Mean-field theories and quasi-stationary simulations of epidemic models on complex networks

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Complex Systems Foundations and Applications
Rio de Janeiro, October 29th - November 1st

Outline

- Introduction
- Pair quenched mean-field theory
- Quasi-stationary simulations
- Nature of epidemics on complex networks
- Prospects

Viçosa - Minas Gerais

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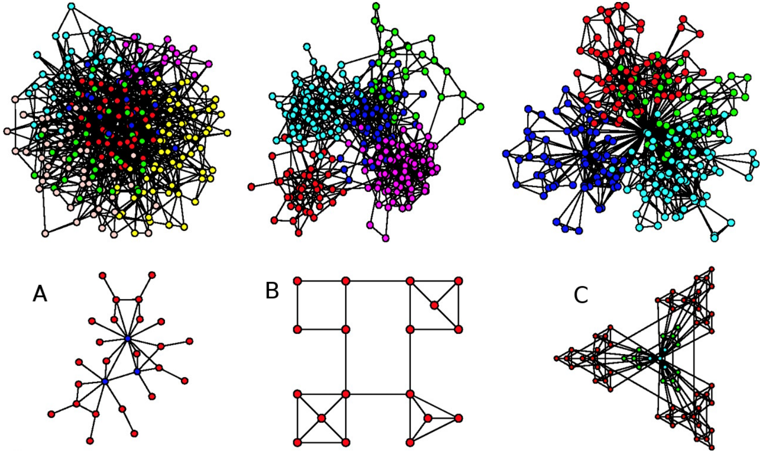
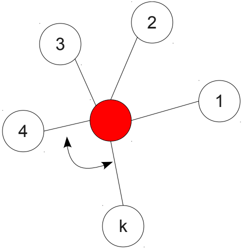
How do we get sick?

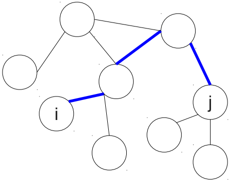
Network



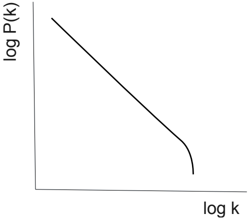
Ravasz et al., Science **297** 1551 (2002).

Basic network properties





$l_{ij} = 3$



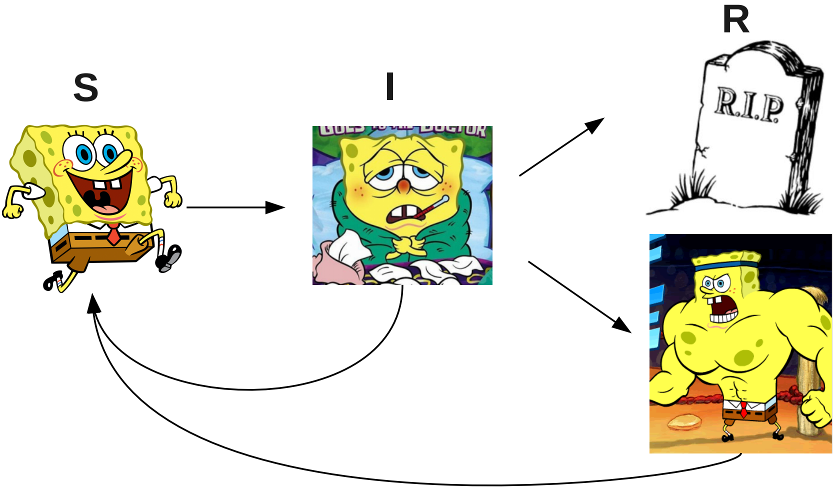
Small-world (SW): $\langle l \rangle \sim \log N$

Scale-free (SF): $P(k) \sim k^{-\gamma}$

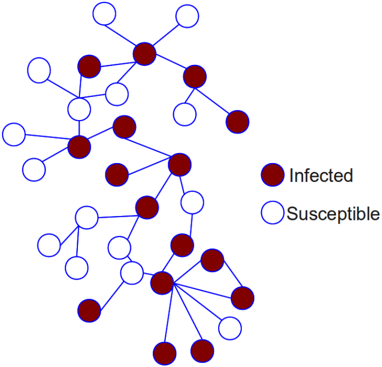
$\langle k^2 \rangle \to \infty$ for $2 < \gamma < 3$ - Scale-free

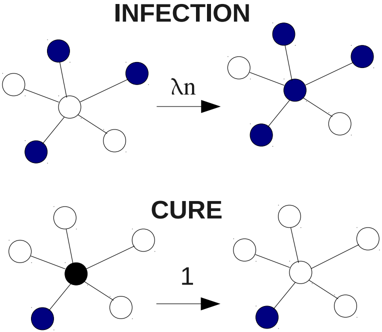
Epidemic dynamics

States: Susceptible (S), Infected (I), Removed (R) and so forth ...

