Paradoxical asymptotic independence for strongly correlated systems

Max Jáuregui and Constantino Tsallis

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3. Probabilistic model with q-Gaussian attractors
4. Some properties of the model
5. Case $q < 1$: Analogy to a quantum spin chain
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Preliminaries

Nonextensive statistical mechanics is based on the S_q entropy, defined for a discrete probability distribution $p = (p_1, p_2, \ldots)$ by

$$S_q(p) = k_{\mathrm{B}} \sum_i p_i \ln_q \frac{1}{p_i}, \quad q \in \mathbb{R}, \tag{1}$$

where $\ln_q x$ denotes the q-logarithm, defined for $x > 0$ by

$$\ln_q x = \int_1^x \frac{\mathrm{d}y}{y^q} = \begin{cases} \dfrac{x^{1-q} - 1}{1 - q} & \text{if } q \neq 1 \\ \ln x & \text{if } q = 1. \end{cases} \tag{2}$$

The inverse of the q-logarithm is the q-exponential, defined by

$$\mathrm{e}_q^x = \begin{cases} [1 + (1-q)x]^{1/(1-q)} & \text{if } q \neq 1 \\ \mathrm{e}^x & \text{if } q = 1. \end{cases} \tag{3}$$

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The *q*-Gaussian distribution with parameters $q < 3$ and $\beta > 0$ is defined by

$$g_{q,\beta}(x)\,\mathrm{d}x = \begin{cases} \dfrac{\sqrt{\beta}}{N_q} e_q^{-\beta x^2}\,\mathrm{d}x & \text{if } 1-(1-q)\beta x^2 > 0 \\ 0 & \text{if } 1-(1-q)\beta x^2 \le 0, \end{cases} \tag{4}$$

where N_q is a normalization constant.

- q-Gaussian distributions have compact support if $q < 1$, namely the interval

$$\left[-\frac{1}{\sqrt{(1-q)\beta}}, \frac{1}{\sqrt{(1-q)\beta}}\right]; \tag{5}$$

  otherwise their support is $\mathbb{R}$.

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q-generalisation of the central limit theorem

The escort distribution of order q of a discrete random variable (r.v.) X with distribution $p_i = P[X = x_i]$ is defined by

$$p_i^{(q)} = \frac{p_i^q}{\sum_j p_j^q}\,. \tag{6}$$

[Beck and Schlögl, *Thermodynamics of chaotic systems*, Cambridge University Press, 1993]

The q-expectation and $(2q-1)$-variance of X are defined by

$$\langle X\rangle_q = \sum_i x_i p_i^{(q)} \quad \text{and} \quad \mathrm{Var}_{2q-1} X = \langle X - \langle X\rangle_q\rangle_{2q-1}\,. \tag{7}$$

The q-Fourier transform of X is defined by

$$\varphi_q[X](t) = \sum_i p_i \exp_q(\mathrm{i}txp_i^{q-1})\,, \quad 1 \le q < 2\,. \tag{8}$$

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Two r.v.'s X_1 and Y_1 are q-independent (of the first type) if

$$\boxed{\varphi_q[X+Y] = \varphi_q[X] \otimes_q \varphi_q[Y]\,, \quad 1 \le q < 2\,,} \tag{9}$$

where $X = X - \langle X_1 \rangle_q$, $Y = Y_1 - \langle Y_1 \rangle_q$ and $\otimes_q$ denotes the q-product, defined by

$$\boxed{a \otimes_q b = \begin{cases} (a^{1-q} + b^{1-q} - 1)^{1/(1-q)} & \text{if } q \neq 1 \\ ab & \text{if } q = 1. \end{cases}} \tag{10}$$

[Borges, Physica A **340**, 95 (2004)]

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q-generalisation of the central limit theorem

