Three deformations of generalized Leibniz triangles

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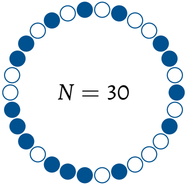
An elementary model

Consider a set of N *identical binary random variables* $\{x_i\}_{i=1,\ldots,N}$, $x_i \in \{0, 1\}$



Let us call $r_{N,n}$ the probability of having $\sum_i x_i = n$ *in a certain configuration*. Due to indistinguishability, the probability of having $\sum_i x_i = n$ is

$$p_{N,n} = \binom{N}{n} r_{N,n}.$$

Pascal triangle

Or the *Meru–prastaara*, the Staircase of Mount Meru

It is well known that the binomial coefficients can be arranged in a triangle:

Piṅgala, II century BC, *India*

Varāhamirhira, 505 AD, *India*

Mahāvīra, 850 AD, *India*

Halayudha, 975 AD, *India*

Bhaṭṭotpala, 1086 AD, *India*

Al-Karaji, XI century AD, *Iran*

Omar Khayyám, XI–XII century AD, *Iran*

Jia Xian, XI century AD, *China*

Yang Hui, XIII century AD, *China*

Levi ben Gershon, XIV century AD, *France*

Peter Apian, 1527 AD, *Germany*

Michael Stifel, 1544 AD, *Germany*

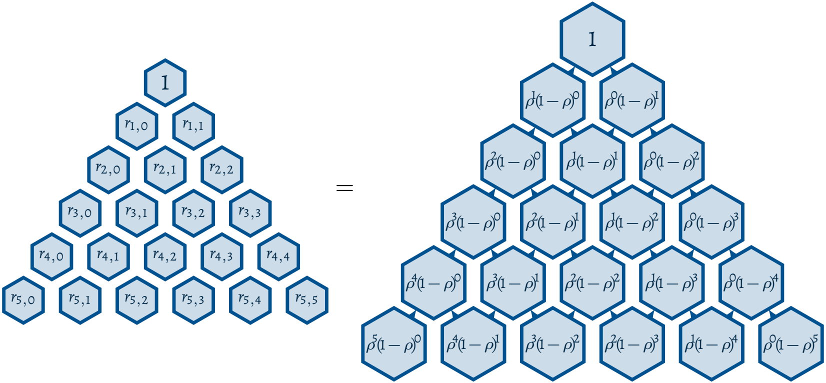
Niccolò Tartaglia, 1556 AD, *Italy*

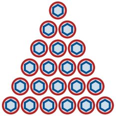
Gerolamo Cardano, 1570 AD, *Italy*

Blaise Pascal, 1665 AD, *France*

Independent variables

For *independent variables*, $r_{N,n} = \rho^n (1-\rho)^{N-n}$ for $\rho \in (0,1)$ and therefore $p_{N,n}$ is the binomial distribution.



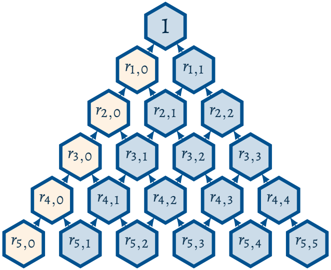
$p_{N,n} =$  $\overset{N \gg 1}{\sim} \frac{1}{\sqrt{2\pi N \rho(1-\rho)}} e^{-\frac{N}{2\rho(1-\rho)}\left(\frac{n}{N}-\rho\right)^2}.$

Introducing correlations: Leibniz triangle

Leibniz considered the set of triangles satisfying the following *scale–invariance property*:

$$r_{N,n+1} + r_{N,n} = r_{N-1,n}$$

Moreover, this constraint *uniquely* determines the entries of the triangle *if* the values $r_{N,0}$ are given.



