Summary of part I

- Structures are dominated by *collisionless* dark matter
  - It's properties are testable (direct, indirect, astrophysics)
  - Many models currently
- Redshift

$$z := \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}} - 1 \approx v/c, \quad \text{for} \quad v << c$$

- Luminosity distance, standard candles
- Hubble diagram ("velocity" x "distance")

$$v \simeq H_0 d_L \qquad H_0 = 100 h Km/s/Mpc$$

- The Universe is expanding. Homogeneous and isotropic in the mean

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Connecting Kinematics with Dynamics

The Friedmann Equation and the Content of the Universe

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Newtonian Cosmology I: The Friedmann Equations

- Uniform and spherically symmetric distribution:

$$R(t) = a(t)r$$

- "Energy conservation"

Friedmann Equation

$$\frac{1}{2}m\left(\frac{dR}{dt}\right)^2 - \frac{GMm}{R} = E \quad \Longrightarrow \quad \boxed{\dot{a}^2 = \frac{8\pi G}{3}a^2\rho - K}$$

- Knowing $\rho(a)$, we may obtain $a(t)$
  - Example I: Matter (particles): $\rho \propto a^{-3}$
    - $K = 0 \longrightarrow a \propto t^{2/3}$ Einstein - de Sitter
    - $K \neq 0$

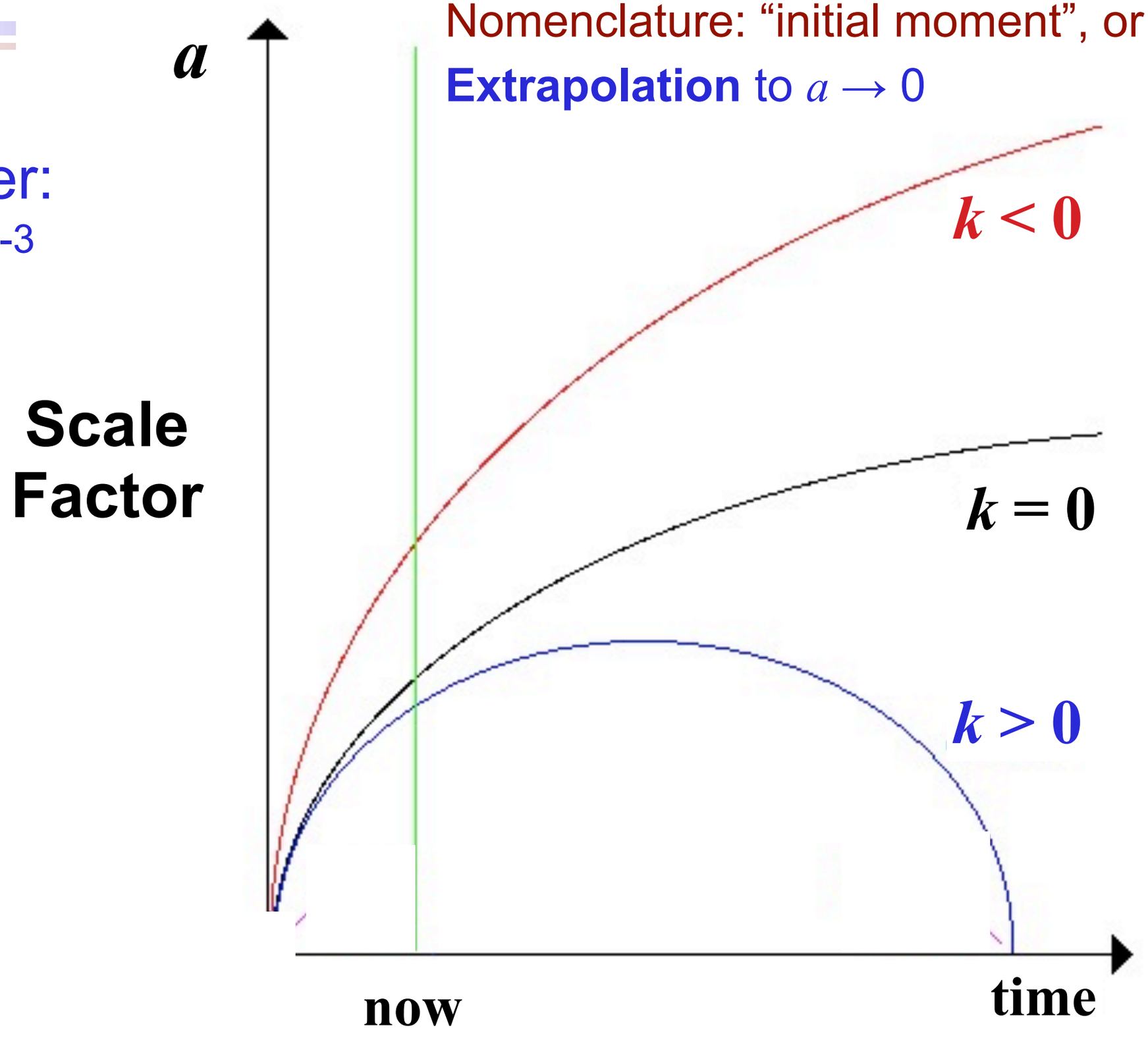
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The Evolution of the Universe ($\Lambda = 0$)

matter:
$\rho \propto a^{-3}$

Nomenclature: “initial moment”, or “Big-Bang”:

