Nonlinear Fokker-Planck Equations and Generalized Entropies for Complex Systems

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1. Stochastic Processes and Variables

- Basic notation and definitions:

Random variable $\equiv$ stochastic variable

Random variable X: number associated with an experiment $\Rightarrow$ represented by x

Probability density function (pdf) for X:
$P(x) > 0$ (probability distribution)

Probability for X to assume a value between x and x+dx: $P(x)dx$

Dimension of P(x): $[P(x)]_{\mathcal{D}} = ([x]_{\mathcal{D}})^{-1}$

Normalization: $$\int_{-\infty}^{\infty} P(x)dx = 1$$

Discrete variables:

$$x_1 \text{ (prob. } P_1), \; x_2 \text{ (prob. } P_2), \cdots$$

$$\Rightarrow P(x) = \sum_i P_i \delta(x - x_i) \; ; \qquad P_i \geq 0 \; ; \qquad \sum_i P_i = 1$$

- Expectation value (or average) of f(X):

$$\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x)P(x)dx$$

- Moment of order m:

$$\mu_m = \langle X^m \rangle = \int_{-\infty}^{\infty} x^m P(x)dx$$

Average: $\mu_1 = \langle X \rangle$

Centered variable: $\mu_1 = 0$

- Variance of X:

$$\sigma^2 = \langle (X - \langle X \rangle)^2 \rangle = \mu_2 - \mu_1^2$$

$$\sigma^2 \geq 0 \; ; \qquad \sigma^2 \to 0 \;\; : \;\; \Rightarrow X \text{ deterministic}$$

Mean-square-root deviation (width measure): σ

- Kurtosis:

$$\kappa = \frac{\langle (X - \langle X \rangle)^4 \rangle}{3 \langle (X - \langle X \rangle)^2 \rangle^2}$$

- Gaussian probability distribution:

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu_1)^2}{2\sigma^2}\right]$$

➡ $\kappa = 1$ for the Gaussian distribution

- Cauchy distribution (or Lorentzian):

$$P(x) = \frac{a}{\pi} \, \frac{1}{(x-x_0)^2 + a^2} \; ; \qquad (a > 0)$$

➡ $\mu_1 = x_0 \; ; \quad \mu_m \to \infty \; (\forall m \geq 2)$

- Distribuição q-gaussiana:

$$P(x) = \frac{1}{\sqrt{\pi A_q}} \exp_q\left[-\frac{(x-\mu_1)^2}{A_q}\right]$$

$$\exp_q(x) = [1 + (1-q)x]_+^{1/(1-q)}$$

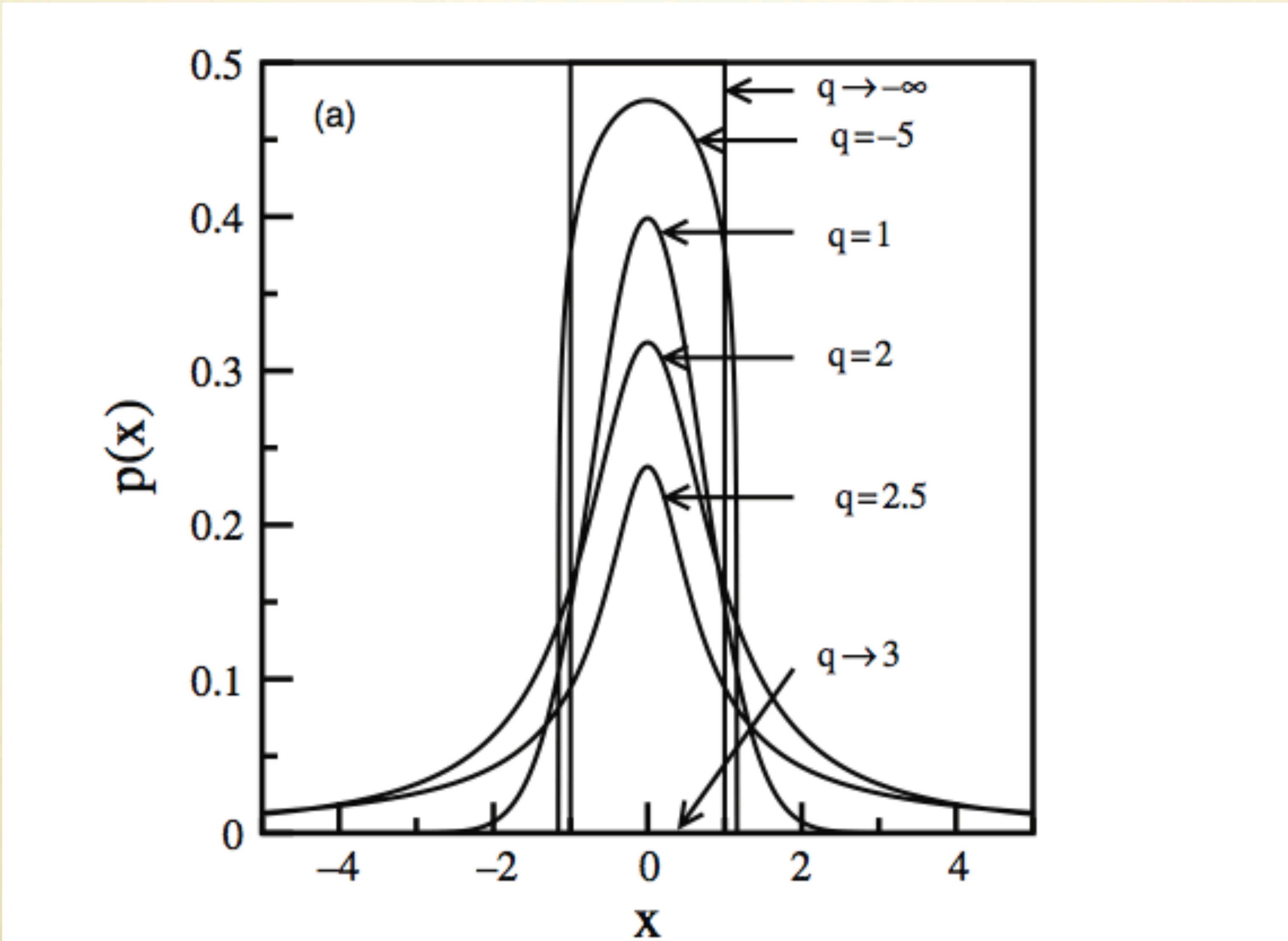
$$[y]_+ = y \ , \quad \text{if} \quad y > 0$$

$$[y]_+ = 0 \ , \quad \text{if} \quad y \leq 0$$

q=1 ➡ Gaussian

q=2 ➡ Lorentzian

$$P(x) = \frac{1}{\sqrt{\pi A_q}} \frac{1}{[1+(q-1)(x^2/A_q)]^{1/(q-1)}}$$



- Characteristic function:

$$G(k) = \langle \exp(ikX) \rangle = \int_{-\infty}^{\infty} \exp(ikx) P(x) dx$$

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) G(k) dk$$

$$G(k=0) = 1 \; ; \qquad |G(k)| \leq 1$$

➡ Generates moments: $G(k) = \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} \mu_m$

- Characteristic function of the Gaussian:

$$G(k) = \exp\left(i\mu_1 k - \frac{1}{2}\,\sigma^2 k^2 \right)$$

PS: G(k) is differentiable at k=0

➡ All moments are finite

All moments are expressed in terms of μ_1 e σ^2

- Characteristic function of the Lorentzian:

$$G(k) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp(ikx)}{(x-x_0)^2 + a^2}\, dx = \exp(-a|k| + ikx_0)$$

➡ Not differentiable at k=0

Moments are not defined

- Independent random variables:

P(x,y): prob. for X and Y to take the values x and y, respectively

If $$P(x,y) = P_X(x)P_Y(y)$$

$$\langle f(X)g(Y)\rangle = \langle f(X)\rangle\langle g(Y)\rangle$$

- Correlated random variables:

In this case, $P(x,y) \neq P_X(x) P_Y(y)$

Marginal probabilities:

$$P_X(x) = \int_{-\infty}^{\infty} P(x,y)dy \;; \qquad P_Y(y) = \int_{-\infty}^{\infty} P(x,y)dx$$

