Tsallis entropy composition: geometric aspects

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1. Tsallis entropy

Tsallis entropy (set $k = 1$)

$$S_q[\rho] \ = \ \frac{1}{q-1}\left\{1 - \int_\Omega [\rho(x)]^q \ dvol_\Omega\right\}$$

Non-extensive (entropic) parameter: $q \in \mathbb{R}$

$$\lim_{q \to 1} S_q[\rho] = S_{BGS}[\rho]$$

For which systems is S_q applicable? (Largely) a topic of this conference.
Composition: for two independent systems $\rho_{1\cup 2} = \rho_1 \rho_2$

$$S_q[\rho_{1\cup 2}] \ = \ S_q[\rho_1] + S_q[\rho_2] + (1-q) S_q[\rho_1] S_q[\rho_2]$$

Define a generalized addition: $x \oplus_q y \ = \ x + y + (1-q)xy$

Differences S_q vs S_{BGS} : differences between $x + y$ and $x \oplus_q y$

2. BGS vs Tsallis: comparison through metrics

Define a generalised multiplication $\otimes_q$ associative with respect to $\oplus_q$

Form the field $(\mathbb{R}_q, \oplus_q, \otimes_q)$

Field isomorphism $\tau_q : \mathbb{R} \to \mathbb{R}_q$, for $q \in [0, 1)$, given by

$$\tau_q(x) = \frac{(2-q)^x - 1}{1-q}$$

Differences between S_q and S_{BGS}: differences between $\mathbb{R}$ and $\mathbb{R}_q$

Differences via metrics: form $\mathbb{R} \times \mathbb{R}_q$, explore its Riemannian geometry.

Then $\mathbb{R}^2$ gets a Riemannian metric with constant (sectional) curvature

$$K = -\{\log(2-q)\}^2$$

So, S_q is the "hyperbolic analogue" of the "Euclidean" S_{BGS}

Obvious: $\oplus_q$ is essentially the exponential function for $|x|, |y| \gg 1$

3. Hyperbolic metric implications

Assumption: this hyperbolic metric is effective in config./phase space.

This is quite strong and non-trivial: well-fit for tensorization (Gaussians?)

Then: the largest Lyapunov exponent of the underlying dynamics is zero.

Justification of using escort distributions ρ^q instead of the "usual" ones ρ.

"Nature" of q starts showing as a "dimension" related to volume/measure.

Tsallis entropy and polynomial rate of phase space volume growth (?)

Metric induced by S_q maps long-range to short-range interactions
(in "real" as opposed to configuration/phase space)?

4. Hyperbolization, obstructions and volumes

Can we put such an effective metric on configuration/phase space?

Probably yes, but unlikely to be sufficiently smooth (for Physics?)

However, look at a "Smooth Hyperbolization" paper (Ontaneda, 2011).

We do not really need a modified/effective metric in config/phase space.

Statistical mechanics (Gibbs) considers a dynamical system and ones "close" to it: same evolution law but slightly different initial conditions.

Then suffices to consider the behaviour of volumes/measures in config/phase space and not necessarily that of metrics (ergodic theory).

Volume evolution is determined by the Ricci curvature (crucial in GR).

Lohkamp's h-principle: no topological obstructions exist to metrics of negative Ricci curvature for n-manifolds with $n \geq 3$.

5. The (N-) Bakry-Emery-Ricci tensor

Thererore: $\oplus_q$ induces a deformation of the config/phase space volume ("ergodicity breaking").

Deformation given by $e^{-f} dvol_M$, for $f : M \to \mathbb{R}$. Manifold with density.

Is there a local quantity that controls the evolution of such deformations?