Extrapolation to $a \to 0$



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Energy Density and Scale Factor

- Energy Conservation

$$dE = -pdV$$

Internal energy

- **Energy density:** $\rho = \rho_0 + \varepsilon = nmc^2 + \varepsilon$

$$dE = d\left(\varepsilon V\right) = a^3 d\varepsilon + \varepsilon 3a^2 da$$

- Conservation of rest mass

$$dMc^2 = d\left(\rho_0 V\right) = a^3 d\rho_0 + \rho_0 3a^2 da = 0$$

$$\Longrightarrow \quad \boxed{d\rho + 3\left(\rho + p\right)\frac{da}{a} = 0}$$

Valid also in General Relavity!

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Evolution of the Components

- Energy density conservation for each component

$$d\rho_i + 3(\rho_i + p_i)\frac{da}{a} = 0$$

Equation of state

$$p = p(\rho)$$

Example 1: baryons (today), dark matter: *p = 0*

$$\Longrightarrow \quad \rho_M \propto a^{-3}$$

Example 2: radiation: *p = ρ/3*

$$\Longrightarrow \quad \rho_r \propto a^{-4} \qquad \rho_\gamma = \sigma T^4 \longrightarrow T_\gamma \propto a^{-1}$$

Example 3: "vacuum": *p = -ρ*

$$\Longrightarrow \quad \rho_v = const.$$

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Solutions of Friedmann's Equation

Friedmann's Equation

$$\dot{a}^2 = \frac{8\pi G}{3} a^2 \rho - K$$

Energy Conservation

$$\frac{d\rho}{da} + \frac{3}{a}(\rho + p) = 0$$

- $K = 0$
  - *Radiation* $\Rightarrow$ $a \propto t^{1/2}$
  - *Matter* $\Rightarrow$ $a \propto t^{2/3}$ (Einstein-de Sitter)
  - "Vacuum" $\Rightarrow$ $a \propto e^{Ht}$ (De Sitter)
- $K \neq 0$
  - "Empty" $\Rightarrow$ $a \propto t$

Exercise:

$p = w \rho$

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The Multicomponent Universe

Friedmann Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{K}{a^2}$$

Non-interacting components

➔ Independent energy conservation for each type

Energy Conservation

$$\frac{d\rho_i}{da} + \frac{3}{a}\left(\rho_i + p_i\right) = 0$$

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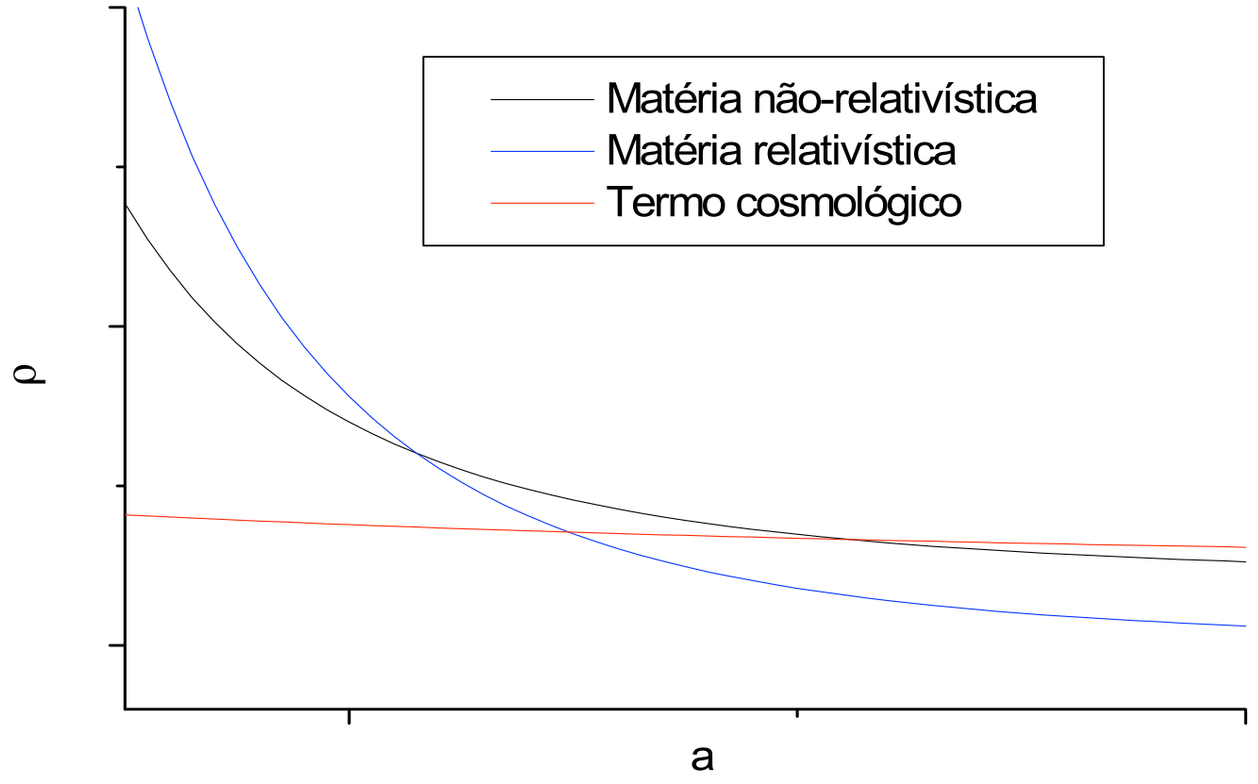
Who "controls" the expansion?