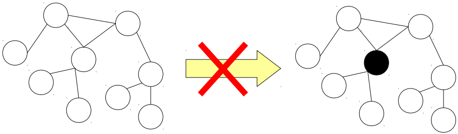


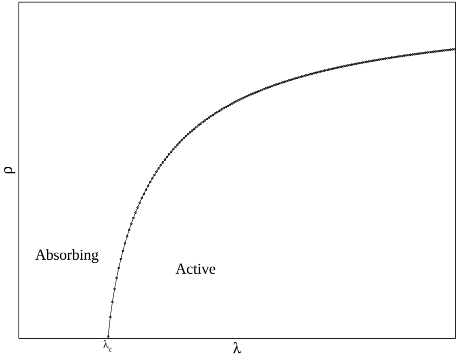
The susceptible-infected-susceptible (SIS) model





Absorbing-state and phase transitions

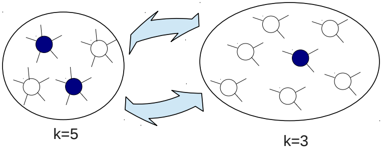




Competing mean-field theories

Heterogeneous mean-field theory (HMF)

[Pastor-Satorras and Vepignani, PRL **86** 3200 (2001)]



$$\frac{d\rho_k}{dt} = -\rho_k + \lambda k(1-\rho_k)\sum_{k'} P(k'|k)\rho'_k \quad \Rightarrow \quad \lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}$$

Quenched mean-field theory (QMF)

[Chakrabarti at al., ACM Trans. Inf. Syst. Secur. **10** 1 (2008)]

$$\frac{d\rho_i}{dt} = -\rho_i + \lambda(1-\rho_i)\sum_{j} A_{ij}\rho_j \quad \Rightarrow \quad \lambda_c = \frac{1}{\Lambda_m},$$

where Λ_m is the largest eigenvalue of the adjacency matrix $A_{i,j}$

Thresholds for $P(k) \sim k^{-\gamma}$

HMF

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle} \rightarrow \begin{cases} const. > 0 & \gamma > 3 \\ 0 & \gamma \leq 3 \end{cases}$$

Conclusion: Scale-free networks ($\gamma < 3$) do not have a threshold in the thermodynamical limit ($N \rightarrow \infty$) while networks with finite $\langle k^2 \rangle$ do.

QMF [Castallano and Pastor-Satorrras, PRL **105**, 218701 (2010)]

$$\lambda_c = \frac{1}{\Lambda_m} \simeq \begin{cases} 1/\sqrt{k_{max}} & \gamma > 2.5 \\ \langle k \rangle / \langle k^2 \rangle & 2 < \gamma \leq 2.5 \end{cases}$$

Conclusion: does not exist a finite threshold for SIS in the limit $N \rightarrow \infty$ for any network with diverging cutoff.

Quenched pair-approximation

*Mata and Ferreira, EPL **103** 48003 (2013))*

- Simple QMF theory assumes that the probability that a vertex is occupied or empty does not depend on the states of its neighbors.

$$\phi_{i,j} \approx (1 - \rho_i)\rho_j$$

- Pair approximation is the simplest MF theory that includes dynamical correlations.
- Dynamical equation for a pair of vertices are investigated.

Notation and normalization conditions

- Infected vertex $\to 1$
- Heath vertex $\to 0$

- $\rho_i = [1_i]$
- $[0_i] = 1 - \rho_i$
- $\psi_{ij} = [1_i, 1_j]$
- $\omega_{ij} = [0_i, 0_j]$
- $\phi_{ij} = [0_i, 1_j]$
- $\bar{\phi}_{ij} = [1_i, 0_j]$

