CPT-EVEN GAUGE SECTOR OF THE STANDARD MODEL EXTENSION

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The CPT-even gauge sector of the Standard Model Extension is composed of 19 components comprised in the tensor $(K_F)^{\mu\nu\alpha\beta}$. This sector is parametrized in terms of four 3×3 matrices, whose properties are here scrutinized. In this note, we examine the Maxwell electrodynamics supplemented by these terms, analysing the new properties induced by them. We achieve some wave equations, which yield the corresponding dispersion relations. In this context, we analyze the existence or absence of birrefringence. We finalize presenting some classical solutions for the Maxwell equations modified by the nonbirefringent parity-odd coefficients.

PACS numbers: 11.30.Cp, 12.60.-i, 11.55.Fv

I. INTRODUCTION

The Standard Model Extension (SME) [1] is a broad framework for investigating Lorentz invariance violation (LIV) in all sectors of the Standard Model. The gauge sector of the SME is composed of a CPT-odd part and CPT-even sector. The CPT-odd gauge sector of the SME is represented by the Carroll-Field-Jackiw (CFJ) electrodynamics [2], with many properties largely examined in literature, [3], [4, 5], [6],[7], [8], [9], [10].

The CPT-even sector of SME has been also much investigated since 2002, mainly after the pioonering contributions by Kostelecky & Mewes in Refs.[12], [13]. This CPT-even sector is represented by the tensor $(K_F)_{\alpha\nu\rho\varphi}$, which has the same symmetries of the Riemann tensor and an double null trace (it has 19 components). The studies about the properties of the CPT-even electrodynamics stipulated the existence of ten linearly independent combinations of the components of $(K_F)_{\alpha\nu\rho\varphi}$ sensitive to birefringence. A broader and interesting study in this respect was performed recently in Ref. [14]. The nonbirefringent components are contained in two 3×3 matrices, $\widetilde{\kappa}_{e+}$ and $\widetilde{\kappa}_{o-}$. Using high-quality spectropolarimetry data of cosmological sources [15], stringent upper bounds $\left(10^{-32}\text{ and }10^{-37}\right)$ were imposed on these birefringent LIV parameters.

In this class, we present the CPT-even gauge sector of the SME, discussing its properties and structure. In the sequel we present the equations of motion, obtaining the modified wave equations. This allows to obtain the dispersion relation of this theory, which is important to give information about the way the electromagnetic waves propagate.

II. CPT-EVEN GAUGE SECTOR OF THE SME

The Lagrangian of the CPT-even electrodynamics of SME is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}V^{\mu}A^{\nu}F^{\alpha\beta}, \tag{1}$$

where $F_{\alpha\nu}$ is the electromagnetic field tensor, V^{μ} is a fixed that does not transform as a four-vector.

is a renormalizable and dimensionless tensor coupling, responsible for Lorentz violation. The tensor $(K_F)_{\alpha\nu\rho\varphi}$ has the same symmetries as the Riemann tensor, that is,

$$(K_F)_{\alpha\nu\rho\varphi}=-(K_F)_{\nu\alpha\rho\varphi},\ \ (K_F)_{\alpha\nu\rho\varphi}=-(K_F)_{\alpha\nu\varphi\rho},\ \ (K_F)_{\alpha\nu\rho\varphi}=(K_F)_{\rho\varphi\alpha\nu}, \tag{2}$$

$$(K_F)_{\alpha\beta\rho\varphi}+(K_F)_{\alpha\rho\varphi\beta}+(K_F)_{\alpha\varphi\beta\rho}=0, \tag{3}$$

and a double null trace,

$$(K_F)^{\rho\varphi}{}_{\rho\varphi}=0. \tag{4}$$

So, the tensor $(K_F)_{\alpha\nu\rho\varphi}$ has 19 independent components, from which nine do not yield birefringence.

Lagrangian can be written in an explicit form

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}\left(2F_{0i}F^{0i} + F_{ij}F^{ij}\right) \\ & -\frac{1}{4}\left[4\left(K_F\right)_{0i0j}F^{0i}F^{0j} + 2\left(K_F\right)_{0ilm}F^{0i}F^{lm} + 2\left(K_F\right)_{lm0i}F^{lm}F^{0i} + \left(K_F\right)_{ablm}F^{ab}F^{lm}\right],\end{aligned} \tag{5}$$

We know that

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A_0 = E^i, \tag{6}$$

$$F_{lm} = \epsilon_{lmk}B_{k,} \tag{7}$$

so that

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \frac{1}{4}\left[4\left(K_F\right)_{0i0j}E^iE^j + 4\left(K_F\right)_{0ilm}\epsilon_{lmp}\left(-E^i\right)B_p + \left(K_F\right)_{ablm}\epsilon_{abq}\epsilon_{lmp}B_qB_p\right], \tag{8}$$

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \frac{1}{4}\left[4\left(K_F\right)_{0i0j}E^iE^j + 4\left(K_F\right)_{0ilm}\epsilon_{lmp}E^iB^p + \left(K_F\right)_{ablm}\epsilon_{abq}\epsilon_{lmp}B_qB_p\right]. \tag{9}$$

III. SIMETRIAS DISCRETAS

Sob operação de paridade ($\mathcal{P}$), temos:

$$A_0 \overset{\mathcal{P}}{\rightarrow} A_0, \ \mathbf{A} \overset{\mathcal{P}}{\rightarrow} -\mathbf{A}; \quad \mathbf{E} \overset{\mathcal{P}}{\rightarrow} -\mathbf{E}; \ \mathbf{B} \overset{\mathcal{P}}{\rightarrow} +\mathbf{B},$$

$$A_\mu \overset{\mathcal{P}}{\rightarrow} A^\mu, \quad J_\mu \overset{\mathcal{P}}{\rightarrow} J^\mu, \quad \partial_\mu \overset{\mathcal{P}}{\rightarrow} \partial^\mu.$$

Sob a operação de inversão temporal ($\mathcal{T}$), onde usamos $\partial_t \overset{\mathcal{T}}{\rightarrow} -\partial_t$, $\partial_i \overset{\mathcal{T}}{\rightarrow} \partial_i$,

$$\mathbf{B} \overset{\mathcal{T}}{\rightarrow} -\mathbf{B}; \quad \mathbf{E} \overset{\mathcal{T}}{\rightarrow} \mathbf{E}; \ A_0 \overset{\mathcal{T}}{\rightarrow} A_0, \quad \mathbf{A} \overset{\mathcal{T}}{\rightarrow} -\mathbf{A};$$

$$A_\mu \overset{\mathcal{T}}{\rightarrow} A^\mu, \quad J_\mu \overset{\mathcal{T}}{\rightarrow} J^\mu, \quad \partial_\mu \overset{\mathcal{T}}{\rightarrow} -\partial^\mu.$$

Sob a operação de conjugação de carga,

$$A_0 \overset{\mathcal{C}}{\rightarrow} -A_0, \quad \mathbf{A} \overset{\mathcal{C}}{\rightarrow} -\mathbf{A}; \ J_\mu \overset{\mathcal{C}}{\rightarrow} -J_\mu,$$

$$\mathbf{E} \overset{\mathcal{C}}{\rightarrow} -\mathbf{E}; \quad \mathbf{B} \overset{\mathcal{C}}{\rightarrow} -\mathbf{B};$$

Portanto, o operador CPT atua da seguinte forma sobre o campo eletromagnético e sobre as derivadas espaço-temporais:

$$\begin{aligned} A_0 &\overset{\mathcal{CPT}}{\rightarrow} -A_0, \quad \mathbf{A} \overset{\mathcal{CPT}}{\rightarrow} -\mathbf{A}, \\ A_\mu &\overset{\mathcal{CPT}}{\rightarrow} -A_\mu, \quad \partial_\mu \overset{\mathcal{CPT}}{\rightarrow} -\partial_\mu, \\ \mathbf{E} &\overset{\mathcal{CPT}}{\rightarrow} \mathbf{E}, \qquad \mathbf{B} \overset{\mathcal{CPT}}{\rightarrow} \mathbf{B}. \end{aligned}$$

Sob CPT-operation:

$$\mathcal{L}\overset{\mathcal{CPT}}{\rightarrow}\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\frac{1}{4}\left[4\left(K_F\right)_{0i0j}\underbrace{E^iE^j}_{+E^iE^j}+4\left(K_F\right)_{0ilm}\epsilon_{lmp}\underbrace{E^iB^p}_{+E^iB^p}+\left(K_F\right)_{ablm}\epsilon_{abq}\epsilon_{lmp}\underbrace{B_qB_p}_{+B_qB_p}\right].$$

Vemos assim que os coeficientes $\left(K_F\right)_{0i0j},\left(K_F\right)_{0ilm},\left(K_F\right)_{ablm}$, são CPT-pares.
Sob o operador paridade:

$$\mathcal{L}\overset{\mathcal{P}}{\rightarrow}\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\frac{1}{4}\left[4\left(K_F\right)_{0i0j}\underbrace{E^iE^j}_{+E^iE^j}+4\left(K_F\right)_{0ilm}\epsilon_{lmp}\underbrace{E^iB^p}_{-E^iB^p}+\left(K_F\right)_{ablm}\epsilon_{abq}\epsilon_{lmp}\underbrace{B_qB_p}_{+B_qB_p}\right].$$

Sob o operador de inversão temporal:

$$\mathcal{L}\overset{\mathcal{T}}{\rightarrow}\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\frac{1}{4}\left[4\left(K_F\right)_{0i0j}\underbrace{E^iE^j}_{+E^iE^j}+4\left(K_F\right)_{0ilm}\epsilon_{lmp}\underbrace{E^iB^p}_{-E^iB^p}+\left(K_F\right)_{ablm}\epsilon_{abq}\epsilon_{lmp}\underbrace{B_qB_p}_{+B_qB_p}\right].$$

Vemos assim que os coeficientes $\left(K_F\right)_{0i0j},\left(K_F\right)_{ablm}$ são P-even e T-even, enquanto $\left(K_F\right)_{0ilm}$ é P-odd e T-odd.

IV. A FIRST MATRIX PARAMETRIZATION

A very useful parametrization for addressing this theory is the one presented in Refs. [12, 13], in which these 19 components are contained in four 3×3 matrices:

$$\left(\kappa_{DE}\right),\ \left(\kappa_{HB}\right),\ \left(\kappa_{DB}\right),\ \left(\kappa_{HE}\right).$$

The components of these matrices are given as

$$\left(\kappa_{DE}\right)^{jk}=-2\left(K_F\right)^{0j0k},\ \left(\kappa_{HB}\right)^{jk}=\frac{1}{2}\epsilon^{jpq}\epsilon^{klm}\left(K_F\right)^{pqlm}, \tag{10}$$

$$\left(\kappa_{DB}\right)^{jk}=-\left(\kappa_{HE}\right)^{kj}=\epsilon^{kpq}\left(K_F\right)^{0jpq}. \tag{11}$$

Note that

$$\begin{aligned}\left(\kappa_{DE}\right)_{jk}&=-2\left(K_F\right)_{0j0k,}\\ \left(K_F\right)_{0ilm}\epsilon_{lmp}&=\left(\kappa_{DB}\right)_{ip},\\ \left(K_F\right)_{ablm}\epsilon_{abq}\epsilon_{lmp}&=2\left(\kappa_{HB}\right)_{qp}.\end{aligned}$$

In accordance with this parametrization, we note the

$$\begin{aligned}&\left(\kappa_{DE}\right)^{jk}=\left(\kappa_{DE}\right)^{kj}\rightarrow\ \text{symmetric matrix}\rightarrow6\ \text{components},\\ &\left(\kappa_{HB}\right)^{jk}=\left(\kappa_{HB}\right)^{kj}\rightarrow\ \text{symmetric matrix}\rightarrow6\ \text{components},\\ &\left(\kappa_{DB}\right)^{jk}\rightarrow\text{has no symmetry}\ \rightarrow9\ \text{components}.\end{aligned}$$

Using this parametrization, we have

$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\frac{1}{4}\left[-2\left(\kappa_{DE}\right)_{ij}E^iE^j+4\left(\kappa_{DB}\right)_{ip}E^iB^p+2\left(\kappa_{HB}\right)_{qp}B_qB_p\right], \tag{12}$$

$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)+\frac{1}{2}\left[\left(\kappa_{DE}\right)_{ij}E^iE^j-2\left(\kappa_{DB}\right)_{ip}E^iB^p-\left(\kappa_{HB}\right)_{qp}B_qB_p\right], \tag{13}$$