- Behavior of the ingredients

$\rho_{radiation} \propto a^{-4}$, $\rho_{matter} \propto a^{-3}$, $\rho_x \propto a^{-3(1+w_x)}$, with $w_x < -1/2$

Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum \rho_i - \frac{K}{a^2}$$



- Radiation dominated era
- Matter dominated era
- Curvature dominated?
- Dark Energy Dominated / accelerated expansion

⇨ **Coincidence problem (x2)**

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Cosmological Parameters

- Friedmann Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{K}{a^2}$$

- Density Parameters

$$\Omega_i := \frac{\rho_{i0}}{\rho_{crit}} \quad \text{with} \quad \rho_{crit} = \frac{3H_0^{\,2}}{8\pi G} \quad \text{and} \quad \Omega_k = -\frac{K}{H_0^{\,2}}$$

- Hubble parameter (ΛCDM model)

$$H^2(a) = H_0^{\,2}\left[\Omega_r a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\right]$$

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Cosmological Parameters

- Friedmann Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{K}{a^2}$$

- Density Parameters

$$\Omega_k = 1 - \Omega_0 = 1 - \Omega_m - \Omega_r - \Omega_\Lambda = -\frac{K}{{H_0}^2}$$

- Hubble parameter (ΛCDM model)

$$H^2(a) = {H_0}^2\left[\Omega_r a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\right]$$

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Cosmological Parameters

- Friedmann Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{K}{a^2} + \frac{\Lambda}{3}$$

- Density Parameters

$$\Omega_i := \frac{\rho_{i0}}{\rho_{crit}} \quad \text{with} \quad \rho_{crit} = \frac{3H_0^{\,2}}{8\pi G} \quad \text{and} \quad \Omega_k = -\frac{K}{H_0^{\,2}}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^{\,2}} = \frac{\rho_\Lambda}{\rho_{\text{crit}}}$$

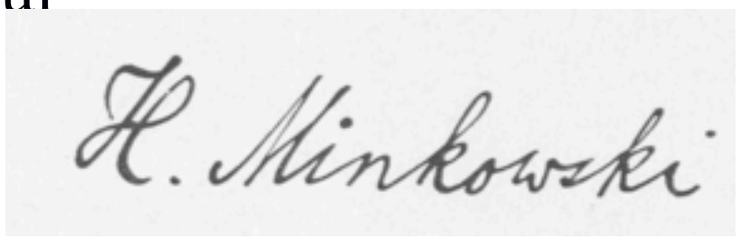
- Hubble parameter

$$H^2(a) = H_0^{\,2}\left[\Omega_r a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\right]$$

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The Geometry of the Cosmos

"The visions of space and time that I will present here originate from experimental physics and this is why they are so powerful"



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The metric

- Invariant in special relativity

$$\Delta S^2 = (\Delta x_0)^2 - (\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- In spherical coordinates

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & \sin^2\theta \end{pmatrix}.$$

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The metric

- Invariant in special relativity

$$\Delta S^2 = (\Delta x_0)^2 - (\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- In curved space-time

$$dS^2 = g_{\mu\nu} dx^\nu dx^\nu$$

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The (Spatial) Metric of the Homogeneous Universe

- Metric: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
- Spatial section, isotropy:

$$\left(d\vec{r}\right)^2 = f(r)dr^2 + g\left(r\right)\left(r^2 d\theta^2 + r^2 sen^2\theta\, d\phi^2\right)$$

