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Restricted random walk model as a new testing ground for the applicability of *q*-statistics

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Reference:

U. Tirnakli, H.J. Jensen and C. Tsallis,
Restricted random walk model as a new testing ground for the applicability of q-statistics
EPL **96**, 40008 (2011) & arXiv:1105.6184.

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Complexity & Networks

Constantino Tsallis - congratulations! ! !

- A privilege to be here amongst all these brilliant scientists celebrating a bold and innovative friend

- Whatever "q-statistics" turns out to be about, it has already proved to be a source of great inspiration and motivated a concerted global effort to try to figure out "what is going on" in strongly correlated statistical mechanics.

A great teacher

Outline:

1. Motivation
2. Model
3. The q-Gaussian and its nature
4. Generality issues
5. Summary
6. Conclusion

The q-Gaussian

$$P(y) = \begin{cases} P(0)\left[1-\beta(1-q)y^2\right]^{\frac{1}{1-q}} & \text{for } \beta(1-q)y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

wide spread occurrence.

Frequently found to represent the probability density function of (appropriately scaled and shifted) sums.

$$y = \sum_{t=1}^{T} x_t$$

I.e. breaking of the CLT.

Question: when does this occur ? ?

why q-expo or q-Gaussians ? ?

excellent fits or more ? ?

For example

Tirnakli, Tsallis and Beck studied the logistic map at the edge of chaos

$$x_{n+1} = 1 - a x_n^2$$

- Focused on sums of the iterates

$$y_N = \sum_{n=1}^{N} (x_n - \langle x \rangle)$$

- Found that the q-Gaussian fits well the distribution of the sum near onset of chaos

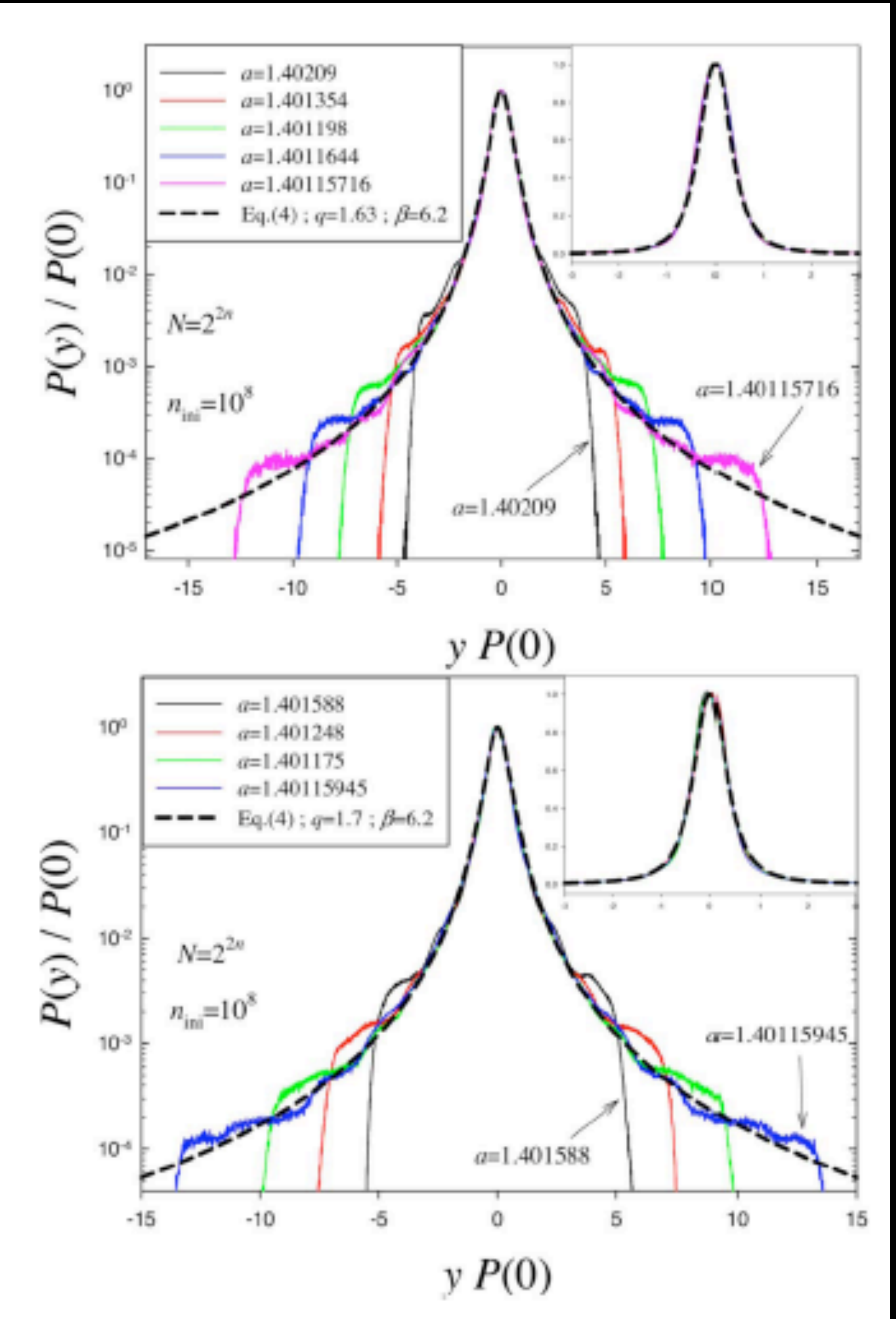


FIG. 2. (Color online) Data collapse of probability density functions for the cases $N=2^{2n}$, where $2n$ is (a) odd and (b) even. As n increases, a good fit using a q-Gaussian with (a) $q=1.68$ and $\beta=6.2$ and (b) $q=1.70$ and $\beta=6.2$ is obtained for regions of increasing size. Inset: the linear-linear plot of the data for a better visualization of the central part.

Reference:
Tirnakli, Tsallis and Beck,
PRE **79** 056209 (2009)

Scaling plots

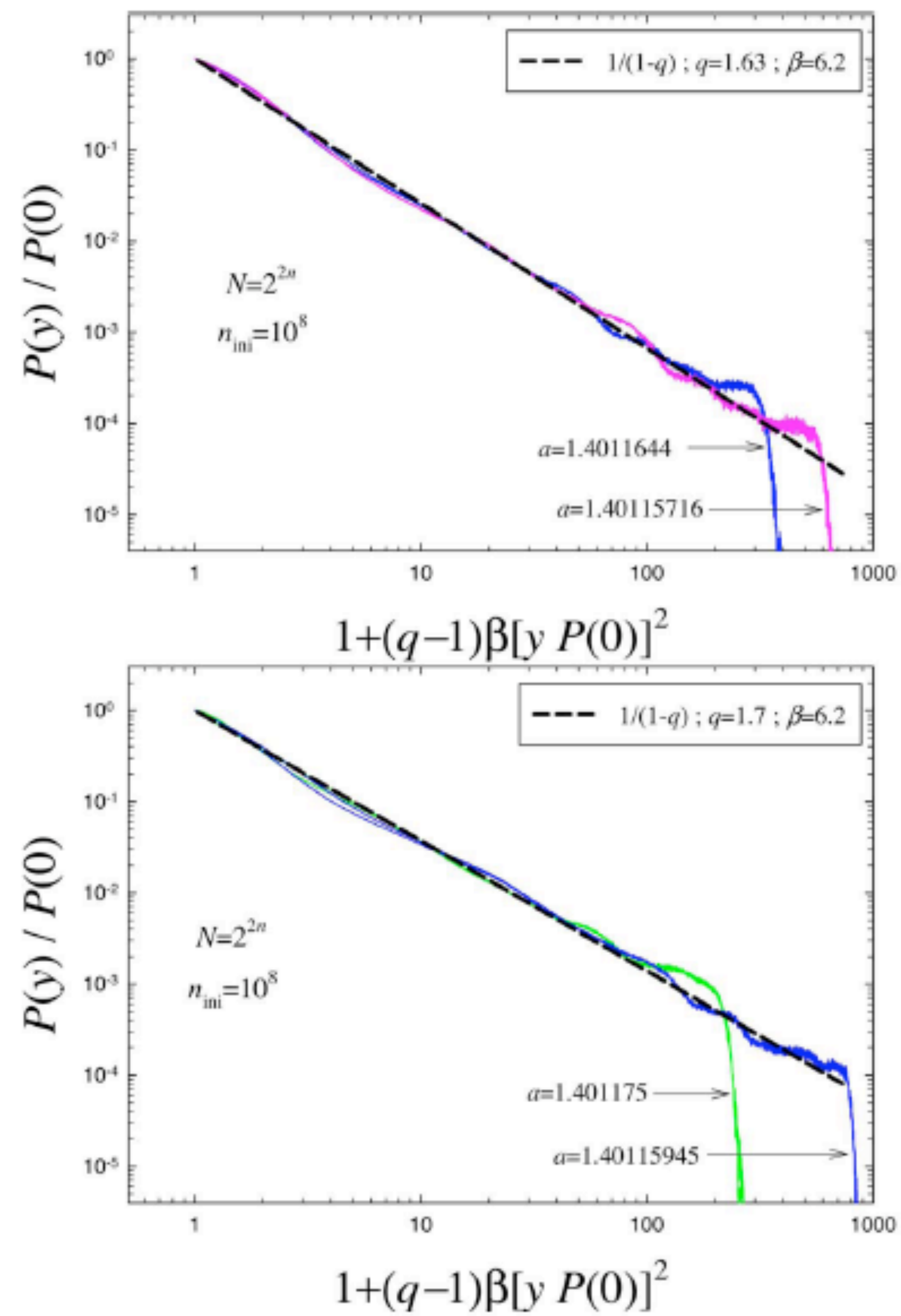


FIG. 3. (Color online) Probability density functions plotted against $1+(q-1)\beta[yP(0)]^2$ on a log-log plot for the cases $N=2^{2n}$, where $2n$ is (a) odd and (b) even. A straight line is expected with a slope $1/(1-q)$ if the curve is a q-Gaussian. It is clearly seen how the straight line is surrounded by the log-periodically modulated curves.