$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)+\frac{1}{2}\left[\left(\kappa_{DE}\right)_{ij}E^iE^j+2\left(\kappa_{DB}\right)_{ip}E^iB^p-\left(\kappa_{HB}\right)_{qp}B_qB_p\right]. \tag{14}$$

This result may also be written as

$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)+\frac{1}{2}\left[\left(\kappa_{DE}\right)_{ij}E^iE^j+\left(\kappa_{DB}\right)_{ip}E^iB^p-\left(\kappa_{HB}\right)_{qp}B^qB^p\right], \tag{15}$$

or

$$\mathcal{L}=\frac{1}{2}\left(\delta_i^j+\left(\kappa_{DE}\right)_{ij}\right)E^iE^j-\frac{1}{2}\left(\delta_p^q-\left(\kappa_{HB}\right)_{pq}\right)B^pB^q+\frac{1}{2}\left(\kappa_{DB}\right)_{ip}E^iB^p. \tag{16}$$

Under parity, we know that

$$\mathbf{E}\quad\overset{\mathcal{P}}{\rightarrow}\quad-\mathbf{E},\quad\mathbf{B}\quad\overset{\mathcal{P}}{\rightarrow}\quad-\mathbf{B}, \tag{17}$$

$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)+\frac{1}{2}\left[\left(\kappa_{DE}\right)_{ij}\underbrace{E^iE^j}_{+E^iE^j}+\left(\kappa_{DB}\right)_{ip}\underbrace{E^iB^p}_{-E^iB^p}-\left(\kappa_{HB}\right)_{qp}\underbrace{B^qB^p}_{+B^qB^p}\right], \tag{18}$$

so we notice that

$$\left(\kappa_{DE}\right)_{ij}\rightarrow\ \text{parity-even}, \tag{19}$$
$$\left(\kappa_{HB}\right)_{pq}\rightarrow\ \text{parity-even}, \tag{20}$$
$$\left(\kappa_{DB}\right)_{ip}\rightarrow\ \text{parity-odd}, \tag{21}$$
$$\left(\kappa_{HE}\right)_{ip}\rightarrow\ \text{parity-odd}. \tag{22}$$

The double null trace property (4) implies

$$\left(K_F\right)^{\rho\varphi}{}_{\rho\varphi}=2\left(K_F\right)^{0i}{}_{0i}+\left(K_F\right)^{ij}{}_{ij}=-\left(\kappa_{DE}\right)^{\ i}_{i}-\left(\kappa_{HB}\right)^{l}{}_{l}=0, \tag{23}$$
$$\left(K_F\right)^{\rho\varphi}{}_{\rho\varphi}=-\left(\kappa_{DE}+\kappa_{HB}\right)^{\ i}_{i}=0, \tag{24}$$

meaning that the matrix $\left(\kappa_{DE}+\kappa_{HB}\right)$ has null trace, that is,

$$\mathrm{tr}\left(\kappa_{DE}\right)+\mathrm{tr}\left(\kappa_{HB}\right)=0\quad\rightarrow\mathrm{tr}\left(\kappa_{HB}\right)=-\mathrm{tr}\left(\kappa_{DE}\right).$$

Here, we have used

$$\left(\kappa_{HB}\right)^{jj}=\frac{1}{2}\epsilon^{jpq}\epsilon^{jlm}\left(K_F\right)^{pqlm}=\frac{1}{2}\left(\delta^{pl}\delta^{qm}-\delta^{pm}\delta^{ql}\right)\left(K_F\right)^{pqlm}, \tag{25}$$
$$\left(\kappa_{HB}\right)^{jj}=\frac{1}{2}\left[\left(K_F\right)^{pqpq}-\left(K_F\right)^{pqqp}\right]=\left(K_F\right)^{pqpq}=\left(K_F\right)^{pq}{}_{pq}. \tag{26}$$
$$-\left(\kappa_{HB}\right)^{j}_{j}=\left(K_F\right)^{pqpq}=\left(K_F\right)^{pq}{}_{pq}. \tag{27}$$

On the other hand, the property (3) yields

$$\left(K_F\right)_{0ilm}+\left(K_F\right)_{0lmi}+\left(K_F\right)_{0mil}=0. \tag{28}$$

Multiplying by ϵ_{lmp},

$$\left(K_F\right)_{0ilm}\epsilon_{lmp}+\left(K_F\right)_{0lmi}\epsilon_{lmp}+\left(K_F\right)_{0mil}\epsilon_{lmp}=0, \tag{29}$$
$$\left(\kappa_{DB}\right)_{ip}+\left(K_F\right)_{0lmi}\epsilon_{lmp}+\left(K_F\right)_{0mil}\epsilon_{lmp}=0, \tag{30}$$

$$\left(\kappa_{DB}\right)_{ip}+\left(\kappa_F\right)_{0lmi}\epsilon_{lmp}+\left(\kappa_F\right)_{0mil}\epsilon_{lmp}=0, \tag{31}$$

Multiplying by $\delta_{ip,}$ and using $(K_F)_{0ilm}\,\epsilon_{lmp} = (\kappa_{DB})_{ip}$, we have

$$\begin{aligned}
(\kappa_{DB})_{ii} + (\kappa_F)_{0lmi}\,\epsilon_{lmi} + (\kappa_F)_{0mil}\,\epsilon_{lmi} = 0,\\
(\kappa_{DB})_{ii} + (\kappa_F)_{0lmi}\,\epsilon_{mil} + (\kappa_F)_{0mil}\,\epsilon_{ilm} = 0,\\
(\kappa_{DB})_{ii} + (\kappa_{DB})_{ll} + (\kappa_{DB})_{mm} = 0 \quad \rightarrow \quad (\kappa_{DB})_{ii} = 0.
\end{aligned}$$

Thus, we notice that the matrices $(\kappa_{DB}) = -(\kappa_{HE})^T$ are traceless, possessing $(9-1) = 8$ components. Hence:

$$\begin{aligned}
\kappa_{DE} \quad \text{e} \quad \kappa_{HB} \quad &\rightarrow \quad (12-1) = 11 \quad \text{components},\\
(\kappa_{DB})^{jk} = -(\kappa_{HE})^{kj} \quad &\rightarrow \quad (9-1) = 8 \quad \text{components},\\
&\text{Total:} \quad 19 \text{ components!}
\end{aligned}$$

This is a sucessful and consistent parametrization!

$$\begin{aligned}
\textbf{Even sector} \rightarrow \quad &(12-1) = 11 \quad \text{components},\\
\textbf{Odd sector} \rightarrow \quad &(9-1) = 8 \quad \text{components},
\end{aligned}$$

Note that we also write

$$\begin{aligned}
(\kappa_{HB})^{jk}\,\epsilon^{jab} &= \frac{1}{2}\epsilon^{jab}\epsilon^{jpq}\epsilon^{klm}\,(K_F)^{pqlm},\\
&= \frac{1}{2}\left(\delta^{ap}\delta^{bq} - \delta^{aq}\delta^{bp}\right)\epsilon^{klm}\,(K_F)^{pqlm} = \frac{1}{2}\left[\epsilon^{klm}\,(K_F)^{ablm} - \epsilon^{klm}\,(K_F)^{balm}\right]\\
(\kappa_{HB})^{jk}\,\epsilon^{jab} &= \epsilon^{klm}\,(K_F)^{ablm}
\end{aligned}$$

$$\begin{aligned}
(K_F)_{0ilm}\,\epsilon_{lmp}\epsilon_{sqp} &= \epsilon_{sqp}\,(\kappa_{DB})_{ip},\\
2\,(K_F)_{0isq} &= \epsilon_{sqp}\,(\kappa_{DB})_{ip}.
\end{aligned}$$

A. A second matrix parametrization

There is another parametrization is the one that explicitly separates the parity-even and parity-odd components of the CPT-even sector in terms of the matrices $(\widetilde{\kappa}_{e+}), (\widetilde{\kappa}_{e-})$:

$$(\widetilde{\kappa}_{e+})^{jk} = \frac{1}{2}(\kappa_{DE} + \kappa_{HB})^{jk}, \quad (\widetilde{\kappa}_{e-})^{jk} = \frac{1}{2}(\kappa_{DE} - \kappa_{HB})^{jk} - \frac{1}{3}\delta^{jk}(\kappa_{DE})^{ii}, \tag{32}$$

$$(\widetilde{\kappa}_{o+})^{jk} = \frac{1}{2}(\kappa_{DB} + \kappa_{HE})^{jk}, \quad (\widetilde{\kappa}_{o-})^{jk} = \frac{1}{2}(\kappa_{DB} - \kappa_{HE})^{jk}, \tag{33}$$

$$\kappa_{\rm tr} = \frac{1}{3}{\rm tr}(\kappa_{DE}) = \frac{1}{3}(\kappa_{DE})^{ii}. \tag{34}$$

where $\widetilde{\kappa}_e$ and $\widetilde{\kappa}_o$ designate parity-even and parity-odd matrices, respectively. Knowing that (κ_{DE}) and (κ_{HB}) are symmetric matrices, we have

$$\begin{aligned}
(\widetilde{\kappa}_{e+})^{jk} = (\widetilde{\kappa}_{e+})^{kj} \rightarrow \text{ symmetric matrix} \rightarrow 6 \text{ components},\\
(\widetilde{\kappa}_{e+})^{jk} = (\widetilde{\kappa}_{e+})^{jk} \rightarrow \text{ symmetric matrix} \rightarrow 6 \text{ components}.
\end{aligned}$$

Note that the matrices $(\widetilde{\kappa}_{e+}), (\widetilde{\kappa}_{e-})$ are traceless.

Remembering that $(\kappa_{DB})^{jk} = -(\kappa_{HE})^{kj}$:

$$\begin{aligned}
(\widetilde{\kappa}_{o+})^{jk} &= \frac{1}{2}\left[(\kappa_{DB})^{jk} + (\kappa_{HE})^{jk}\right] = \frac{1}{2}\left[(\kappa_{DB})^{jk} - (\kappa_{DB})^{kj}\right]\\
(\widetilde{\kappa}_{o+})^{kj} &= \frac{1}{2}\left[(\kappa_{DB})^{kj} - (\kappa_{DB})^{jk}\right]\\
(\widetilde{\kappa}_{o+})^{kj} &= -(\widetilde{\kappa}_{o+})^{jk} \rightarrow \text{ antisymmetric matrix} \rightarrow 3 \text{ components}.
\end{aligned}$$

	(κ_{DE})	(κ_{HB})	(κ_{DB})	(κ_{HE})	$(\widetilde{\kappa}_{e+})$	$(\widetilde{\kappa}_{e-})$	$(\widetilde{\kappa}_{o+})$	$(\widetilde{\kappa}_{o-})$	$\kappa_{\rm tr}$
Symmetry	S	S	no	no	S	S	AS	S	no
Parity	even	even	odd	odd	even	even	odd	odd	even
Components	6	6	8	8	6	6	3	5	1
Trace	$\kappa_{\rm tr}$	$-\kappa_{\rm tr}$	0	0	0	0	0	0	

TABLE I: Summary of the properties of the 3×3 Lorentz-violating matrices. Here, S and AS represents symmetric and antisymmetric.

$$(\widetilde{\kappa}_{o-})^{jk}=\frac{1}{2}\left[(\kappa_{DB})^{jk}-(\kappa_{HE})^{jk}\right]=\frac{1}{2}\left[(\kappa_{DB})^{jk}+(\kappa_{DB})^{kj}\right],$$
$$(\widetilde{\kappa}_{o-})^{kj}=(\widetilde{\kappa}_{o-})^{jk}\rightarrow \text{ symmetric traceless matrix}\rightarrow 5 \text{ components}.$$

$$(\widetilde{\kappa}_{e+}),(\widetilde{\kappa}_{e-})\rightarrow \text{ parity-even symmetric and traceless matrix}\rightarrow (6+6-1)=11 \text{ components},$$
$$(\widetilde{\kappa}_{o+}),(\widetilde{\kappa}_{o-})\rightarrow \text{ parity-odd symmetric and traceless matrix}\rightarrow (3+5)=8 \text{ components}.$$