Leibniz chose

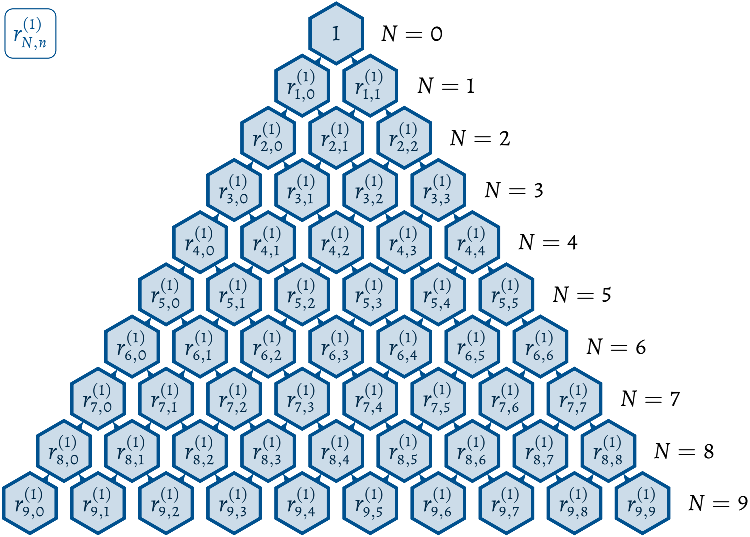
$$r^{(1)}_{N,0} = \frac{1}{N+1} \Longrightarrow r^{(1)}_{N,n} = \frac{1}{N+1}\,\frac{1}{\binom{N}{n}}.$$

Obviously, the limiting distribution is the *uniform distribution*.

A model for q–Gaussians

Rodríguez, Schwämmle, Tsallis – J. Stat. Mech., **2008**(09), P09006 (2008).

$r^{(1)}_{N,n}$



A model for q–Gaussians

Rodríguez, Schwämmle, Tsallis – J. Stat. Mech., **2008**(09), P09006 (2008).

$$r^{(3)}_{N,n} = \frac{r^{(1)}_{N+4,n+2}}{r^{(1)}_{4,2}}$$

→ *scale–invariant*

and in general

$$r^{(\nu)}_{N,n} = \frac{r^{(1)}_{N+2(\nu-1),n+\nu-1}}{r^{(1)}_{2(\nu-1),\nu-1}}$$

↓

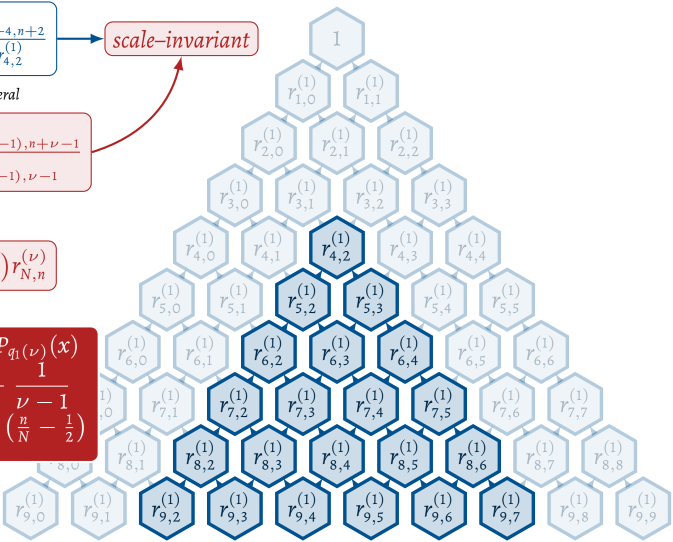
$$p^{(\nu)}_{N,n} = \binom{N}{n} r^{(\nu)}_{N,n}$$

↓

q–Gaussian $P_{q_1(\nu)}(x)$

$$q_1(\nu) = 1 - \frac{1}{\nu - 1}$$

$$x = \frac{2}{\sqrt{1-q_1(\nu)}}\left(\frac{n}{N} - \frac{1}{2}\right)$$

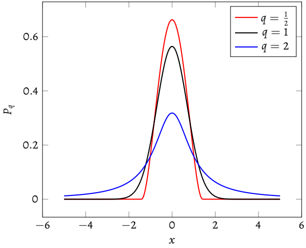


The q–Gaussian distribution is defined as follows:

$$P_q(x) := \begin{cases} \frac{3-q}{2}\sqrt{\frac{1-q}{\pi}}\frac{\Gamma\left(\frac{3-q}{2-2q}\right)}{\Gamma\left(\frac{1}{1-q}\right)}\left[1-(1-q)x^2\right]_+^{\frac{1}{1-q}} & \text{for } q<1, \\ \frac{e^{-x^2}}{\sqrt{\pi}} & \text{for } q=1, \\ \sqrt{\frac{q-1}{\pi}}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{3-q}{2q-2}\right)}\left[1-(1-q)x^2\right]^{\frac{1}{1-q}} & \text{for } 1<q<3. \end{cases}$$

