Combinatorial basis for Tsallis entropy and its applications

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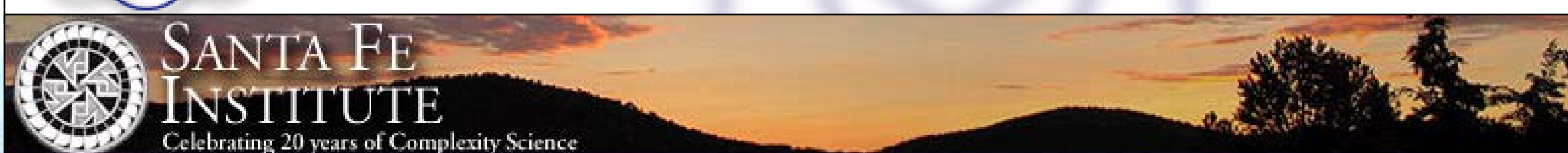
Congratulation! Constantino!



My brief history on recent activities

1. Chief Organizer (Kyoto, 2009)

2. Chief Organizer (NEXT2012Nara)

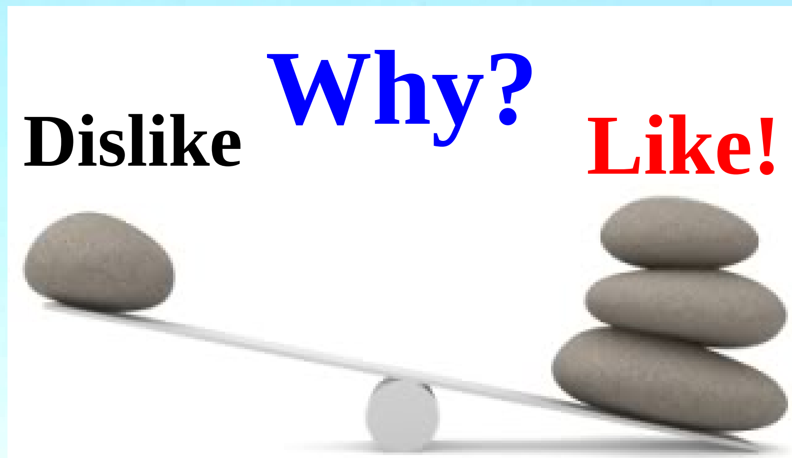
3. My Book in Japanese (2010)

someday English ver.

Why I became interested in Tsallis entropy S_q

Confession

At first glance (2000), I was not so interested in S_q.



What is my tipping point?

= This is my talk.

Conversion

Late in 2003, I became very interested in S_q .

Natural question for me (2000)

Why is Tsallis entropy a staring point for MaxEnt?

Around that time, many papers on Tsallis statistics were published

If Tsallis entropy is riate generalized entropy,
I believed there exis fundamental approach
along the line of the Boltzmann's original ideas.

= counting numbers of states

Outline of my approach

q-exponential • q-product • Tsallis entropy

Characterization of the exponential function

$$\frac{dy}{dx} = y$$

(Tsallis, 2004)

q-exponential

$$\frac{y}{A} = \exp_q\left(\frac{x}{A^{1-q}}\right)$$

(Borges, 2004)

q-product

$$\exp_q(x+y) = \exp_q(x) \otimes_q \exp_q(y)$$

(Suyari, 2006)

q-Stirling's formula, q-multinomial coefficient

(Suyari, 2006)

Tsallis entropy

$$S_q = \frac{1 - \sum_{i=1}^{n} p_i^q}{q-1}$$

Characterization of the *q*-exponential function

$$\frac{dy}{dx} = y^q$$

Tsallis (2004)

$$\Rightarrow \int \frac{1}{y^q}\, dy = \int dx$$

$$\Rightarrow \ln_q y = x + C$$

q-logarithm

ANY constant

$$\Rightarrow y = \exp_q(x + C)$$

q-exponential

$$\Rightarrow \frac{y}{\exp_q(C)} = \exp_q\left(\frac{x}{\left(\exp_q(C)\right)^{1-q}}\right)$$