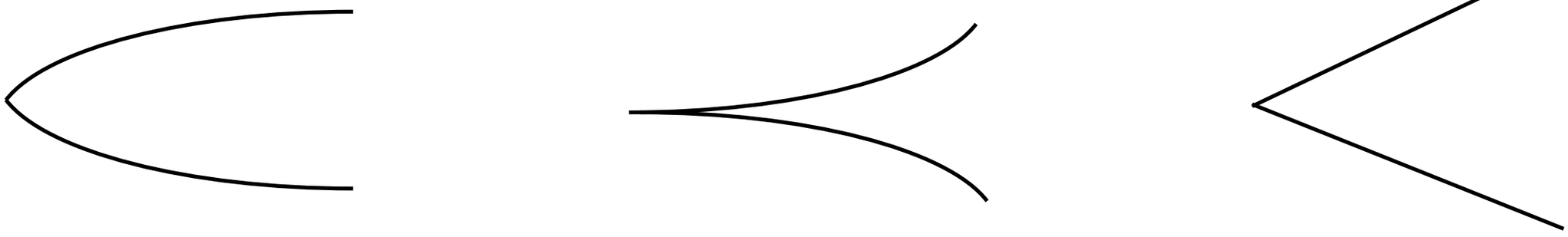
- r' defined by the area
- Constant curvature: $f = \dfrac{1}{1 - Kr^2}$

$$\boxed{\left(d\vec{r}\right)^2 = \frac{1}{1 - Kr^2} dr^2 + r^2 d\theta^2 + r^2 sen^2\theta\, d\phi^2}$$

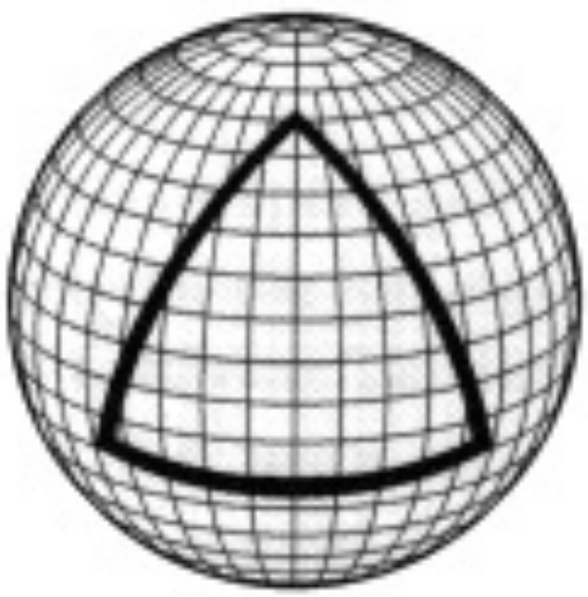
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Geometry

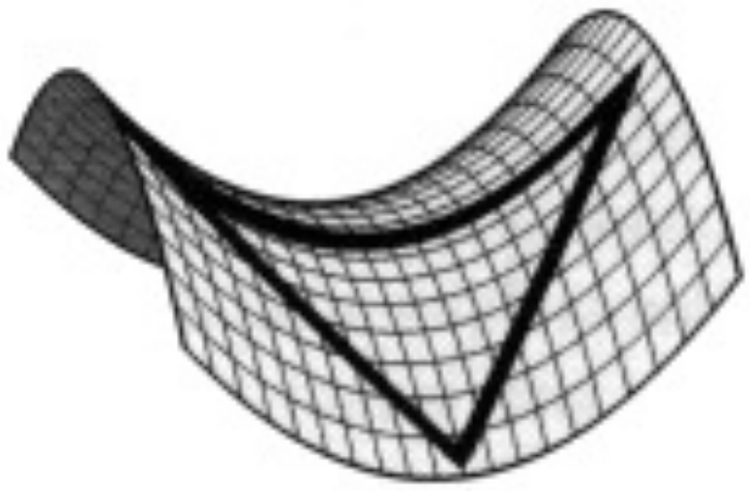
Angles:



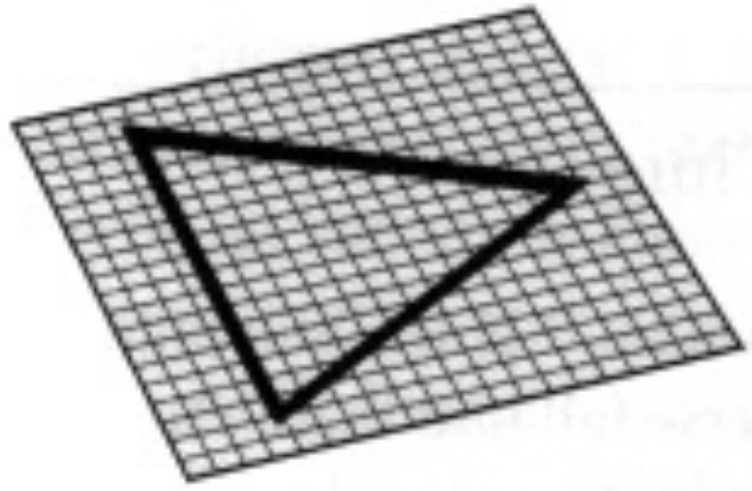
2D analogy:



Spherical



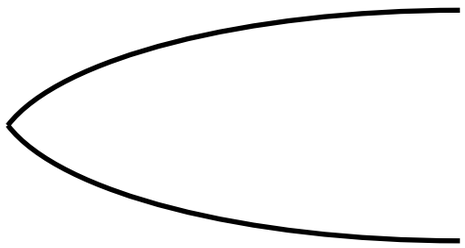
Hyperbolic



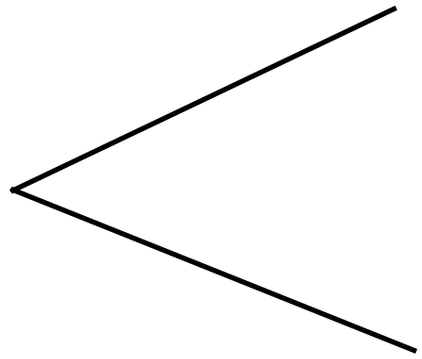
Flat

Geometry

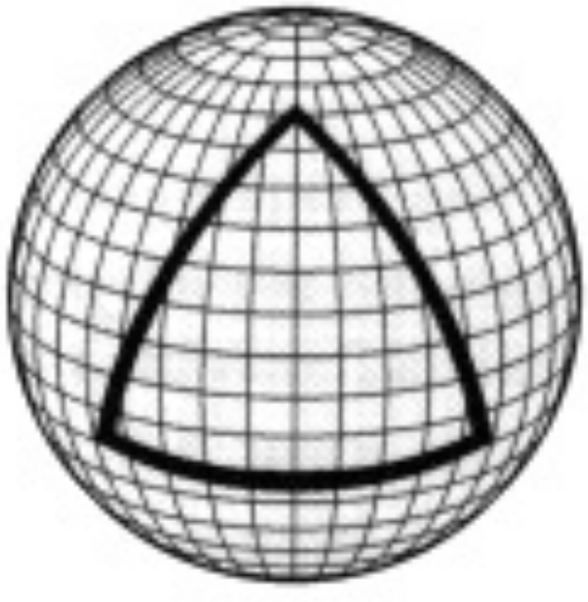
Angles:

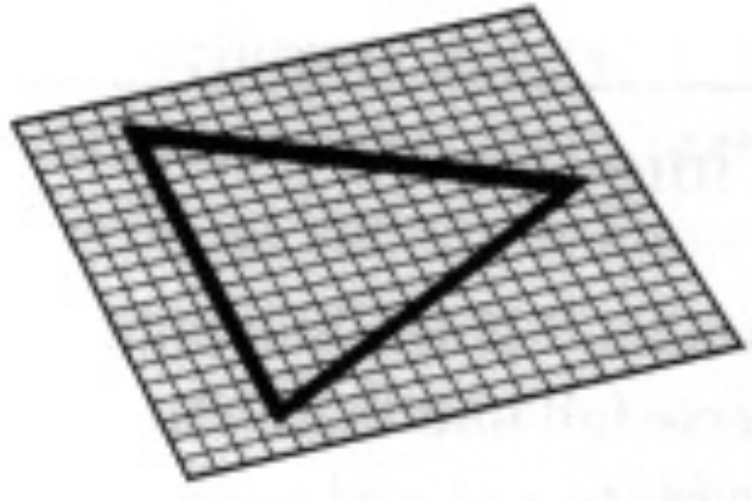






2D analogy:





Spherical **Hyperbolic** **Flat**

Space-time Geometry of the Homogeneous Universe

- Invariant: $ds^2 = dt^2 - \left(d\vec{x}\right)^2$
- All scales expand with $a(t)$:

$$\left(d\vec{x}\right)^2 = a^2\left(t\right)\left(d\vec{r}\right)^2 = a^2\left(t\right)\left(\frac{1}{1-Kr^2}dr^2 + r^2 d\theta^2 + r^2 sen^2\theta\, d\phi^2\right)$$

$$ds^2 = dt^2 - a^2\left(t\right)\left(\frac{1}{1-Kr^2}dr^2 + r^2 d\theta^2 + r^2 sen^2\theta\, d\phi^2\right)$$