Afsar and Tirnakli did similar study for the sine-circle map

$$x_{n+1} = x_n + \Omega - \frac{K}{2\pi}\sin(2\pi x_n)$$

and studied again

$$y_N = \sum_{n=1}^{N}(x_n - \langle x \rangle)$$

at the edge of chaos.

Found evidence of q-Gaussian with

q<1

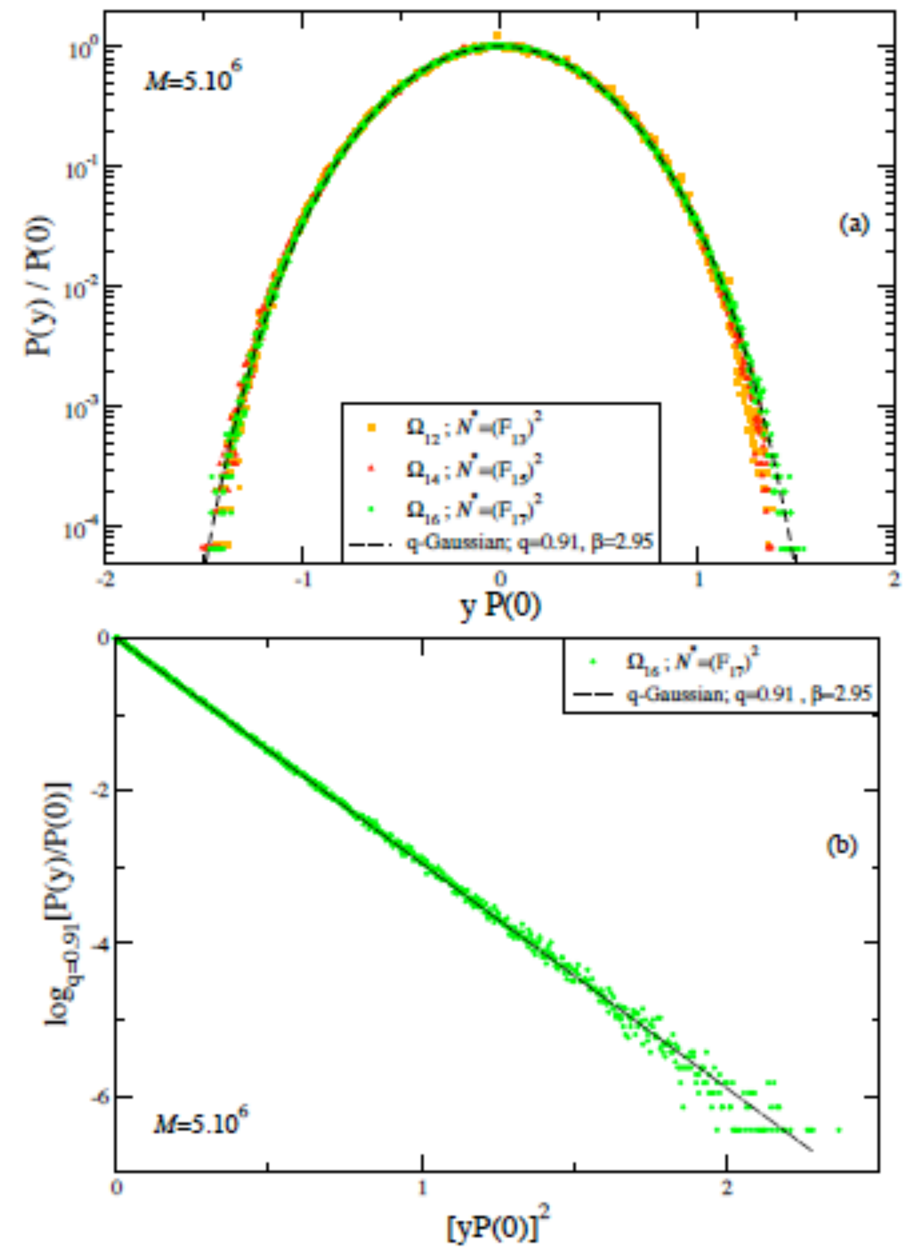


FIG. 3: (a) Probability distributions of the cases where the critical point is approached from below. (b) q-logarithmic plot of the same data for the case Ω_{16} which is the closest to the critical point that we can obtain numerically in our simulations.

Reference:
Afsar and Tirnakli
arXiv:1001.2689

Why is this?

Can we improve our understanding by turning to simpler situations?

Preferably solvable models.

Use the **Random Walker**

The random walker - the workhorse of stat mech

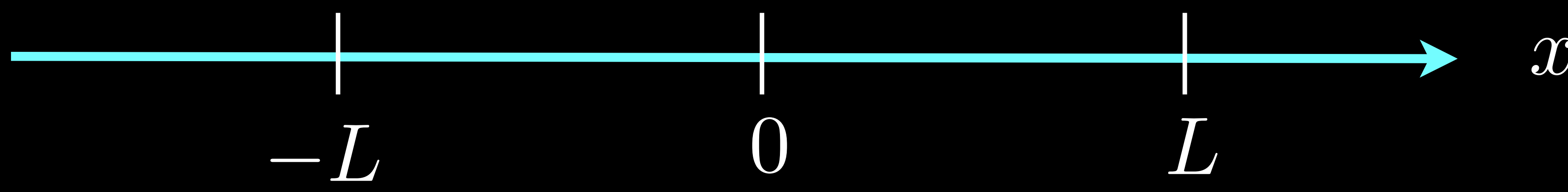
$$x_{t+1} = \begin{cases} x_t + 1 & \text{with probability } \gamma/2 \\ x_t - 1 & \text{with probability } \gamma/2 \\ x_t & \text{with probability } 1 - \gamma. \end{cases}$$

the sum of the position $y = \sum_{t=1}^{T} x_t$

is Gaussian distributed - when shifted and scaled

We expect the q-Gaussian to be related to significant correlations

Consider *Restricted* Random Walker



$$x_{t+1} = \begin{cases} x_t + 1 & \text{with probability } g(x)/2 \\ x_t - 1 & \text{with probability } g(x)/2 \\ x_t & \text{with probability } 1 - g(x). \end{cases}$$

with $g(x) = \begin{cases} |\frac{x}{L}|^a & \text{if } x \neq 0 \\ p & \text{if } x = 0 \end{cases}$ with $p << 1$

How is $y = \sum_{t=1}^{T} x_t$ distributed?

The exponent *a*

Since q-stat should essentially be a matter of correlations would expect applicability for a range of *a* values

According to e.g. Tsallis and Thurner & Hanel

Anomalous statistics occur when $W(N) \neq \mu^N$

The (effective) number of possible paths visited after *T* time step may for the Restricted Random Walker very likely be different from

$$W(T) \neq 2^T$$

because of the position dependent transition probability.

But this should happen for all values of the exponent *a* (?)

