From Multiplicity to Entropy

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SECTION FOR SCIENCE OF COMPLEX SYSTEMS

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When I first visited CBPF...

What is the probability of finding a **probability distribution function**?

In other words:

What is the probability to find a **histogram**?

What is the probability to find a histogram?

Sequence $x = (x_1, x_2, \cdots, x_N)$ with $x_n \in \{1, 2\}$ (coin tosses)

$$1\,q_1^0 q_2^0$$

$$1\,q_1^1 q_2^0 \qquad 1\,q_1^0 q_2^1$$

$$1\,q_1^2 q_2^0 \qquad 2\,q_1^1 q_2^1 \qquad 1\,q_1^0 q_2^2$$

$$1\,q_1^3 q_2^0 \qquad 3\,q_1^2 q_2^1 \qquad 3\,q_1^1 q_2^2 \qquad 1\,q_1^1 q_2^3$$

$$1\,q_1^4 q_2^0 \qquad 4\,q_1^3 q_2^1 \qquad 6\,q_1^2 q_2^2 \qquad 4\,q_1^1 q_2^3 \qquad 1\,q_1^0 q_2^4$$

- q_1 and q_2 ... probabilities to toss 1 (heads) or 2 (tails) in a trial x_n.
- $k = (k_1, k_2)$ ... is the **histogram** of x;
  i.e. k_1 (k_2) ... number of times 1 (2) appears in x.

What is the probability to find a histogram?
Binomial

The probability to find the histogram k is:

$$P(k; q) = \frac{N!}{k_1!k_2!} q_1^{k_1} q_2^{k_2}$$

- N ... length of sequence x
- 2 states
- $x_n \in \{1, 2\}$

What is the probability to find a histogram?

Multinomial

The probability to find the histogram k is:

$$P(k;q) = \frac{N!}{k_1!k_2!\cdots k_W!} q_1^{k_1} q_2^{k_2} \cdots q_W^{k_W}$$

- N ... length of sequence x
- W states
- $x_n \in \{1, 2, \cdots, W\}$

What is the probability to find a histogram?

Multinomial

The probability to find the histogram k is:

$$P(k;q) = \underbrace{\frac{N!}{k_1!k_2!\cdots k_W!}}_{\text{multiplicity } M(k)} \underbrace{q_1^{k_1} q_2^{k_2} \cdots q_W^{k_W}}_{\text{probability } G(k;q)}$$

What is the probability to find a histogram?
Multinomial

We make **two** observations ...

Observation 1: The factorization property

Factorization

$$P(k;q) = M(k) \ \ G(k;q)$$

- Multiplicity $M(k)$ does **not** depend on q!
- **All** dependencies on q are captured by $G(k;q)$

Observation 1: The factorization property

Take the log on both sides and scale with $1/N$...

$$
\begin{aligned}
P(k;q) &= {\color{red}M(k)} \qquad {\color{blue}G(k;q)} \\
\frac{1}{N}\log P(k;q) &= {\color{red}\frac{1}{N}\log M(k)} + {\color{blue}\frac{1}{N}\log G(k;q)}
\end{aligned}
$$

Observation 2: From factorization property to MEP

$$
\begin{aligned}
\frac{1}{N}\log M(k) &= \frac{1}{N}\log\left(\frac{N!}{k_1!k_2!\cdots k_W!}\right) \\
&\sim \frac{1}{N}\log\left(\frac{N^N}{k_1^{k_1}k_2^{k_2}\cdots k_W^{k_W}}\right) \quad \cdots \quad \text{Stirling} \\
&= \log N - \frac{1}{N}\sum_{i=1}^{W} k_i \log k_i \\
&= -\sum_{i=1}^{W} \frac{k_i}{N}\log\frac{k_i}{N} \\
&= -\sum_{i=1}^{W} p_i \log p_i \quad \cdots \quad \text{frequencies } p_i = \frac{k_i}{N} \\
&= S_{\text{Shannon}}[p]
\end{aligned}
$$

Observation 2: From factorization property to MEP

$$
\begin{aligned}
\frac{1}{N}\log G(k;q) &= \frac{1}{N}\log\left(q_1^{k_1} q_2^{k_2}\cdots q_W^{k_W}\right)\\
&= \frac{1}{N}\sum_{i=1}^{W} k_i \log q_i\\
&= \sum_{i=1}^{W} p_i \log q_i\\
&= -S_{\mathrm{cross}}[p|q]
\end{aligned}
$$

Observation 2: From factorization property to MEP

The identified terms:

$$\frac{1}{N}\log P(k;q) = \frac{1}{N}\log M(k) + \frac{1}{N}\log G(k;q)$$

$$= -\sum_{i=1}^{W} p_i \log(p_i) + \sum_{i=1}^{W} p_i \log(q_i)$$

The Maximum Entropy Principle

Variable transformation ... $q_i = \exp(-\alpha - \beta\varepsilon_i)$

$$\frac{1}{N}\log P(k;q) = \underbrace{-\sum_{i=1}^{W} p_i \log(p_i)}_{\text{constraint independent}} - \underbrace{\alpha\sum_{i=1}^{W} p_i - \beta\sum_{i=1}^{W} p_i\varepsilon_i}_{constraints}$$

What is the most likely histogram k?

$$0 = \frac{\partial}{\partial k_i}\log P(k;q)$$

for fixed $N \Rightarrow$

$$0 = \frac{\partial}{\partial p_i}\left(-\sum_{i=1}^{W} p_i \log(p_i) - \alpha\sum_{i=1}^{W} p_i - \beta\sum_{i=1}^{W} p_i\varepsilon_i\right)$$

Principle

If a factorization

$$P(k;q) = M(k)G(k;q)$$

exists then there exists a MEP with

$$\frac{1}{\phi(N)}\log P(k;q) = \underbrace{\frac{1}{\phi(N)}\log M(k)}_{\text{generalized entropy}} + \underbrace{\frac{1}{\phi(N)}\log G(k;q)}_{\textit{constraints}}$$

When does factorization exist?

