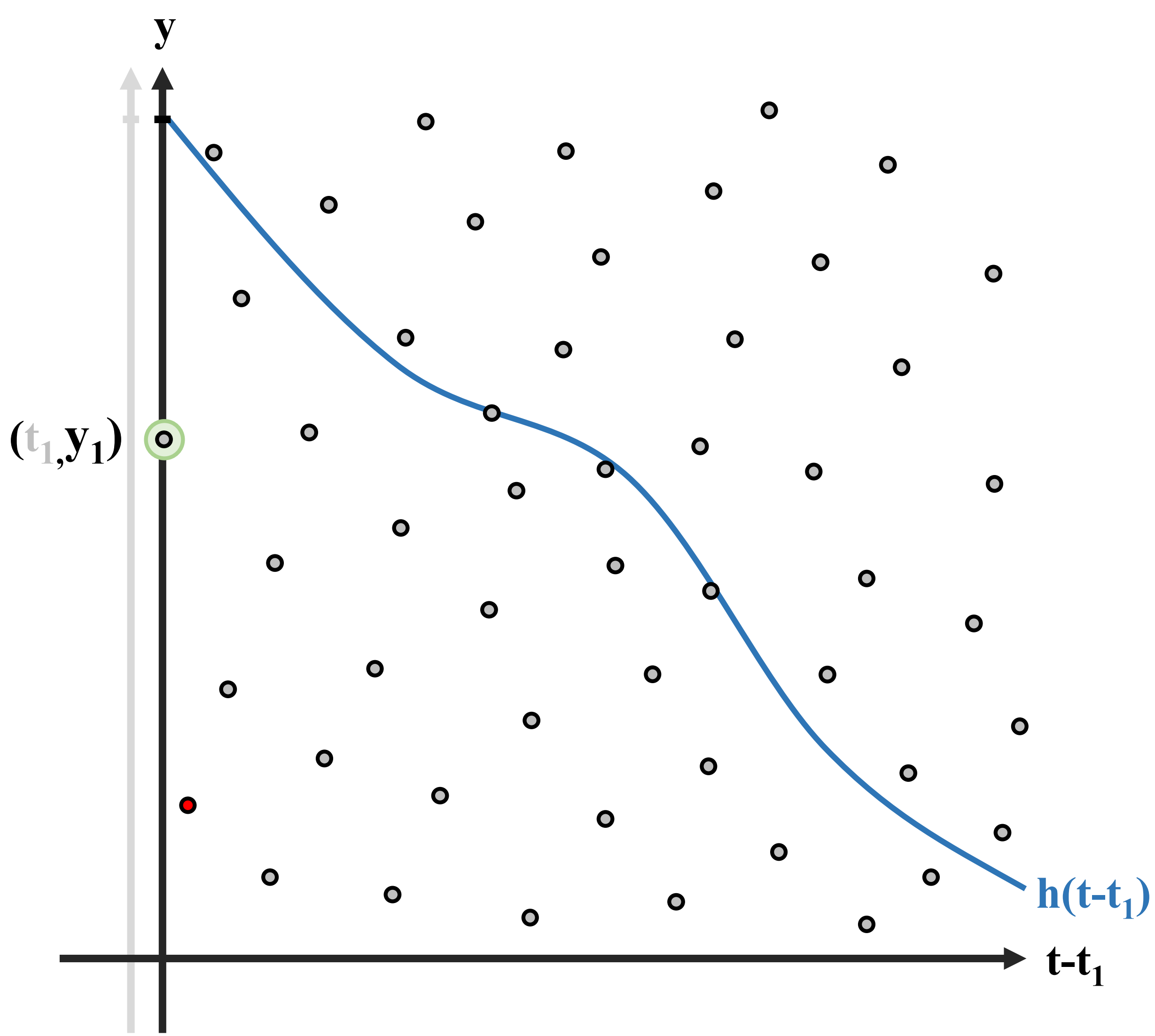

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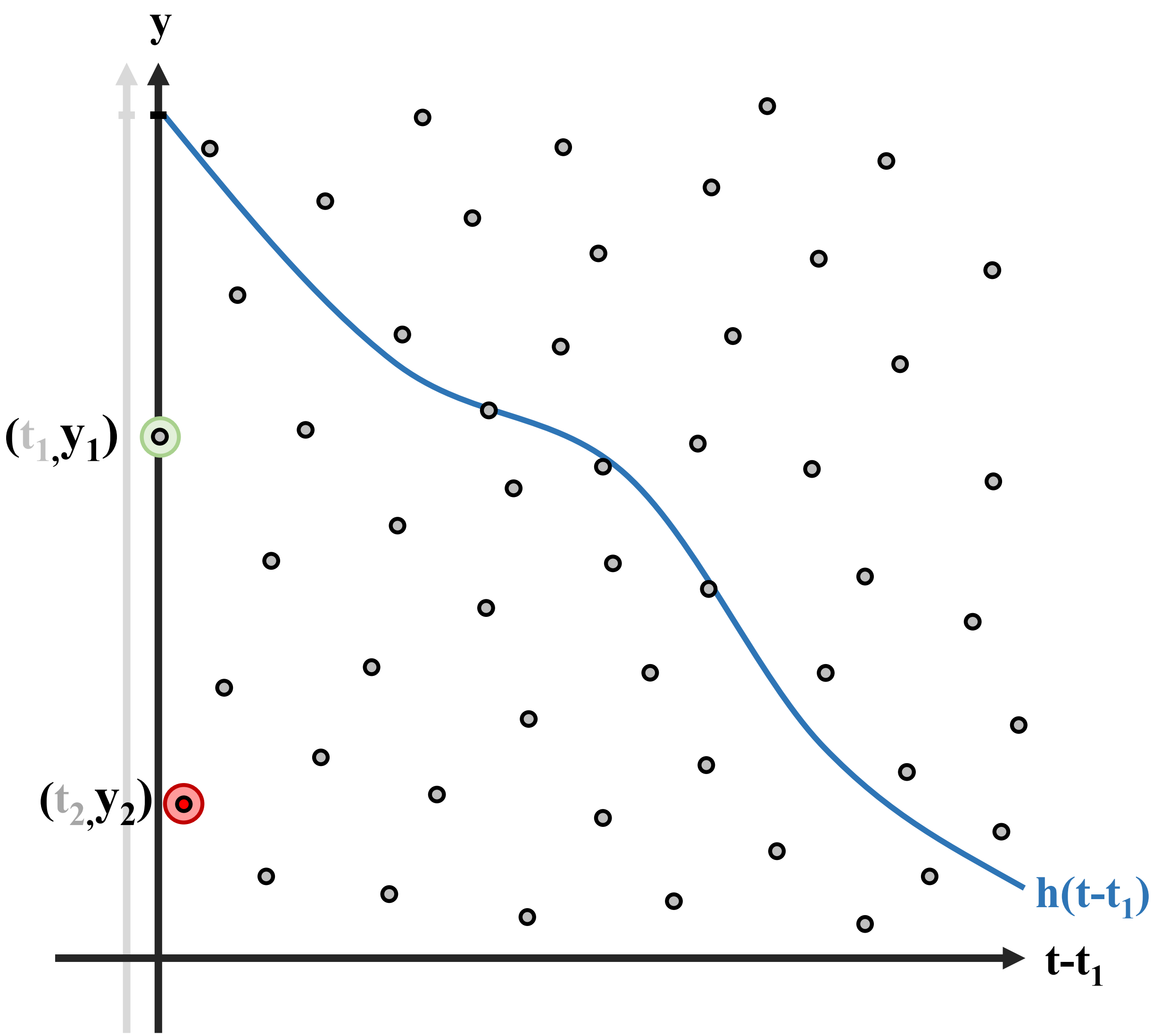