The properties of the matrices can be exhibited in a Table.In terms of this parametrization, it holds

$$(\kappa_{DE})^{jk}=(\widetilde{\kappa}_{e+})^{jk}+(\widetilde{\kappa}_{e-})^{jk}+\delta^{jk}\kappa_{\rm tr}, \tag{35}$$
$$(\kappa_{HB})^{jk}=(\widetilde{\kappa}_{e+})^{jk}-(\widetilde{\kappa}_{e-})^{jk}-\delta^{jk}\kappa_{\rm tr}, \tag{36}$$
$$(\kappa_{DB})^{jk}=(\widetilde{\kappa}_{o+})^{jk}+(\widetilde{\kappa}_{o-})^{jk}, \tag{37}$$
$$(\kappa_{HE})^{jk}=(\widetilde{\kappa}_{o+})^{jk}-(\widetilde{\kappa}_{o-})^{jk}. \tag{38}$$

Using these relations for evaluating the terms of Lagrangian (15), we attain:

$$(\kappa_{DE})_{ij}\,E^iE^j=\left[(\widetilde{\kappa}_{e+})_{ij}+(\widetilde{\kappa}_{e-})_{ij}+\delta_{ij}\kappa_{\rm tr}\right]E^iE^j=\left((\widetilde{\kappa}_{e+})_{ij}+(\widetilde{\kappa}_{e-})_{ij}\right)E^iE^j+\kappa_{\rm tr}E^iE^i$$
$$(\kappa_{DB})_{ip}\,E^iB^p=\left[(\widetilde{\kappa}_{o+})_{ip}+(\widetilde{\kappa}_{o-})_{ip}\right]E^iB^p=(\widetilde{\kappa}_{o+})_{ip}\,E^iB^p+(\widetilde{\kappa}_{o-})_{ip}\,E^iB^p,$$
$$(\kappa_{HB})_{qp}\,B^qB^p=\left[(\widetilde{\kappa}_{e+})_{qp}-(\widetilde{\kappa}_{e-})_{qp}-\delta_{qp}\kappa_{\rm tr}\right]B^qB^p=\left((\widetilde{\kappa}_{e+})_{qp}-(\widetilde{\kappa}_{e-})_{qp}\right)B^qB^p-\kappa_{\rm tr}B^pB^p,$$

Thus, Lagrangian (15) takes the form:

$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)+\frac{1}{2}\left[\left((\widetilde{\kappa}_{e+})_{ij}+(\widetilde{\kappa}_{e-})_{ij}\right)E^iE^j+\kappa_{\rm tr}E^iE^i+(\widetilde{\kappa}_{o+})_{ip}\,E^iB^p+(\widetilde{\kappa}_{o-})_{ip}\,E^iB^p\right. \tag{39}$$
$$\left.-\left((\widetilde{\kappa}_{e+})_{qp}-(\widetilde{\kappa}_{e-})_{qp}\right)B^qB^p+\kappa_{\rm tr}B^pB^p\right], \tag{40}$$
$$\mathcal{L}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)+\frac{1}{2}\left[\left((\widetilde{\kappa}_{e+})_{ij}+(\widetilde{\kappa}_{e-})_{ij}\right)E^iE^j+\kappa_{\rm tr}\mathbf{E}^2+(\widetilde{\kappa}_{o+})_{ip}\,E^iB^p+(\widetilde{\kappa}_{o-})_{ip}\,E^iB^p\right. \tag{41}$$
$$\left.-\left((\widetilde{\kappa}_{e+})_{qp}-(\widetilde{\kappa}_{e-})_{qp}\right)B^qB^p+\kappa_{\rm tr}\mathbf{B}^2\right], \tag{42}$$

$$\mathcal{L}=\frac{1}{2}(1+\kappa_{\rm tr})\mathbf{E}^2-\frac{1}{2}(1-\kappa_{\rm tr})\mathbf{B}^2+\frac{1}{2}\left[\left((\widetilde{\kappa}_{e+})_{ij}+(\widetilde{\kappa}_{e-})_{ij}\right)E^iE^j+\left((\widetilde{\kappa}_{o+})_{ip}+(\widetilde{\kappa}_{o-})_{ip}\right)E^iB^p\right. \tag{43}$$
$$\left.-\left((\widetilde{\kappa}_{e+})_{qp}-(\widetilde{\kappa}_{e-})_{qp}\right)B^qB^p\right]. \tag{44}$$

V. EQUATIONS OF MOTION, WAVE EQUATIONS, AND BIREFRINGENCE

The equations of motion of the electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\left(K_F\right)_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} - A_\mu J^\mu, \tag{45}$$

come from the Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)} = 0. \tag{46}$$

We first carry out the two terms of Eq.(46):

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = -\frac{\partial\left(J_\alpha A^\alpha\right)}{\partial A_\mu} = -J^\mu.$$

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)} = -\frac{1}{4}\frac{\partial\left(F_{\alpha\beta}F^{\alpha\beta}\right)}{\partial\left(\partial_\nu A_\mu\right)} - \frac{1}{4}\left(K_F\right)_{\alpha\beta\rho\varphi}\frac{\partial(F^{\alpha\beta})}{\partial\left(\partial_\nu A_\mu\right)}F^{\rho\varphi} - \frac{1}{4}\left(K_F\right)_{\alpha\beta\rho\varphi}F^{\alpha\beta}\frac{\partial\left(F^{\rho\varphi}\right)}{\partial\left(\partial_\nu A_\mu\right)},$$
$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)} = -F^{\nu\mu} - \left[\left(K_F\right)^{\nu\mu}{}_{\rho\varphi}\right]F^{\rho\varphi} = -F^{\nu\mu} - \left(K_F\right)^{\nu\mu\rho\varphi}F_{\rho\varphi}.$$

Replacing these results in Eq.(46):

$$-J^\mu + \partial_\nu F^{\nu\mu} + \left(K_F\right)^{\nu\mu\rho\varphi}\partial_\nu F_{\rho\varphi} = 0,$$
$$\partial_\nu F^{\nu\mu} + \left(K_F\right)^{\nu\mu\rho\varphi}\partial_\nu F_{\rho\varphi} = J^\mu,$$

we obtain the modified Maxwell equation:

$$\partial_\nu F^{\nu\mu} - \left(K_F\right)^{\mu\nu\rho\varphi}\partial_\nu F_{\rho\varphi} = J^\mu. \tag{47}$$

The other Maxwell equation comes from the Bianchi identity,

$$\partial_\nu F_{\alpha\beta} + \partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha} = 0, \tag{48}$$

yielding the homogeneous Maxwell equation - the Gauss´s law for the magnetic field ($\partial_a B_a = 0$) and the Faraday´s law, that is,

$$\partial_t B_a + \epsilon_{abc}\partial_b E_c = 0. \tag{49}$$

The details of this calculation are below:

$$\frac{\partial\left(F_{\alpha\beta}F^{\alpha\beta}\right)}{\partial\left(\partial_\nu A_\mu\right)} = 2\frac{\partial(F_{\alpha\beta})}{\partial\left(\partial_\nu A_\mu\right)}F^{\alpha\beta} = 2\left(\delta^\nu_\alpha\delta^\mu_\beta - \delta^\nu_\beta\delta^\mu_\alpha\right)F^{\alpha\beta} = 2\left(F^{\nu\mu} - F^{\mu\nu}\right) = 4F^{\nu\mu},$$

$$\frac{\partial(F^{\alpha\beta})}{\partial\left(\partial_\nu A_\mu\right)}F^{\rho\varphi} = \frac{\partial(\partial^\alpha A^\beta - \partial^\beta A^\alpha)}{\partial\left(\partial_\nu A_\mu\right)}F^{\rho\varphi} = \left(g^{\alpha\nu}g^{\beta\mu} - g^{\beta\nu}g^{\alpha\mu}\right)F^{\rho\varphi},$$
$$F^{\alpha\beta}\frac{\partial\left(F^{\rho\varphi}\right)}{\partial\left(\partial_\nu A_\mu\right)} = F^{\alpha\beta}\frac{\partial(\partial^\rho A^\varphi - \partial^\varphi A^\rho)}{\partial\left(\partial_\nu A_\mu\right)} = F^{\alpha\beta}\left(g^{\rho\nu}g^{\varphi\mu} - g^{\varphi\nu}g^{\rho\mu}\right).$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)} &= -\frac{1}{4}\frac{\partial\left(F_{\alpha\beta}F^{\alpha\beta}\right)}{\partial\left(\partial_\nu A_\mu\right)} - \frac{1}{4}\left(K_F\right)_{\alpha\beta\rho\varphi}\frac{\partial(F^{\alpha\beta})}{\partial\left(\partial_\nu A_\mu\right)}F^{\rho\varphi} - \frac{1}{4}\left(K_F\right)_{\alpha\beta\rho\varphi}F^{\alpha\beta}\frac{\partial\left(F^{\rho\varphi}\right)}{\partial\left(\partial_\nu A_\mu\right)},\\
&= -\frac{1}{4}2F^{\nu\mu} - \frac{1}{4}\left(K_F\right)_{\alpha\beta\rho\varphi}\left(g^{\alpha\nu}g^{\beta\mu} - g^{\beta\nu}g^{\alpha\mu}\right)F^{\rho\varphi} - \frac{1}{4}\left(K_F\right)_{\alpha\beta\rho\varphi}F^{\alpha\beta}\left(g^{\rho\nu}g^{\varphi\mu} - g^{\varphi\nu}g^{\rho\mu}\right),\\
&= -\frac{1}{2}F^{\nu\mu} - \frac{1}{4}\left[\left(K_F\right)^{\nu\mu}{}_{\rho\varphi} - \left(K_F\right)^{\mu\nu}{}_{\rho\varphi}\right]F^{\rho\varphi} - \frac{1}{4}\left[\left(K_F\right)_{\alpha\beta}{}^{\nu\mu} - \left(K_F\right)_{\alpha\beta}{}^{\mu\nu}\right]F^{\alpha\beta},\\
&= \frac{1}{2}F^{\nu\mu} - \frac{1}{2}\left[\left(K_F\right)^{\nu\mu}{}_{\rho\varphi}\right]F^{\rho\varphi} - \frac{1}{2}\left[\left(K_F\right)_{\alpha\beta}{}^{\nu\mu}\right]F^{\alpha\beta},\\
\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)} &= \frac{1}{2}F^{\nu\mu} - \left[\left(K_F\right)^{\nu\mu}{}_{\rho\varphi}\right]F^{\rho\varphi} = \frac{1}{2}F^{\nu\mu} - \left(K_F\right)^{\nu\mu\rho\varphi}F_{\rho\varphi}.
\end{aligned}$$

Taking $\mu = 0$ in Eq. (47), we obtain the Gauss's law, $\partial_i F^{i0} - (K_F)^{0\nu\rho\varphi}\,\partial_\nu F_{\rho\varphi} = J^0$, that is:

$$\partial_i F^{i0} - 2\,(K_F)^{0l0j}\,\partial_l F_{0j} - (K_F)^{0lmn}\,\partial_l F_{mn} = J^0, \tag{50}$$

$$\partial_i E^i - 2\,(K_F)^{0l0j}\,\partial_l E^j - (K_F)^{0lmn}\,\epsilon_{mnk}\partial_l B_k = J^0, \tag{51}$$

$$\partial_i E^i - 2\,(K_F)_{0l0j}\,\partial_l E^j + (K_F)_{0lmn}\,\epsilon_{mnk}\partial_l B_k = J^0, \tag{52}$$

Using the parametrization (10,11):

$$\partial_i E^i + (\kappa_{DE})_{lj}\,\partial_l E^j + (\kappa_{DB})_{lk}\,\partial_l B_k = \rho. \tag{53}$$