C is **ANY** constant satisfying $1 + (1-q)C > 0$

q-product

Fundamental functions derived from $\frac{dy}{dx} = y^q$

q- logarithm $\ln_q x := \frac{x^{1-q}-1}{1-q} \xrightarrow{q \to 1} \ln x$

q- exponential $\exp_q x := [1+(1-q)x]^{\frac{1}{1-q}} \xrightarrow{q \to 1} \exp x$

The new product $\otimes_q$ is introduced to satisfy the *q*-exponential law:

$$\exp_q x \otimes_q \exp_q y = \exp_q (x+y)$$

$\otimes_q$: *q*-product

$$x \otimes_q y := \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}$$

Nivanen, Mehaute, Wang (*Rep. Math. Phys.*, vol. 52, 2003), Borges (*Physica A*, vol. 340, 2004.)

q-Stirling's formula

Suyari (*Physica A*, 368, 2006.)

q-factorial $n!_q := 1 \otimes_q \cdots \otimes_q n$

tight q-Stirling's formula

$$\ln_q(n!_q) \cong \begin{cases} \left(n+\frac{1}{2}\right)\ln n - n + \theta_{n,1} + (1-\delta_1) & \text{if } q=1 \\ n - \frac{1}{2n} - \ln n - \frac{1}{2} + \theta_{n,2} - \delta_2 & \text{if } q=2 \\ \left(\frac{n}{2-q}+\frac{1}{2}\right)\frac{n^{1-q}-1}{1-q} - \frac{n}{2-q} + \theta_{n,q} + \left(\frac{1}{2-q}-\delta_q\right) & \text{if } q>0 \; q\neq 1,2 \end{cases}$$

More useful for applications

***rough* q-Stirling's formula**

$$\ln_q(n!_q) = \begin{cases} n - \ln n + O(1) & \text{if } q=2 \\ \frac{n}{2-q}\ln_q n - \frac{n}{2-q} + O(\ln_q n) & \text{if } q>0 \; q\neq 2 \end{cases}$$

q-multinomial coefficient

Suyari (*Physica A*, 368, 2006.)

In order to define the *q*-multinomial coefficient,
the new ratio $\oslash_q$ is introduced to satisfy

$$\exp_q x \oslash_q \exp_q y = \exp_q (x - y)$$

$$\Rightarrow \quad x \oslash_q y := \left[x^{1-q} - y^{1-q} + 1 \right]^{\frac{1}{1-q}}$$

$\oslash_q$: *q*-ratio

q-multinomial coefficient (contin.)

standard multinomial coefficient

$$\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix} = \frac{n\,!}{n_1\,! \cdots n_k\,!}$$

q-multinomial coefficient

$$\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_q := (n\,!_q) \oslash_q \left[(n_1\,!_q) \otimes_q \cdots \otimes_q (n_k\,!_q) \right]$$

$$= \left[\sum_{\ell=1}^{n} \ell^{1-q} - \sum_{i_1=1}^{n_1} i_1^{1-q} \cdots - \sum_{i_k=1}^{n_k} i_k^{1-q} + 1 \right]^{\frac{1}{1-q}}$$

Derivation of Tsallis entropy & additive duality

Suyari (*Physica A*, 368, 2006.)

$$\ln \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix} \cong n\, S_1\left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right)$$

$q \rightarrow 1$

$$\ln_q \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_q \cong \frac{n^{2-q}}{2-q} S_{2-q}\left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right)$$

q-logarithm

q-multinomial coefficient

Tsallis entropy

approximation by the *q*-Stirling's formula

additive duality: $q \leftrightarrow 2-q$

$$\ln_{1-(1-q)} \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{1-(1-q)} \cong \frac{n^{1+(1-q)}}{1+(1-q)} S_{1+(1-q)}\left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right)$$

This discovery is my tipping point!

~~some entropies~~

Tsallis entropy only!

~~Tsal... y,~~
and

MaxEnt

BUT

***q*-exponentail function**

Counting Number of States

The unique entropy corresponding to the *q*-exponential function is Tsallis entropy!