In the previous definition

$$[x]_+ := \begin{cases} x & x>0, \\ 0 & x\leq 0. \end{cases}$$



Let us now **deform** the generalized triangles above introducing **two deformation of the natural numbers**.

The α–numbers

Given $n \in \mathbb{N} \cup 0$, and $\alpha > 0$, $\alpha \neq 1$,

$$\{n\}_\alpha := (n+1)\left(1 - \frac{1-\alpha}{1-\alpha^{n+1}}\right).$$

We define the α–binomial as

$$\begin{Bmatrix} N \\ n \end{Bmatrix}_\alpha := \frac{\prod_{k=1}^{N} \{k\}_\alpha}{\prod_{k=1}^{N-n} \{k\}_\alpha \prod_{k=1}^{n} \{k\}_\alpha}.$$

$$\lim_{\alpha \to 1} \{n\}_\alpha = n.$$

The β–numbers

Given $n \in \mathbb{N} \cup 0$, and $\beta > 0$, $\beta \neq 1$,

$$[n]_\beta := n\left(1 - \frac{1-\beta}{1-\beta^{n}}\right) + 1.$$

We define the β–binomial as

$$\begin{bmatrix} N \\ n \end{bmatrix}_\beta := \frac{\prod_{k=1}^{N} [k]_\beta}{\prod_{k=1}^{N-n} [k]_\beta \prod_{k=1}^{n} [k]_\beta}.$$

$$\lim_{\beta \to 1} [n]_\beta = n.$$

$$\lim_{n \to \infty} \frac{[n]_\alpha}{\{n\}_\alpha} = 1.$$

The Leibniz α–triangles

$$r^{(1)}_{N,n,\alpha} := \frac{1}{\{N+1\}_\alpha \left\{ {N \atop n} \right\}_\alpha}$$

For $\nu \in \mathbb{N}$:

$$r^{(\nu)}_{N,n,\alpha} := \frac{r^{(1)}_{N+2(\nu-1),n+\nu-1,\alpha}}{\sum_{k=0}^{N} \underbrace{\binom{N}{k} r^{(1)}_{N+2(\nu-1),k+\nu-1,\alpha}}_{p^{(\nu)}_{N,k,\alpha}}}$$

The Leibniz β–triangles

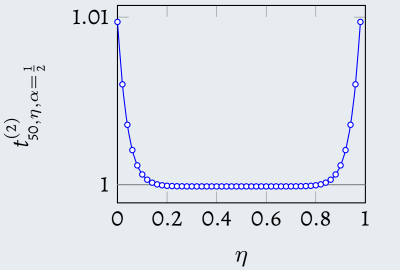
$$r^{(1)}_{N,n,\beta} := \frac{1}{[N+1]_\beta \left[{N \atop n} \right]_\beta}$$

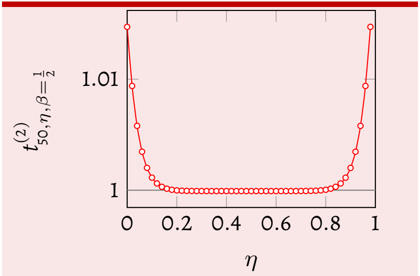
For $\nu \in \mathbb{N}$:

$$r^{(\nu)}_{N,n,\beta} := \frac{r^{(1)}_{N+2(\nu-1),n+\nu-1,\beta}}{\sum_{k=0}^{N} \underbrace{\binom{N}{k} r^{(1)}_{N+2(\nu-1),k+\nu-1,\beta}}_{p^{(\nu)}_{N,k,\beta}}}$$

Both the deformed triangles satisfy the scale invariance condition **asymptotically**!