Taking $\mu = i$ in Eq. (47), we obtain the Ampère's law,

$$\partial_0 F^{0i} + \partial_j F^{ji} - (K_F)^{i\nu\rho\varphi}\,\partial_\nu F_{\rho\varphi} = J^i, \tag{54}$$

that is:

$$\partial_0 F^{0i} + \partial_j F^{ji} - 2\,(K_F)^{i00j}\,\partial_0 F_{0j} - (K_F)^{i0lm}\,\partial_0 F_{lm} - (K_F)^{iplm}\,\partial_p F_{lm} - 2\,(K_F)^{ip0m}\,\partial_p F_{0m} = J^i, \tag{55}$$

$$-\partial_0 E^i + \epsilon_{jik}\partial_j B_k + 2\,(K_F)^{0i0j}\,\partial_0 E^j + (K_F)^{0ilm}\,\epsilon_{lmk}\partial_0 B_k - (K_F)^{iplm}\,\epsilon_{lmk}\partial_p B_k - 2\,(K_F)^{0mip}\,\partial_p E^m = J^i, \tag{56}$$

$$-\partial_t E^i - \epsilon_{ijk}\partial_j B_k - (\kappa_{DE})_{ij}\,\partial_t E^j - (K_F)_{0ilm}\,\epsilon_{lmk}\partial_t B_k - (K_F)_{iplm}\,\epsilon_{lmk}\partial_p B_k + 2\,(K_F)_{0mip}\,\partial_p E^m = J^i, \tag{57}$$

$$-\partial_t E^i - \epsilon_{ijk}\partial_j B_k - (\kappa_{DE})_{ij}\,\partial_t E^j - (\kappa_{DB})_{ik}\,\partial_t B_k - (K_F)_{iplm}\,\epsilon_{lmk}\partial_p B_k + 2\,(K_F)_{0mip}\,\partial_p E^m = J^i, \tag{58}$$

$$(\kappa_{HB})^{jk}\,\epsilon^{jab} = \epsilon^{klm}\,(K_F)^{ablm} \rightarrow (\kappa_{HB})_{jk}\,\epsilon_{jab} = \epsilon_{klm}\,(K_F)_{ablm}\,,$$
$$2\,(K_F)_{0isq} = \epsilon_{sqp}\,(\kappa_{DB})_{ip}\,.$$

$$-\partial_t E^i - \epsilon_{ijk}\partial_j B_k - (\kappa_{DE})_{ij}\,\partial_t E^j - (\kappa_{DB})_{ik}\,\partial_t B_k - \underbrace{(K_F)_{iplm}\,\epsilon_{lmk}}_{(\kappa_{HB})_{jk}\epsilon_{jip}}\partial_p B_k + \underbrace{2\,(K_F)_{0mip}}_{\epsilon_{ipk}(\kappa_{DB})_{mk}}\partial_p E^m = J^i,$$

$$-\partial_t E^i - \epsilon_{ijk}\partial_j B_k - (\kappa_{DE})_{ij}\,\partial_t E^j - (\kappa_{DB})_{ik}\,\partial_t B_k - (\kappa_{HB})_{jk}\,\epsilon_{jip}\partial_p B_k + \epsilon_{ipk}\,(\kappa_{DB})_{mk}\,\partial_p E^m = J^i,$$

Finally, the Ampère´s law reads

$$-\partial_t E^i + \epsilon_{ijk}\partial_j B^k - (\kappa_{DE})_{ij}\,\partial_t E^j + (\kappa_{DB})_{ik}\,\partial_t B^k + (\kappa_{HB})_{jk}\,\epsilon_{jip}\partial_p B^k + \epsilon_{kip}\,(\kappa_{DB})_{mk}\,\partial_p E^m = J^i, \tag{59}$$

or

$$\partial_t E_i - \epsilon_{ijk}\partial_j B_k + (\kappa_{DE})_{ij}\,\partial_t E_j - (\kappa_{DB})_{ik}\,\partial_t B_k - (\kappa_{HB})_{jk}\,\epsilon_{jip}\partial_p B_k - \epsilon_{ipk}\,(\kappa_{DB})_{mk}\,\partial_p E_m = -J_i. \tag{60}$$

VI. WAVE EQUATIONS

With Eqs. (53), (59), (60), we can obtain second order equations (wave equations) for the electric and magnetic fields. We begin rewriting Eq.(60) (in the absence of surces) as

$$\partial_t E_k - \epsilon_{kja}\partial_j B_a + (\kappa_{DE})_{kj}\,\partial_0 E_j - (\kappa_{DB})_{kb}\,\partial_t B_b - (\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j E_a - (\kappa_{HB})_{ab}\,\epsilon_{bkj}\partial_j B_a = 0. \tag{61}$$

Now, we apply the operator ∂_t on Eq. (61) :

$$\partial_t^2 E_k - \epsilon_{kja}\partial_j\,(\partial_t B_a) + (\kappa_{DE})_{kj}\,\partial_t^2 E_j - (\kappa_{DB})_{ka}\,\partial_t^2 B_a - (\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a - (\kappa_{HB})_{ap}\,\epsilon_{pkj}\partial_j\,(\partial_t B_a) \;= 0,$$

Using the Faraday's law, $\partial_t B_a = -\epsilon_{abc}\partial_b E_c$, we have

$$\begin{aligned}&\partial_t^2 E_k - \epsilon_{kja}\partial_j\,(-\epsilon_{abc}\partial_b E_c) + (\kappa_{DE})_{kj}\,\partial_t^2 E_j - (\kappa_{DB})_{ka}\,\partial_t\,(-\epsilon_{abc}\partial_b E_c)\\ &- (\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a - (\kappa_{HB})_{ap}\,\epsilon_{pkj}\partial_j\,(-\epsilon_{abc}\partial_b E_c) = 0,\end{aligned} \tag{62}$$

$$\partial_t^2 E_k + \underbrace{\epsilon_{kja}\epsilon_{abc}\partial_j\partial_b E_c}_{T1} + (\kappa_{DE})_{kj}\,\partial_t^2 E_j + (\kappa_{DB})_{ka}\,\epsilon_{abc}\partial_t\partial_b E_c$$
$$-(\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a + \underbrace{(\kappa_{HB})_{ap}\,\epsilon_{pkj}\epsilon_{abc}\partial_j\partial_b E_c}_{T2} = 0. \tag{63}$$

After some algebraic work for evaluating the terms $T1$ and $T2$, we obtain

$$\partial_t^2 E_k + \partial_k\left(\partial_c E_c\right) - \left(\partial_c\partial_c\right) E_k + (\kappa_{DE})_{kj}\,\partial_t^2 E_j + (\kappa_{DB})_{ka}\,\epsilon_{abc}\partial_t\partial_b E_c - (\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a + (\kappa_{HB})_{aa}\left[\partial_k\partial_b E_b - \partial_b\partial_b E_k\right]$$
$$+ (\kappa_{HB})_{ab}\,\partial_a\partial_b E_k - (\kappa_{HB})_{ka}\,\partial_a\partial_b E_b + (\kappa_{HB})_{ka}\,\partial_b\partial_b E_a - (\kappa_{HB})_{ab}\,\partial_a\partial_k E_b = 0, \tag{64}$$

$$\partial_t^2 E_k + \partial_k\left(\partial_c E_c\right) - \partial_c^2 E_k + (\kappa_{DE})_{kj}\,\partial_t^2 E_j + (\kappa_{DB})_{ka}\,\epsilon_{abc}\partial_t\partial_b E_c - (\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a + (\kappa_{HB})_{aa}\,\partial_k\partial_b E_b + (\kappa_{DE})_{aa}\,\partial_b^2 E_k$$
$$+ (\kappa_{HB})_{ab}\,\partial_a\partial_b E_k - (\kappa_{HB})_{ka}\,\partial_a\partial_b E_b + (\kappa_{HB})_{ka}\,\partial_b^2 E_a - (\kappa_{HB})_{ab}\,\partial_a\partial_k E_b = 0. \tag{65}$$

where we have used $\mathrm{tr}(\kappa_{HB}) = -\mathrm{tr}(\kappa_{DE})$. Thus, the wave equation for the electric field reads as

$$\partial_t^2 E_k + (\kappa_{DE})_{kj}\,\partial_t^2 E_j - \partial_c^2 E_k + (\kappa_{DE})_{aa}\,\partial_b^2 E_k + \partial_k\left(\partial_c E_c\right) + (\kappa_{DB})_{ka}\,\epsilon_{abc}\partial_t\partial_b E_c$$
$$-(\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a + (\kappa_{HB})_{aa}\,\partial_k\partial_b E_b + (\kappa_{HB})_{ab}\,\partial_a\partial_b E_k - (\kappa_{HB})_{ka}\,\partial_a\partial_b E_b$$
$$+(\kappa_{HB})_{ka}\,\partial_b^2 E_a - (\kappa_{HB})_{ab}\,\partial_a\partial_k E_b = 0. \tag{66}$$

$$\left[\delta_{jk} + (\kappa_{DE})_{kj}\right]\partial_t^2 E_j - \left[1 - (\kappa_{DE})_{aa}\right]\nabla^2 E_k + \partial_k\left(\partial_c E_c\right) + (\kappa_{DB})_{ka}\,\epsilon_{abc}\partial_t\partial_b E_c$$
$$-(\kappa_{DB})_{ab}\,\epsilon_{bkj}\partial_j\partial_t E_a + (\kappa_{HB})_{aa}\,\partial_k\partial_b E_b + (\kappa_{HB})_{ab}\,\partial_a\partial_b E_k - (\kappa_{HB})_{ka}\,\partial_a\partial_b E_b$$
$$+(\kappa_{HB})_{ka}\,\nabla^2 E_a - (\kappa_{HB})_{ab}\,\partial_a\partial_k E_b = 0. \tag{67}$$

Here, we have used the results:

$$T1 \to \epsilon_{kja}\epsilon_{abc}\partial_j\partial_b E_c = \left[\delta_{kb}\delta_{jc} - \delta_{kc}\delta_{jb}\right]\partial_j\partial_b E_c = \partial_k\left(\partial_c E_c\right) - \left(\partial_c\partial_c\right) E_k, \tag{68}$$

$$\epsilon_{pkj}\epsilon_{abc} = \delta_{pa}\delta_{kb}\delta_{jc} - \delta_{pa}\delta_{kc}\delta_{jb} + \delta_{pb}\delta_{kc}\delta_{ja} - \delta_{pb}\delta_{ka}\delta_{jc} + \delta_{pc}\delta_{ka}\delta_{jb} - \delta_{pc}\delta_{kb}\delta_{ja}, \tag{69}$$

$$T2 \to (\kappa_{HB})_{ap}\,\epsilon_{pkj}\epsilon_{abc}\partial_j\partial_b E_c = \delta_{pa}\delta_{kb}\delta_{jc}\,(\kappa_{HB})_{ap}\,\partial_j\partial_b E_c - \delta_{pa}\delta_{kc}\delta_{jb}\,(\kappa_{HB})_{ap}\,\partial_j\partial_b E_c + \delta_{pb}\delta_{kc}\delta_{ja}\,(\kappa_{HB})_{ap}\,\partial_j\partial_b E_c$$
$$-\delta_{pb}\delta_{ka}\delta_{jc}\,(\kappa_{HB})_{ap}\,\partial_j\partial_b E_c + \delta_{pc}\delta_{ka}\delta_{jb}\,(\kappa_{HB})_{ap}\,\partial_j\partial_b E_c - \delta_{pc}\delta_{kb}\delta_{ja}\,(\kappa_{HB})_{ap}\,\partial_j\partial_b E_c,$$

$$T2 = (\kappa_{HB})_{aa}\,\partial_k\partial_b E_b - (\kappa_{HB})_{aa}\,\partial_b\partial_b E_k + (\kappa_{HB})_{ab}\,\partial_a\partial_b E_k$$
$$-(\kappa_{HB})_{ka}\,\partial_a\partial_b E_b + (\kappa_{HB})_{ka}\,\partial_b\partial_b E_a - (\kappa_{HB})_{ab}\,\partial_a\partial_k E_b.$$

A. Wave equation for the parity-even parameters

Given the complexity of Eq. (67), we should analyze it by different sectors. We begin focusing in the parity-even sector, so that we should take the parity-odd sector as null (setting $\kappa_{DB} = 0$). With it, Eq. (67) becomes:

$$\left[\delta_{jk} + (\kappa_{DE})_{kj}\right]\partial_t^2 E_j - \left[1 - (\kappa_{DE})_{aa}\right]\nabla^2 E_k + \partial_k\left(\partial_c E_c\right) + (\kappa_{HB})_{aa}\,\partial_k\partial_b E_b$$
$$+(\kappa_{HB})_{ab}\,\partial_a\partial_b E_k - (\kappa_{HB})_{ka}\,\partial_a\partial_b E_b + (\kappa_{HB})_{ka}\,\nabla^2 E_a - (\kappa_{HB})_{ab}\,\partial_a\partial_k E_b = 0. \tag{70}$$

In order to simplify our analysis, we should retain only a LV matrix. In this sense we set $\kappa_{DE} = -\kappa_{HB}$, implying

$$(\widetilde{\kappa}_{e+})^{jk} = 0, \tag{71}$$
$$(\widetilde{\kappa}_{e-})^{jk} = (\kappa_{DE})^{jk} - n\delta^{jk}, \tag{72}$$
$$n = \frac{1}{3}\mathrm{tr}(\kappa_{DE}), \tag{73}$$

or

$$(\kappa_{DE})_{ab} = (\widetilde{\kappa}_{e-})_{ab} + n\delta_{ab}, \qquad (\kappa_{HB})_{ab} = -(\widetilde{\kappa}_{e-})_{ab} - n\delta_{ab}. \tag{74}$$