The Restricted Random Walk

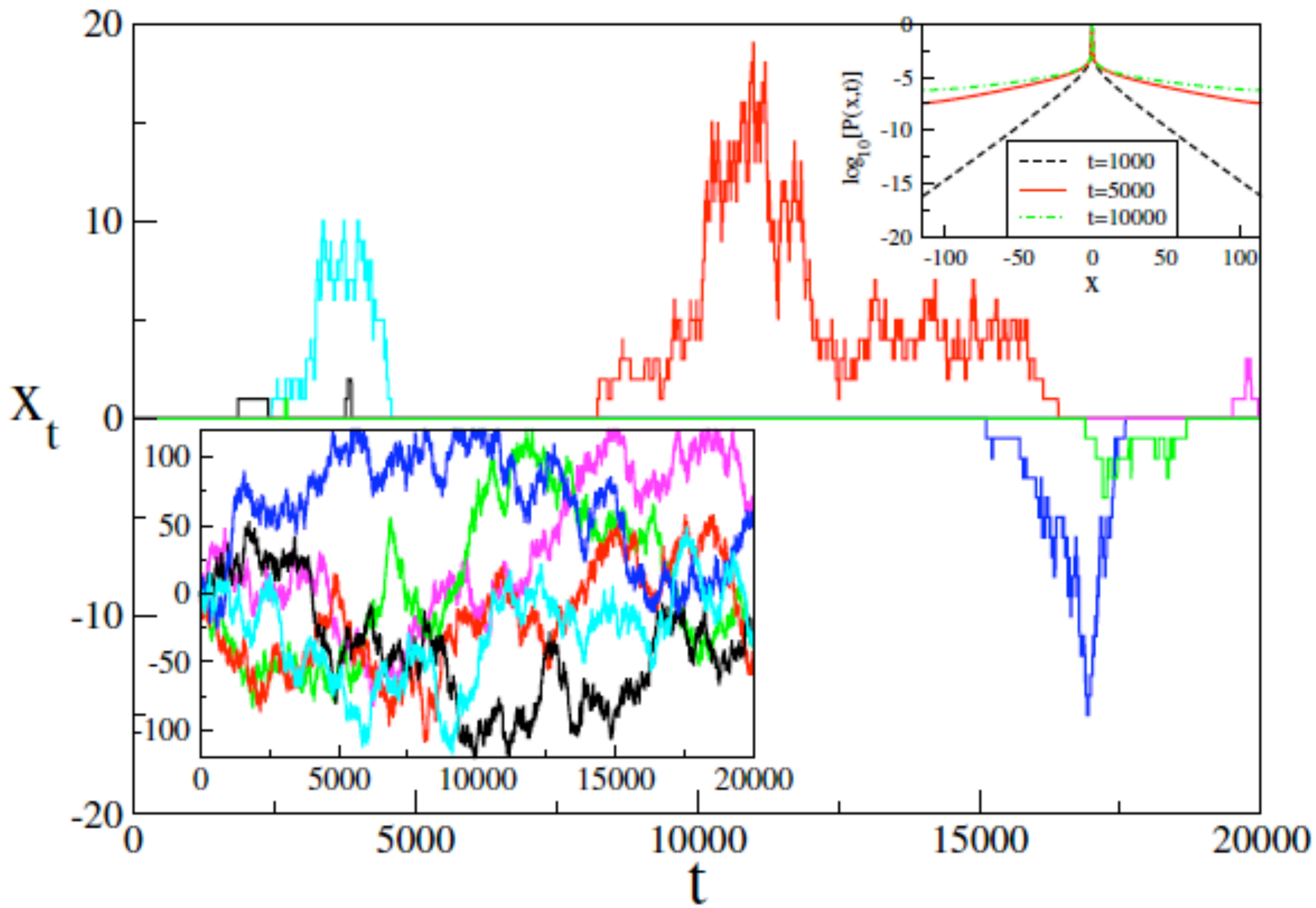
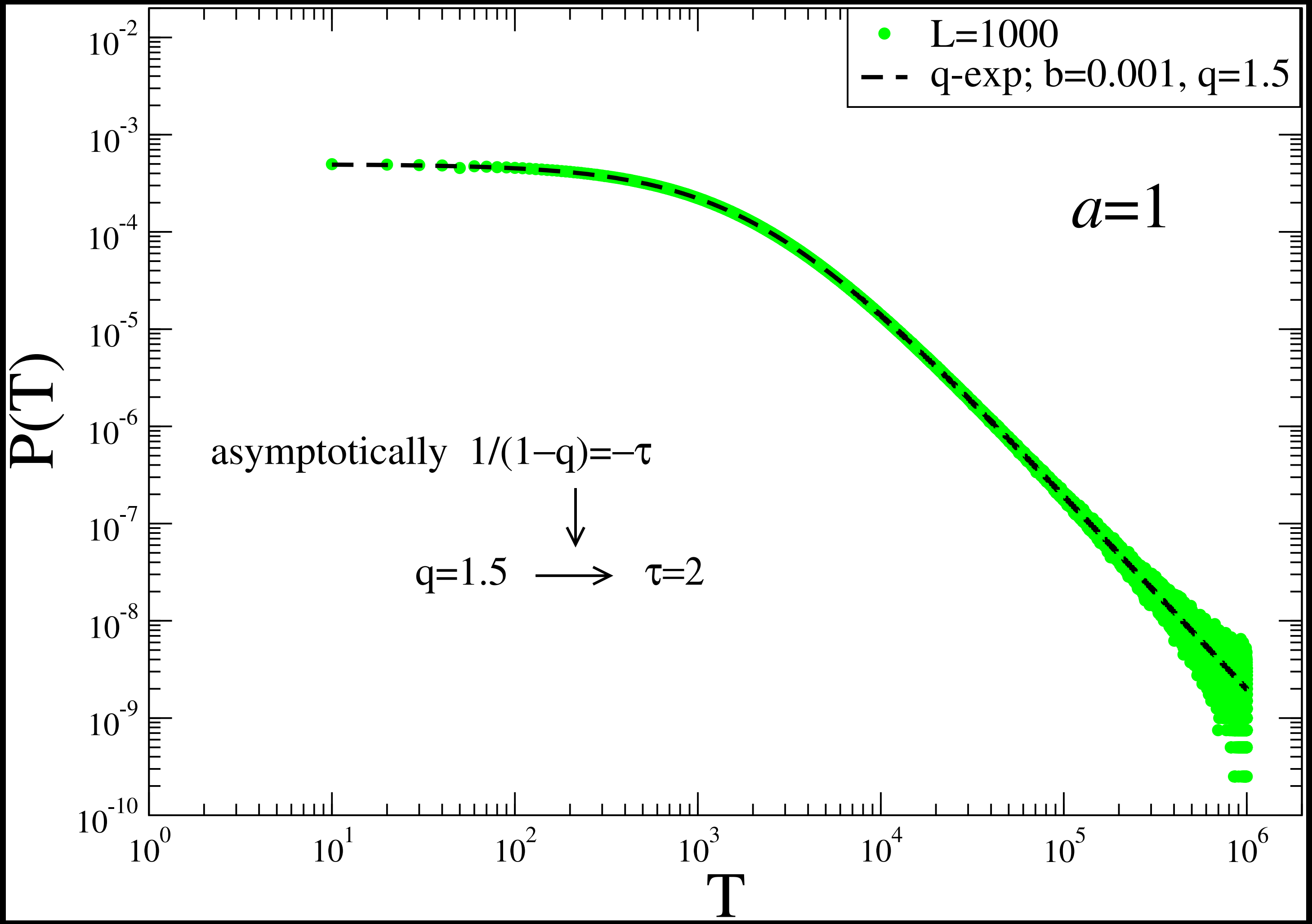


Fig. 1: (Colour on-line) Six representative trajectories for $L = 120$. Main panel: The restricted RW model ($p = 5 \cdot 10^{-6}$). Lower Inset: The standard RW model ($p = 1$). Non-ergodic behavior of the restricted RW model can easily be seen. Upper Inset: The time evolution of $P(x,t)$ at three different t values ($t = 1000$, 5000 and 10000 from bottom to top).