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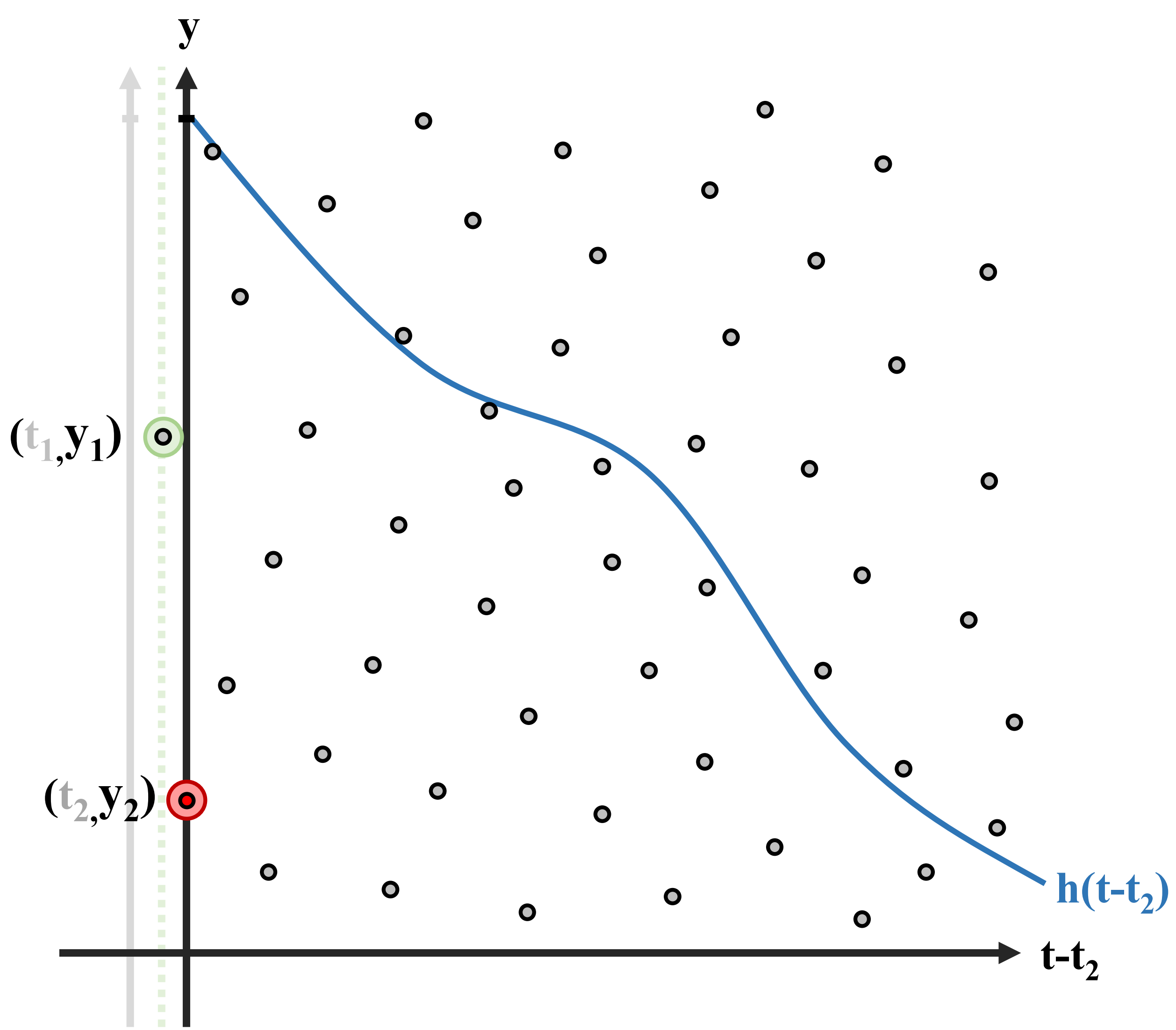