Let $X_1, X_2, \ldots$ be discrete q-independent r.v.'s with a common distribution $p_i = P[X_1 = x_i]$. If $0 < \mathrm{Var}_{2q-1} X_1 < \infty$, then there exists a r.v. Z having a q'-Gaussian distribution with parameters

$$q' = \frac{3q-1}{1+q} \quad e \quad \beta = \left(\frac{3-q'}{4qN_{q'}^{2(q'-1)}} \right)^{1/(2-q')} , \tag{11}$$

such that $\varphi_q[Z_n] \to \varphi_{q'}[Z]$, where

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\langle X_1 \rangle_q}{\sqrt{n \,\mathrm{Var}_{2q-1} X_1 \sum_j p_j^{2q-1}}} . \tag{12}$$

[Umarov, Tsallis and Steinberg, Milan J. Math. **76**, 307 (2008)]

- The intended proof given by Umarov et al. was based on the hypothesis of the invertibility of the q-Fourier transform.
- Hilhorst proved that the q-Fourier transform is not invertible for $q \neq 1$.

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- The following articles give sufficient conditions for the invertibility of the q-Fourier transform, and their results may be used in the elaboration of a correct proof for the q-generalised central limit theorem.

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Probabilistic model with q-Gaussian attractors

We consider the following triangular array of r.v.'s taking values in $\{0, 1\}$:

$$\begin{array}{lllll} X_{q,1,1} & & & & \\ X_{q,2,1} & X_{q,2,2} & & & \\ \vdots & \vdots & \ddots & & \\ X_{q,n,1} & X_{q,n,2} & \cdots & X_{q,n,n} & \\ \vdots & \vdots & & \vdots & \ddots \end{array} \quad (13)$$

The r.v.'s in each line are exchangeable, i.e.

$$P[X_{q,n,1} = x_1, \ldots, X_{q,n,n} = x_n] = P[X_{q,n,1} = x_{\pi(1)}, \ldots, X_{q,n,n} = x_{\pi(n)}] \quad (14)$$

for any permutation π of $\{1, \ldots, n\}$.

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The distribution of $S_{q,n} := X_{n,1} + \cdots + X_{n,n}$ is defined by

$$\boxed{P[S_{q,n} = k] = \frac{1}{Z_{q,n}} \exp_q(-x_{q,n,k}^2)\,, \quad q \le 2\,,\ k = 0, 1, \ldots, n\,,} \tag{15}$$

where

$$x_{q,n,k} = \begin{cases} \sqrt{n+1}\left(\dfrac{k+1}{n+2} - \dfrac{1}{2}\right) & \text{if } 1 \le q \le 2 \\[2ex] \dfrac{1}{\sqrt{1-q}}\left[1 - 2\left(\dfrac{k+1}{n+2}\right)\right] & \text{if } q < 1 \end{cases} \tag{16}$$

and

$$Z_{q,n} = \sum_{k=0}^{n} \exp_q(-x_{q,n,k}^2)\,. \tag{17}$$

[Rodríguez, Schwämmle and Tsallis, J. Stat. Mech P09006 (2008)]

Some properties of the model

- $P[X_{q,n,i} = 1] = P[X_{q,n,i} = 0] = 1/2$.
- For $q < 1$, we have (analitically)

$$\frac{n}{2}\sqrt{1-q}P[S_{q,n} = k] \sim g_{q,1}(x_{q,n,k}), \quad k = 0, 1, \ldots, n. \quad (18)$$

- If $1 \leq q \leq 2$, we have (numerically)

$$\sqrt{n}P[S_{q,n} = k] \approx g_{q,1}(x_{q,n,k}), \quad k = 0, 1, \ldots, n. \quad (19)$$

- The r.v.'s $X_{q,n,1}, \ldots, X_{q,n,n}$ are strongly correlated.