Conditional probability: X, for given Y

$$P(x|y) = \frac{P(x,y)}{\int_{-\infty}^{\infty} P(x,y)dx} = \frac{P(x,y)}{P_Y(y)}$$

- Correlations:

$$\langle XY \rangle \neq \langle X \rangle \langle Y \rangle$$

Correlation coefficient:

$$C_{XY} = \frac{\langle XY \rangle - \langle X \rangle \langle Y \rangle}{(\sigma_X^2 \sigma_Y^2)^{1/2}}$$

$C_{XY} = 0:$ uncorrelated variables

$C_{XY} = 1:$ maximum correlation

$C_{XY} = -1:$ maximum anticorrelation

- Central limit theorem (Laplace, 1812):

$\{X_1, X_2, \cdots, X_N\}$: independent random variables

$$Y_N = \frac{1}{\sqrt{N}}(X_1 + X_2 + \cdots + X_N)$$

⇒ P(y): Gaussian distribution

⇒ Gaussian distribution is so common !

Theorem applies to:

(a) Distinct individual laws

(b) Weakly correlated variables

- Theorem does not apply:

(a) $\{X_i\}$ with diverging moments

(b) Strongly correlated variables

2. Diffusion and Fokker-Planck Equations

- One-dimensional diffusion equation:

$$\frac{\partial P(x,t)}{\partial t} = D\,\frac{\partial^2 P(x,t)}{\partial x^2}$$

Form of a continuity equation:

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0 \; ; \qquad J(x,t) = -D\,\frac{\partial P(x,t)}{\partial x}$$

Conditions for a finite norm (finite times):

$$P(x,t)\big|_{x\to\pm\infty} = 0 \; ; \qquad \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0 \; ; \quad (\text{finite } t)$$

Preservation of norm:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(x,t)dx + J(x,t)|_{-\infty}^{\infty} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(x,t)dx - D\ \left(\frac{\partial P(x,t)}{\partial x}\right)_{-\infty}^{\infty}$$

$$\left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0\ ; \qquad \Rightarrow \quad \int_{-\infty}^{\infty} P(x,t)dx = \mathrm{cte}$$

➡ choosing $P(x,t_0)$ normalized, norm will be preserved for <u>finite times</u>

- Solution: Gaussian distribution

$$P(x,t) = \frac{1}{(4\pi D t)^{1/2}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right]$$

Einstein (Brownian motion): $D = \mu k T$

μ : medium mobility

Moments: $\mu_0 = \int_{-\infty}^{\infty} P(x,t)dx = 1$

$$\mu_1 = \langle X \rangle = \int_{-\infty}^{\infty} xP(x,t)dx = x_0$$

$$\mu_2 = \langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x,t)dx = x_0^2 + 2Dt$$

$$\sigma^2 = \mu_2 - \mu_1^2 = 2Dt = 2\mu kTt$$

Linear (normal) diffusion

$$t \to \infty \quad \Rightarrow \quad \sigma^2 \to \infty$$

(a) No stationary state

(b) Norm is lost as $t \to \infty$

- Porous medium equation:

$$\frac{\partial P(x,t)}{\partial t} = D\ \frac{\partial^2 P^\nu(x,t)}{\partial x^2}$$

Form of a continuity equation:

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0\ ; \qquad J(x,t) = -D\ \frac{\partial P^\nu(x,t)}{\partial x}$$

Conditions for a finite norm (finite times):

$$P(x,t)\big|_{x\to\pm\infty} = 0\ ; \qquad \left.\frac{\partial P^\nu(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0\ ; \qquad (\text{finite } t)$$

Solution: q-Gaussian distribution

$$P(x,t) = B(t)\left[1 - \beta(t)(\nu - 1)(x - x_0)^2\right]_+^{\frac{1}{\nu-1}}$$

PS: $\nu = 2 - q$

$$\sigma^2 \propto t^{2/(\nu+1)} \qquad (\nu > 1/3)$$

Nonlinear (anomalous) diffusion

(superdiffusion / subdiffusion)

Second moment diverges for $\nu \leq 1/3$

Linear Fokker-Planck equation

Diffusion under an external potential:

$$\phi(x) \quad \Rightarrow \quad A(x) = -d\phi(x)/dx$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + D\,\frac{\partial^2 P(x,t)}{\partial x^2}$$