$$\lim_{\substack{N\to\infty \\ \frac{n}{N}\equiv\eta}} \frac{r^{(\nu)}_{N-1,n,\bullet}}{\underbrace{r^{(\nu)}_{N,n+1,\bullet}+r^{(\nu)}_{N,n,\bullet}}_{t^{(\nu)}_{N,\eta,\bullet}}} = 1$$





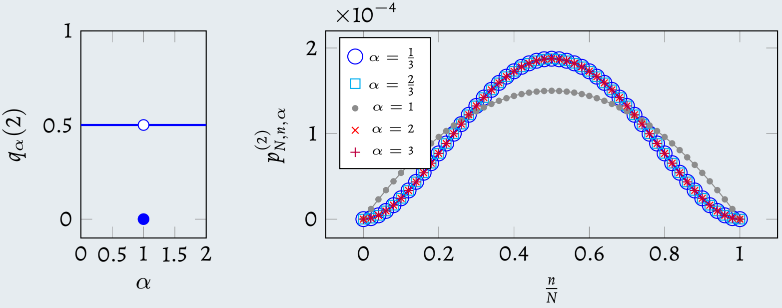
Robustness of α–triangles

We have that

$$\frac{N}{2\sqrt{\nu-\delta_{\alpha,1}}} p^{(\nu)}_{N,n,\alpha} \overset{N\gg 1}{\sim} P_{q_\alpha(\nu)}(x), \quad x := 2\sqrt{\nu-\delta_{\alpha,1}}\left(\frac{n}{N}-\frac{1}{2}\right),$$

where

$$q_\alpha(\nu) := \begin{cases} 1-\frac{1}{\nu} & \text{for } \alpha \neq 1, \\ 1-\frac{1}{\nu-1} & \text{for } \alpha = 1. \end{cases}$$



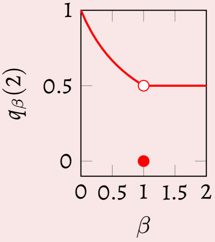
Robustness of β–triangles

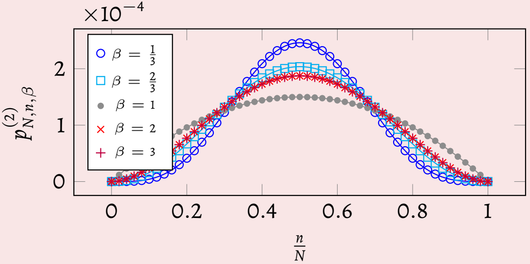
We have that

$$\frac{N}{2\sqrt{\nu-\chi(\beta)}}p^{(\nu)}_{N,n,\beta} \stackrel{N\gg 1}{\sim} P_{q_\beta(\nu)}(x), \quad x := 2\sqrt{\nu-\delta_{\beta,1}+\frac{1-\min\{\beta,1\}}{\min\{\beta,1\}}}\left(\frac{n}{N}-\frac{1}{2}\right),$$

where

$$q_\beta(\nu) = \begin{cases} 1-\frac{1}{\nu} & \text{for } \beta > 1, \\ 1-\frac{1}{\nu-1} & \text{for } \beta = 1, \\ 1-\frac{\beta}{\beta\nu+1-\beta} & \text{for } 0 < \beta < 1. \end{cases}$$





α–triangles

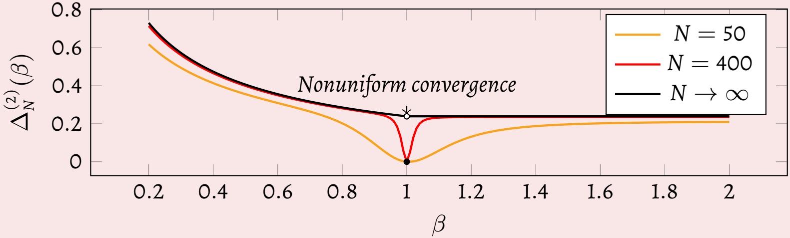
$$\frac{1}{1-q_\alpha(\nu)} = \frac{1}{1-q_1(\nu)} + 1, \quad \alpha \neq 1$$

β–triangles

$$\frac{\min\{\beta,1\}}{1-q_\beta(\nu)} = \frac{\min\{\beta,1\}}{1-q_1(\nu)} + 1, \quad \beta \neq 1$$