More general formulra

$q \leftrightarrow 2-q$

$$\ln_q \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_q \simeq \frac{n^{2-q}}{2-q} S_{2-q}\left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right)$$

(μ, ν, q)-relation

$$\nu(1-\mu)+1 = q$$

$$\frac{1}{\nu} \ln_\mu \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,\nu)} \simeq \frac{n^q}{q} S_q\left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right)$$

(μ,ν)-Stirling's formula

Suyari&Wada (*Physica A*, 387, 2008.)

q-factorial $\quad n!_q := 1 \otimes_q 2 \otimes_q \cdots \otimes_q n$

(μ,ν)-factorial $\quad n!_{(\mu,\nu)} := 1^{\nu} \otimes_\mu 2^{\nu} \otimes_\mu \cdots \otimes_\mu n^{\nu}$

$$\ln_\mu n!_{(\mu,\nu)} \cong \int_1^n \ln_\mu x^\nu dx$$

(μ,ν)-Stirling's formula

$$= \begin{cases} \nu(n - \ln n) + O(1) & \text{if } \nu(1-\mu)+1 = 0 \\ \dfrac{n \ln_\mu n^\nu - \nu n}{\nu(1-\mu)+1} + O(\ln_\mu n) & \text{if } \nu(1-\mu)+1 \neq 0 \end{cases}$$

Similarly as before, (μ,ν)-multinomial coefficient is defined.

Generalized correspondence between (μ,ν)-multinomial coefficient and Tsallis entropy S_q

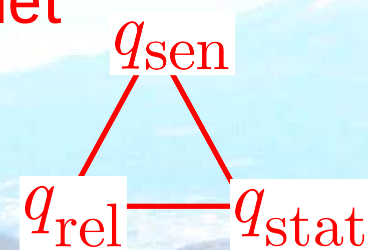
$$\frac{1}{\nu}\ln_\mu \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,\nu)} \cong \frac{n^q}{q} S_q\left(\frac{n_1}{n},\cdots,\frac{n_k}{n}\right)$$

(μ,ν,q)-relation $\quad \nu(1-\mu)+1=q$

Suyari&Wada
(*Physica A*, 387, 2008.)

This correspondence recovers the 4 typical mathematical structures as special cases:

1. additive duality $\quad q \leftrightarrow 2-q$
2. multiplicative duality $\quad q \leftrightarrow \dfrac{1}{q}$
3. q-triplet



4. multifractal triplet

$$\frac{1}{1-q_{\rm sen}} = \frac{1}{q-1} - \frac{1}{q}$$

$$\frac{1}{\nu}\ln_{\mu}\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,\nu)} \cong \frac{n^q}{q} S_q\left(\frac{n_1}{n},\dots,\frac{n_k}{n}\right)$$

(μ,ν,q)-relation $\quad \nu(1-\mu)+1=q$

(i) additive duality $\quad q \leftrightarrow 2-q \qquad \nu=1 \;\Rightarrow\; \mu=2-q$

$$\ln_{2-q}\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(2-q,1)} \cong \frac{n^q}{q} S_q\left(\frac{n_1}{n},\dots,\frac{n_k}{n}\right)$$

(ii) multiplicative duality $\quad q \leftrightarrow \frac{1}{q} \qquad \nu=q \;\Rightarrow\; \mu=\frac{1}{q}$

$$\frac{1}{q}\ln_{\frac{1}{q}}\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{\left(\frac{1}{q},q\right)} \cong \frac{n^q}{q} S_q\left(\frac{n_1}{n},\dots,\frac{n_k}{n}\right)$$

(μ, ν, q)-relation

$$\nu(1-\mu)+1=q$$

(iii) *q*-triplet

$$\nu = 2 - q \;\Rightarrow\; \mu = \frac{3-2q}{2-q}$$

Tsallis et al (2005) conjectured

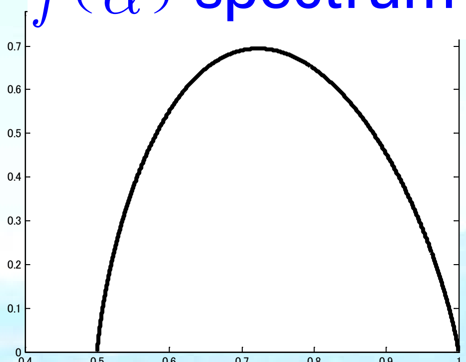
$$q_{\mathrm{rel}} + \frac{1}{q_{\mathrm{sen}}} = q_{\mathrm{stat}} + \frac{1}{q_{\mathrm{rel}}} = 2 \;\Rightarrow\; \frac{1}{q_{\mathrm{sen}}} = \frac{3-2q_{\mathrm{stat}}}{2-q_{\mathrm{stat}}}$$

$$(\mu, \nu, q) = \left(\frac{1}{q_{\mathrm{sen}}}, \frac{1}{q_{\mathrm{rel}}}, q_{\mathrm{stat}}\right)$$