Return time distribution - q-exponentials

$a = 1$



$P(T_r) \sim T_r^{-\tau}$

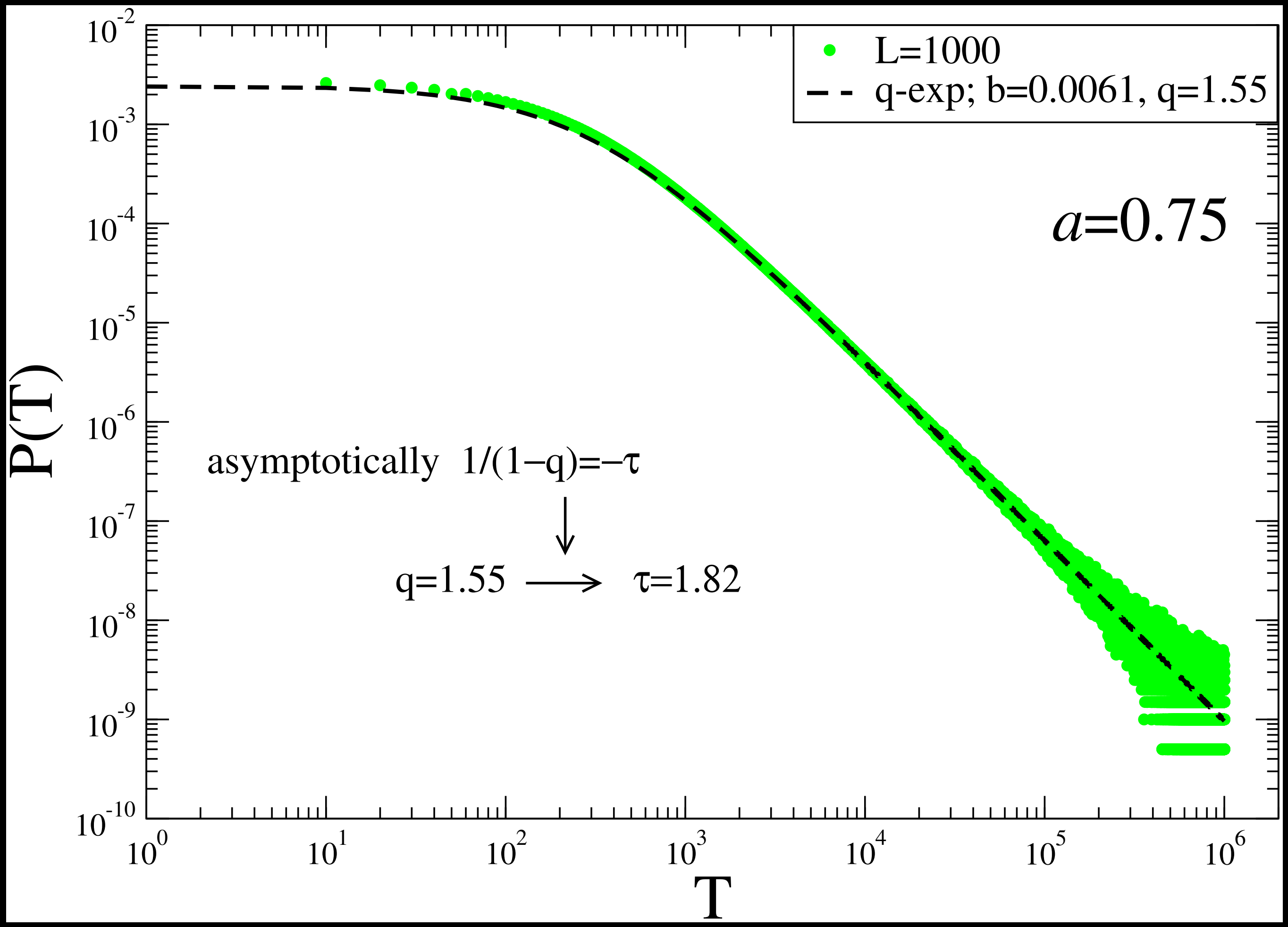
with $\tau = 2$

rather than the usual $\tau = 3/2$

Ugur Tirnakli - 30.10.13

Return time distribution

$a = 0.75$



$$P(T_r) \sim T_r^{-\tau}$$

with $\tau = 1.82$

rather than the usual $\tau = 3/2$

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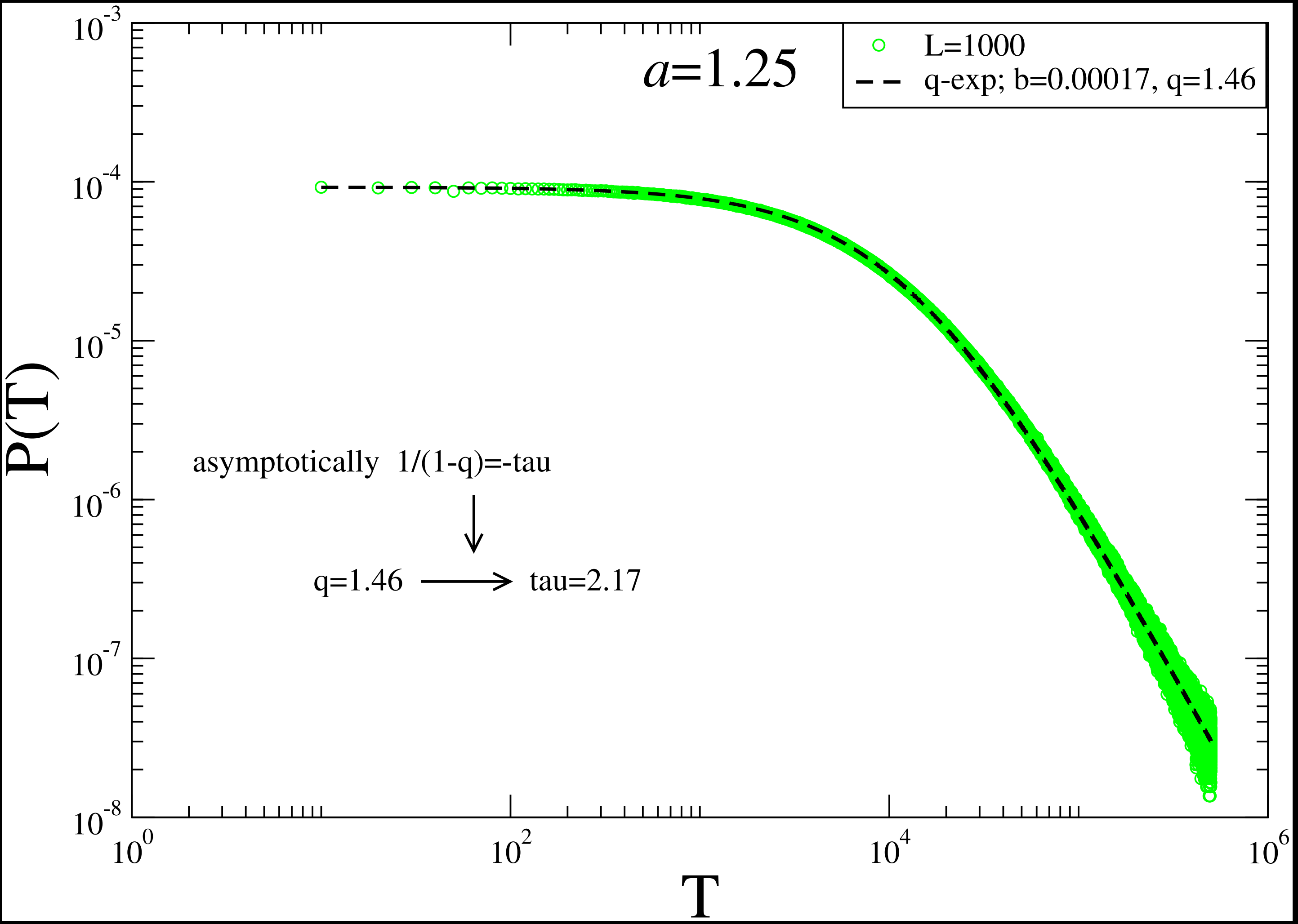
Return time distribution

$a = 1.25$



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$P(T_r) \sim T_r^{-\tau}$

with $\tau = 2.17$

rather than the usual $\tau = 3/2$

Summary

a	0.75	1	1.25
q	1.55	1.5	1.46
τ	1.82	2	2.17

How about P(x,t)

P(x,t) = prob{walker is at position x at time t}

Obviously a Gaussian for the ordinary Random Walker

The *Restricted* Random Walker

$$x_{t+1} = \begin{cases} x_t + 1 & \text{with probability } g(x)/2 \\ x_t - 1 & \text{with probability } g(x)/2 \\ x_t & \text{with probability } 1 - g(x). \end{cases}$$

with $$g(x) = \begin{cases} \left|\frac{x}{L}\right|^a & \text{if } x \neq 0 \\ p & \text{if } x = 0 \end{cases}$$ with $p << 1$

Master Equation for P(x,t)

$$
\begin{aligned}
P_X(x,t+1) &= P_X(x,t) \\
&+ \frac{1}{2}g(x-1)P_X(x-1,t) + \frac{1}{2}g(x+1)P_X(x+1,t) - g(x)P_X(x,t)
\end{aligned}
$$

from

$$
x_{t+1} = \begin{cases} x_t + 1 & \text{with probability } g(x)/2 \\ x_t - 1 & \text{with probability } g(x)/2 \\ x_t & \text{with probability } 1 - g(x). \end{cases}
$$

$$
g(x) = \begin{cases} |\frac{x}{L}|^a & \text{if } x \neq 0 \\ p & \text{if } x = 0 \end{cases}
$$

Continuum approximation for P(x,t)

$$\partial_t P(x,t) = \kappa \partial_x^2 [g(x) P(x,t)]$$

Stationary solution is $$P(x) = \frac{\text{constant}}{g(x)}$$

which is in very good agreement with the numerical solution of the discrete Master Equation

For $a = 1 \text{ hence } g(x) = |x/L|$ the continuum equation becomes

$$\partial_t P(x,t) = \gamma \partial_x^2 [x P(x,t)]$$

$$\partial_t P(x,t) = \gamma \partial_x^2 [x P(x,t)]$$

is solved (to a good approximation) by (see van Kampen)