Replacing the parametrization (74) in Eq. (70), we achieve:

$$\left[\delta_{jk} + (\widetilde{\kappa}_{e-})_{kj} + n\delta_{kj}\right]\partial_t^2 E_j - [1 - (\widetilde{\kappa}_{e-})_{aa} - n\delta_{aa}]\nabla^2 E_k + \partial_k(\partial_c E_c) - [(\widetilde{\kappa}_{e-})_{aa} + n\delta_{aa}]\partial_k\partial_b E_b$$
$$-[(\widetilde{\kappa}_{e-})_{ab} + n\delta_{ab}]\partial_a\partial_b E_k - [-(\widetilde{\kappa}_{e-})_{ka} - n\delta_{ka}]\partial_a\partial_b E_b - [(\widetilde{\kappa}_{e-})_{ka} + n\delta_{ka}]\nabla^2 E_a - [-(\widetilde{\kappa}_{e-})_{ab} - n\delta_{ab}]\partial_a\partial_k E_b = 0, \tag{75}$$

$$\left[(1+n)\delta_{jk} + (\widetilde{\kappa}_{e-})_{kj}\right]\partial_t^2 E_j - [1-3n]\nabla^2 E_k + \partial_k(\partial_c E_c) - [3n]\partial_k\partial_b E_b - (\widetilde{\kappa}_{e-})_{ab}\partial_a\partial_b E_k$$
$$-n\partial_a^2 E_k + (\widetilde{\kappa}_{e-})_{ka}\partial_a\partial_b E_b + n\partial_k\partial_b E_b - (\widetilde{\kappa}_{e-})_{ka}\nabla^2 E_a - n\nabla^2 E_k + (\widetilde{\kappa}_{e-})_{ab}\partial_a\partial_k E_b + n\partial_a\partial_k E_a = 0, \tag{76}$$

$$\left[(1+n)\delta_{jk} + (\widetilde{\kappa}_{e-})_{kj}\right]\partial_t^2 E_j - [1-n]\nabla^2 E_k + \partial_k(\partial_c E_c) - n\partial_k\partial_b E_b$$
$$-(\widetilde{\kappa}_{e-})_{ab}\partial_a\partial_b E_k + (\widetilde{\kappa}_{e-})_{ka}\partial_a\partial_b E_b - (\widetilde{\kappa}_{e-})_{ka}\nabla^2 E_a + (\widetilde{\kappa}_{e-})_{ab}\partial_a\partial_k E_b = 0, \tag{77}$$

finally attaining:

$$\left[(1+n)\delta_{jk} + (\widetilde{\kappa}_{e-})_{kj}\right]\partial_t^2 E_j - [[1-n]\delta_{ka} + (\widetilde{\kappa}_{e-})_{ka}]\nabla^2 E_a + \partial_k(\partial_c E_c) - n\partial_k\partial_b E_b$$
$$-(\widetilde{\kappa}_{e-})_{ab}\partial_a\partial_b E_k + (\widetilde{\kappa}_{e-})_{ka}\partial_a\partial_b E_b + (\widetilde{\kappa}_{e-})_{ab}\partial_a\partial_k E_b = 0. \tag{78}$$

For this case, the Maxwell equation (53) reads

$$\partial_i E^i + (\kappa_{DE})_{lj}\partial_l E^j = \rho, \tag{79}$$
$$\partial_j E_j + \left[n\delta_{ja} + (\widetilde{\kappa}_{e-})_{ja}\right]\partial_j E_a = 0, \tag{80}$$

$$(1+n)\partial_a E_a + (\widetilde{\kappa}_{e-})_{ba}\partial_b E_a = 0. \tag{81}$$

1. *Isotropic case:* $(\widetilde{\kappa}_{e-}) = 0$ *and* $n \neq 0$

In this very particular case, there remains only one isotropic LV parameter, n. The wave equation (78) is reduced to:

$$[(1+n)]\partial_t^2 E_k - [1-n]\nabla^2 E_k + \partial_k(\partial_c E_c) - n\partial_k\partial_b E_b = 0, \tag{82}$$

and the Gauss's law is simply

$$(1+n)\partial_a E_a = 0. \tag{83}$$

Replacing (83) in (82) leads to

$$\left[(1+n)\partial_t^2 - (1-n)\nabla^2\right]E_k = 0. \tag{84}$$

In Fourier space it holds

$$\mathbf{E} = \tilde{\mathbf{E}} \exp(-ip \cdot r), \tag{85}$$

so that

$$\left[(1+n)\, p_0^2 - (1-n)\, \mathbf{p}^2\right] \tilde{E}_k = 0. \tag{86}$$

The corresponding dispersion relation is

$$(1+n)\, p_0^2 - (1-n)\, \mathbf{p}^2 = 0, \tag{87}$$

$$p_0 = \pm \sqrt{\frac{(1-n)}{(1+n)}}\, |\mathbf{p}|\,. \tag{88}$$

This dispersion relation yields equal phase and group velocities,

$$u_{ph} = \frac{p_0}{|\mathbf{p}|} = \sqrt{\frac{(1-n)}{(1+n)}}, \tag{89}$$

$$u_g = \frac{dp_0}{d\,|\mathbf{p}|} = \sqrt{\frac{(1-n)}{(1+n)}}. \tag{90}$$

As the the right and left modes propagate with the same phase velocity, this represents a **nonbirefringent electrodynamics**. Note that it should hold $0 \leq n \leq 1$ for assuring a group velocity minor or equal than 1. We can show that the relation (88) describes the propagation of electromagnetical waves in a dieletric medium, where the light propagates with velocity c/n_{index}, being n_{index} the refraction index. Note that the LV isotropic parameter can be written in terms of the refraction index:

$$\frac{1}{n_{index}} = \sqrt{\frac{(1-n)}{(1+n)}} \rightarrow n = \frac{n_{index}^2 - 1}{n_{index}^2 + 1}.$$

In this case, the effect of Lorentz violation is to modify the propagation velocity, such as it might occur in a dieletric continuum.

2. *Anisotropic case:* $n = 0$ *and* $(\widetilde{\kappa}_{e-}) \neq 0$.

In this case, we keep as non null the matrix coefficient $(\widetilde{\kappa}_{e-})$, able to induce anisotropies in the theory. The wave equation (78) becomes

$$\left[\delta_{jk} + (\widetilde{\kappa}_{e-})_{kj}\right] \partial_t^2 E_j - \left[\delta_{ka} + (\widetilde{\kappa}_{e-})_{ka}\right] \nabla^2 E_a + \partial_k \left(\partial_c E_c\right) - (\widetilde{\kappa}_{e-})_{ab}\, \partial_a \partial_b E_k + (\widetilde{\kappa}_{e-})_{ka}\, \partial_a \partial_b E_b + (\widetilde{\kappa}_{e-})_{ab}\, \partial_a \partial_k E_b = 0, \tag{91}$$

$$\partial_t^2 E_k + (\widetilde{\kappa}_{e-})_{kj}\, \partial_t^2 E_j - \nabla^2 E_k - (\widetilde{\kappa}_{e-})_{ka}\, \nabla^2 E_a + \partial_k \left(\partial_c E_c\right) - (\widetilde{\kappa}_{e-})_{ab}\, \partial_a \partial_b E_k + (\widetilde{\kappa}_{e-})_{ka}\, \partial_a \partial_b E_b + (\widetilde{\kappa}_{e-})_{ab}\, \partial_a \partial_k E_b = 0. \tag{92}$$

The Maxwell equation (80) now is

$$\partial_a E_a + (\widetilde{\kappa}_{e-})_{ba}\, \partial_b E_a = 0. \tag{93}$$

Combining Eqs. (93) and (92) leads to

$$\partial_t^2 E_k + (\widetilde{\kappa}_{e-})_{kj}\, \partial_t^2 E_j - \nabla^2 E_k - (\widetilde{\kappa}_{e-})_{ka}\, \nabla^2 E_a + \partial_k \left(\partial_c E_c\right) - (\widetilde{\kappa}_{e-})_{ab}\, \partial_a \partial_b E_k + (\widetilde{\kappa}_{e-})_{ka}\, \partial_a \partial_b E_b - \partial_k \partial_a E_b = 0, \tag{94}$$

$$\left(\partial_t^2 - \nabla^2\right) E_k + (\widetilde{\kappa}_{e-})_{kj} \left(\partial_t^2 - \nabla^2\right) E_j - (\widetilde{\kappa}_{e-})_{ab}\, \partial_a \partial_b E_k + (\widetilde{\kappa}_{e-})_{ka}\, \partial_a \partial_b E_b = 0, \tag{95}$$

$$\Box E_k + (\widetilde{\kappa}_{e-})_{ka} \Box E_a - (\widetilde{\kappa}_{e-})_{ab} \partial_a \partial_b E_k + (\widetilde{\kappa}_{e-})_{kb} \partial_b \partial_a E_a = 0,$$
$$[\Box - (\widetilde{\kappa}_{e-})_{cb} \partial_c \partial_b] E_k + (\widetilde{\kappa}_{e-})_{kb} [\Box \delta_{ab} + \partial_b \partial_a] E_a = 0, \tag{96}$$

attaining the result:

$$\{[\Box - (\widetilde{\kappa}_{e-})_{cb} \partial_c \partial_b] \delta_{ka} + (\widetilde{\kappa}_{e-})_{kb} [\Box \delta_{ab} + \partial_b \partial_a]\} E_a = 0, \tag{97}$$

In the momentum space,

$$\left\{ \left[p^2 - (\widetilde{\kappa}_{e-})_{cb} p_c p_b\right] \delta_{ka} + (\widetilde{\kappa}_{e-})_{kb} \left[p^2 \delta_{ab} + p_b p_a\right] \right\} E_a = 0. \tag{98}$$

As the matrix $(\widetilde{\kappa}_{e-})$ is symmetric and traceless, it can be parameterized in terms of two orthogonal 3D vectors, $\mathbf{a}$ and $\mathbf{b}$, as

$$(\widetilde{\kappa}_{e-})_{jk} = \frac{1}{2} \left(a_j b_k + a_k b_j\right), \tag{99}$$

with $\mathbf{a} \cdot \mathbf{b} = 0$ and $\det(\widetilde{\kappa}_{e-}) = 0$. Then

$$(\widetilde{\kappa}_{e-})_{cb} p_c p_b = (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) \tag{100}$$

$$(\widetilde{\kappa}_{e-})_{kc} \left[p^2 \delta_{jc} + p_c p_j\right] = \frac{1}{2} \left(a_k b_c + a_c b_k\right) \left[p^2 \delta_{jc} + p_c p_j\right] \tag{101}$$
$$= \frac{1}{2} \left(a_k b_j + a_j b_k\right) p^2 + \frac{1}{2} a_k p_j (\mathbf{b} \cdot \mathbf{p}) + \frac{1}{2} b_k p_j (\mathbf{a} \cdot \mathbf{p}) \tag{102}$$

Replacing these terms in Eq. (98):

$$\left\{ \left[p^2 - (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p})\right] \delta_{kj} + \frac{1}{2} \left(a_k b_j + a_j b_k\right) p^2 + \frac{1}{2} a_k p_j (\mathbf{b} \cdot \mathbf{p}) + \frac{1}{2} b_k p_j (\mathbf{a} \cdot \mathbf{p}) \right\} E_j = 0, \tag{103}$$

which can be read as a matrix equation,

$$M_{kj} E_j = 0, \tag{104}$$

with

$$M_{kj} = \left[p^2 - (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p})\right] \delta_{kj} + \frac{1}{2} \left(a_k b_j + a_j b_k\right) p^2 + \frac{1}{2} a_k p_j (\mathbf{b} \cdot \mathbf{p}) + \frac{1}{2} b_k p_j (\mathbf{a} \cdot \mathbf{p}) \tag{105}$$

The dispersion relations are obtained by setting $\det M = 0$. Computing the determinant, we achieve

$$\det \mathbb{M} = \left(1 - \frac{1}{4} \mathbf{a}^2 \mathbf{b}^2\right) p^2 \left[p^2 - (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p})\right] \left[p^2 - \frac{4 (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) + (\mathbf{a} \cdot \mathbf{p})^2 \mathbf{b}^2 + \mathbf{a}^2 (\mathbf{b} \cdot \mathbf{p})^2}{4 - \mathbf{a}^2 \mathbf{b}^2}\right] \tag{106}$$

Thus, the dispersion relation are the following:

$$p^2 - (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) = 0, \tag{107}$$

$$p^2 - \frac{4 (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) + (\mathbf{a} \cdot \mathbf{p})^2 \mathbf{b}^2 + \mathbf{a}^2 (\mathbf{b} \cdot \mathbf{p})^2}{4 - \mathbf{a}^2 \mathbf{b}^2} = 0. \tag{108}$$

The expression (107) leads to

$$p_0^2 - \mathbf{p}^2 - (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) = 0, \tag{109}$$

$$p_0^2 = \mathbf{p}^2 + (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) \quad \rightarrow \quad p_0 = \pm |\mathbf{p}| \sqrt{1 + |\mathbf{a}| |\mathbf{b}| \cos\theta_a \cos\theta_b}, \tag{110}$$

which represents a nonbirefringent expression. Note that the left mode remains equal to right mode under the operation $\mathbf{p} \rightarrow -\mathbf{p}$. This assures non birefringence.