- $\psi_{ij} = \psi_{ji}$
- $\omega_{ij} = \omega_{ji}$
- $\phi_{ij} = \bar{\phi}_{ji}$
- $\psi_{ij} + \phi_{ij} = \rho_j$
- $\psi_{ij} + \bar{\phi}_{ij} = \rho_i$
- $\omega_{ij} + \phi_{ij} = 1 - \rho_i$
- $\omega_{ij} + \bar{\phi}_{ij} = 1 - \rho_j$

Dynamical equations

$$\frac{d\rho_i}{dt} = -\rho_i + \lambda \sum_j \phi_{ij} A_{ij}$$

$$\frac{d\phi_{ij}}{dt} = -\phi_{ij} - \lambda\phi_{ij} + \psi_{ij} + \lambda \sum_{\substack{l \in \mathcal{N}(j) \\ l \neq i}} [0_i 0_j 1_l] - \lambda \sum_{\substack{l \in \mathcal{N}(i) \\ l \neq j}} [1_l 0_i 1_j]$$

Pair approximation [ben Avraham and Kohler, PRA **45**, 8358 (1992)]

$$[A_i, B_j, C_l] \approx \frac{[A_i, B_j][B_j, C_l]}{[B_j]}$$

$$\frac{d\phi_{ij}}{dt} = -(1+\lambda)\phi_{ij} + \psi_{ij} + \lambda \sum_l \frac{\omega_{ij}\phi_{jl}}{1-\rho_j}(A_{jl} - \delta_{il}) - \lambda \sum_l \frac{\phi_{ij}\bar{\phi}_{li}}{1-\rho_i}(A_{il} - \delta_{lj})$$

Thresholds

- **Performing a linear stability** analysis around the fixed point $\rho_i = \phi_{ij} = \psi_{ij} = 0$ and using a **quasi-static approximation** for long times we find

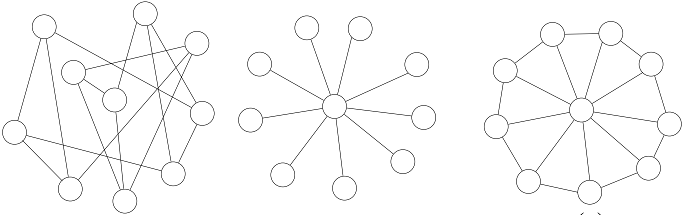
$$\frac{d\rho_i}{dt} = \sum_j L_{ij}\rho_j$$

with the Jacobian

$$L_{ij} = -\left(1 + \frac{\lambda^2 k_i}{2\lambda + 2}\right)\delta_{ij} + \frac{\lambda(2+\lambda)}{2\lambda + 2}A_{ij}.$$

- The critical point is obtained when absorbing state $\rho_i = 0$ loses stability or, equivalently, when the largest eigenvalue of L_{ij} vanishes.

Analytical results for simple networks



(a) (b) (c)

	RRN	Star	Wheel
QMF	$\lambda_c = \frac{1}{m}$	$\lambda_c = \sqrt{\frac{1}{N}}$	$\lambda_c \simeq \sqrt{\frac{1}{N}}$
PQMF	$\lambda_c = \frac{1}{m-1}$	$\lambda_c \simeq \sqrt{\frac{2}{N} - \frac{1}{N}}$	$\lambda_c \simeq \sqrt{\frac{2}{N} - \frac{3}{N}}$

Absorbing states and finite sizes

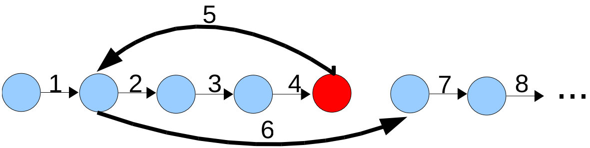
The absorbing configuration is the **unique actual stationary state** for finite size systems.

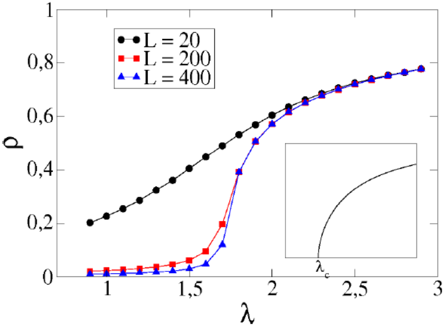
Quasi-stationary approach: the averages are restricted to an ensemble of active configurations.

$$\bar{P}(\sigma) = \frac{P(\sigma, t)}{P_s(t)}, \quad t \to \infty$$

Quasi-stationary simulations

Oliveira and Dickman, PRE **71** 016129 (2005)