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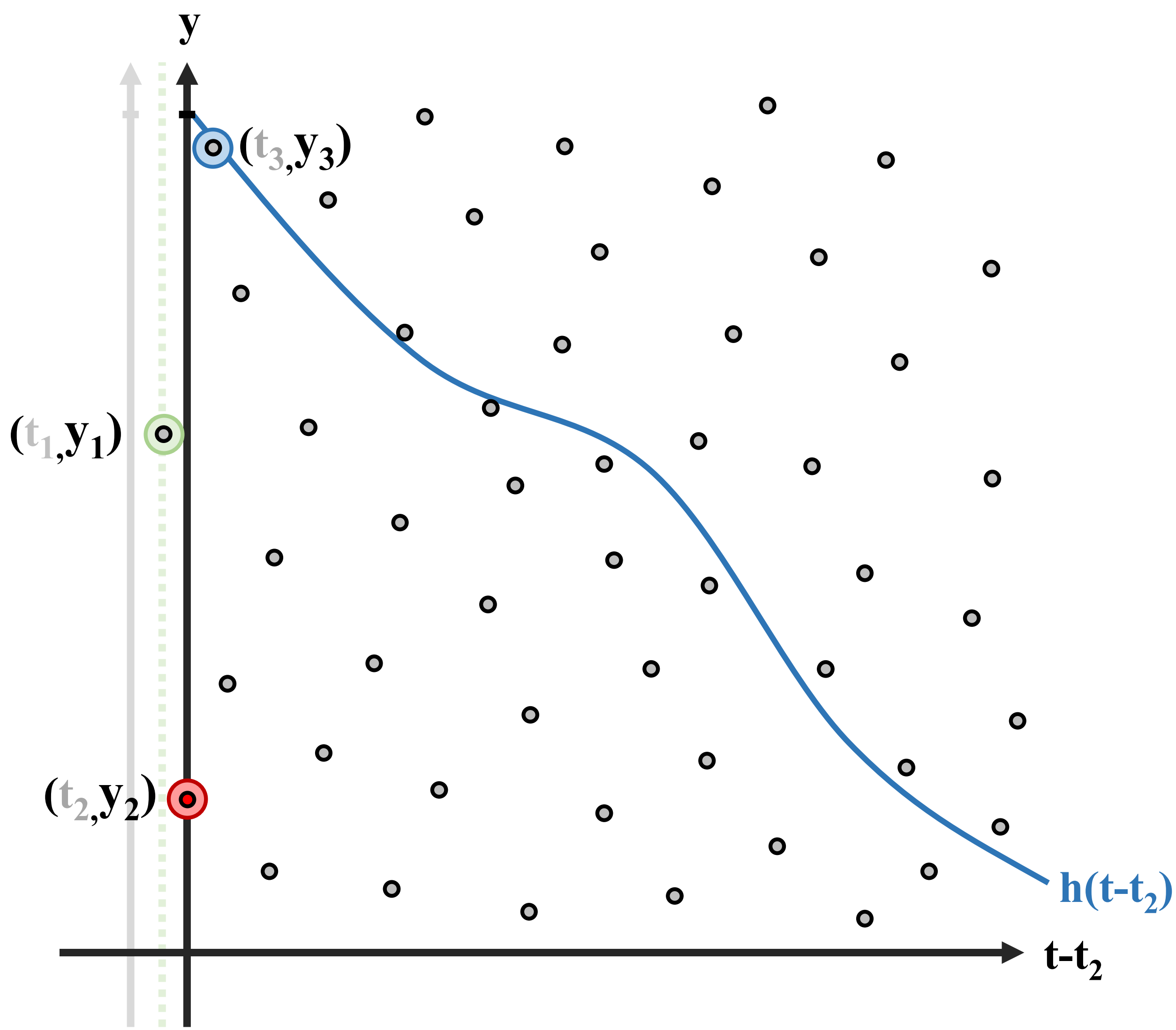