$\phi(x)$ confining: stationary solution for $t \to \infty$

$$\frac{\partial P_{\mathrm{st}}(x)}{\partial t} = 0 \Rightarrow -\frac{\partial[A(x)P_{\mathrm{st}}(x)]}{\partial x} + D\,\frac{\partial^2 P_{\mathrm{st}}(x)}{\partial x^2} = 0$$

Conditions for preservation of norm:

$$P(x,t)|_{x\to\pm\infty} = 0 \ ; \ \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0 \ ; \ A(x)P(x,t)|_{x\to\pm\infty} = 0 \ (\forall t)$$

Form of a continuity equation:

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0$$

$$J(x,t) = A(x)P(x,t) - D\,\frac{\partial P(x,t)}{\partial x}$$

Preservation of norm:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(x,t)dx = -J(x,t)|_{-\infty}^{\infty}$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(x,t)dx = -A(x)P(x,t)|_{-\infty}^{\infty} + D\left(\frac{\partial P(x,t)}{\partial x}\right)_{-\infty}^{\infty} = 0$$

$\Rightarrow$ $$\int_{-\infty}^{\infty} P(x,t)dx = \text{ cte } \quad (\forall t)$$

$\Rightarrow$ choosing $P(x,t_0)$ normalized, norm will be preserved for all times

Stationary-state solution:

$$\frac{\partial P_{\mathrm{st}}(x)}{\partial t} = 0 \Rightarrow -\frac{\partial[A(x)P_{\mathrm{st}}(x)]}{\partial x} + D\ \frac{\partial^2 P_{\mathrm{st}}(x)}{\partial x^2} = 0$$

$$\frac{\partial}{\partial x}\left[-A(x)P_{\mathrm{st}}(x) + D\ \frac{\partial P_{\mathrm{st}}(x)}{\partial x}\right] = 0$$

$\Rightarrow \quad J_{\mathrm{st}}(x) = \mathrm{cte} \ = 0 \quad (\forall x)$ (norm conditions)

$$\frac{A(x)}{D}\ dx = \frac{dP_{\mathrm{st}}}{P_{\mathrm{st}}} \qquad \Rightarrow \qquad -\frac{d\phi}{D} = \frac{dP_{\mathrm{st}}}{P_{\mathrm{st}}}$$

$$\ln\left(\frac{P_{\mathrm{st}}(x)}{P_{\mathrm{st}}(0)}\right) = -\frac{\phi(x) - \phi_0}{D}$$

$$P_{\rm st}(x) = B\exp\left(-\frac{\phi(x)}{D}\right) = B\exp\left(-\frac{\phi(x)}{kT}\right)$$

normalization

Harmonic potential: $\phi(x) = \alpha x^2/2 \qquad (\alpha > 0)$

$P_{\rm st}(x)$ is a Gaussian distribution

$$P_{\rm st}(x) = B\exp\left(-\frac{\alpha x^2}{2kT}\right)$$

Nonlinear Fokker-Planck equation

Nonlinear diffusion of porous medium type:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + D\,\frac{\partial^2 P^\nu(x,t)}{\partial x^2}$$

$\phi(x)$ confining: stationary solution for $t \to \infty$

Conditions for preservation of norm:

$$P(x,t)|_{x\to\pm\infty} = 0\ ;\ \left.\frac{\partial P^\nu(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0\ ;\ A(x)P(x,t)|_{x\to\pm\infty} = 0\ (\forall t)$$

Stationary-state solution:

$$\frac{\partial P_{\rm st}(x)}{\partial t} = 0 \Rightarrow -\frac{\partial [A(x) P_{\rm st}(x)]}{\partial x} + D\ \frac{\partial^2 P_{\rm st}^{\nu}(x)}{\partial x^2} = 0$$

$$\frac{\partial}{\partial x}\left[-A(x)P_{\rm st}(x) + D\ \frac{\partial P_{\rm st}^{\nu}(x)}{\partial x}\right] = 0$$