QS simulations in complex networks

- Finite-size effect are strongly enhanced by the small-world property.
- QS state is a suitable framework to handle absorbing states on complex networks:

  Contact process
  Ferreira, Ferreira, Pastor-Satorras PRE **83**, 066113 (2011)
  Ferreira, Ferreira, Castellano, Pastor-Satorras PRE **84**, 066102 (2011)
  Ferreira and Ferreira EPJB (2013) [arXiv:1307.6186 (2013)]

  Epidemic spreading
  Ferreira, Castellano, Pastor-Satorras *et al*. PRE **86** 041125 (2012)
  Mata and Ferreira EPL **103** 48003 (2013)

  Infinitely many absorbing states
  Sander, Ferreira, Pastor-Satorras *et al*. PRE **87**, 022820 (2013)

  Metapopulation dynamics
  Mata,Ferreira, Pastor-Satorras PRE **88** 042820 (2013).

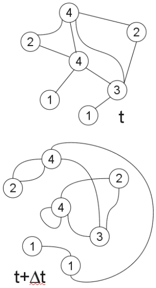
Numerical determination of the thresholds

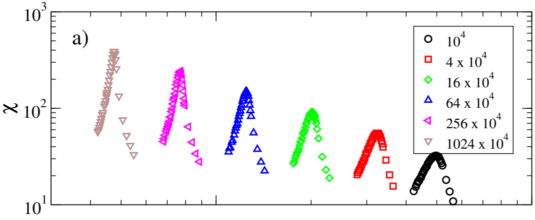
*Ferreira, Castellano, Pastor-Satorras, PRE **86** 041125 (2012).*

Modified susceptibility:

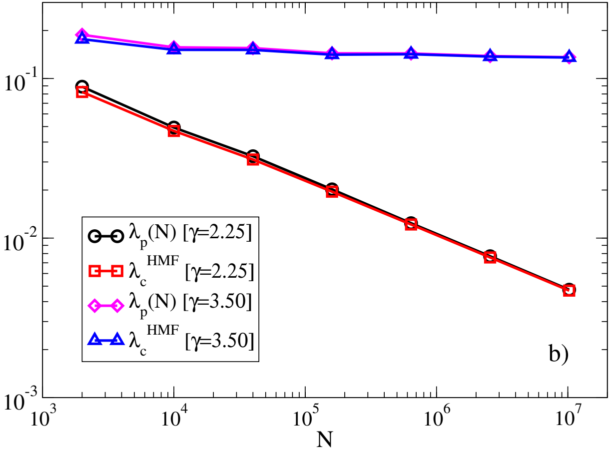
$$\chi = N\frac{\langle \rho^2 \rangle - \langle \rho \rangle^2}{\langle \rho \rangle} \sim N^{\vartheta}, \qquad \vartheta > 0$$