(iv) multifractal triplet

$$\nu = \frac{1}{q} \;\Rightarrow\; \frac{1}{1-\mu} = \frac{1}{q-1} - \frac{1}{q}$$

$f(\alpha)$ spectrum



α_{min} α_{max}

Rescale $|\alpha_{\mathrm{max}} - \alpha_{\mathrm{min}}| = 1$

Lyra&Tsallis (PRL80, 1998.)

$$\frac{1}{1-q_{\mathrm{sen}}} = \frac{1}{\alpha_{\mathrm{min}}} - \frac{1}{\alpha_{\mathrm{max}}}$$

(μ, ν, q) -relation

$$\nu(1-\mu)+1=q$$

$$\frac{1}{\nu}\ln_{\mu}\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,\nu)} \cong \frac{n^q}{q} S_q\left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right)$$

This general formula recovers the 4 important mathematical structures as special cases.

Up to now, this is the unique general formula to recover the 4 important mathematical structures in Tsallis statistics.

Law of error
(pioneer: Gauss(1809))

Consider repeated measurements of an observable.

- **observed values** $x_1, \cdots, x_n$ **: given**
- **true value** $\hat{x}$ **: unknown**

Of course!

Typical errors in measurement:

1. **additive error** $e_a := x_i - \hat{x}$
2. **multiplicative error** $e_m := \frac{x_i}{\hat{x}} \left(> 0 \right)$

$\left(\hat{x} \neq 0 \right)$

Law of additive error

(Suyari, IEEE Inform.Theory, 2005.)

$x_1, \cdots, x_n$: n observed values of an observable

$\theta^* := \dfrac{x_1 + \cdots + x_n}{n}$: true value of an observable

Gauss' law of error

additive error

Likelihood function $L(\theta) := f(x_1 - \theta) \cdots f(x_n - \theta)$

$L(\theta)$ takes the maximum at $\theta = \theta^*$ ⇒ f :Gaussian

Law of error in Tsallis statistics

Likelihood function $L_q(\theta) := f(x_1 - \theta) \otimes_q \cdots \otimes_q f(x_n - \theta)$