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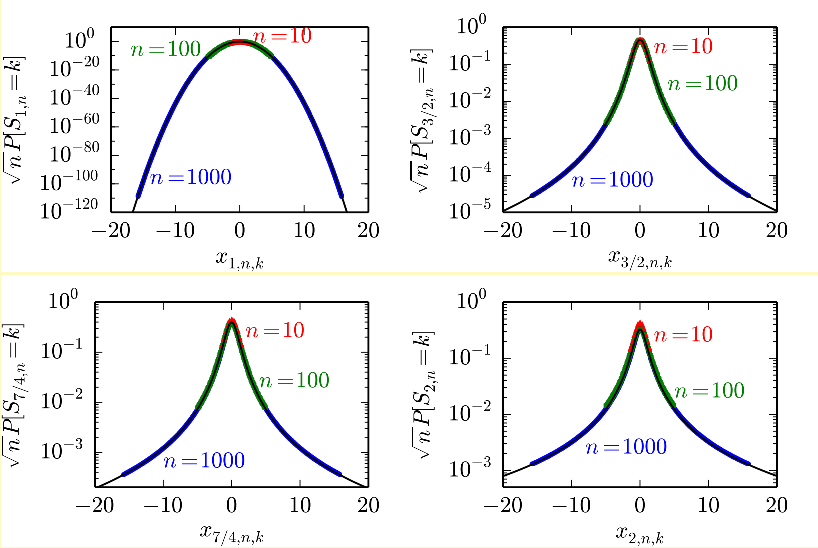

$\sqrt{n}P[S_{1,n}=k]$
$x_{1,n,k}$
$n=10$
$n=100$
$n=1000$
10^{0}
10^{-20}
10^{-40}
10^{-60}
10^{-80}
10^{-100}
10^{-120}
−20 −10 0 10 20
$\sqrt{n}P[S_{3/2,n}=k]$
$x_{3/2,n,k}$
$n=10$
$n=100$
$n=1000$
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
−20 −10 0 10 20
$\sqrt{n}P[S_{7/4,n}=k]$
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Case $q < 1$: Analogy to a quantum spin chain

We are going to study the asymptotic behaviour of the joint distribution of $m < n$ r.v.'s of the set $\{X_{q,n,1}, \ldots, X_{q,n,n}\}$. For simplicity, we will be only interested in the probabilities

$$\boxed{p_{q,n,m} = P[X_{q,n,1} = 1, \ldots, X_{q,n,m} = 1]\,.} \tag{20}$$

It can be verified that

$$p_{q,n,m} = \frac{1}{Z_{q,n}} \sum_{k=0}^{n-m} \left(\prod_{i=1}^{m} \frac{k+i}{n-i+1} \right) \exp_q(-x_{q,n,k+m}^2)\,. \tag{21}$$

If $q < 1$, we obtain, for instance, that

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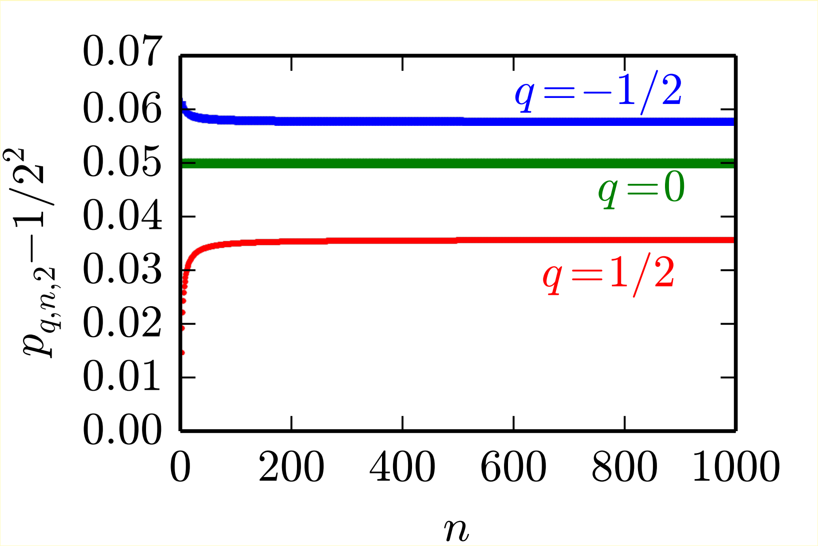

$p_{q,n,2}^{-1/2^2}$
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0
200
400
600
800
1000
n
$q=-1/2$
$q=0$
$q=1/2$