B. Wave equation for the parity-odd parameters

In this case, we set $\kappa_{DE} = \kappa_{HB} = 0$ in Eq. (67), keeping only the parity-odd coeffcients $[(\kappa_{DB})_{ka} \neq 0]$. We thus have:

$$\partial_t^2 E_k - \nabla^2 E_k + \partial_k \left(\partial_c E_c\right) + \left(\kappa_{DB}\right)_{ka} \epsilon_{abc} \partial_t \partial_b E_c - \left(\kappa_{DB}\right)_{ab} \epsilon_{bkj} \partial_j \partial_t E_a = 0. \tag{111}$$

In order to facilitate even more our analysis, we choose $\kappa_{DB} = \kappa_{HE}$, so that

$$\widetilde{\kappa}_{o-} = \frac{1}{2}\left(\kappa_{DB} - \kappa_{HE}\right) = 0,$$
$$\widetilde{\kappa}_{o+} = \frac{1}{2}\left(\kappa_{DB} + \kappa_{HE}\right) = \kappa_{DB}.$$

Replacing it in Eq. (111),

$$\partial_t^2 E_k - \nabla^2 E_k + \partial_k \left(\partial_c E_c\right) + \left(\widetilde{\kappa}_{o+}\right)_{ka} \epsilon_{abc} \partial_t \partial_b E_c - \left(\widetilde{\kappa}_{o+}\right)_{ab} \epsilon_{bkj} \partial_j \partial_t E_a = 0. \tag{112}$$

Note that the Maxwell equation now reads

$$\partial_a E_a - \left(\widetilde{\kappa}_{o+}\right)_{ba} \partial_b B_a = 0. \tag{113}$$

The condition $\kappa_{DB} = \kappa_{HE}$, associated with $\left(\kappa_{DB}\right)^{jk} = -\left(\kappa_{HE}\right)^{kj}$, yields

$$\left(\kappa_{DB}\right)^{jk} = -\left(\kappa_{DB}\right)^{kj}. \tag{114}$$

The matrix $\kappa_{DB} = \widetilde{\kappa}_{o+}$ is now antisymmetric and traceless, possessing only 3 components. It can be then parameterized in terms of one 3-vector, κ,

$$\left(\widetilde{\kappa}_{o+}\right)_{jk} = \epsilon_{jki} \kappa_i, \tag{115}$$

and $\det\left(\widetilde{\kappa}_{o+}\right) = 0$. Using relation (115) in Eq. (112),

$$\partial_t^2 E_k - \nabla^2 E_k + \partial_k \left(\partial_c E_c\right) + \epsilon_{kai} \epsilon_{abc} \kappa_i \partial_t \partial_b E_c - \epsilon_{abi} \epsilon_{bkj} \kappa_i \partial_j \partial_t E_a = 0. \tag{116}$$

In the momentum space, Eq.(116) is written as

$$p_0^2 E_k - \mathbf{p}^2 E_k + p_k p_c E_c + \epsilon_{kai} \epsilon_{abc} \kappa_i p_0 p_b E_c - \epsilon_{abi} \epsilon_{bkj} \kappa_i p_j p_0 E_a = 0,$$

whose algebraic development leads to

$$\begin{aligned}
p_0^2 E_k - \mathbf{p}^2 E_k + p_k p_c E_c - \left[\delta_{kb}\delta_{ic} - \delta_{kc}\delta_{ib}\right] \kappa_i p_0 p_b E_c + \left[\delta_{ak}\delta_{ij} - \delta_{aj}\delta_{ik}\right] \kappa_i p_j p_0 E_a = 0, \\
p_0^2 E_k - \mathbf{p}^2 E_k + p_k p_c E_c - \kappa_c p_0 p_k E_c + \kappa_b p_0 p_b E_k + \kappa_j p_j p_0 E_k - \kappa_k p_j p_0 E_j = 0, \\
\left(p_0^2 - \mathbf{p}^2\right) E_k + p_k p_c E_c - p_0 \left[\kappa_c p_k E_c - 2\kappa_b p_b E_k + \kappa_k p_j E_j\right] = 0, \\
p^2 E_k + 2p_0 \left(\kappa \cdot \mathbf{p}\right) E_k + p_k p_j E_j - p_0 \left[\kappa_j p_k E_j + \kappa_k p_j E_j\right] = 0.
\end{aligned}$$

This odd wave equation takes the form:

$$\left\{\left[p^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right)\right] \delta_{kj} + p_k p_j - p_0 \left(\kappa_j p_k + \kappa_k p_j\right)\right\} E_j = 0, \tag{117}$$

$$M_{kj} E_j = 0, \tag{118}$$

with

$$M_{kj} = \left[p^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right)\right] \delta_{kj} + p_k p_j - p_0 \left(\kappa_j p_k + \kappa_k p_j\right). \tag{119}$$

The dispersion relations are determined from the condition $\det M = 0$. Computing the determinant we have

$$\det M = p_0^2 \left[p^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right)\right] \left[p^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right) - \kappa^2 \mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2\right].$$

Thus, the physical dispersion relation are

$$p^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right) = 0, \tag{120}$$

$$p^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right) - \kappa^2 \mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2 = 0. \tag{121}$$

We can analyze the relation (120), which may be written as

$$p_0^2 - \mathbf{p}^2 + 2p_0 \left(\kappa \cdot \mathbf{p}\right) = 0, \tag{122}$$

which provides the solutions

$$p_0 = -\left(\kappa \cdot \mathbf{p}\right) \pm \sqrt{\mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2}, \tag{123}$$

conveniently written as

$$p_0 = \pm \left[\sqrt{\mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2} \mp \left(\kappa \cdot \mathbf{p}\right)\right]. \tag{124}$$

Note that the two modes are:

$$p_{0+} = \sqrt{\mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2} - \left(\kappa \cdot \mathbf{p}\right), \tag{125}$$

$$p_{0-} = \sqrt{\mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2} + \left(\kappa \cdot \mathbf{p}\right). \tag{126}$$

In order to investigate about birefringence, we should compare the phase velocities of a right wave and a left wave (both moving to the right).

$$p_{0+}(+\mathbf{p}) = \sqrt{\mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2} - \left(\kappa \cdot \mathbf{p}\right), \tag{127}$$

$$p_{0-}(-\mathbf{p}) = \sqrt{\mathbf{p}^2 + \left(\kappa \cdot \mathbf{p}\right)^2} - \left(\kappa \cdot \mathbf{p}\right). \tag{128}$$

These two modes become equal, yielding the same phase velocity

$$u_{ph} = \sqrt{1 + \kappa^2 \cos^2 \theta_p} - |\kappa| \cos \theta_p.$$

We thus conclude that the coefficients κ are nonbirefringent.

VII. CLASSICAL SOLUTIONS IN THE CPT-EVEN AND PARITY-ODD ELECTRODYNAMICS OF THE SME

Fisrt, we will use some information coming from birefringence data to impose restrictions on the LV parameters. Considering the experimental data supporting absence of birefringence in the light coming from far away galaxies, the following upper bounds

$$(\kappa_{DE} + \kappa_{HB}) \leq 10^{-37}, \ (\kappa_{DB} - \kappa_{HE}) \leq 10^{-37}, \tag{129}$$

are stated [see Refs. [12, 13]].

Such bounds allow to choose $\kappa_{DE}=\kappa_{HB}=0$, $\kappa_{DB}=\kappa_{HE}\neq 0$, and $\widetilde{\kappa}_{tr}=0$ (in order to keep only the parity-even parameters). With it, we have:

$$(\widetilde{\kappa}_{e+})^{jk}=0,\ \ (\widetilde{\kappa}_{e-})^{jk}=0, \tag{130}$$

$$(\widetilde{\kappa}_{o+})^{jk}=(\kappa_{DB})^{jk},\ \ (\widetilde{\kappa}_{o-})^{jk}=0. \tag{131}$$

This represents the same situation summarized in Eq. (115), in which the three components of the matrix κ_{DB} are expressed in terms of a vector

$$\kappa^{j}=\frac{1}{2}\epsilon^{jpq}\left(\kappa_{DB}\right)^{pq}. \tag{132}$$

This approach is equivalent to considering $(\widetilde{\kappa}_{e+})^{jk}=(\widetilde{\kappa}_{e-})^{jk}=(\widetilde{\kappa}_{o-})^{jk}=0, (\widetilde{\kappa}_{o+})^{jk}=\epsilon^{jpq}\left(\kappa_{DB}\right)^{pq}$, $\widetilde{\kappa}_{tr}=0$. Nowadays, these four nonbirefringent coefficients are constrained by microwave cavity experiments, $\kappa^{j}\leq 10^{-12}$ [12], by the absence of vacuum Cerenkov radiation for ultrahigh-energy cosmic rays (UHECRs), $\kappa^{j}\leq 10^{-17}-10^{-18}$[16], and by tree-level decay rates in a modified quantum electrodynamics, $\widetilde{\kappa}_{tr}<10^{-16}$[17].

In this present section, we aim at evaluating the classical solutions of the Maxwell electrodynamics supplemented by the LV $\boldsymbol{\kappa}-$vector, in an extension to the developments of ref. [23]. We take as starting point the modified Maxwell equations and the wave equations for the potentials and field strengths. Such equations show that charges contribute to the magnetic sector as well as currents contribute to the electric field. Such contributions are explicitly carried out by means of the Green method, which provides solutions for point-like and spatially extended sources. The key-point is the expression for the scalar potential, written for a general source $(\rho,\mathbf{j})$. From it, we obtain the electric and magnetic field strength at second order in κ. A dipolar expansion is evaluated for these fields, revealing that the current contributions for the electric field and the charge contributions for the magnetic field are first order ones. We finalize establishing an upper bounds for the LV parameter as stringent as $k<10^{-16}$, a nice value for an earth based laboratory experiment.