$$P(x,t) = \frac{1}{t\sqrt{x}} \exp[-\frac{x+1}{t}] I_1(2\sqrt{x}/t)$$

for an initial
delta peak at x=1

First return time

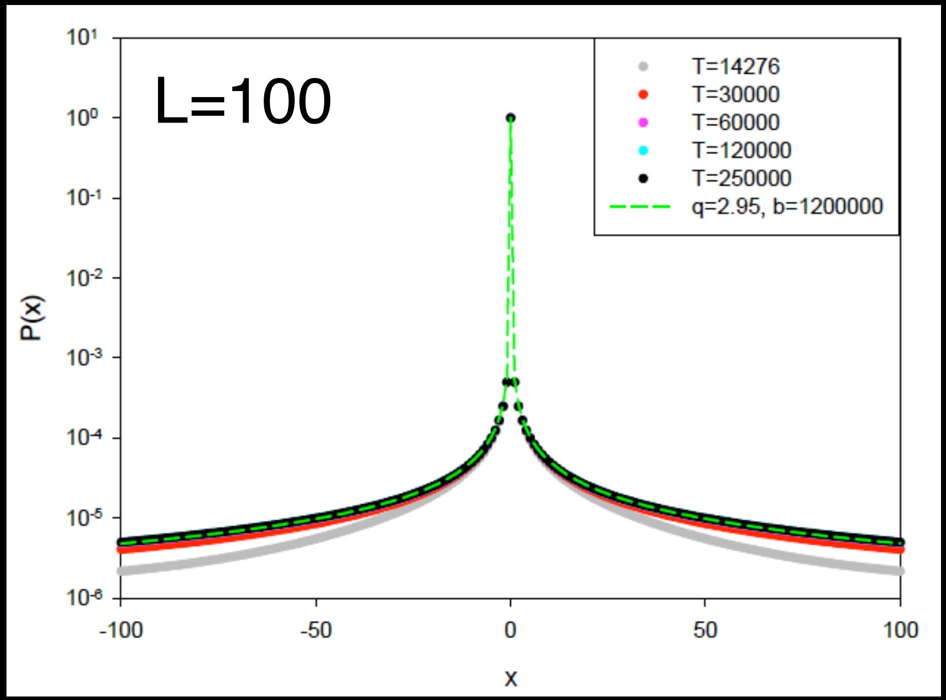
From current at the origin

$$P(T) = \text{Prob}\{\text{first returnat } t = T\} = \partial_x P(x=1,T)$$

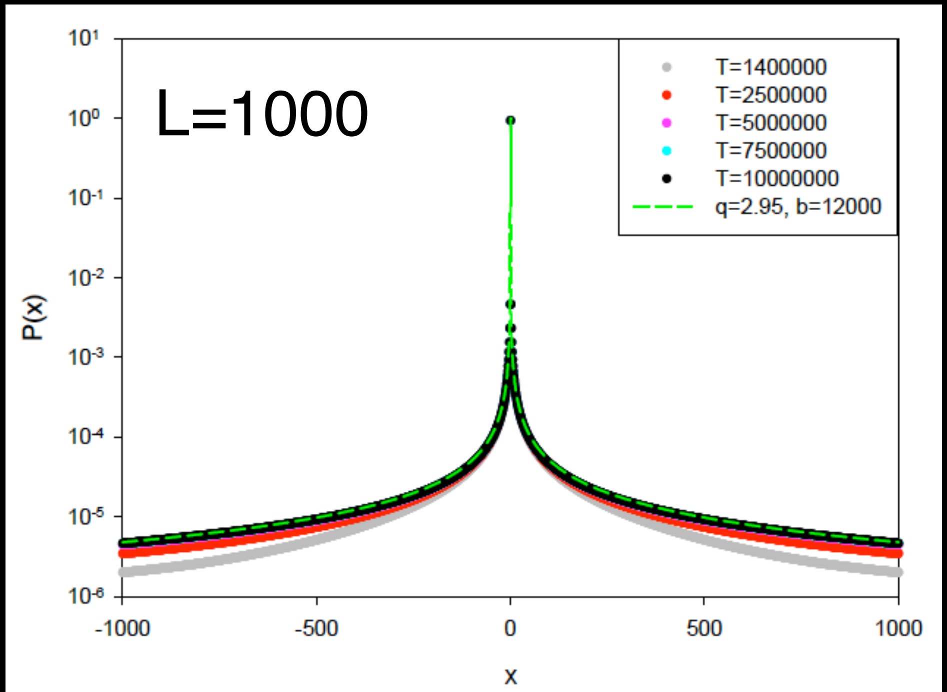
$$\propto 1/T^2$$

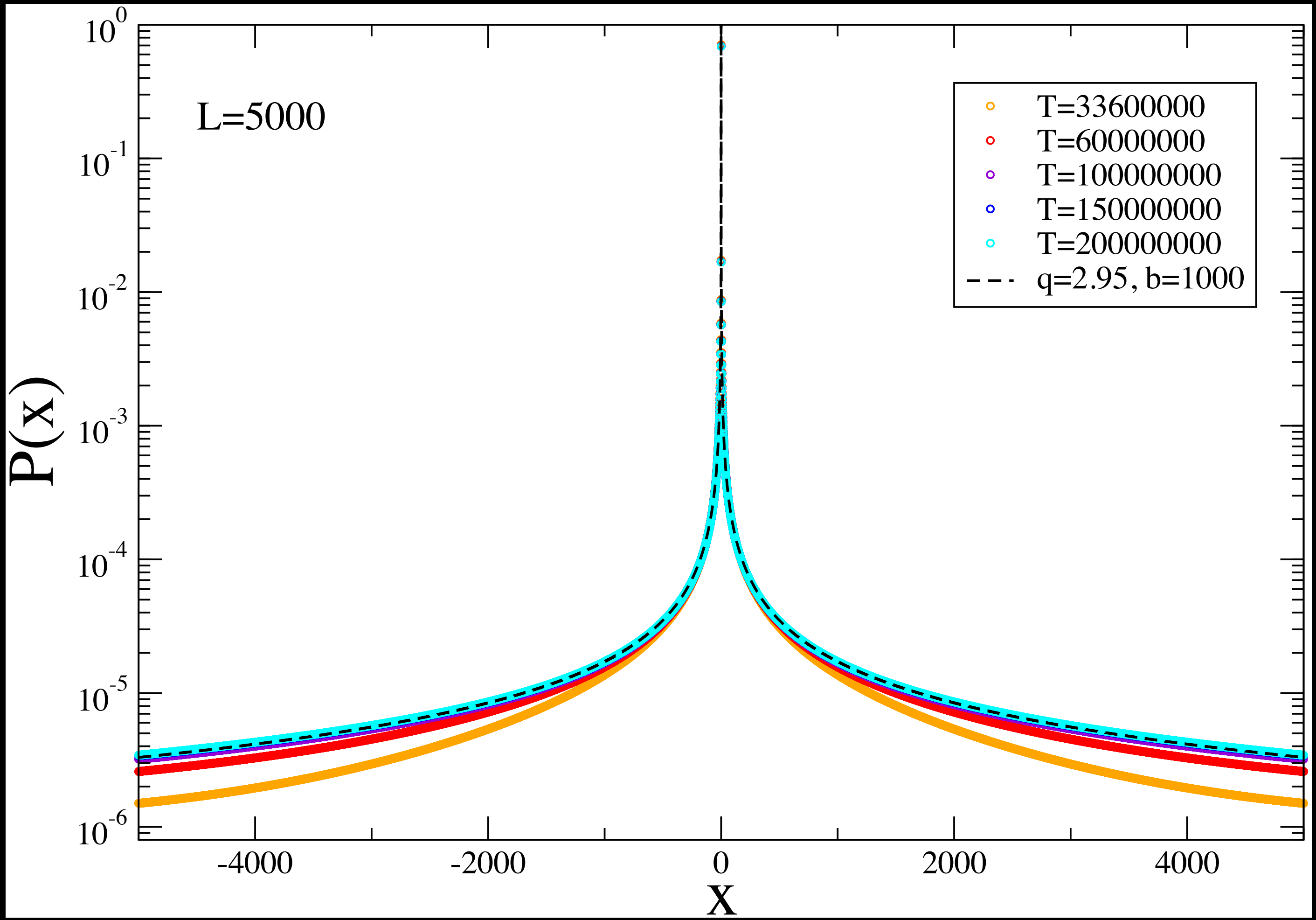
In agreement with Ugur's simulation

Solutions for P(x) fitted to q-Gaussian



$$a = 1 \text{ hence } g(x) = |x/L|$$





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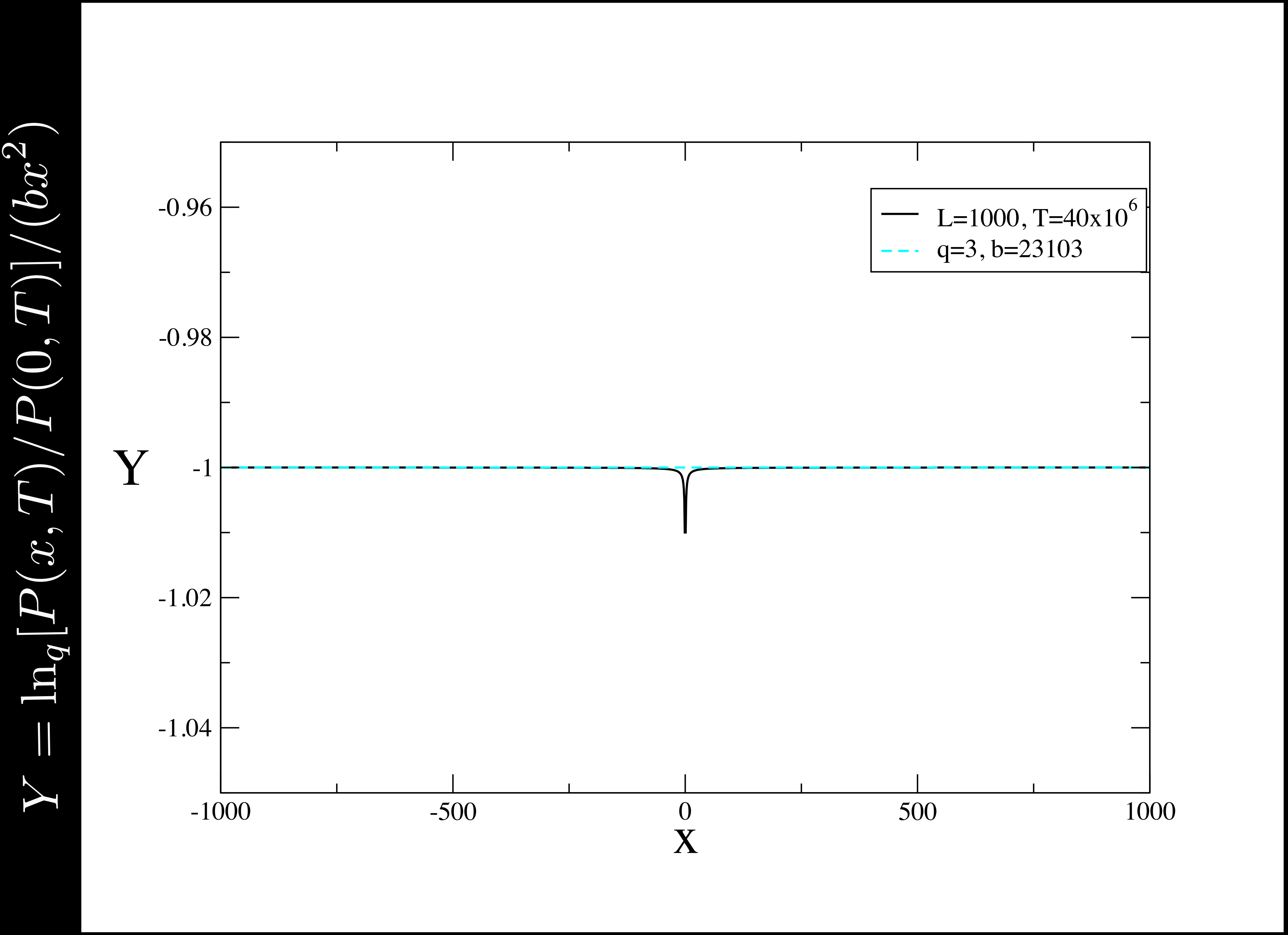
and now the asymptotic fit behaviour



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Recall $q = 3 \Rightarrow P_q(x)/P(0) = 1/\sqrt{1+2bx^2} \quad \propto 1/x \propto 1/g(x)$

Situation $a \neq 1$

Continuum approximation then

$$\partial_t P(x,t) = \gamma \partial_x^2 [x^a P(x,t)]$$

Work in progress ....