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Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

Filippo Caruso[1] and Constantino Tsallis[2,3]
[1] *NEST CNR-INFM and Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126 Pisa, Italy*
[2] *Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
[3] *Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*
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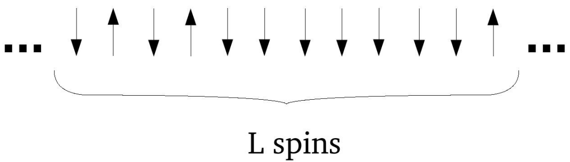
The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size L) of some (much larger) d-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to L^{d-1}. Here we show, for $d=1,2$, that the (nonadditive) entropy S_q satisfies, for a special value of $q\neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. **102** 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index q.

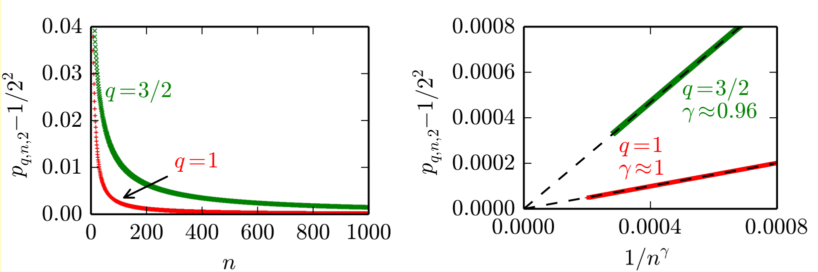
DOI: 10.1103/PhysRevE.78.021102 PACS number(s): 05.70.Jk, 05.30.−d



Case $q \geq 1$: A paradoxical result

If $q \geq 1$, we obtain that

$$\boxed{p_{q,n,2} \to \frac{1}{2^2}\,, \quad n \to \infty\,,} \tag{23}$$



Any two r.v.'s of the set $\{X_{q,n,1}, \dots, X_{q,n,n}\}$ become independent when $n \to \infty$.

It can be verified that, for a fixed $m < n$,

$$\mathrm{Var}(X_{n,1} + \cdots + X_{n,m}) = m \left(\frac{1}{2} - p_{q,n,2} \right) + m^2 \left(p_{q,n,2} - \frac{1}{4} \right) . \quad (24)$$

We see that, since $p_{q,n,2} \to 1/4$,

$$\boxed{\mathrm{Var}(X_{n,1} + \cdots + X_{n,m}) \to \frac{m}{4} , \quad n \to \infty .} \quad (25)$$

This suggests that any fixed number of r.v.'s of the set $\{X_{q,n,1}, \dots, X_{q,n,n}\}$ become independent when $n \to \infty$.