When does factorization exist?

There is little one can say in general but ...

⇒ focus on processes **out of equilibrium** that are **aging**, **non-Markovian**, **path-dependent**

- process remembers its entire past in terms of the histogram k
- Consider growing sequences $x = (x_1, \cdots, x_N) \to x' = (x_1, \cdots, x_N, x_{N+1})$
- $\hat{p}(x_{N+1}|k;q)$ ... conditional probability to find x_{N+1} given histogram k of x

IF process follows $\hat{p}(x_{N+1}|k;q)$ and

$$\frac{d}{dN} k_i(N) = \hat{p}(i|k;q)$$

has stable solutions

$\Rightarrow$ There exists a **recursive definition** of how $M(k)$ (similar to the multinomial coefficients)

$\Rightarrow$ Asymptotically $M(k)$ is given by a **deformed multinomial** ...

Deformed factorials and deformed multinomials

$$\underbrace{M_{u,T}(k) \equiv \frac{N\,!_u}{\prod_i NT\left(\frac{k_i}{N}\right)!_u}}_{\text{Deformed multinomial}} \quad , \quad \underbrace{N\,!_u \equiv \prod_{n=1}^{N} u(n)}_{\text{Deformed factorial}}$$

- T monotonic with $T(0) = 0$, $T(1) = 1$
- deformed multinomials lead to trace-form entropies

$$S[p] = \frac{1}{\phi(N)} \log M_{u,T}(k) = \ldots = -\sum_{i=1}^{W} \int_0^{p_i} dz\, \Lambda(z)$$

$$\Lambda(z) = \frac{1}{a}\left(T'(z)\left[\frac{N}{\phi(N)} \log u(NT(z))\right] - \log \lambda\right)$$

General solution for deformed multinomials

Asymptotic identity for constants a and λ

$$\Lambda(z) = \frac{1}{a}\left(T'(z)\left[\frac{N}{\phi(N)}\log u(NT(z))\right] - \log\lambda\right)$$

$\Lambda(z)$ does not depend on $N \rightarrow$ separation of variables with **separation constant** $\nu \rightarrow$

$$\begin{aligned}
\Lambda(z) &= \frac{T'(z)T(z)^{\nu} - T'(1)}{T''(1) + \nu T'(1)^2} \\
u(N) &= \frac{\lambda^{(N^{\nu})} - 1}{\lambda - 1} \qquad \text{...q} - \text{shifted factorials } (\nu = 1) \\
\phi(N) &= N^{1+\nu}
\end{aligned}$$

Happy Birthday

The MEP with Tsallis entropy derived for a class of aging processes

for the simplest (and probably the generic) case $T(z) = z$ ...

$$\Lambda(z) = \frac{T'(z)T(z)^{\nu} - T'(1)}{T''(1) + \nu T'(1)^2} \quad = \quad \frac{z^{\nu} - 1}{\nu}$$

and

$$S[p] = \sum_{i=1}^{W} \int_0^{p_i} dz \Lambda(z) = K \frac{1 - \sum_{i=1}^{W} p_i^Q}{Q - 1}$$

where $Q \equiv 1 + \nu$ and K a constant

Remark on Extensive vs MEP

- Are the extensive (c, d)-entropies of a process equivalent to the (c, d)-entropy of the MEP ?

No they are NOT – but they are related

$$d_{\mathrm{extensive}} = \frac{1}{c_{\mathrm{MEP}}}$$

R. Hanel & S. Thurner, Generalized (c,d)-entropy and aging random walks, arXiv:1310.5959

Example

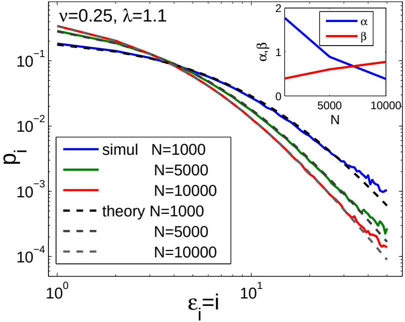
- Consider for example the conditional probability

$$\hat{p}(x_{N+1} = i|k; q) = \frac{1}{Z(k)} q_i \prod_{j=i+1}^{W} \lambda^{-(k_j^\nu)}$$

- plug $p(i|k; q)$ into the recursive definition of $M(k)$ ...
- ... [do some not so easy math] ...
- Use $S[p] = \frac{1}{\phi(N)} \log M(k)$ and solve the MEP to get theoretical predictions for the p_i for fixed N
- generate sequences x of length N on the computer (with $q_i = 1/W$) and produce the histograms k_i of the sequences (full lines)

Example

$$\hat{p}(i|k;q) = \frac{1}{Z(k)} q_i \prod_{j=i+1}^{W} \lambda^{-(k_j^{\nu})}$$



(i) **Simulation**: generate sequences x on the computer with $q_i = 1/W$ and $W = 50$;

(ii) **MEP** prediction: $p_i = \exp_q(-\alpha - \beta\varepsilon_i)$ with $q = 1 + \nu$ and $\varepsilon_i = i$.

FIN

Happy birthday, Constantino!