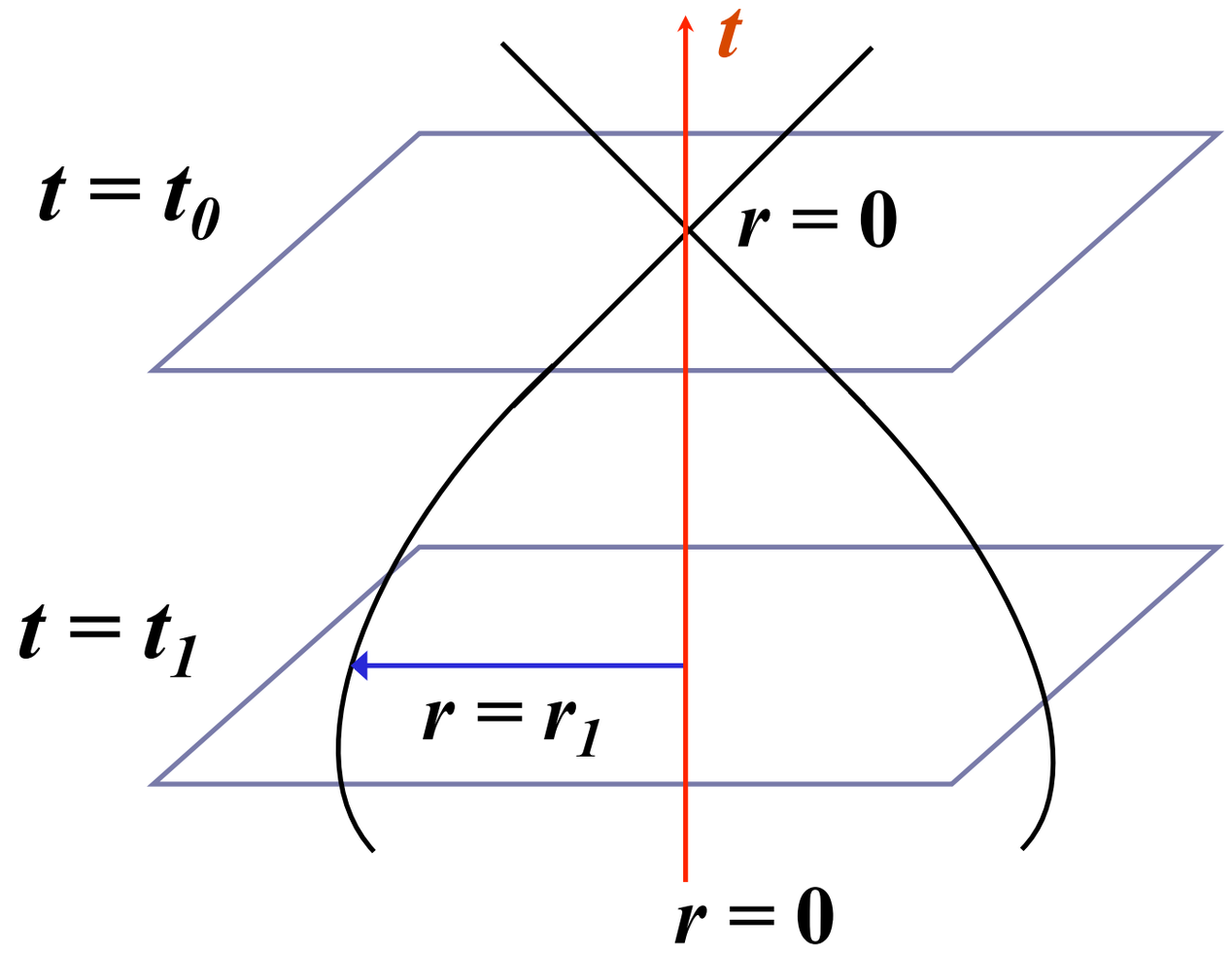
Friedmann metric (Robertson-Walker)

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Light Propagation

- Integrating a null geodesic in the radial direction:

$$dt^2 = a^2(t)\left(\frac{1}{1-Kr^2}dr^2\right)$$



Thus

$$\int_{t_1}^{t_0}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{\sqrt{1-Kr^2}} \equiv f(r_1)$$