Validating the method: annealed networks

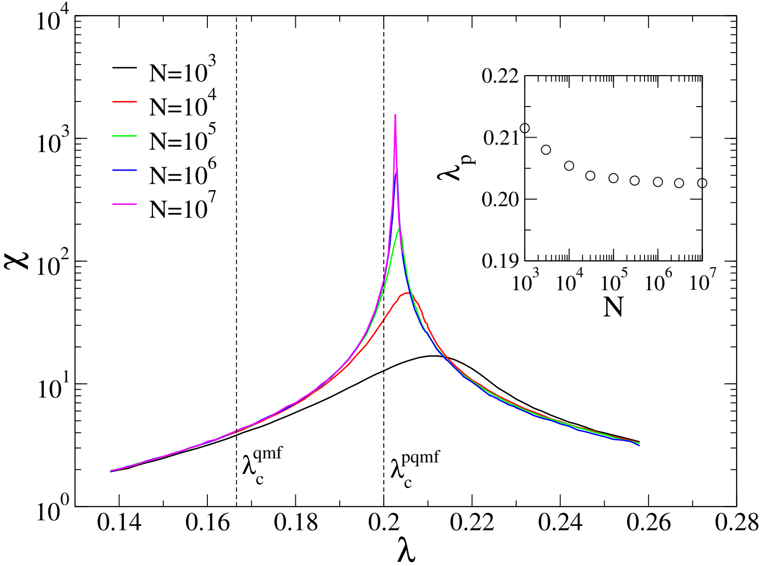




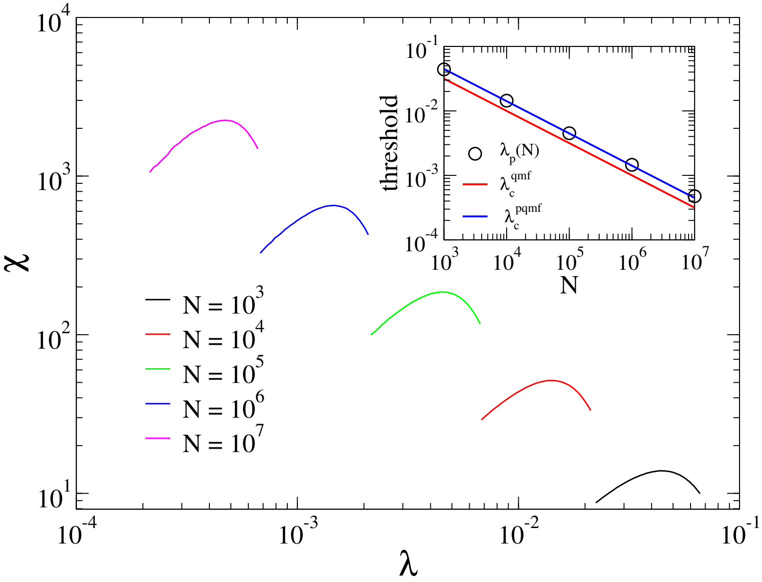
Thresholds for annealed networks



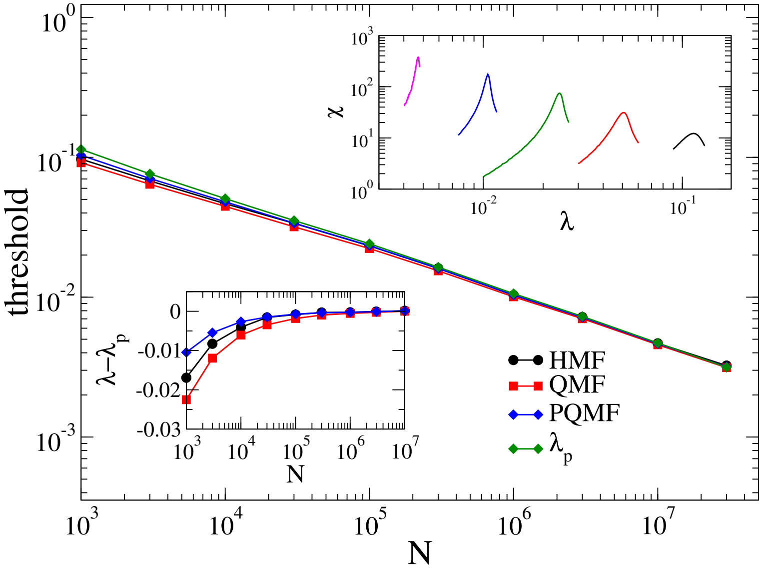
Simple homogeneous case: RRN



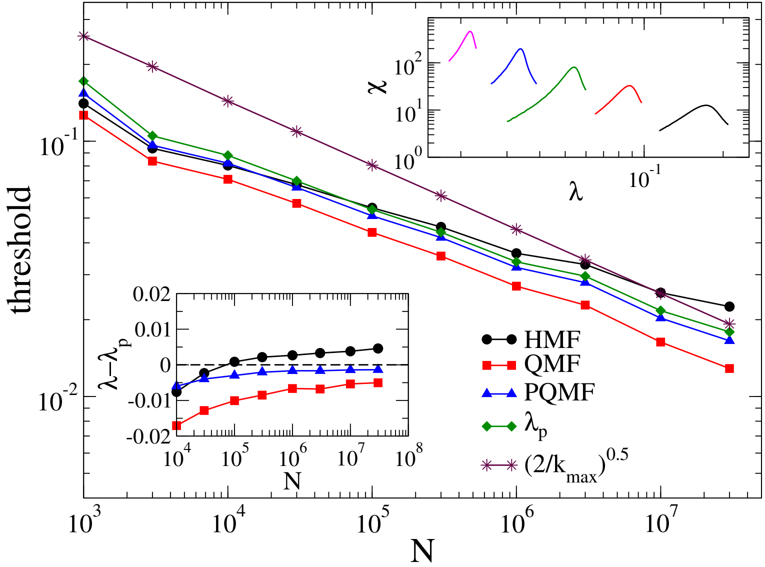
Star/Wheel graphs



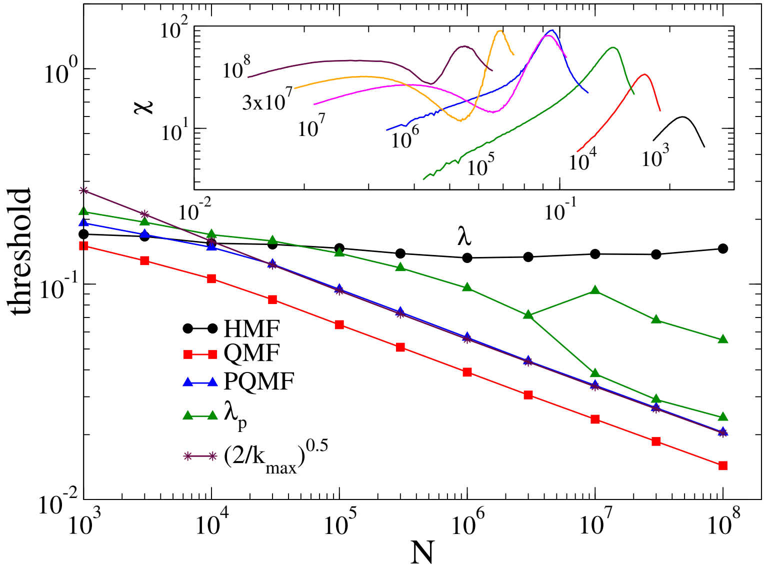
Quenched networks: $\gamma < 2.5$



Quenched networks: $2.5 < \gamma < 3$



Quenched networks: $\gamma > 3$ *and* $k_{max} = \langle k_{max} \rangle$



The nature of the epidemic thresholds

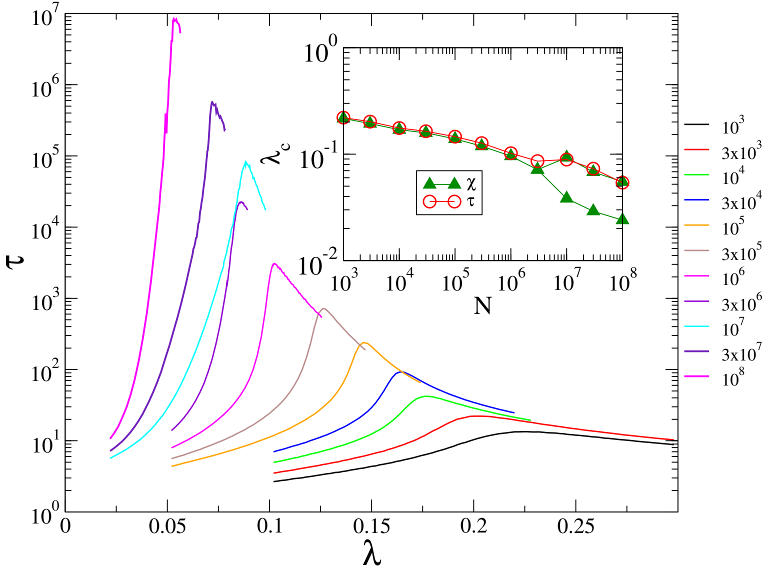
- The peak at low infection rate reproduced by the PQMF theory is associated to a localized epidemics.

  Goltsev et al., PRL **109** 128702 (2012).

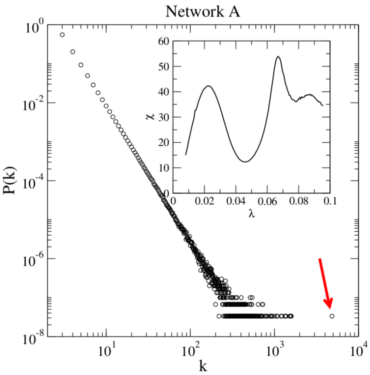
- The second peak if associated to an endemic phase where all vertices are infected at some time.

  Boguña, Castellano, Pastor-Satorras PRL **111**, 068701 (2013).

Lifespan method *[B,C,P-S, op cit.]*



Prospects



Acknowledgments

- Collaborators
- Students
- Applied statistical physics group
- GISC
- Supporting agencies

CNPq
Conselho Nacional de Desenvolvimento
Científico e Tecnológico

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