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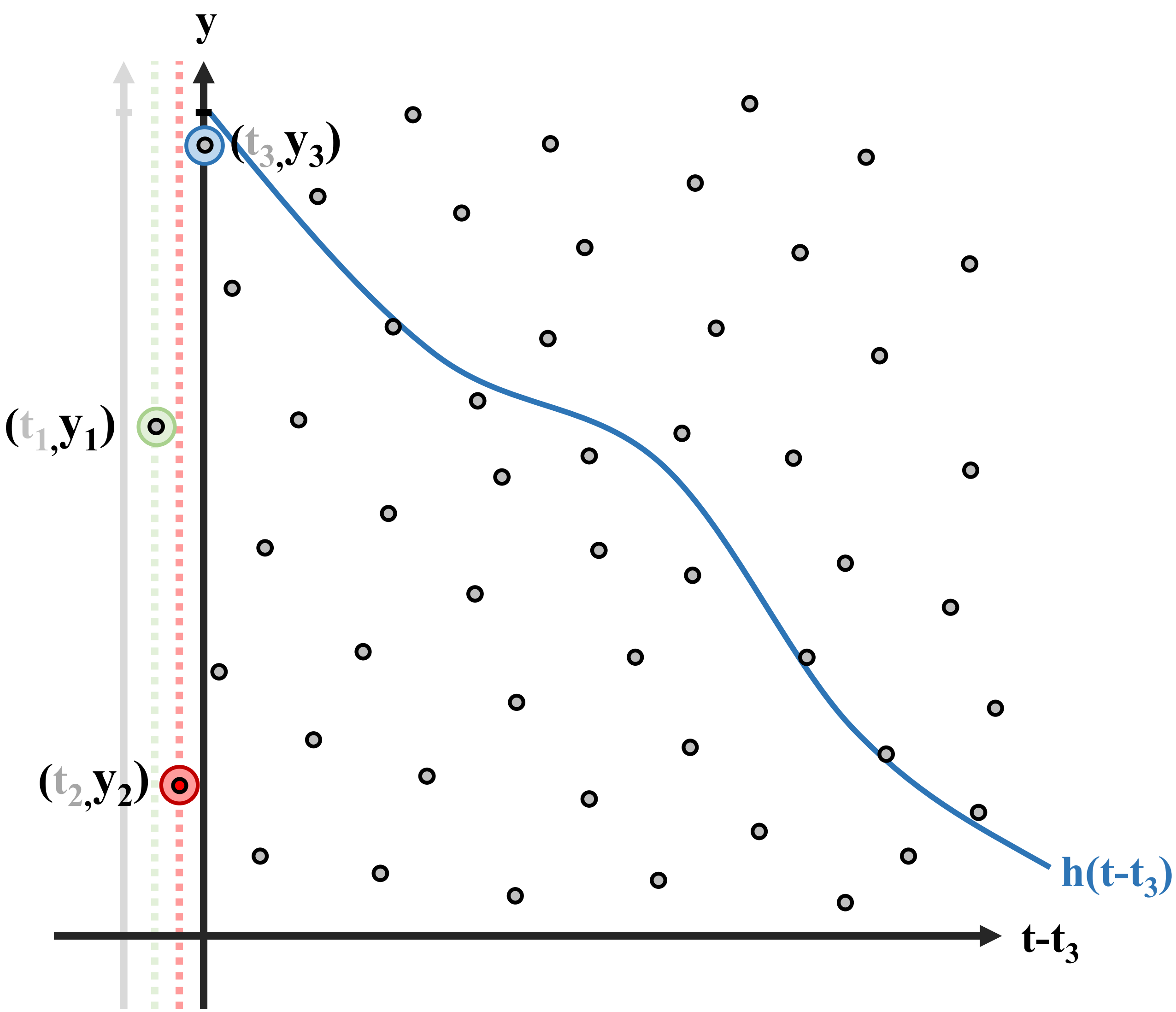