Surprisingly, there is: the generalised (N-)Bakry-Emery-Ricci tensor $(R_N)_{ij}$: (Bakry-Emery, Qian, von Renesse-Sturm, Sturm, Lott, Lott-Villani etc)

$$(R_N)_{ij} = \begin{cases} R_{ij} + (Hess\ f)_{ij} & , \text{ if } N = \infty \\ R_{ij} + (Hess\ f)_{ij} - \frac{1}{N-n}(df \otimes df)_{ij} & , \text{ if } n < N < \infty \\ R_{ij} + (Hess\ f)_{ij} - \infty \cdot (df \otimes df)_{ij} & , \text{ if } N = n \\ -\infty & , \text{ if } N < n \end{cases}$$

R_{ij}: Ricci tensor, $(Hess\ f)_{ij} = \partial_i \partial_j f - \Gamma_{ij}{}^k \partial_k f$, $N \in [0, \infty]$, $\infty \cdot 0 = 0$

6. Isoperimetric interpretation of q

What is N ? The "dimension" of the deformed measure $e^{-f} dvol_M$

Why? Generalized Bishop-Gromov volume comparison theorem:
Let $Ric_N = (R_N)_{ij} X^i X^j$ for $\{X\}$ orthonormal w.r.t. g in M.
If $Ric_N \geq (N-1)H$, then $\frac{Vol_f(B_p(r))}{Vol_H^N(r)}$ is non-increasing in $r > 0$.
where $p \in M$ and $Vol_f(B_p(R)) = \int_{B_p(R)} e^{-f} dvol_M$

Relation between N and q: $N = \frac{1}{1-q} \iff q = \frac{N-1}{N}$

For $N \to \infty$ we get $q = 1$, S_{BGS}. So S_q: a "finite-dim" version of S_{BGS}

Alternatively: $\oplus_q$ maps the linear into a power-law isoperimetric function.
OR: an exponential volume growth rate mapped to a doubling measure.

Interpretation of N: isoperimetric dimension of $e^{-f} dvol_M$

Motivated by: the convexity of entropy functionals in Wassestein space
(cf. Bakry-Emery, Qian, Brenier, McCann, Otto, Sturm, Lott-Villani etc)

7. Heat baths, scalings and all that ...

Consider an auxiliary manifold (F, h) with $dim\ F = N - n$. Then, for $\phi \in C^\infty(M)$ (Fukaya, Cheeger-Colding)

$$\lim_{t\to 0} \Big(M \times F,\ g \oplus (te^{-\frac{\phi}{N-n}})^2 h \Big) \ = \ \Big(M, g, e^{-f} dvol \Big)$$

in the measured Gromov-Hausdorff sense.

Geometry of heat bath? Embed M into $\tilde{M}$ with $dim\ \tilde{M} > dim\ M$.
Easiest: trivial fiber bundle $\tilde{M} = M \times F$, with (warped) product metric.
Then (Lott): $Ric(\bar{X}, \bar{X}) \ = \ Ric_N(X, X)$ for $\bar{X} = \pi_* X$ with $X \in TM$
$N \to \infty \Leftrightarrow q = 1$: S_q "finite version" of S_{BGS}. $N - n$: dim. of heat bath.

Adib et al: let heat bath Hamiltonian $H(\lambda^{\frac{1}{a_1}} r_1, \lambda^{\frac{1}{a_2}} r_2, \ldots) = \lambda H(r_1, r_2, \ldots)$

$$\frac{1}{q-1} \ = \ \sum_{i=1,2,\ldots} \frac{n_i}{a_i} - 1$$

with configuration/phase space volumes $d^{n_1} r_1,\ d^{n_2} r_2, \ldots$

8. Current and near future work...

- Tsallis entropy composition, (Gromov) hyperbolic spaces and rigidity (boundary determines bulk: hyperbolicity, Chern-Simons, AdS/CFT).
- "Dualities" of q as (quasi-) conformal / Möbius transformations?
- Long-range interactions, A_p weights and doubling measures.
- Generalization of ergodicity, Birkhoff's thm, asymptotic invariants...
- Highly entropic objects (black holes) detecting S_q vs S_{BGS}.
- Tsallis entropy in QFT and QG ? (also in black hole stat. mech.)
- Perel'man's reduced volume/entropy with S_q instead of S_{BGS} ?
- Concrete models of many d.o.f. that are analytically tractable?

HAPPY BIRTHDAY ♡ ΧΡΟΝΙΑ ΠΟΛΛΑ

OBRIGADO THANK YOU