Statistics - continued

Want to determine the the distributions

P(y,T) = prob{ $\sum_{t=1}^{T} x_t = y$ at time T}

note $\langle y \rangle = 0$

so shift not needed - scaling included in fitting to q-Gaussian

Master Equation for *P(y,x;t)*

$(x_t, y_t) \mapsto (x_{t+1}, y_{t+1})$ **where** $y_{t+1} = \sum_{i=1}^{t} x_i + x_{t+1} = y_t + x_{t+1}$

$$(x_t, y_t) \mapsto \begin{cases} (x_t + 1\ , y_t + x_t + 1) & \text{with probability } g(x_t)/2 \\ (x_t - 1\ , y_t + x_t - 1) & \text{with probability } g(x_t)/2 \\ (x_t\ , y_t + x_t) & \text{with probability } 1 - g(x_t). \end{cases}$$

$$P(y, x; t+1) = P(y, x; t) + \sum_{\Delta \in \{-1,0,1\}} [W(y, x; y - (x - \Delta), x - \Delta) P(y - (x - \Delta), x - \Delta; t)) - W(y + x + \Delta, x + \Delta; y, x) P(y, x; t)].$$

The transition probabilities simplify

$$W(y, x; y-(x-\Delta), x-\Delta) = w(x-\Delta, \Delta)$$

$$W(y+x+\Delta, x+\Delta; y, x) = w(x, \Delta)$$

Where

$$w(z,\Delta) = \begin{cases} g(z)/2 & \text{if } \Delta = \pm 1 \\ 1-g(z) & \text{if } \Delta = 0 \end{cases}$$

Master Equation for *P(y,x;t)* reduces to

$$P(y,x;T+1) = \frac{1}{2}[g(x+1)P(y-(x+1),x+1;T)$$
$$+g(x-1)P(y-(x-1),x-1;T)]$$
$$+(1-g(x))P(y-x,x;T)$$

Not found analytic solution
Solve by numerical iteration

and obtain

$$P(y,T) = \sum_{x=-L}^{L} P(y,x;T)$$

The probability density of

$$P(y,T) \text{ with } y = \sum_{t=1}^{T}$$

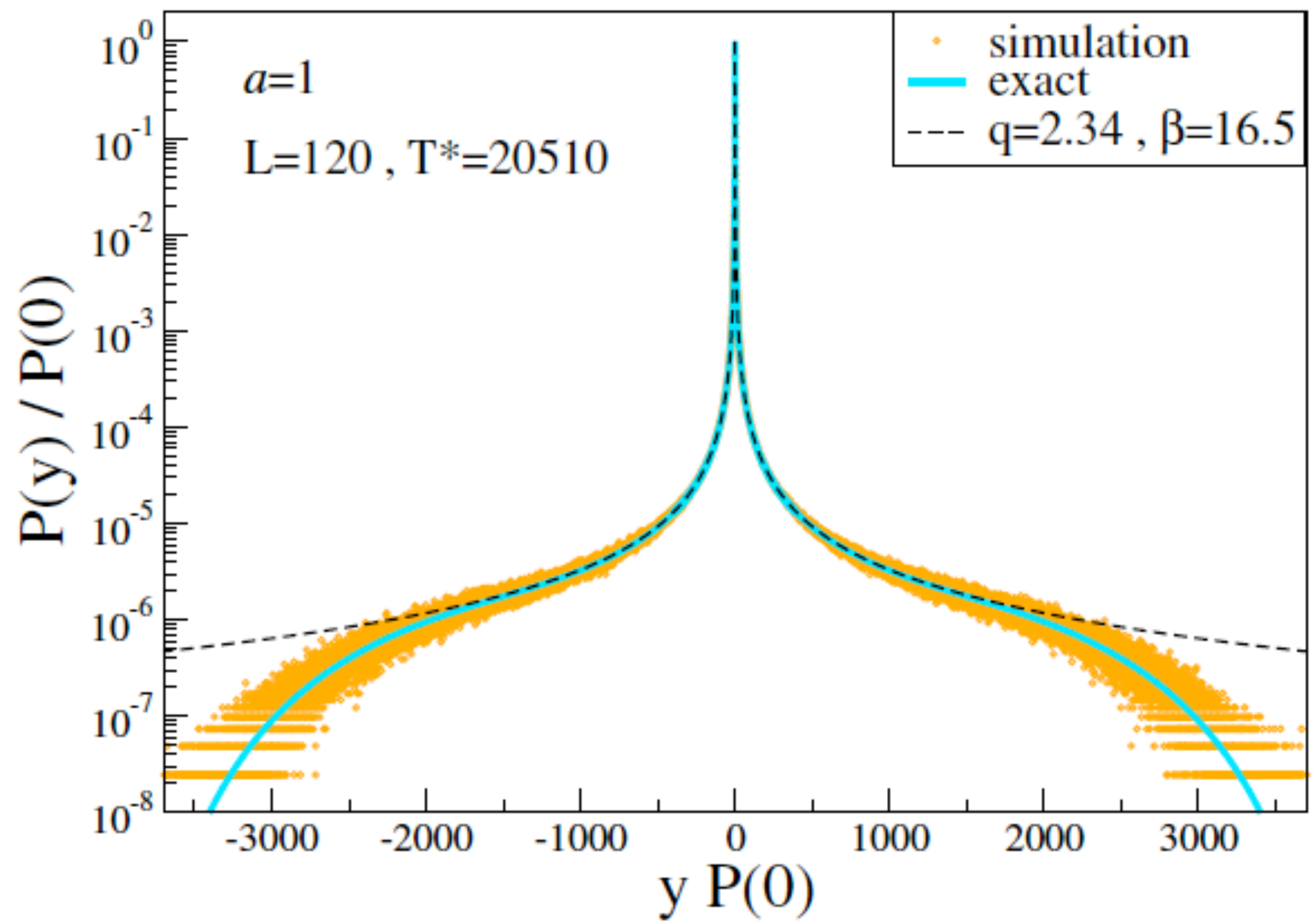

Fig. 2: (Colour on-line) Exact and simulation results of the case $a=1$, $L=120$ and $p=5\cdot10^{-6}$. It is clearly seen that the probability function $P(y,T^*)$ obtained from simulations is completely in accordance with the exact results. The number of experiments used in our simulations are $2\cdot10^8$.