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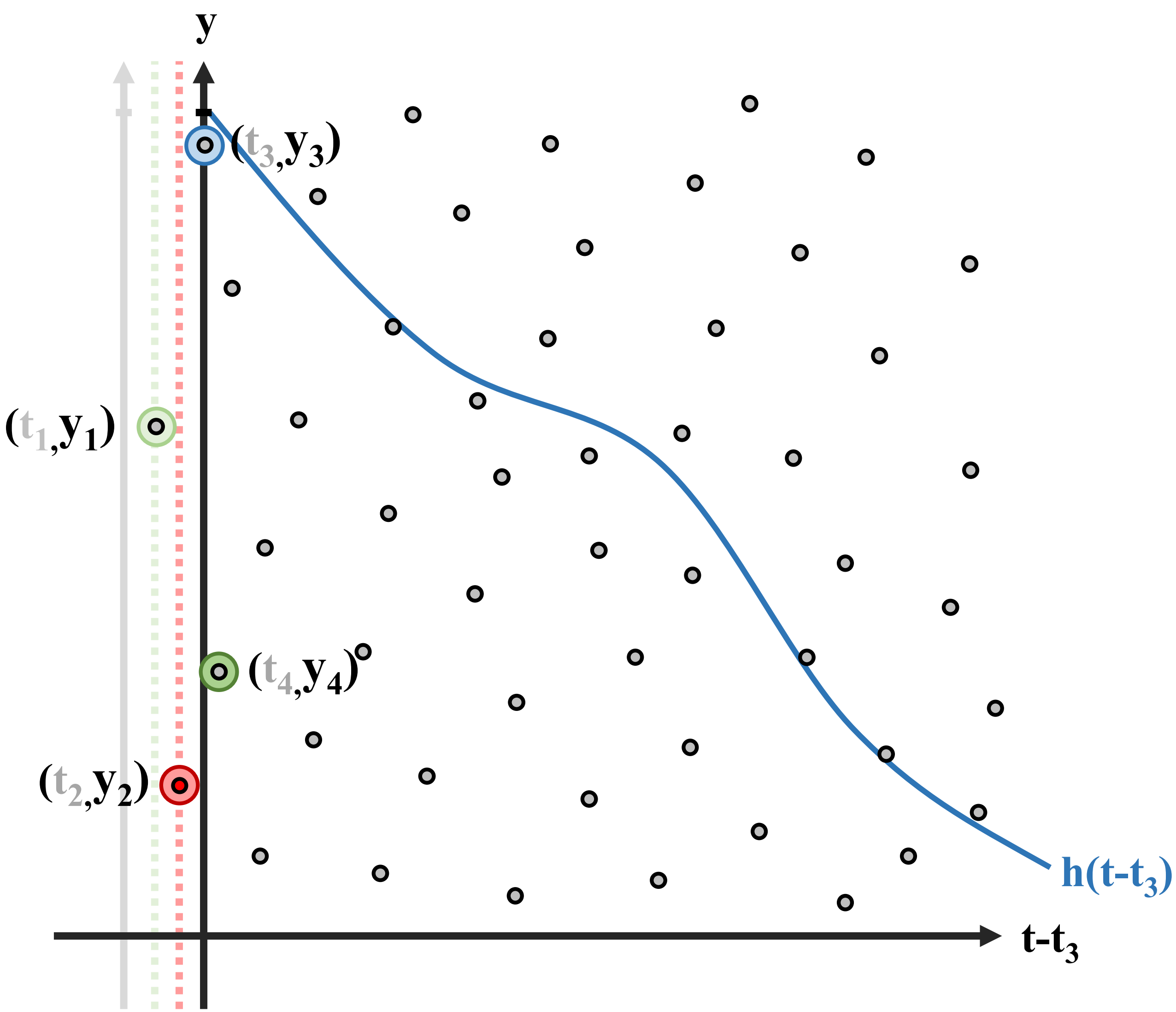