A. Modified Maxwell equations

Focusing specifically on the CPT-even term $\left(V^{\beta}=0\right)$, the Euler-Lagrange equation leads to the following motion equation:

$$\partial_{\nu}F^{\nu\alpha}-\left(K_{F}\right)^{\alpha\nu\rho\lambda}\partial_{\nu}F_{\rho\lambda}=J^{\alpha}, \tag{133}$$

which contains the two modified inhomogeneous Maxwell equations, while the two homogeneous ones come from the Bianchi identity $(\partial_{\nu}F^{\nu\alpha *}=0)$, with $F^{\alpha\beta *}=\frac{1}{2}\epsilon^{\alpha\beta\lambda\mu}F_{\lambda\mu}$ being the dual tensor. The Maxwell equations

$$\nabla\cdot\mathbf{E}+\kappa\cdot\left(\nabla\times\mathbf{B}\right)=+\rho, \tag{134}$$

$$\nabla\times\mathbf{B}-\partial_{t}\left(\mathbf{B}\times\kappa\right)-\partial_{t}\mathbf{E}+\nabla\times\left(\mathbf{E}\times\kappa\right)=\mathbf{j}, \tag{135}$$

$$\nabla\cdot\mathbf{B}=0\ , \tag{136}$$

$$\nabla\times\mathbf{E}+\partial_{t}\mathbf{B}=\mathbf{0}. \tag{137}$$

are the starting point for searching classical solutions. In order to solve such equations, we should achieve wave equations for the vector and scalar potentials and field strengths. Working at the stationary regime, we attain the following equations:

$$\nabla^{2}A_{0}-\kappa\cdot\left(\nabla\times\mathbf{B}\right)=-\rho, \tag{138}$$

$$\nabla^{2}\mathbf{A}+\left[\left(\kappa\cdot\nabla\right)\nabla-\kappa\nabla^{2}\right]A_{0}=-\mathbf{j}. \tag{139}$$

Using $\nabla\cdot\mathbf{A}=0$, a consequence of the stationary condition on the Lorentz condition $(\partial_{\mu}A^{\mu}=0)$, Eq. (138) takes the form:

$$\nabla^{2}A_{0}+\kappa\cdot\left(\nabla^{2}\mathbf{A}\right)=-\rho. \tag{140}$$

The curl operator, when applied on Eq. (139), implies:

$$\nabla^2\mathbf{B}+(\kappa\times\nabla)\,\nabla^2 A_0=-\nabla\times\mathbf{j}. \tag{141}$$

Taking the scalar product of the vector κ with the entire expression (139) and replacing it on Eq. (140), we attain a wave equation for the scalar potential, namely:

$$\left[\left(1+\kappa^2\right)\nabla^2-(\kappa\cdot\nabla)^2\right]A_0=-\rho+\kappa\cdot\mathbf{j}. \tag{142}$$

Now, applying the full differential operator $\left[\nabla^2-(\kappa\cdot\nabla)^2+\kappa^2\nabla^2\right]$ on Eqs. (139) and (141), it leads to intricate wave equations for the vector potential and magnetic field strength:

$$\nabla^2\left[(1+\kappa^2)\nabla^2-(\kappa\cdot\nabla)^2\right]\mathbf{A}=\left[(\kappa\cdot\nabla)\,\nabla-\kappa\nabla^2\right][\rho-\kappa\cdot\mathbf{j}]-[(1+\kappa^2)\nabla^2-(\kappa\cdot\nabla)^2]\mathbf{j}, \tag{143}$$

$$\nabla^2\left[(1+\kappa^2)\nabla^2-(\kappa\cdot\nabla)^2\right]\mathbf{B}=(\kappa\times\nabla)\,\nabla^2\left[\rho-(\kappa\cdot\mathbf{j})\right]-\left[(1+\kappa^2)\nabla^2-(\kappa\cdot\nabla)^2\right]\nabla\times\mathbf{j}. \tag{144}$$

An alternative and simpler relation for the magnetic field can be derived from Eq. (141)**,** which implies:

$$\mathbf{B}=\kappa\times\mathbf{E}+\frac{1}{4\pi}\nabla\times\int d^3\mathbf{r}'\,\frac{\mathbf{j}\,(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{145}$$

which relates the magnetic field with the electric field and the current. The term $-\left|\mathbf{r}-\mathbf{r}'\right|^{-1}/4\pi$ is by the usual Green function of the Laplacian operator ∇^2. It gives a easy way to evaluate the magnetic field generated by a generic sources $(\rho,\mathbf{j})$, once the electric field is known.

All these wave equations reveal that the electric and magnetic sectors are closely entwined in the sense that both charge and current generate both magnetic and electric field strengths. Such connection in this model was discussed in Refs. [20, 23] and also appears in the case of the Carroll-Field-Jackiw electrodynamics for a pure spacelike background. The difference is that in the present case this connection is really manifest in the solutions for any background configuration, whereas the electric and magnetic CFJ solutions remain uncoupled for the case of a purely timelike background.

A general solution for Eq. (142) can be given by the integral expression,

$$A_0\,(\mathbf{r})=\int G(\mathbf{r}-\mathbf{r}')[-\rho(\mathbf{r}')+\kappa\cdot\mathbf{j}(\mathbf{r}')]d^3\mathbf{r}', \tag{146}$$

where $G(\mathbf{r}-\mathbf{r}')$ is the associated Green function which fulfills the differential equation,

$$\left[(1+\kappa^2)\nabla^2-(\kappa\cdot\nabla)^2\right]G(\mathbf{r}-\mathbf{r}')=\delta^3(\mathbf{r}-\mathbf{r}'). \tag{147}$$

In order to achieve $G(\mathbf{r})$, we use the Fourier transform:

$$G(\mathbf{r}-\mathbf{r}')=\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\;\tilde{G}\,(\mathbf{p})\exp\left[-i(\mathbf{r}-\mathbf{r}')\right], \tag{148}$$

which replaced in Eq.(147) leads to

$$\left[-(1+\kappa^2)\mathbf{p}^2+(\kappa\cdot\mathbf{p})^2\right]\tilde{G}\,(\mathbf{p})=1, \tag{149}$$

$$\left[(1+\kappa^2)\mathbf{p}^2-\kappa^2\mathbf{p}^2\cos\alpha^2\right]\tilde{G}\,(\mathbf{p})=-1, \tag{150}$$

$$\left[1+\kappa^2\mathbf{p}^2\sin\alpha^2\right]\tilde{G}\,(\mathbf{p})=-1, \tag{151}$$

$$\tilde{G}\,(\mathbf{p})=-\frac{1}{[\mathbf{p}^2(1+\kappa^2\sin^2\alpha)]}, \tag{152}$$

with α being the angle defined by the background vector (κ) and the vector $\mathbf{p}$, so that $\kappa\cdot\mathbf{p}=\kappa p\cos\alpha$.

Here, we need to define the spherical coordinates of the momentum vector, $\mathbf{p} = (p, \theta, \phi)$, and the coordinates of the fixed background, $\kappa = (\kappa, \theta_1, \phi_1)$. For calculation purpose, we align the vector $(\mathbf{r} - \mathbf{r}')$ with the z-axis, so that θ_1 is the angle defined by the vectors κ and $(\mathbf{r} - \mathbf{r}')$, and θ is the angle defined by the vectors $\mathbf{p}$ and $(\mathbf{r} - \mathbf{r}')$:

$$[\kappa \cdot (\mathbf{r} - \mathbf{r}') = \kappa \left|\mathbf{r} - \mathbf{r}'\right| \cos\theta_1], \tag{153}$$

$$[\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}') = p \left|\mathbf{r} - \mathbf{r}'\right| \cos\theta], \tag{154}$$

In this case, the angle α is given by

$$\cos\alpha = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi - \phi_1). \tag{155}$$

The Fourier transform of $\tilde{G}(\mathbf{p})$ can not be solve exactly, but an explicit solution can be achieved for the case $\kappa^2 << 1$, for which it holds

$$\frac{1}{(1 + \kappa^2 \sin^2\alpha)} \simeq (1 - \kappa^2 \sin^2\alpha), \tag{156}$$

so that:

$$\tilde{G}(\mathbf{p}) = -\frac{1}{\mathbf{p}^2}\left[1 - \kappa^2 \sin^2\alpha\right], \tag{157}$$

The Green function than takes the form:

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\ \left[-\frac{1}{\mathbf{p}^2} + \frac{\kappa^2 \sin^2\alpha}{\mathbf{p}^2}\right] \exp\left[-i\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')\right], \tag{158}$$

whose result is:

$$G(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi}\left\{\left(1 - \frac{\kappa^2}{2}\right)\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{\left(\kappa \cdot (\mathbf{r} - \mathbf{r}')\right)^2}{2\left|\mathbf{r} - \mathbf{r}'\right|^3}\right\}. \tag{159}$$

Here, we have used the following Fourier results:

$$\int d^3\mathbf{p}\ \frac{1}{\mathbf{p}^2} e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \frac{dp}{(2\pi)^3}\sin\theta\ e^{-ipr\cos\theta},$$

$$\int d^3\mathbf{p}\ \frac{1}{\mathbf{p}^2} e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} = 2\pi^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|},$$

$$\sin^2\alpha = 1 - \cos^2\theta\cos^2\theta_1 - \frac{1}{2}\sin 2\theta \sin 2\theta_1 \cos(\phi - \phi_1) - \sin^2\theta\sin^2\theta_1\cos^2(\phi - \phi_1).$$

Using the Green function (159) and Eq. (146), the scalar potential due to general sources (at order κ^2) is:

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\left\{c(\kappa)\int d^3\mathbf{r}'\ \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{2}\int d^3\mathbf{r}'\ \frac{\left[\kappa\cdot(\mathbf{r} - \mathbf{r}')\right]^2}{\left|\mathbf{r} - \mathbf{r}'\right|^3}\rho(\mathbf{r}') - \int d^3\mathbf{r}'\ \frac{\kappa\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right\}. \tag{160}$$

with $c(\kappa) = \left(1 - \kappa^2/2\right)$. Such expression reveals that the Lorentz-violating charge corrections to A_0 are proportional to κ^2, while the current corrections are of first order in κ. With this expression, we may immediately evaluate the scalar potential for a point-like charge at rest $[\rho(\mathbf{r}') = e\delta(\mathbf{r}')]$ and a point-like charge at stationary motion with velocity $\mathbf{u}$, $[\mathbf{j}(\mathbf{r}') = e\mathbf{u}\delta(\mathbf{r}')]$. Direct integration yields:

$$A_0(\mathbf{r}) = \frac{e}{4\pi}\left\{\frac{c(\kappa)}{r} - \frac{\kappa\cdot\mathbf{u}}{r} - \frac{1}{2}\frac{(\kappa\cdot\mathbf{r})^2}{r^3}\right\}, \tag{161}$$

This potential leads to the following expressions for the electric field of a static and stationary charge:

$$\mathbf{E}\left(\mathbf{r}\right)=\frac{e}{4\pi}\left\{c\left(\kappa\right)\frac{\mathbf{r}}{r^{3}}-\frac{3\left(\kappa\cdot\mathbf{r}\right)^{2}}{2r^{5}}\mathbf{r}+\frac{\left(\kappa\cdot\mathbf{r}\right)}{r^{3}}\kappa\right\}, \tag{162}$$

$$\mathbf{E}\left(\mathbf{r}\right)=-\frac{e}{4\pi}\left(\kappa\cdot\mathbf{u}\right)\frac{\mathbf{r}}{r^{3}}, \tag{163}$$

respectively**.** Here, both fields present a $1/r^{2}$ decaying behavior. Although the static field (162) decays as $1/r^{2}$, its behavior is non-Coulombian once the magnitude of the second term changes with direction and the third term points in the $\kappa-$direction. The presence of the coefficient $c\left(\kappa\right)$ in the Coulombian term reveals that the LV background also induces a screening in the magnitude of the electric charge. Such effects may be contrasted with the ones induced by the Carroll-Field-Jackiw background $\left(V^{\beta}\right)$ on the Maxwell theory. Indeed, it is known that the electric field engendered by a static (or moving stationary charge) remains exactly Coulombian for the case of a timelike background, $V^{\beta}=\left(\mathrm{v}_{0},0\right)$.

The electric field generated by the sources $(\rho,\mathbf{j})$, read off from Eq.(160):

$$\mathbf{E}\left(\mathbf{r}\right)=\frac{1}{4\pi}\left\{c\left(\kappa\right)\int\frac{\rho\left(\mathbf{r}'\right)\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}d^{3}\mathbf{r}'\ +\frac{1}{2}\nabla\int d^{3}\mathbf{r}'\ \frac{\left[\kappa\cdot\left(\mathbf{r}-\mathbf{r}'\right)\right]^{2}}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}\rho\left(\mathbf{r}'\right)-\int\ \frac{\kappa\cdot\mathbf{j}\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}\left(\mathbf{r}-\mathbf{r}'\right)d^{3}\mathbf{r}'\right\}. \tag{164}$$

The magnetic field stemming from Eq.(145) is then given by (at κ^{2} order):

$$\mathbf{B}\left(\mathbf{r}\right)=\frac{1}{(4\pi)}\left\{\int\frac{\rho\left(\mathbf{r}'\right)\kappa\times\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}d^{3}\mathbf{r}'\ -\int\ \frac{\kappa\times\left(\mathbf{r}-\mathbf{r}'\right)\left(\kappa\cdot\mathbf{j}\left(\mathbf{r}'\right)\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^{3}}d^{3}\mathbf{r}'+\nabla\times\int d^{3}\mathbf{r}'\ \frac{\mathbf{j}\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}\right\}. \tag{165}$$

This expression shows that the charges yield a first order LV contribution to the magnetic field while the currents provide only a second order contribution. The last term of the expression above is the usual contribution of the Maxwell theory. For the case of point-like sources $[\rho(\mathbf{r}')=q\delta(\mathbf{r}'),\mathbf{j}=q\mathbf{u}\delta(\mathbf{r}')]$, the resulting magnetic field (at κ^{2} order) is:

$$\mathbf{B}\left(\mathbf{r}\right)=\frac{q}{4\pi}\left\{\left[1-\left(\kappa\cdot\mathbf{u}\right)\right]\frac{\kappa\times\mathbf{r}}{r^{3}}+\frac{\mathbf{u}\times\mathbf{r}}{r^{3}}\right\}. \tag{166}$$