Scaling behaviour

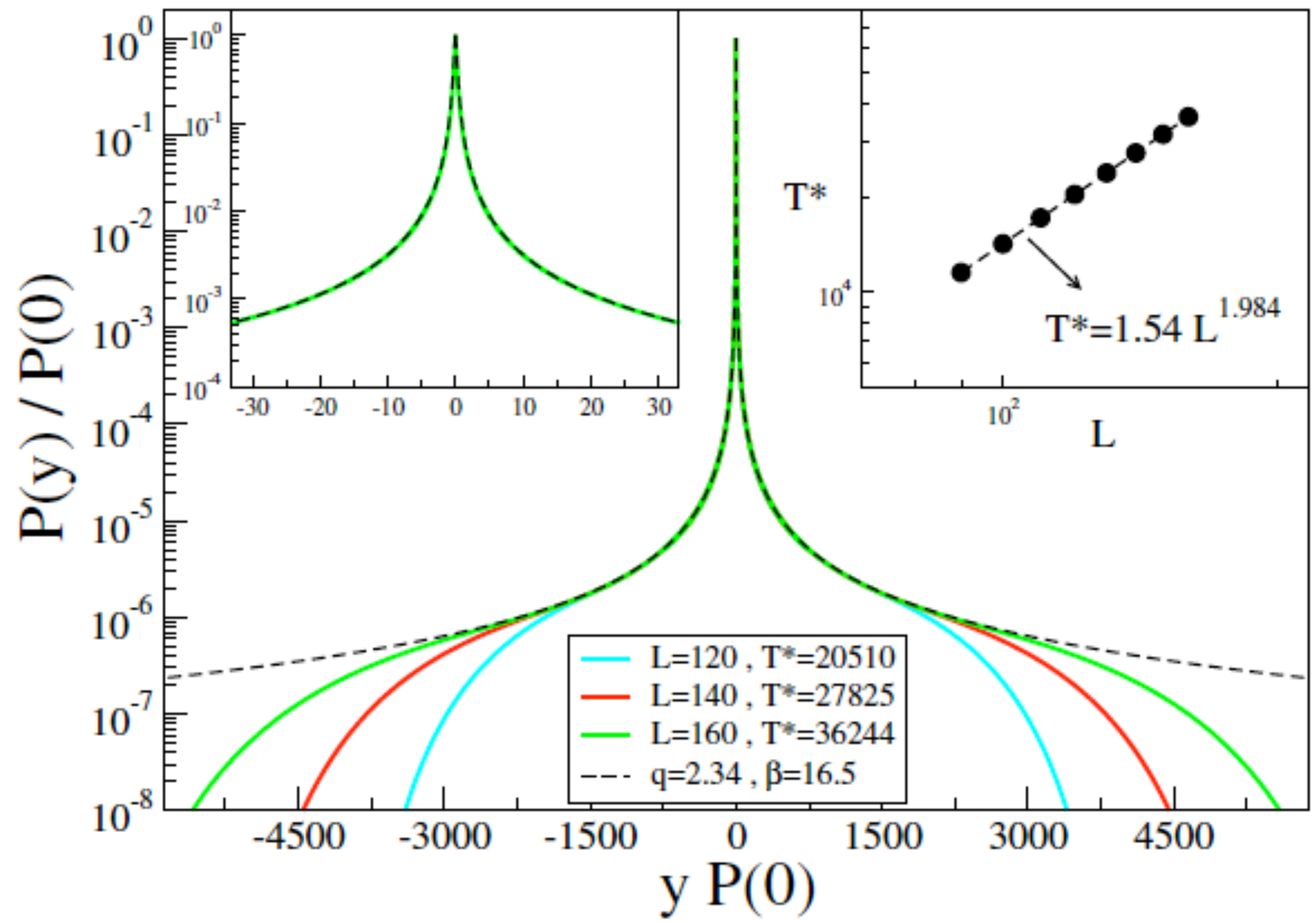

Fig. 3: (Colour on-line) The case $a = 1$ for $L = 120$, 140 and 160 with $p = 5 \cdot 10^{-6}$. The main panel shows the probability function $P(y, T^*)$. The center of the function is shown in detail in the left inset. The time T^* is chosen to optimize the fit to the q-Gaussian. The scaling of T^* is given in the right inset.

Quality of the fit to q-Gaussian

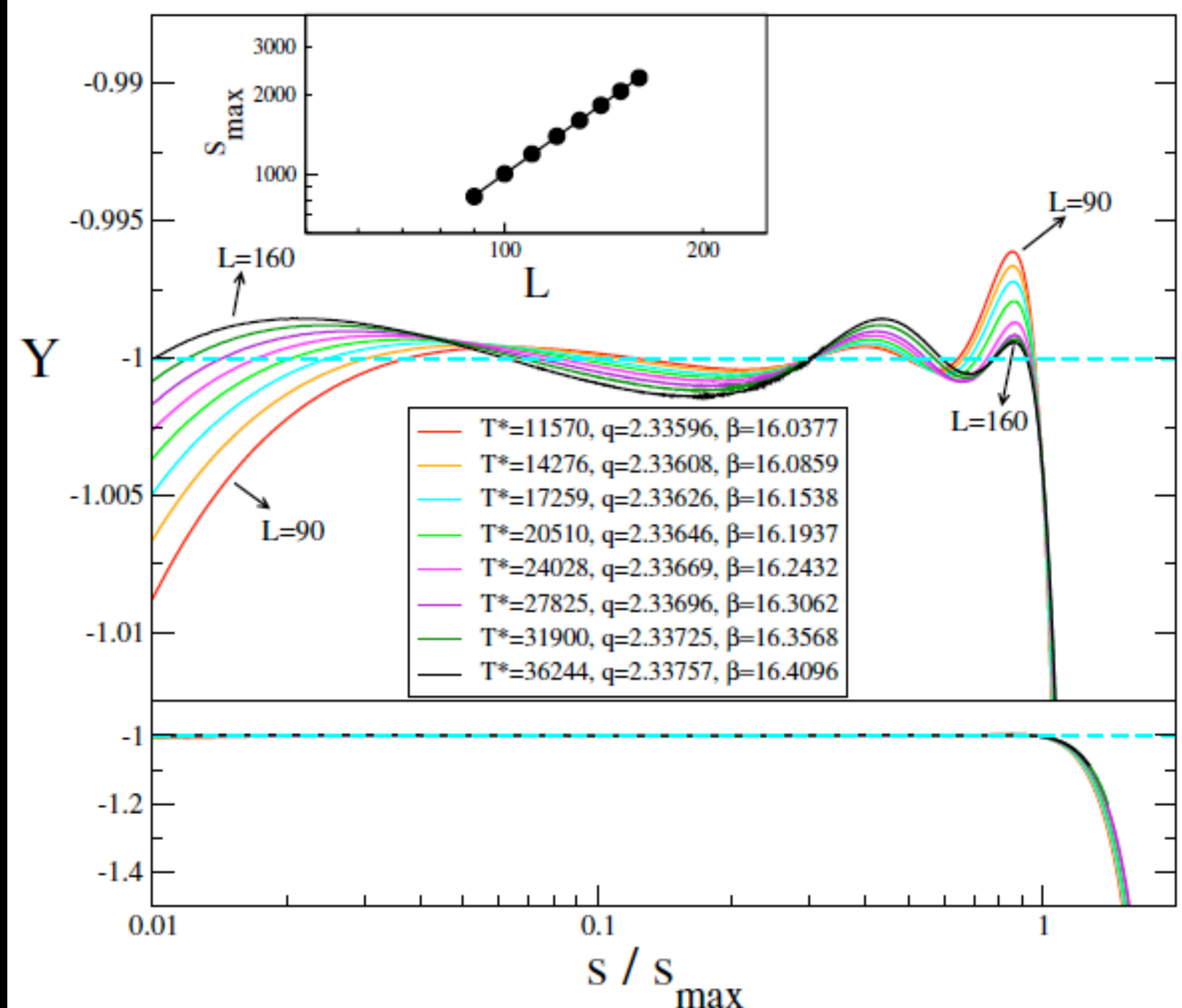

Fig. 4: (Colour on-line) The case $a = 1$ and $p = 5 \cdot 10^{-6}$ with L values between $L = 90$ to 160. The lower panel shows the data collapse when the x-axis is appropriately scaled. The upper panel shows a zoomed region around the $Y = -1$ line. In the inset the scaling of s_{max} with L is given. The straight line is $s_{max} = AL^C$ with $A = 0.2726$ and $C = 1.7828$.

Scaling of *q* with systems size *L*

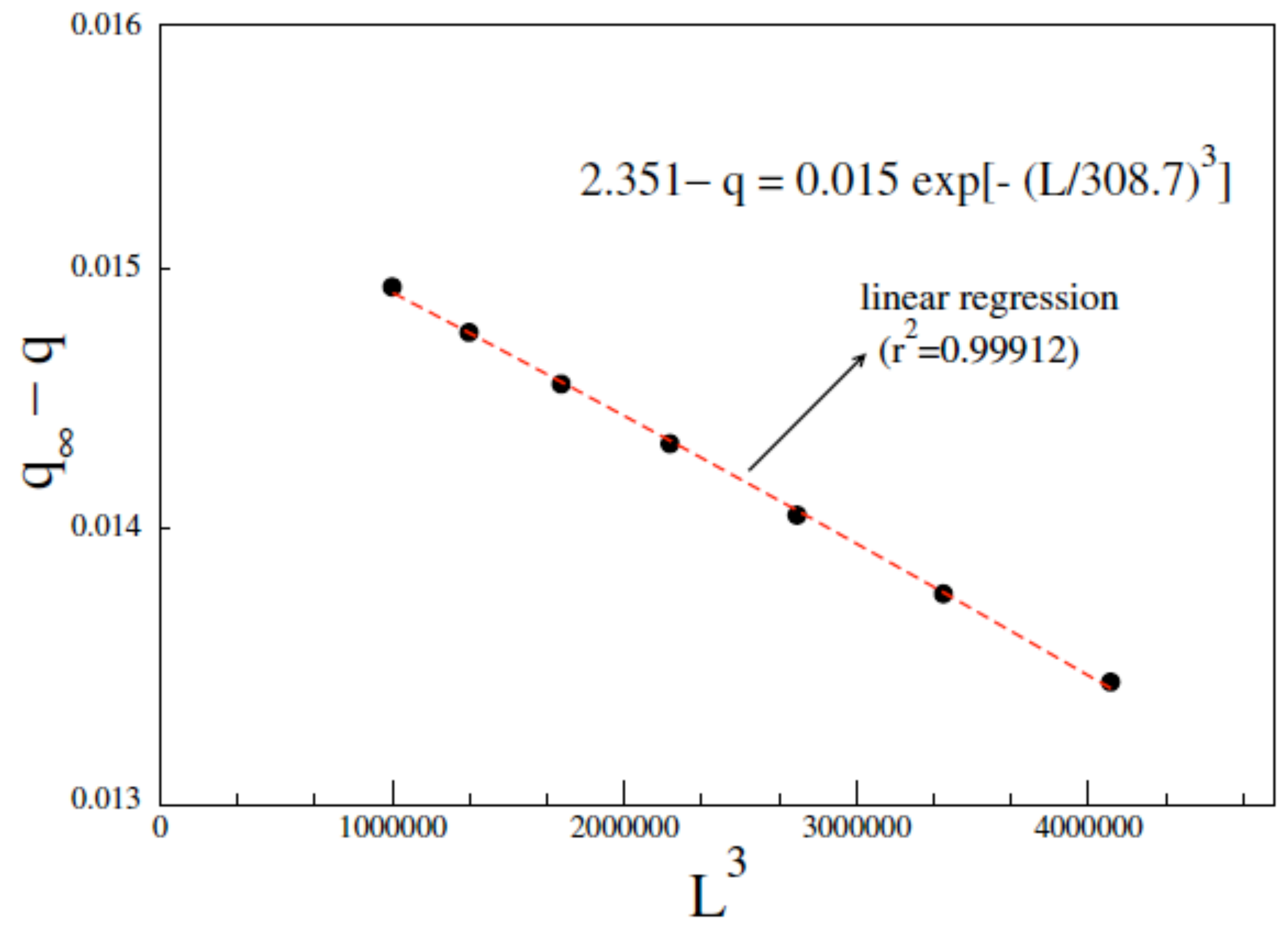

Fig. 5: (Colour on-line) Linear-log representation for the L dependence of q values. This exponential dependence suggests an asymptotic value around $q_\infty \simeq 2.351$.