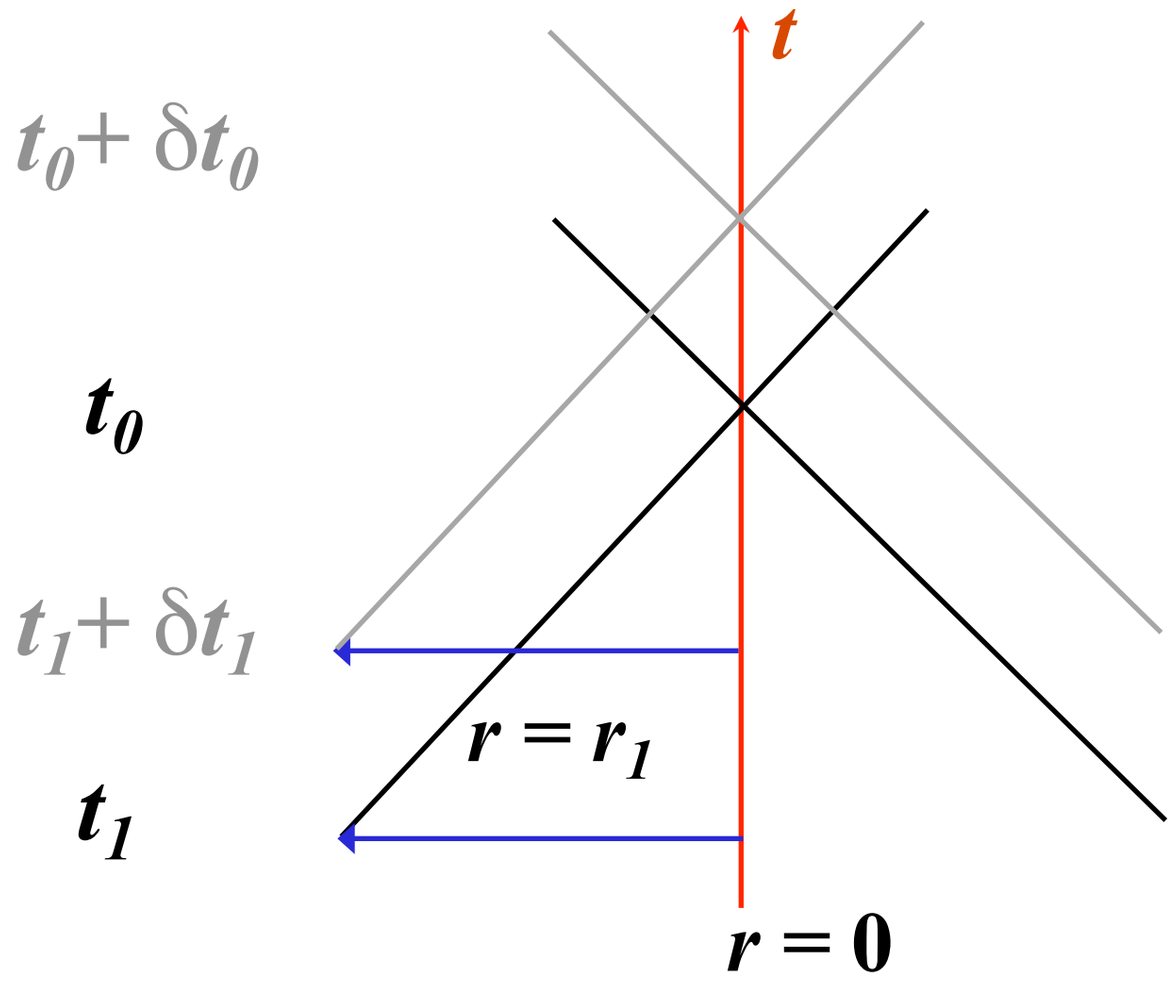
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Light propagation and redshift

- Two “light pulses” emitted by a galaxy:

$$\int_{t_1}^{t_0} \frac{\mathrm{dt}}{a(t)} = f(r_1) = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{dt}}{a(t)}$$

$$\Rightarrow \quad \frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)}$$



Redshift ⇒

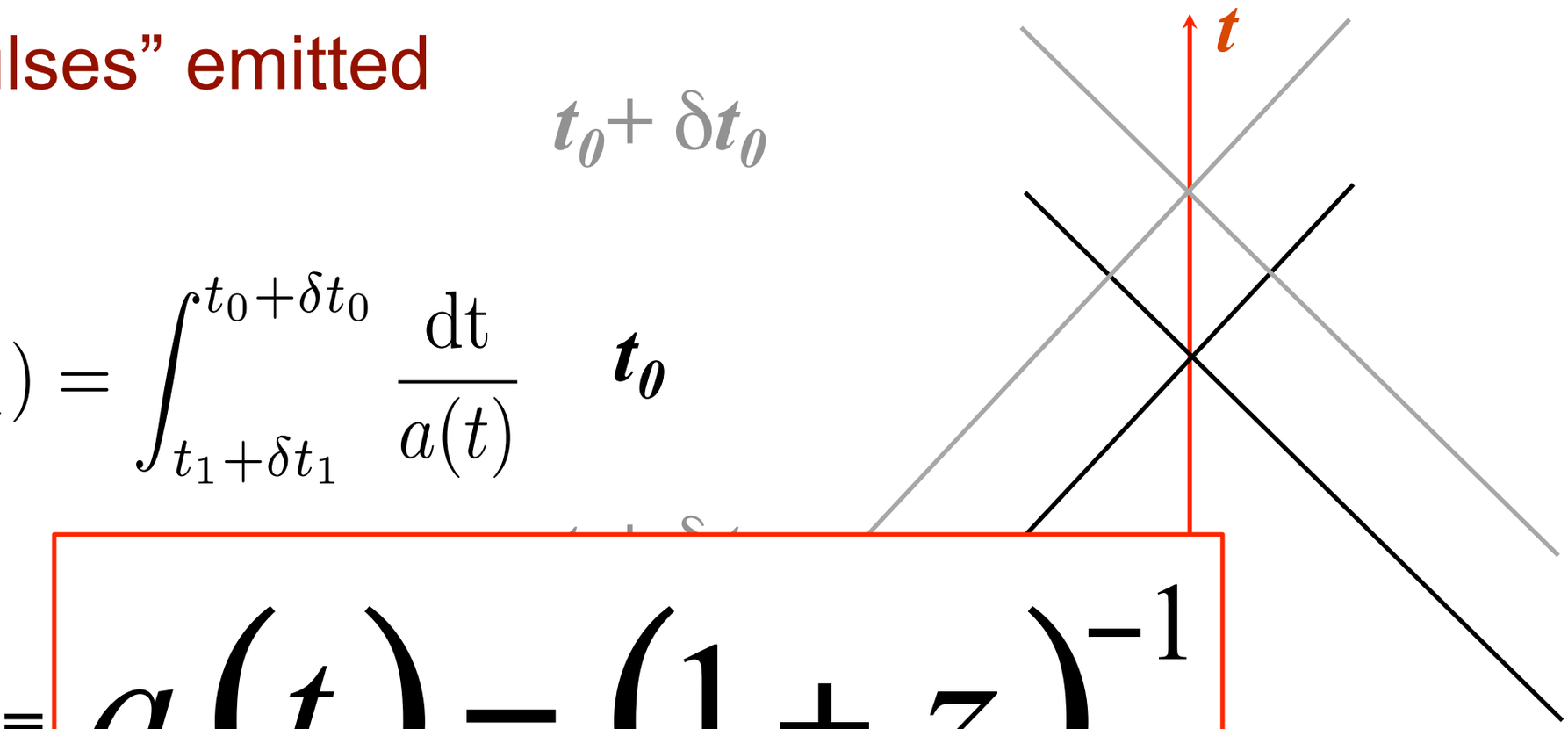
$$z := \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1$$