Such a solution has two components: one pointing in the direction $\kappa\times\mathbf{r}$**,** other in the direction $\mathbf{u}\times\mathbf{r}$**,** revealing that the magnetic field is always orthogonal to the position vector $\mathbf{r}$**.** For a static point-like charge, the associate magnetic field is:

$$\mathbf{B}\left(\mathbf{r}\right)=\frac{q}{4\pi}\frac{\kappa\times\mathbf{r}}{r^{3}}. \tag{167}$$

For a consistency issue, it should be mentioned that the result (166) can be obtained directly from Eq. (144) for point-like sources. In fact, proposing a Fourier transform expression, $B\left(\mathbf{r}\right)=\left(2\pi\right)^{-3}\int\tilde{B}\left(\mathbf{p}\right)\exp\left(-i\mathbf{p}\cdot\mathbf{r}\right)d^{3}\mathbf{p}$, and replacing it in the full expression (144), we achieve:

$$\tilde{B}\left(\mathbf{p}\right)=-q\left[i\frac{(1-\kappa\cdot\mathbf{u})\kappa\times\mathbf{p}}{\mathbf{p}^{2}(1+\kappa^{2})-\left(\kappa\cdot\mathbf{p}\right)^{2}}+i\frac{\mathbf{u}\times\mathbf{p}}{\mathbf{p}^{2}}\right], \tag{168}$$

whose Fourier transform, at order κ^{2}, provides exactly the outcome of Eq. (166)**.**

B. The dipole approximation

In the case of spatially distributed sources, we can work in the dipole approximation, $\left|\mathbf{r}-\mathbf{r}'\right|^{-1}=r^{-1}+\left(\mathbf{r}\cdot\mathbf{r}'\right)/r^{3}$. With it, Eq. (160) provides:

$$A_{0}\left(\mathbf{r}\right)=\frac{1}{4\pi}\left\{c\left(\kappa\right)\frac{q}{r}-\frac{\left(\kappa\cdot\mathbf{r}\right)^{2}}{2r^{3}}q+\left[\frac{c\left(\kappa\right)}{r^{3}}-\frac{3\left(\kappa\cdot\mathbf{r}\right)^{2}}{2r^{5}}\right]\left(\mathbf{r}\cdot\mathbf{p}\right)+\frac{\left(\kappa\cdot\mathbf{r}\right)}{r^{3}}\left(\kappa\cdot\mathbf{p}\right)-\frac{1}{r^{3}}\mathbf{r}\cdot\left(\kappa\times\mathbf{m}\right)\right\}. \tag{169}$$

Here, we have used $q = \int \rho(\mathbf{r}')d^3\mathbf{r}'$ as the electric charge, $\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}'$ as the electric dipole moment, and $\mathbf{m} = \frac{1}{2}\int \mathbf{r}' \times \mathbf{j}(\mathbf{r}')d^3\mathbf{r}'$ as the magnetic dipole moment associated with the current $\mathbf{j}$ (considering a localized and divergenceless current density).

The corresponding electric field is obtained from (160) via $\mathbf{E} = -\nabla A_0$, or introducing the dipolar approximation directly in (164):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi}\left\{c(\kappa)\left[\frac{q}{r^3}\mathbf{r} - \frac{\mathbf{p}}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{p})}{r^5}\mathbf{r}\right] - \frac{3q(\kappa\cdot\mathbf{r})^2}{2r^5}\mathbf{r} - \frac{15(\kappa\cdot\mathbf{r})^2(\mathbf{r}\cdot\mathbf{p})}{2r^7}\mathbf{r} + \frac{3(\kappa\cdot\mathbf{r})(\kappa\cdot\mathbf{p})}{r^5}\mathbf{r}\right. \\ \left. + \frac{3(\kappa\cdot\mathbf{r})^2}{2r^5}\mathbf{p} + \frac{q(\kappa\cdot\mathbf{r})}{r^3}\kappa + \frac{3(\kappa\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{p})}{r^5}\kappa - \frac{(\kappa\cdot\mathbf{p})}{r^3}\kappa - \frac{3[\mathbf{r}\cdot(\kappa\times\mathbf{m})]}{r^5}\mathbf{r} + \frac{\kappa\times\mathbf{m}}{r^3}\right\}, \tag{170}$$

It exhibits a $1/r^2$ behavior. The first three terms (into brackets) at zeroth order represent the usual Coulombian behavior (the ones of the usual Maxwell theory at first order in κ), whereas the following nine terms represent the non Coulombian electric character. The last two terms are the corrections stemming from the magnetic moment of the current. All these terms could originate new interesting phenomena potentially observable both at microscopic (atomic) and macroscopic level.

In the dipole approximation, the general expression (145) is reads as

$$\mathbf{B} = \kappa\times\mathbf{E} + \frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right]. \tag{171}$$

with the two last terms coming from the usual Maxwell theory.

The magnetic field in the dipole approximation can be obtained directly from Eq.(165) or using the general expression (171) with (170), thus it amounts at:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi}\left\{\left[\frac{q}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{p})}{r^5} - \frac{3\mathbf{r}\cdot(\kappa\times\mathbf{m})}{r^5}\right]\kappa\times\mathbf{r} - \frac{\kappa\times\mathbf{p}}{r^3} + \frac{\kappa\times(\kappa\times\mathbf{m})}{r^3} + \left[\frac{3(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right]\right\}. \tag{172}$$

The non Maxwellian terms are induced by the LV background. As already noticed, the LV first order effects are induced by the charge distribution.

C. Some applications

Now, we can make some interesting applications. We begin with the case of the (magnetic) contribution to the scalar potential due to a circular ring of current (I_0) of radius R, confined in the $x-y$ plane, described by the following current density $j(\mathbf{r}') = I_0[\delta(\cos\theta')\delta(r'-R)/R]\,\widehat{\mathbf{e}}_{\phi'}$, with $\widehat{\mathbf{e}}_{\phi} = -\sin\phi'\widehat{i} + \cos\phi'\widehat{j}$. Since the geometry is cylindrically symmetric, we may choose the observation point in the $x-z$ plane ($\phi = 0$) for purposes of calculation. Replacing such current density (with $\rho = 0$) in Eq.(160), we achieve:

$$A_0(\mathbf{r}) = -\frac{I_0R}{(4\pi)}\kappa\cdot\left[-\int_0^{2\pi}\frac{\sin\phi'd\phi'}{\sqrt{a-b\cos\phi'}}\widehat{i} + \int_0^{2\pi}\frac{\cos\phi'd\phi'}{\sqrt{a-b\cos\phi'}}\widehat{j}\right], \tag{173}$$

with

$$a = (r^2+R^2)\ ,\ b = 2rR\sin\theta. \tag{174}$$

While the first integral is null, the second integral yields a non null result:

$$A_0(\mathbf{r}) = -\frac{I_0R}{(4\pi)}\frac{4}{\sqrt{a+b}}\left[\frac{a}{b}K(\alpha) - \frac{a+b}{b}E(\alpha)\right]\ \kappa\cdot\widehat{j}, \tag{175}$$

where K and E represent the complete elliptic functions of first and second kind, respectively, with $\alpha = \sqrt{2b/(a+b)}$. The result of Eq.(175) can be expressed as an expansion of the ratio $(b/2a) = [rR\sin\theta/\left(r^2+R^2\right)]$:

$$A_0(\mathbf{r}) = -\frac{1}{4\pi}\frac{mr\sin\theta}{(r^2+R^2)^{3/2}}\ \left[1 + 15\frac{R^2r^2\sin^2\theta}{8(r^2+R^2)^2} + ...\right]\kappa\cdot\widehat{j}, \tag{176}$$

where $m = \pi R^2 I_0$ is the magnitude of the dipolar moment associated with the current, given as $\mathbf{m} = m\widehat{k}$. Now, it is important to show that this result can be reconciliated with the dipolar expansion of Eq.(169). Using the identification $mr\sin\theta\widehat{j} = \mathbf{m}\times\mathbf{r}$ (valid for the configuration of evaluation), we rewrite the scalar potential as:

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\frac{1}{\left(r^2+R^2\right)^{3/2}}\left[1+15\frac{R^2r^2\sin^2\theta}{8\left(r^2+R^2\right)^2}+...\right]\mathbf{r}\cdot(\mathbf{m}\times\kappa), \tag{177}$$

which in the limit $r \gg R$ recovers the behavior predicted in Eq.(169), namely:

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\left[\frac{\mathbf{r}\cdot(\mathbf{m}\times\kappa)}{r^3}+\cdots\right]. \tag{178}$$

In this limit the associated electric field reveals a typical dipolar behavior as well, as can be verified by simple inspection.

Another example is the magnetic field generated by a ring of radium R containing charge (Q), located at plane $x-y$, whose charge density is read as $\rho(r') = (Q/2\pi R^2)\delta(r'-R)\delta(\cos\theta')$. The magnetic field generated by such charge is given by the first term of Eq.(165):

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\frac{Q}{2\pi R^2}\left[\int\frac{\kappa\times(\mathbf{r}-\mathbf{r}')\,\delta(r'-R)\delta(\cos\theta')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'\right], \tag{179}$$

which implies:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\frac{Q}{2\pi}\left[(\kappa\times\mathbf{r})\int_0^{2\pi}\frac{1}{[a-b\cos\phi']^{3/2}}d\phi' - \left(\kappa\times\widehat{i}\right)R\int_0^{2\pi}\frac{\cos\phi'}{[a-b\cos\phi']^{3/2}}d\phi'\right], \tag{180}$$

with the parameters a, b given by Eq.(174), and it was used $\mathbf{R} = R(\cos\phi'\widehat{i}+\sin\phi'\widehat{j})$. The corresponding solution is:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\frac{Q}{2\pi}\frac{\mathbf{4}}{\sqrt{a+b}(a-b)}\left\{(\kappa\times\mathbf{r})\,E(\alpha) - R\left(\kappa\times\widehat{i}\right)\left[\frac{(a-b)}{b}K(\alpha)-\frac{a}{b}E(\alpha)\right]\right\}. \tag{181}$$

An expansion in terms of the ratio $[rR\sin\theta/\left(r^2+R^2\right)]$ can be performed, implying:

$$\mathbf{B}(\mathbf{r}) = \frac{Q}{2\pi^2}\frac{1}{\left(r^2+R^2\right)^{3/2}}\left\{(\kappa\times\mathbf{r})\,\frac{\pi}{2}\left[1+\frac{15}{4}\frac{(rR\sin\theta)^2}{\left(r^2+R^2\right)^2}\right]+\right. \tag{182}$$

$$\left.-R\left(\kappa\times\widehat{i}\right)\frac{\pi}{2}\left[-\frac{3}{2}\frac{(rR\sin\theta)}{(r^2+R^2)}-\frac{105}{16}\frac{(rR\sin\theta)^3}{(r^2+R^2)^3}\right]\right\}.$$

In the limit $r \gg R$, we have:

$$\mathbf{B}(\mathbf{r}) = \frac{Q}{4\pi}\frac{(\kappa\times\mathbf{r})}{r^3}, \tag{183}$$

This outcome coincides with the one of the dipolar expansion (172), once $\mathbf{p} = \int\mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}' = 0$ for the charge distribution here considered.

Finally, we shall employ the expression for the magnetic field generated by a static charge [see Eq. (167)] to obtain an upper bound on the Lorentz-violating vector. First, we consider the magnetic field created by the electric charge confined in an atomic nucleus (Z).

Another case that can provide a better bound consists of a conducting sphere of radium equal to R, and endowed with a large electric charge Q. Once the magnetic field from a point-like charge goes as r^{-2} - according to Eq. (167), a charged sphere should engender a magnetic field proportional to $Q\kappa/r^2$. A $R = 0.9$ m sphere charged with $1C$ (maintained in vacuum) generates a magnetic field at $r = 1$ m equal to $|\mathbf{B}(\mathbf{r})| = 2\times10^4\kappa\ (\mathrm{eV})^2$. Remembering that SQUID devices are able to detect magnetic field variations as small as $10^{-10}G$, an upper limit as stringent as $\kappa < 10^{-16}$ can be set up. This is a more restrictive bound when compared with some other results stemming from birefringence [12, 13].

Acknowledgments

The authors are grateful for FAPEMA, CAPES and CNPq (Brazilian research agencies) for invaluable financial support. The authors also thank Jose A. Helayel-Neto for relevant comments on this work.

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