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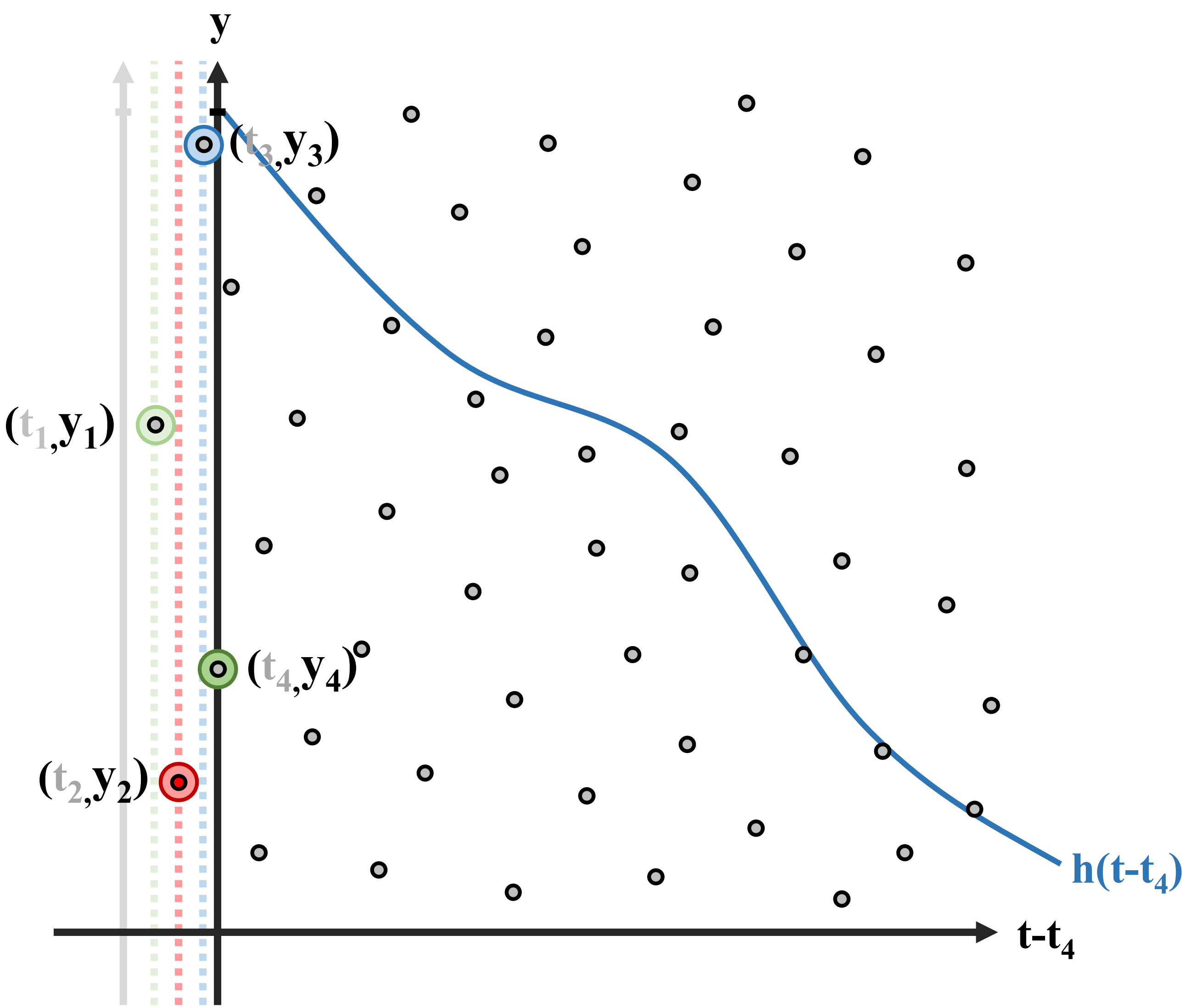

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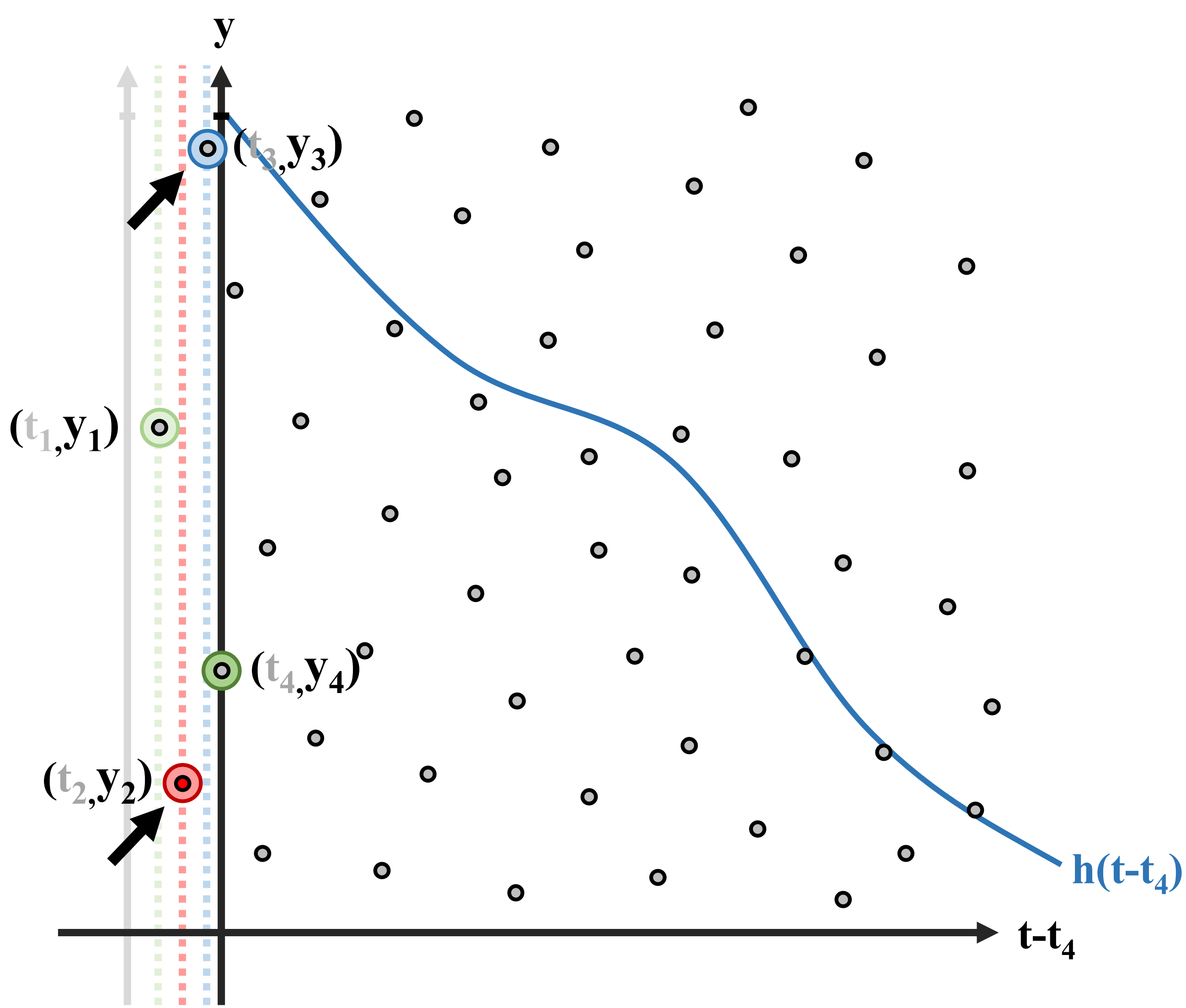