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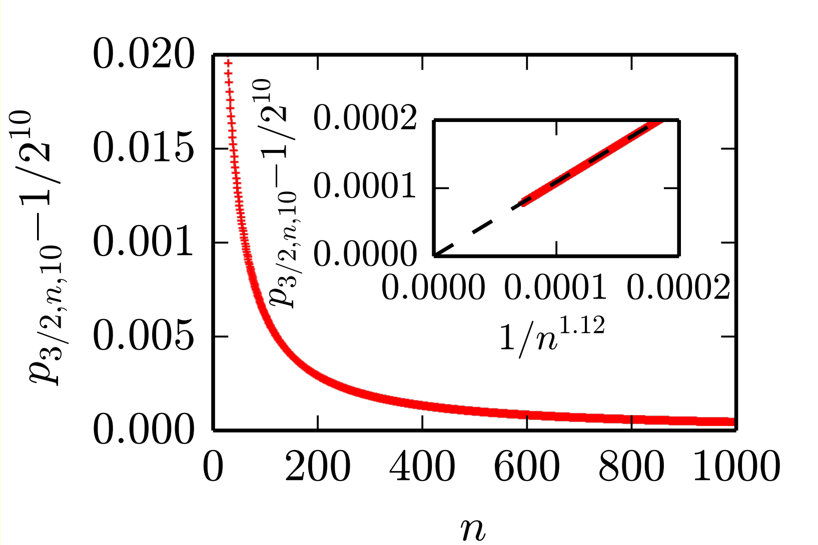
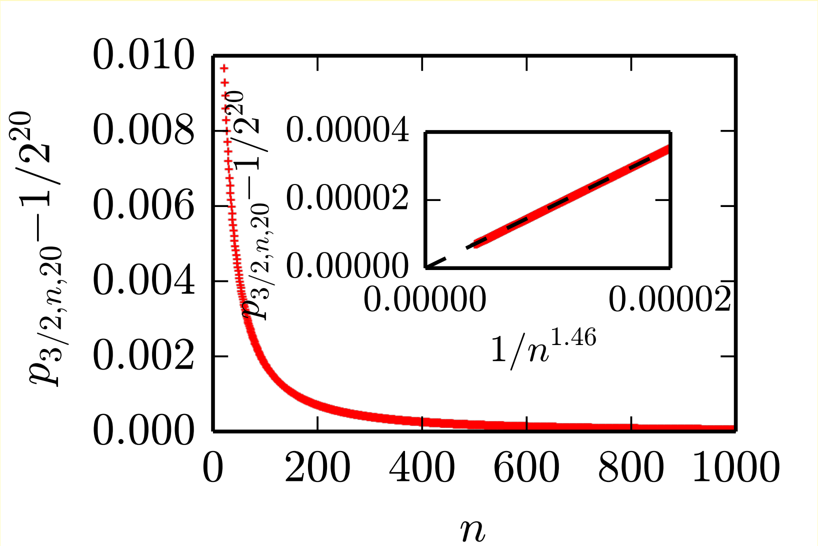
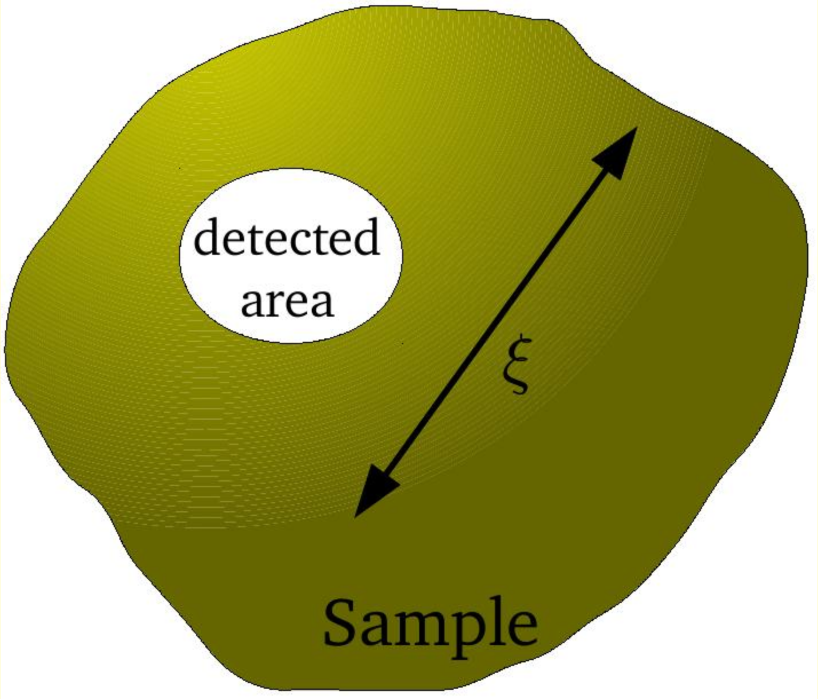
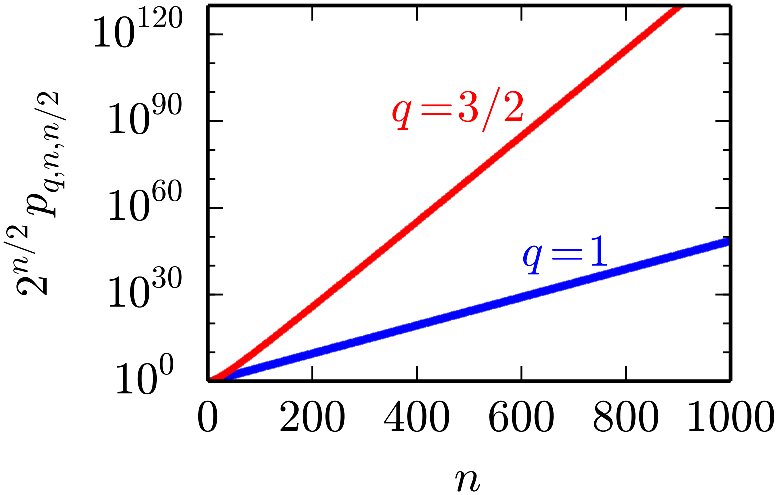