Different values of *a*

$$g(x) = \min\left\{\left|\frac{x}{L}\right|^a + p, 1\right\}$$ and $$x_{t+1} = \begin{cases} x_t + 1 & \text{with probability } g(x)/2 \\ x_t - 1 & \text{with probability } g(x)/2 \\ x_t & \text{with probability } 1 - g(x). \end{cases}$$

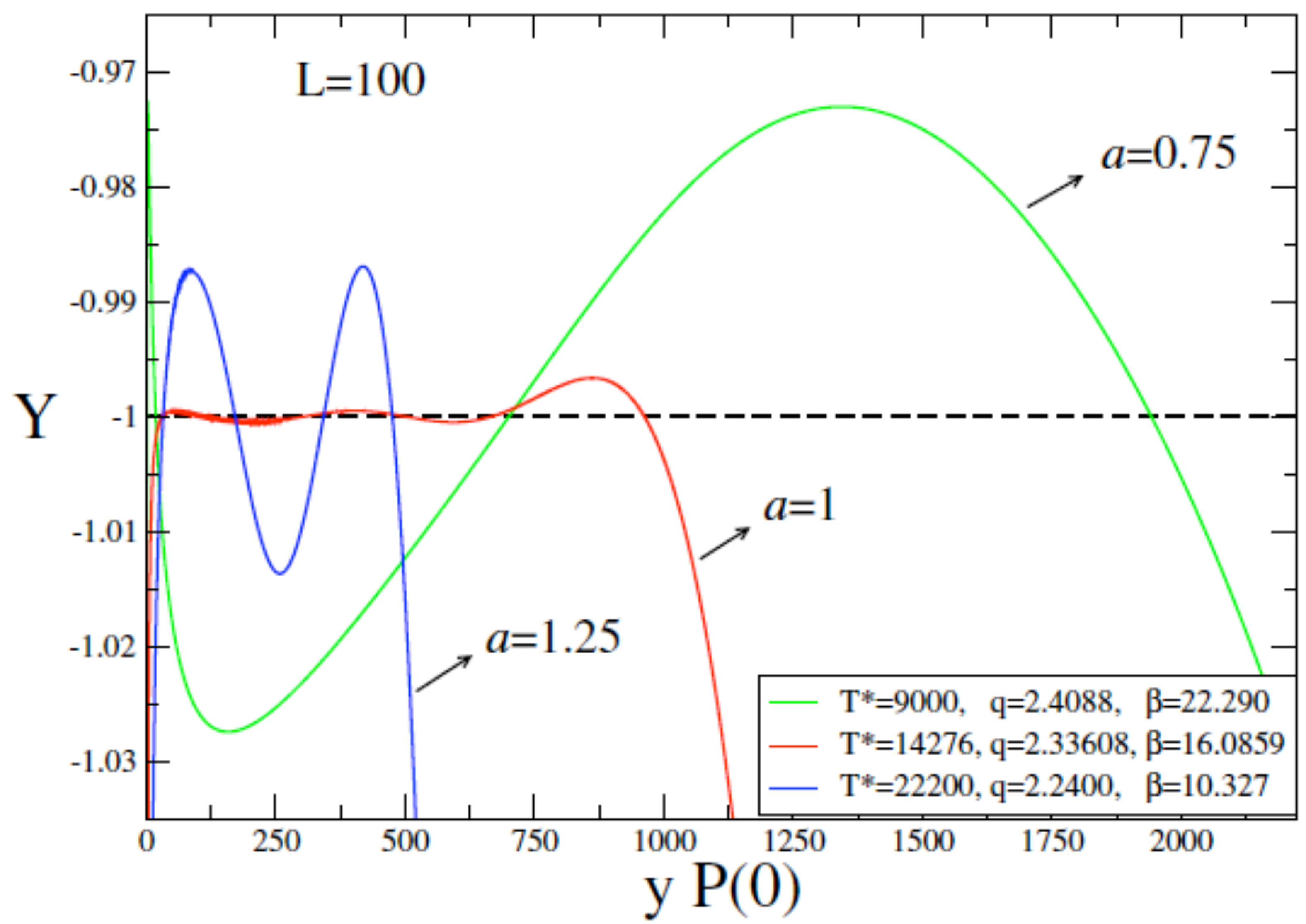

Fig. 6: (Colour on-line) Y plot of cases $a = 0.75$, $a = 1$ and $a = 1.25$ for a representative L value. Whenever $a \neq 1$, increasing order of deviation from -1 line is evident.

Correlated versus uncorrelated

Does the peculiar shape of P(x) produce the q-Gaussian like behaviour?

Correlated | Uncorrelated

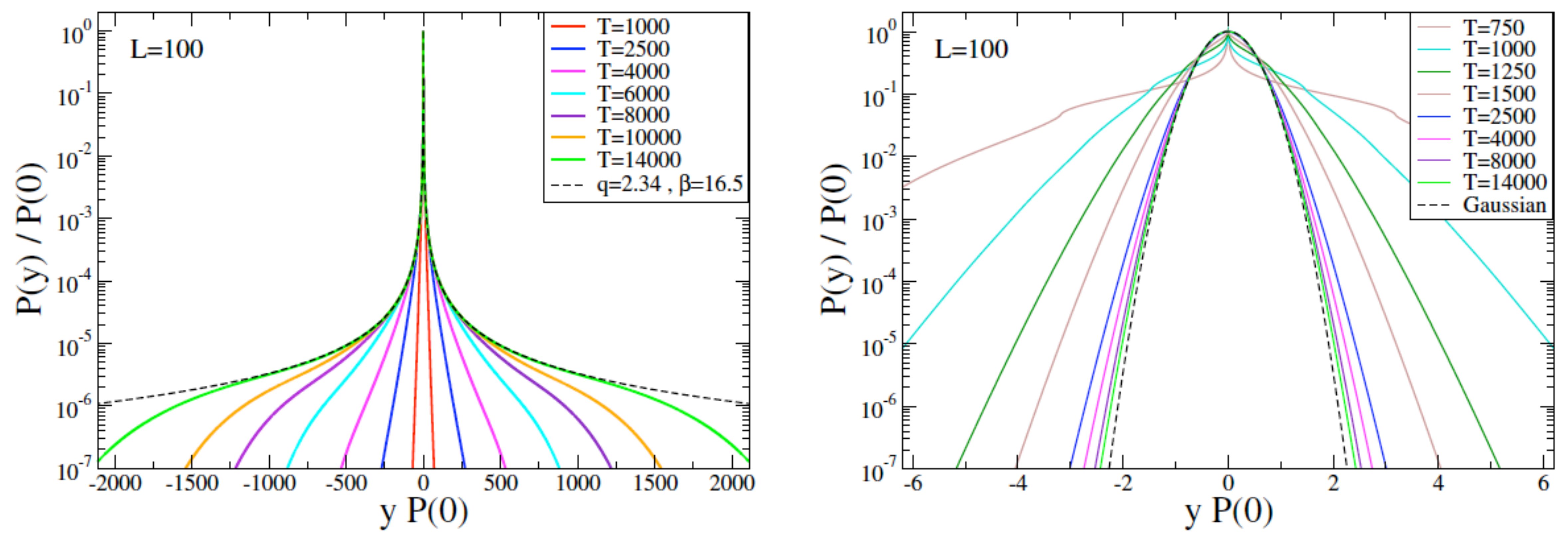

Fig. 7: (Colour on-line) Comparison of the T dependence for fixed values of L with $p = 5 \cdot 10^{-6}$ for the correlated (left panel) and the uncorrelated (right panel) $P(y, T)$ distribution (see text for the details).

Summary

The convergence towards q-Gaussian appears fragile

- *L* and *T* needs to scale: $T = L^2$
- *q* and *L* scale with: $q_\infty = 2.351 - 0.015 \exp[-(L/308.7)^3]$
- why does only *a* = 1 lead to q-Gaussian ?

$$g(x) = \begin{cases} |\frac{x}{L}|^a & \text{if } x \neq 0 \\ p & \text{if } x = 0 \end{cases}$$

Conclusion

The simplicity of the (restricted) random walker was helpful:

→ Allows derivation of exact Master Equations

→ q-Gaussian does describe one asymptotic limit (for $a = 1$)

→ But deeper analytic understanding is needed to discover the significance of the appearance of the q-Gaussian in the case when $T \sim L^2$

If q-Gaussians are not the analytic attractors why are they so efficient ? ? ?

Thank You

Collaborators:
Ugur Tirnakli and Constantino Tsallis