$\Rightarrow \quad J_{\rm st}(x) = \mathrm{cte}\ = 0 \quad (\forall x)$ (norm conditions)

$$\frac{A(x)}{D}\ dx = \frac{dP_{\rm st}^{\nu}}{P_{\rm st}} \quad \Rightarrow \quad -\frac{d\phi}{D} = \nu P_{\rm st}^{\nu-2} dP_{\rm st}$$

$$\frac{\nu}{\nu-1}\left[P_{\mathrm{st}}^{\nu-1}(x)-P_{\mathrm{st}}^{\nu-1}(0)\right]=-\frac{\phi(x)-\phi_0}{D}$$

$$P_{\mathrm{st}}(x)=\left[P_{\mathrm{st}}^{\nu-1}(0)-(\nu-1)\frac{\phi(x)-\phi_0}{\nu D}\right]_+^{1/(1-\nu)}$$

$$P_{\mathrm{st}}(x)=B\left[1-(\nu-1)\frac{\phi(x)}{kT}\right]_+^{1/(\nu-1)}$$

Considering $\nu=2-q$:

$$P_{\mathrm{st}}(x)=B\left[1-(1-q)\frac{\phi(x)}{kT}\right]_+^{1/(1-q)}=B\exp_q\left(-\frac{\phi(x)}{kT}\right)$$

3. Statistical Entropy and MaxEnt Principle

- Statistical Entropy

$$S = S[P(x,t)] \quad \text{or} \quad S = S[\{P_i(t)\}]$$

Maximum Statistical Entropy Principle (MaxEnt):

"Among all statistical distributions, compatible with the data, we must consider as the equilibrium distribution the one which yields the largest value of the statistical entropy."

Lagrange Multipliers

Extremize (maximum or minimum) the function

$$f(x_1, x_2, \cdots, x_n)$$

with the constraint $\quad g(x_1, x_2, \cdots, x_n) = 0$

$$\Phi(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) + \lambda g(x_1, x_2, \cdots, x_n)$$

$\delta\Phi/\delta x_k = 0 \quad :$

$$\Rightarrow \quad \frac{\partial f}{\partial x_k} + \lambda \frac{\partial g}{\partial x_k} = 0 \qquad (k = 1, 2, \cdots, n)$$

Lagrange multiplier

Ex.: Boltzmann-Gibbs entropy (discrete states)

$$S_{\mathrm{BG}}(\{P_i\}) = -k \sum_{i=1}^{W} P_i \ln P_i$$

Constraints: normalization and energy definition

➡ Lagrange multipliers λ_0 and β

$$\sum_{i=1}^{W} P_i = 1 \qquad U = \sum_{i=1}^{W} P_i \varepsilon_i$$

$$\Phi(\{P_i\}) = \frac{S_{\mathrm{BG}}(\{P_i\})}{k} - \lambda_0 \left(\sum_{i=1}^{W} P_i - 1 \right) - \beta \left(\sum_{i=1}^{W} P_i \varepsilon_i - U \right)$$

$$\frac{\delta \Phi}{\delta P_k} = 0 \quad \Rightarrow \quad -\ln P_k^{\mathrm{eq}} - 1 - \lambda_0 - \beta \varepsilon_k = 0$$

$$\Rightarrow \quad P_k^{\mathrm{eq}} = e^{-1-\lambda_0} e^{-\beta \varepsilon_k} = \frac{e^{-\beta \varepsilon_k}}{Z}$$

Ex.: BG entropy (continuous states)

$$S_{\mathrm{BG}}[P] = -k \int_{-\infty}^{\infty} P(x,t) \ln[P(x,t)]\, dx$$

$$\int_{-\infty}^{\infty} P(x,t)\, dx = 1 \; ; \qquad U = \int_{-\infty}^{\infty} E(x) P(x,t)\, dx$$

$$\Phi[P] = \frac{S_{\mathrm{BG}}[P]}{k} - \lambda_0 \left(\int_{-\infty}^{\infty} P(x,t)\; dx - 1 \right)$$

$$-\beta \left(\int_{-\infty}^{\infty} E(x) P(x,t)\; dx - U \right)$$

$$\frac{\delta \Phi[P]}{\delta P(x)} = 0 \quad \Rightarrow \quad -\ln P_{\mathrm{eq}}(x) - 1 - \lambda_0 - \beta E(x) = 0$$

$$\Rightarrow \quad P_{\mathrm{eq}}(x) = e^{-1-\lambda_0} e^{-\beta E(x)} = \frac{e^{-\beta E(x)}}{Z}$$