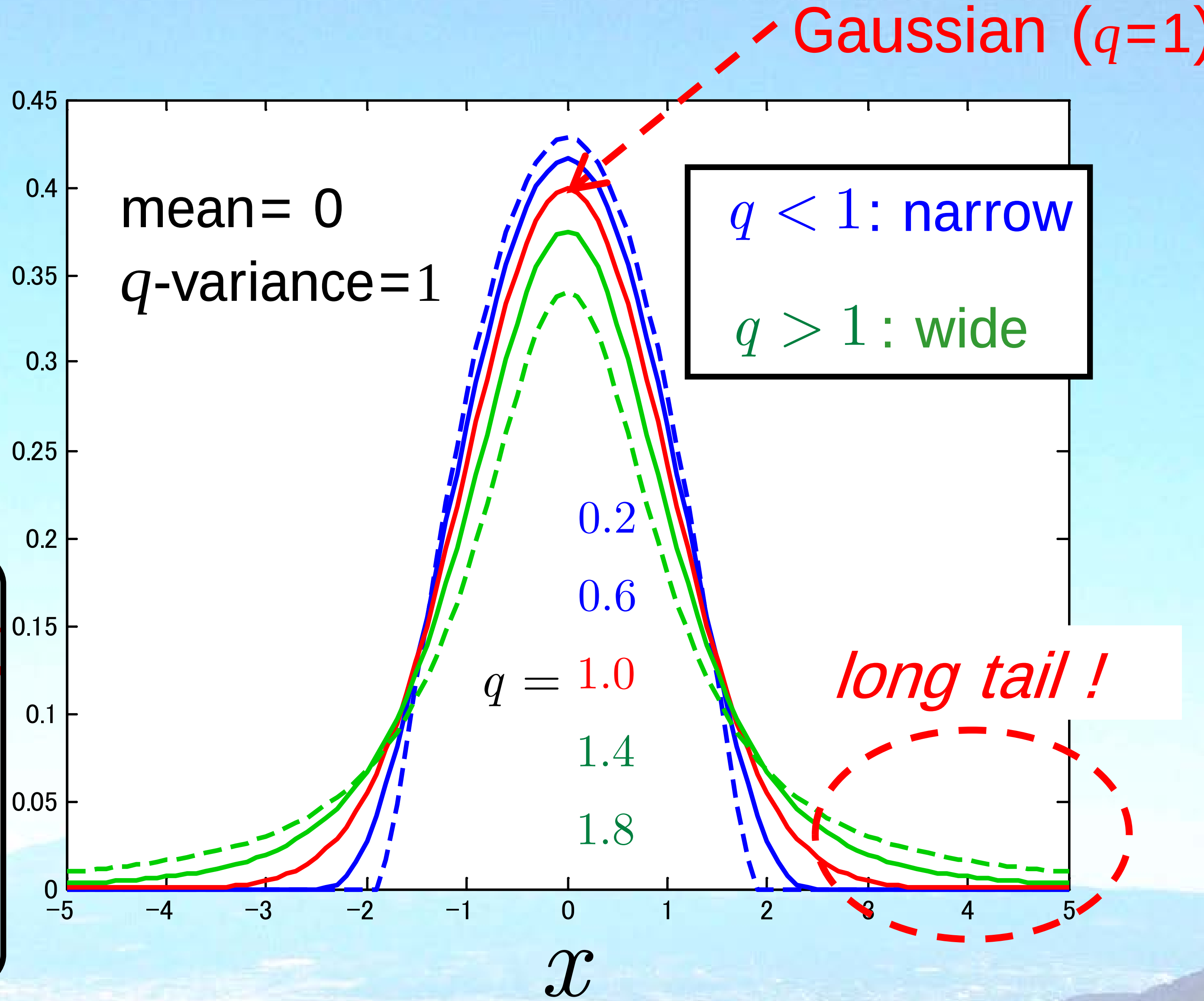
$L_q(\theta)$ takes the maximum at $\theta = \theta^*$ ⇒ f: q-Gaussian

q-Gaussian

$$f_q(x) = \frac{\exp_q\left[-\beta_q x^2\right]}{Z_q} = \frac{\left[1-\beta_q(1-q)x^2\right]^{\frac{1}{1-q}}}{Z_q}$$

$$(\beta_q > 0)$$

↑ Power function

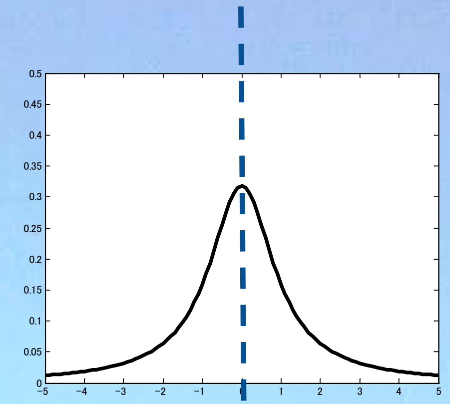


q-Gaussian Family

$q=2$

Cauchy dist.

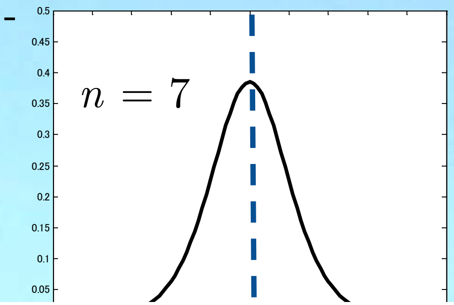
$$f(x)=\frac{1}{\pi\sigma}\left[1+\frac{(x-\mu)^2}{\sigma^2}\right]^{-1}$$



$$q=1+\frac{2}{n+1}$$

***t* -dist.**

$$f(x)=\frac{1}{\sqrt{n}B\left(\frac{n}{2},\frac{1}{2}\right)\sigma}\left[1+\frac{1}{n}\frac{(x-\mu)^2}{\sigma^2}\right]^{-\frac{n+1}{2}}$$