## Hypothesis A and FKG inequalities

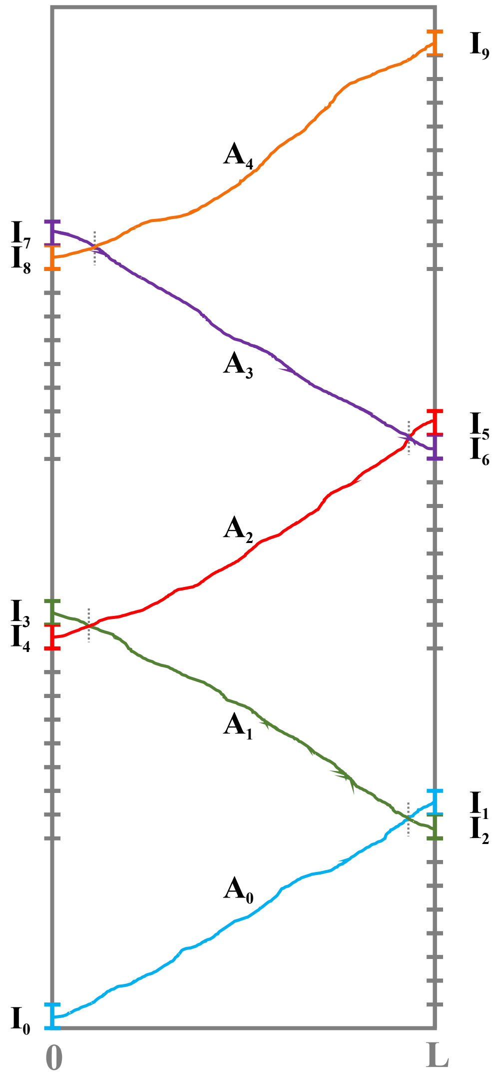
**Definition:** We say that an event depending on renewal and $\lambda$ Poisson process points in a finite space time rectangle is *increasing* if it is increasing with respect to the $\lambda$ Poisson processes of arrows, and decreasing with the renewal processes.

As a consequence:

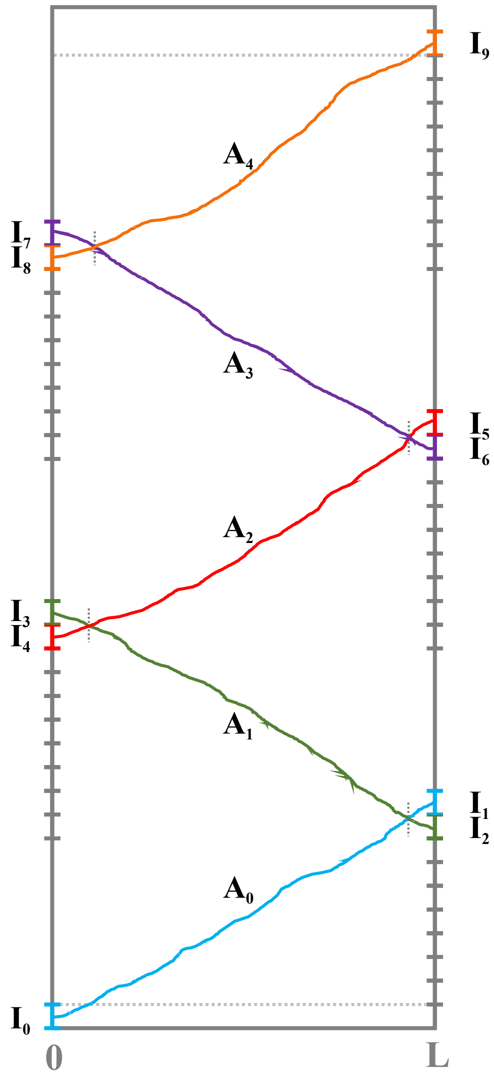
**Proposition** Assume hypothesis A. Let events $A_1, A_2, \ldots, A_n$ be increasing events on a finite space time rectangle. Then

$$P(\cap_{i=1}^{n} A_i) \;\geq\; \prod_{i=1}^{n} P(A_i).$$

- Allows to use arguments that show similarity with known RSW estimates in percolation.