$E(x)$ harmonic : $E(x) = \alpha x^2/2 \qquad (\alpha > 0)$

$\Rightarrow$ $P_{\mathrm{eq}}(x)$ is a Gaussian distribution

Ex.: Tsallis entropy (continuous states)

$$S_q = \frac{k}{q-1}\left[1 - \int_{-\infty}^{\infty} P^q(x,t)\; dx\right]$$

$$\int_{-\infty}^{\infty} P(x,t)\; dx = 1\;; \qquad U = \int_{-\infty}^{\infty} E(x)P(x,t)\; dx$$

$$\Phi[P] = \frac{S_q[P]}{k} - \lambda_0 \left(\int_{-\infty}^{\infty} P(x,t)\; dx - 1\right)$$

$$-\beta \left(\int_{-\infty}^{\infty} E(x)P(x,t)\; dx - U\right)$$

$$\frac{\delta\Phi[P]}{\delta P(x)} = 0 \quad \Rightarrow \quad -\frac{q P_{\mathrm{eq}}^{q-1}(x)}{q-1} - \lambda_0 - \beta E(x) = 0$$

$$P_{\mathrm{eq}}^{q-1}(x) = -\frac{(q-1)\lambda_0}{q} - \frac{(q-1)\beta E(x)}{q}$$

$$P_{\mathrm{eq}}(x) = \left[-\frac{(q-1)\lambda_0}{q} - \frac{(q-1)\beta E(x)}{q}\right]_+^{1/(q-1)}$$

$$P_{\mathrm{eq}}(x) = \frac{1}{Z}\ [1-(q-1)\beta' E(x)]_+^{1/(q-1)}$$

PS: $\exp_q(x) = [1+(1-q)x]_+^{1/(1-q)}$

$$\Rightarrow \qquad P_{\mathrm{eq}}(x) = \frac{1}{Z'}\ \exp_{(2-q)}[-\beta' E(x)]$$

- Duality q $\Longleftrightarrow$ (2-q)

$$q' = 2 - q \quad \Rightarrow \quad 1 - q' = q - 1$$

$$\ln_q x = \frac{x^{1-q} - 1}{1-q} \; ; \qquad e_q^x = [1 + (1-q)x]^{1/(1-q)}$$

$$\Rightarrow \quad \ln_q [e_q^x] = x$$

$$\ln_{(2-q)} x = \frac{x^{q-1} - 1}{q-1} = -\ln_q(1/x)$$

$$e_{(2-q)}^x = [1 + (q-1)x]^{1/(q-1)} = 1/[e_q^{-x}]$$

Tsallis entropy (discrete states):

$$S_q = \frac{k}{q-1}\left(1 - \sum_{i=1}^{W} P_i^q\right) = \frac{k}{q-1} \sum_{i=1}^{W} P_i \left(1 - P_i^{q-1}\right)$$

$$S_q = k \sum_{i=1}^{W} P_i \ln_q(1/P_i) = -k \sum_{i=1}^{W} P_i \ln_{(2-q)} P_i$$

Tsallis entropy (continuous states):

$$S_q = \frac{k}{q-1}\left[1 - \int_{-\infty}^{\infty} P^q(x,t)\, dx\right] = \frac{k}{q-1} \int_{-\infty}^{\infty} dx\; P(x,t)\left[1 - P^{q-1}(x,t)\right]$$

$$S_q = k \int_{-\infty}^{\infty} dx\; P(x,t) \ln_q[1/P(x,t)] = -k \int_{-\infty}^{\infty} dx\; P(x,t) \ln_{(2-q)} P(x,t)$$

- Extremizing Tsallis entropy with

$$\int_{-\infty}^{\infty} P(x,t)\, dx = 1\ ; \qquad U = \int_{-\infty}^{\infty} E(x)P(x,t)\, dx$$

$$P_{\mathrm{eq}}(x) = \frac{1}{Z}\ [1-(q-1)\beta E(x)]_{+}^{1/(q-1)}$$

$$\Rightarrow \quad P_{\mathrm{eq}}(x) = \frac{1}{Z_{(2-q)}}\ \exp_{(2-q)}[-\beta E(x)]$$