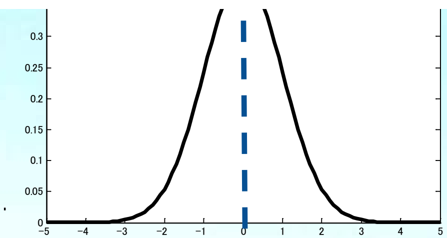


Interpolation by one-parameter (q)

$q=1$

Gauss dist.

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

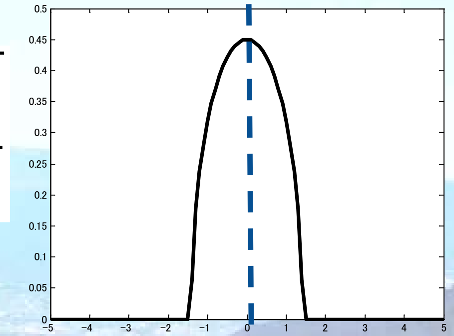


$q=-1$

Wigner semicircle dist.

$$f(x)=\frac{\sqrt{2}}{\pi\sigma}\sqrt{1-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

(Most important dist. in Random Matrix Theory)



Law of multiplicative error

(Suyari, Entropy(2013))

$x_1, \cdots, x_n \ (>0)$: n observed values of an observable

$\theta^* := \frac{\ln x_1 + \cdots + \ln x_n}{n}$: true value of an observable

Gauss' law of error — **multiplicative error**

Likelihood function $L(\theta) := f(\ln x_1 - \theta) \cdots f(\ln x_n - \theta)$

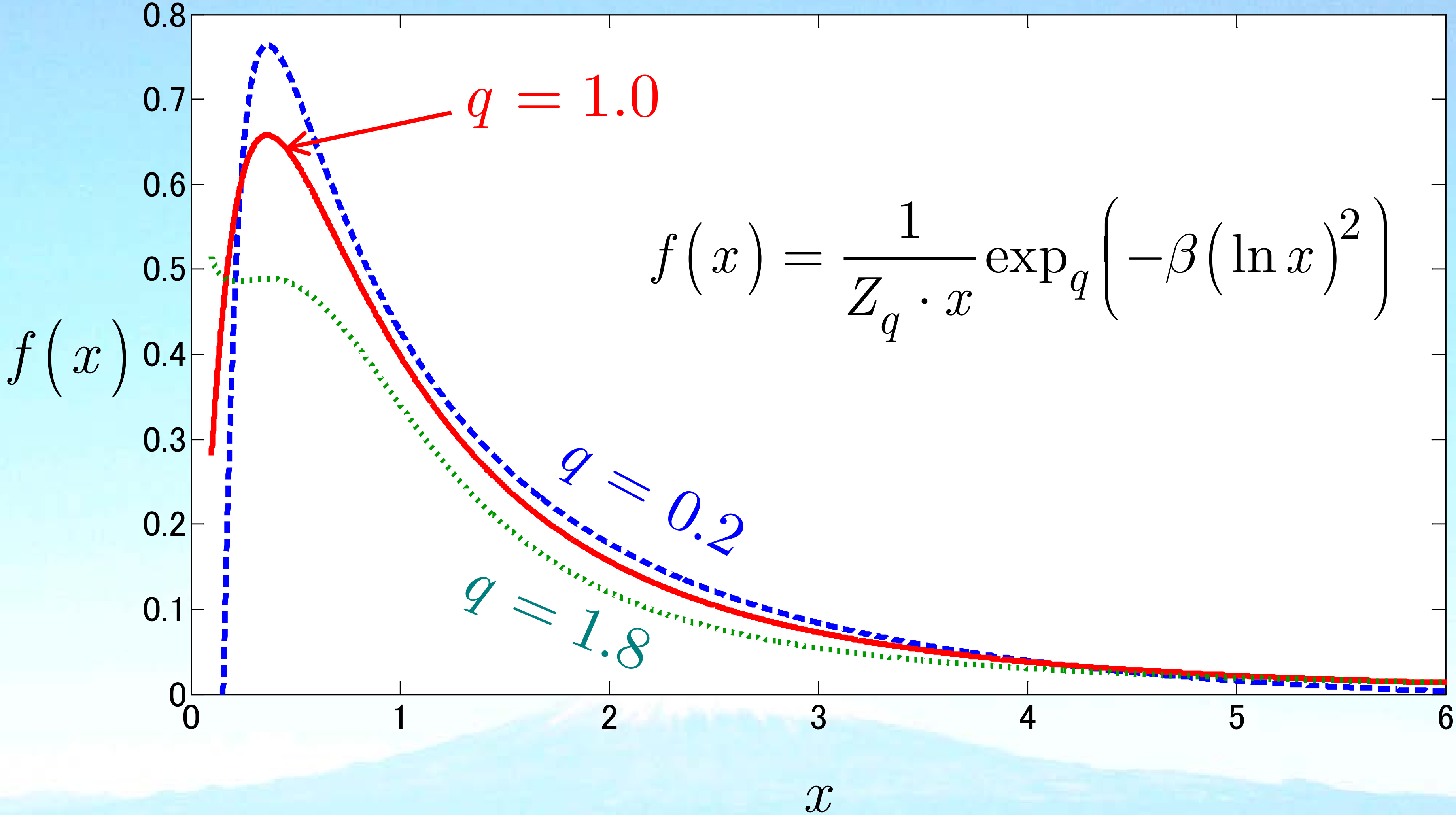
$L(\theta)$ takes the maximum at $\theta = \theta^*$ ⇒ f : log-normal