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Light propagation and redshift

- Two “light pulses” emitted by a galaxy:

$$\int_{t_1}^{t_0} \frac{\mathrm{dt}}{a(t)} = f(r_1) = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{dt}}{a(t)}$$



$$\Rightarrow \quad \frac{\delta t_1}{a(t_1)} = \boxed{a(t) = (1+z)^{-1}} = 0$$

Redshift $\Rightarrow$ $\boxed{z := \dfrac{\lambda_0 - \lambda_1}{\lambda_1} = \dfrac{a(t_0)}{a(t_1)} - 1}$

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Luminosity Distance

$$d_L^2 := \frac{L}{4\pi F}$$

F: energy per unit area per unit time

Area of S^2 centered on the source at the time of detection t_0: $4\pi a^2(t_0) r_1^2$

Variation of the photon energy: $h\nu_1 / h\nu_0 = (1+z)^{-1}$

Time difference: $\delta t_0 / \delta t_1 = a(t_0) / a(t_1) = 1 + z$

Thus: $$F = \frac{L'}{4\pi a^2(t_0) r_1^2} = \frac{L}{4\pi a^2(t_0) r_1^2 (1+z)^2}$$

$$\Rightarrow \quad \boxed{d_L^2 = a^2(t_0) r_1^2 (1+z)^2}$$

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Luminosity Distance

$$d_L = a(t_0) r_1 (1+z)$$

As

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-Kr^2}} = \begin{cases} \sin^{-1}\left(\sqrt{|K|}r_1\right)\Big/\sqrt{|K|}, \text{ para } K>0 \\ r_1, \text{ for } K=0 \\ \sinh^{-1}\left(\sqrt{|K|}r_1\right)\Big/\sqrt{|K|}, \text{ para } K<0 \end{cases}$$

and

$$\int_{t_1}^{t_0} \frac{dt}{a} = \int_{t_1}^{t_0} \frac{da}{\dot{a}a} = -\int_z^0 \frac{dz'}{H}$$

and $$|K| = \sqrt{|1-\Omega_0|}H_0$$

we have

$$d_L = (1+z)\,\mathrm{sen}_K\left(H_0\sqrt{|1-\Omega_0|}\int_0^z \frac{dz'}{H(z')}\right)\Big/ H_0\sqrt{|1-\Omega_0|}$$

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Luminosity Distance

Flat case:

$$d_L = (1+z)\int_0^z \frac{dz'}{H(z')}$$

Recall that (in the ΛCDM model)

$$H^2(a) = H_0{}^2\left[\Omega_r a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\right]$$

then

$$d_L = \frac{(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_r(1+z')^4 + \Omega_M(1+z')^3 + \Omega_\Lambda}}$$

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Luminosity Distance

Series Expansion (**model independent**):

To 1st order

$$H_0 d_L(z) = cz \quad \Leftarrow \quad \textbf{Hubble's "law"}$$

Exercise: obtain, at 2nd order

$$H_0 d_L(z) = cz\left[1 + \frac{1}{2}(1 - q_0)z\right]$$

Hint: expand $\dfrac{1}{a(t)}$ in power series

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The Deceleration Parameter

At 2nd order

$$H_0 d_L(z) = cz\left[1 + \frac{1}{2}(1 - q_0)z\right]$$

where

$$q_0 = -\frac{\ddot{a}_0}{a_0 H_0^2}$$

Exercise:

1) obtain the equation for $\ddot{a}$ and connect to the cosmological parameters

2) derive the series expansion to 4th order in z

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