β–triangles

$$\Delta_N^{(\nu)}(\beta) := \sqrt{N \sum_{k=0}^{N} \left| p_{N,k,\beta}^{(\nu)} - p_{N,k}^{(\nu)} \right|^2}$$



Inspired by the *Q–calculus*, we considered an alternative deformation of the Leibniz triangle based on the so called *Q–numbers*:

$$[\![n]\!]_Q := \frac{1-Q^n}{1-Q}, \quad Q \in (0,\infty)\setminus\{1\}, \quad \lim_{Q\to 1}[\![n]\!]_Q = n.$$

We can introduce also the *Gauss binomial coefficients*

$$\begin{bmatrix} N \\ n \end{bmatrix}_Q := \frac{\prod_{k=1}^{N}[\![k]\!]_Q}{\prod_{k=1}^{N-n}[\![k]\!]_Q \prod_{k=1}^{n}[\![k]\!]_Q} \xrightarrow{Q\to 1} \binom{N}{n}.$$

Q–triangles

$$r^{(1)}_{N,n,Q} := \frac{1}{[\![N+1]\!]_Q}\frac{1}{\begin{bmatrix} N \\ n \end{bmatrix}_Q} \Longrightarrow r^{(\nu)}_{N,n,Q} := \frac{r^{(1)}_{N+2(\nu-1),n+\nu-1,Q}}{\sum_{k=0}^{N}\underbrace{\binom{N}{k} r^{(1)}_{N+2(\nu-1),k+\nu-1,Q}}_{p^{(\nu)}_{N,k,Q}}}$$

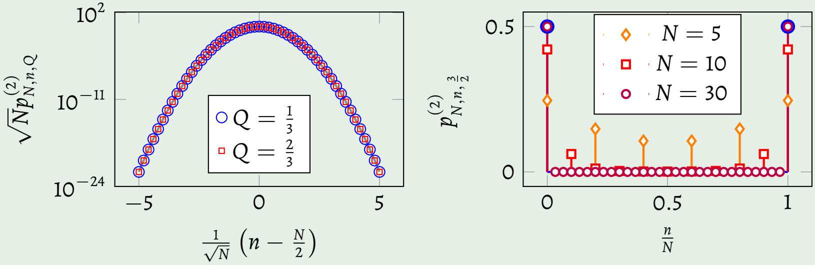
The Q–triangles are **not** asymptotically scale invariant! Indeed

$$\lim_{\substack{N\to\infty \\ \frac{n}{N}\equiv\eta \text{ fixed}}} \frac{r^{(\nu)}_{N-1,n,Q}}{r^{(\nu)}_{N,n,Q}+r^{(\nu)}_{N,n+1,Q}} = \begin{cases} \frac{1}{2} & \text{for } Q<1, \\ 0 & \text{for } Q>1. \end{cases}$$

Robustness of Q–triangles

We have that

$$p^{(\nu)}_{N,n,Q} \overset{N\gg 1}{\sim} \begin{cases} \sqrt{\frac{2}{\pi N}}\, e^{-2N\left(\eta-\frac{1}{2}\right)^2} & \text{if } 0<Q<1, \\ \frac{\delta_{n,0}+\delta_{n,N}}{2} & \text{if } Q>1. \end{cases}$$



We deformed the generalized Leibniz triangle to evaluate the robustness of the limiting distributions in three different ways. We obtained the following results

- The *asymptotically scale invariant deformed triangles* have still q–Gaussians as limiting distribution, but the limiting value of q depends in general on the form of the perturbation, even if two *asymptotically equivalent* perturbations are considered.
- Switching from exact scale invariance to asymptotically scale invariance can make a *discontinuity* appear in the limiting value of q as function of the perturbation parameter.
- The *not-asymptotically scale invariant deformed triangle* can have a limiting distribution outside the family of q–Gaussians.

These results suggest that the (asymptotic) scale–invariance property plays a central role in the robustness of the set of q–Gaussian distributions as limiting distributions. More specifically, the set of q–Gaussians appears to be *robust* under asymptotically scale–invariant deformations.

Thank you
for your attention!