$A_0 = \{$ exists crossing from $\{0\} \times [0, \epsilon]$ to $\{L\} \times [\frac{2}{3}T, \frac{2}{3}T + \epsilon]$ in $[0, L] \times [0, \infty)\}$.




$P($ temporal crossing of $[0, L] \times [\epsilon, \epsilon + mT]) \geq P(A_0)^{\frac{8m}{3}+2}$.

## Sketch of the proof of Theorem 3

Let $0 < \beta < \alpha - 1$.

- $P_r$: the supremum over the probabilities for the space-time rectangle

$$[0, 2^{\beta r}] \times [0, 2^r]$$

of either a spatial or a temporal crossing.

The supremum is over all product renewal measures with inter-arrival distribution $\mu$, for the death points starting at times points strictly less than $0$. (The starting points or times need not be the same.)

The main ingredient is the following

**Proposition** Assume $\beta \in (0, \alpha - 1)$, with $\alpha$ as in the statement of main theorem. There exists $\lambda_0 > 0$ so that for $0 \leq \lambda < \lambda_0$

$$P_r \stackrel{r \to \infty}{\longrightarrow} 0.$$

The theorem follows (quite easily) from the Proposition.

$$\{\xi_{2^r}^{\{0\}} \neq \varnothing\} \subseteq (I) \cup (II) \cup (III),$$

where, letting $R = [-2^{r\beta}/2, 2^{r\beta}/2] \times [0, 2^r]$:

(I) $\{\exists$ path from $(0,0)$ to $\mathbb{Z} \times \{2^r\}$ in $R\}$

(II) $\{\exists$ path from $(0,0)$ to $\{2^{r\beta}/2\} \times [0, 2^r]\}$

(III) $\{\exists$ path from $(0,0)$ to $\{-2^{r\beta}/2\} \times [0, 2^r]\}$.

- $P((I)) \leq P_r.$

Using the previous FKG type estimate

- $P(II) = P(III) \leq K^2 (P_r)^{1/(2^{\lceil 1/\beta \rceil + 1})} \; + \; K P_{r - \lceil 1/\beta \rceil},$

  where $K$ is suitably large (depending on $\beta$ but not on $r$).

• (Key estimate) Control $P_r$ through an iterative procedure.

Consider the probability of temporal crossing $(X(s))_{0\leq s\leq 2^n}$ of $[0, 2^{n\beta}] \times [0, 2^n]$.

• Take $k$ suitably large (but not depending on $n$) and consider the restriction each of the $2^{k-1}$ (even) rectangles $[0, 2^{\beta n}] \times [2i2^{n-k}, (2i+1)2^{n-k}]$

• Show there must be a crossing (space or time) of smaller but very "similar" scales $2^{\beta(n-k-i)} \times 2^{n-k-i}$ $(i \leq 4)$

• Conditioning on the existence of a renewal (cross mark) for each $x \in [0, 2^{\beta n}]$ in the previous time interval $[(2i-1)2^{n-k}, 2i2^{n-k}]$, we have that the probability of the vertical crossing is bounded

$$C(k)\Big(\sup_{n-k-4\leq r\leq n-k} P_r\Big)^{1/10}$$

• $0 < \beta < \alpha - 1$ guarantees that the probability of not having a cross for at least one $x$ and at least one of the odd time intervals is small. The renewals bring some sort of 'independence'. We can combine all time intervals to beat power $1/10$.

Taking $2^{k-1} > 20$ and arguing similarly for the space crossings (easier) one gets if $n-k$ is large

$$P_n \leq 2^{-n\frac{\epsilon_0}{2}} + C(k)(\sup_{n-k-4 \leq n-k} P_r)^2$$

Out of this it is simple to conclude the result.

Indeed, if we have

$$P_r \leq 2^{-r\frac{\epsilon_0}{5}} \text{ for each } n-k-4 \leq r \leq n-k \quad (*)$$

then

$$P_n \leq 2^{-n\frac{\epsilon_0}{2}} + C''2^{-2n\frac{\epsilon_0}{5}}\, 2^{2(k+4)\frac{\epsilon_0}{5}} \leq 2^{-n\frac{\epsilon_0}{5}}$$

for all $n$ large.

Choose $\lambda_0$ small so that (*) holds for $n = n_0$ and $\lambda \in (0, \lambda_0)$.

# THANKS