Law of multiplicative error in Tsallis statistics

Likelihood function $L_q(\theta) := f(\ln x_1 - \theta) \otimes_q \cdots \otimes_q f(\ln x_n - \theta)$

$L_q(\theta)$ takes the maximum at $\theta = \theta^*$ ⇒ f: log-q-normal

Log-*q*-normal



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
$f(x)$
x
$q = 1.0$
$q = 0.2$
$q = 1.8$
$f(x) = \frac{1}{Z_q \cdot x} \exp_q\left(-\beta(\ln x)^2\right)$

Summary

- **The unique entropy corresponding to the *q*-exponential function is Tsallis entropy.**

- **General formula to recover the 4 important mathematical structures as special cases**

$$\frac{1}{\nu}\ln_{\mu}\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,\nu)} \cong \frac{n^q}{q} S_q\left(\frac{n_1}{n},\cdots,\frac{n_k}{n}\right)$$

- **Law of error in Tsallis statistics**

  - **additive error** ⇒ ***q*-Gaussian**
  - **multiplicative error** ⇒ **log-*q*-normal (log-*q*-Gaussian)**

Log-*q*-normal and *Q*-log-normal

Law of error consists of the two parts:

1. Repeated measurements:

Independent or Correlated ?

2. Kinds of error:

Additive, Multiplicative or others ?

Log-*q*-normal (Suyari(2013))

$$f(x) \propto \frac{1}{x} \exp_q \left(-\beta (\ln x)^2 \right)$$

Correlated measurement & Multiplicative error

***Q*-log-normal (Queiros(2009,2012))**

$$f(x) \propto \frac{1}{x^q} \exp \left(-\beta (\ln_q x)^2 \right)$$

Independent measurement & *q*-Multiplicative(?) error

Derivation of Queiros' *q*-log-normal dist. in the framework of the law of error

1. Repeated measurements: independent
2. Kinds of error: *q*-Multiplicative(?) error

- Standard multiplicative error $e_m := \frac{x_i}{\hat{x}} (> 0)$ $(\hat{x} \neq 0)$

$$\ln x_i = \ln \hat{x} + \ln e_m$$

($x_i > 0$ is assumed without loss of generality)

- *Q*-multiplicative(?) error

$$\ln_q x_i = \ln_q \hat{x} + \ln_q e_m$$

$$\ln_q \frac{x_i}{e_m} = \frac{\ln_q \hat{x}}{e_m^{1-q}}$$

Scaling effect