$E(x)$ harmonic : $E(x) = \alpha x^2/2 \qquad (\alpha > 0)$

$\Rightarrow$ $P_{\mathrm{eq}}(x)$ is a (2-q)-Gaussian distribution

Ref.: C. Tsallis, J. Stat. Phys. 52, 479 (1988)

- Extremizing Tsallis entropy with

$$\int_{-\infty}^{\infty} P(x,t)\ dx = 1\ ; \qquad U = \frac{\int_{-\infty}^{\infty} E(x) P^q(x,t)\ dx}{\int_{-\infty}^{\infty} P^q(x,t)\ dx}$$

$$\Rightarrow \qquad P_{\mathrm{eq}}(x) = \frac{e_q^{-\beta_q [E(x)-U]}}{\int_{-\infty}^{\infty} e_q^{-\beta_q [E(x)-U]}\ dx}$$

$$\beta_q = \frac{\beta}{\int_{-\infty}^{\infty} P^q(x,t)\ dx}$$

Ref.: C. Tsallis, R.S. Mendes, and A.R. Plastino, Physica A 261, 534 (1998)

4. H-Theorem and Generalizations

- H-Theorem ➡ Statistical Entropy
- Boltzmann H-Theorem

Boltzmann H functional (1872):

$$H(t) = \int \rho(\vec{r}, \vec{p}, t) \ln[\rho(\vec{r}, \vec{p}, t)] d\vec{r}\, d\vec{p}$$

$$S_{\mathrm{BG}}(t) = -kH(t) + \mathrm{const}$$

$\rho(\vec{r}, \vec{p}, t) d\vec{r}\, d\vec{p}$: number of particles in the element of volume $d\vec{r}\, d\vec{p}$ at time t

$\rho(\vec{r}, \vec{p}, t)$ follows Boltzmann equation:

$$\frac{\partial \rho}{\partial t} + (\vec{v} \cdot \vec{\nabla}_{\vec{r}})\rho + (\vec{F} \cdot \vec{\nabla}_{\vec{p}})\rho = \left(\frac{\partial \rho}{\partial t}\right)_{\text{col}}$$

$(\partial\rho/\partial t)_{\text{col}}$: collisions term

General form: $$\left(\frac{\partial \rho}{\partial t}\right)_{\text{col}} = \left(\frac{\partial \rho}{\partial t}\right)^{(+)}_{\text{col}} - \left(\frac{\partial \rho}{\partial t}\right)^{(-)}_{\text{col}}$$

$(\partial\rho/\partial t)^{(+)}_{\text{col}}$: particles into the element $d\vec{r}\, d\vec{p}$ between t and $t + dt$

$(\partial\rho/\partial t)^{(-)}_{\text{col}}$: particles out of the element $d\vec{r}\, d\vec{p}$ between t and $t + dt$

Using Boltzmann equation for $\partial\rho(\vec{r},\vec{p},t)/\partial t$:

$$\frac{dH(t)}{dt} \leq 0 \quad \Rightarrow \quad \frac{dS_{\mathrm{BG}}(t)}{dt} \geq 0$$

➡ out-of-equilibrium dilute classical gas

- Relevance of the H-Theorem

a) Approach to equilibrium

b) Second law of thermodynamics

c) Entropy production (irreversible processes)

- System in contact with a thermal reservoir:

$$\frac{dF(t)}{dt} \leq 0 \qquad (F = U - TS)$$

- $$\frac{\partial P(x,t)}{\partial t} = 0 \quad \Rightarrow \quad P_{\mathrm{st}}(x)$$

- $$\frac{\partial P(x,t)}{\partial t} = 0 \quad \text{and} \quad (\mathrm{H-Theorem}) \quad \Rightarrow \quad P_{\mathrm{eq}}(x)$$

$$\Rightarrow \quad S_{\mathrm{BG}} = -k \int_{-\infty}^{\infty} dx \; P_{\mathrm{eq}}(x) \ln P_{\mathrm{eq}}(x)$$

$$U = \int_{-\infty}^{\infty} dx \; P_{\mathrm{eq}}(x) \phi(x)$$

$$\frac{1}{T} = \frac{\partial S(U)}{\partial U}$$

- Linear Fokker-Planck Equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)P(x,t)\}}{\partial x} + D\,\frac{\partial^2 P(x,t)}{\partial x^2}$$