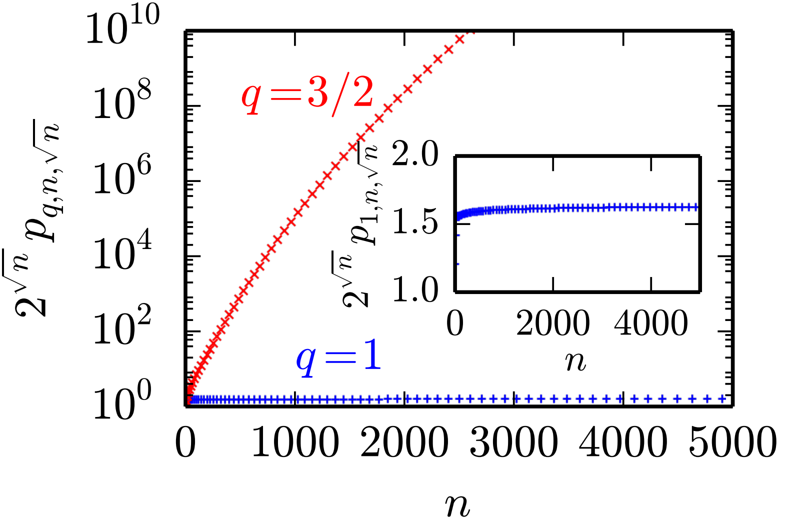
0.020
0.015
0.001
0.005
0.000
0
200
400
600
800
1000
n
$p_{3/2,n,10} - 1/2^{10}$
0.0002
0.0001
0.0000
0.0000
0.0001
0.0002
$1/n^{1.12}$
$p_{3/2,n,10} - 1/2^{10}$



0.010
0.008
0.006
0.004
0.002
0.000
0
200
400
600
800
1000
n
$p_{3/2,n,20} - 1/2^{20}$
0.00004
0.00002
0.00000
0.00000
0.00002
$1/n^{1.46}$
$p_{3/2,n,20} - 1/2^{20}$


detected area
ξ
Sample


10^{120}
10^{90}
10^{60}
10^{30}
10^{0}
$2^{n/2} p_{q,n,n/2}$
$q = 3/2$
$q = 1$
0
200
400
600
800
1000
n



10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{0}
$2^{\sqrt{n}} p_{q,n,\sqrt{n}}$
$q = 3/2$
$q = 1$
0
1000
2000
3000
4000
5000
n
2.0
1.5
1.0
$2^{\sqrt{n}} p_{1,n,\sqrt{n}}$
0
2000
4000
n

Conclusions

- We have studied the behaviour of marginal joint distributions in a probabilistic model involving strongly correlated r.v.'s which presents q-Gaussian distributions as limiting distributions.
- We conclude that, for $q < 1$ (compact support), the correlations are preserved in any subset of r.v.'s.
- For $q \geq 1$ (unbounded support), we have seen that the correlations in any finite subset of r.v.'s become negligible as the system grows. However, if the "size" of the subset of r.v.'s rapidly grows as the system grows, then the correlations are preserved in the subsystem.

[MJ and Tsallis, Phys. Lett. A **379**, 1816 (2015)]

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