$A(x) = -d\phi(x)/dx$ (confining potential)

Conditions for preservation of norm:

$$P(x,t)|_{x\to\pm\infty} = 0\ ;\ \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0\ ;\ A(x)P(x,t)|_{x\to\pm\infty} = 0\ (\forall t)$$

LFPE $\Longleftrightarrow$ S_{BG}

H-Theorem and LFPE:

$$S_{\mathrm{BG}} = -k \int_{-\infty}^{\infty} P(x,t) \ln[P(x,t)]\, dx$$

$$F = U - TS\ ; \qquad U = \int_{-\infty}^{\infty} dx\ \phi(x) P(x,t)$$

$$\frac{dF}{dt} = \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} dx\ \phi(x) P(x,t) + kT \int_{-\infty}^{\infty} dx\ P(x,t) \ln[P(x,t)] \right)$$

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} dx\ \left[\phi(x) + kT(\ln P(x,t) + 1)\right] \frac{\partial P(x,t)}{\partial t}$$

Normalization $\Rightarrow \displaystyle\int_{-\infty}^{\infty} \frac{\partial P(x,t)}{\partial t}\, dx = 0$

Now use LFPE:

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} dx \;\; [\phi(x) + kT \ln P(x,t)]$$
$$\times \left[-\frac{\partial [A(x) P(x,t)]}{\partial x} + D \;\frac{\partial^2 P(x,t)}{\partial x^2} \right]$$

Integration by parts:

$$u = \phi(x) + kT \ln P(x,t) \quad \Rightarrow \quad du = \frac{d\phi(x)}{dx}\; dx + \frac{kT}{P}\frac{\partial P}{\partial x} dx$$

$$dv = \left[-\frac{\partial [A(x) P(x,t)]}{\partial x} + D \;\frac{\partial^2 P(x,t)}{\partial x^2} \right] dx$$

$$v = -A(x) P(x,t) + D \;\frac{\partial P(x,t)}{\partial x}$$

$$uv|_{-\infty}^{\infty} = \Big[(\phi(x) + kT \ln P(x,t)) \\ \times \left(-A(x)P(x,t) + D \; \frac{\partial P(x,t)}{\partial x} \right) \Big]_{-\infty}^{\infty} = 0$$

use norm conditions

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \; \left[-A(x)P(x,t) + D \, \frac{\partial P}{\partial x} \right] \left[\frac{d\phi(x)}{dx} + \frac{kT}{P} \frac{\partial P}{\partial x} \right]$$

Use $\;A(x) = -d\phi(x)/dx\;$ and $\;D = kT\;$:

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \; P(x,t) \left[-A(x) + \frac{kT}{P} \; \frac{\partial P}{\partial x} \right]^2 \leq 0$$

Nonlinear Fokker-Planck Equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + D\frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

$\Psi[P(x,t)]$ and $\Omega[P(x,t)]$: positive, finite, integrable, differentiable (at least once)

$$A(x) = -d\phi(x)/dx$$

Conditions for preservation of norm:

$$P(x,t)|_{x\to\pm\infty} = 0\ ;\ \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0\ ;\ A(x)\Psi[P(x,t)]|_{x\to\pm\infty} = 0\ (\forall t)$$

H-Theorem and NLFPE:

$$S[P] = k\int_{-\infty}^{\infty} dx\ g[P(x,t)]\ ;\ g(0) = g(1) = 0\ ;\ \frac{d^2 g}{dP^2} \leq 0$$

$$F = U - \theta S\ ; \quad U = \int_{-\infty}^{\infty} dx\ \phi(x) P(x,t)$$

θ : positive parameter with temperature dimensions

$$\frac{dF}{dt} = \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} dx\ \phi(x)P(x,t) - k\theta \int_{-\infty}^{\infty} dx\ g[P(x,t)] \right)$$

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} dx\ \left[\phi(x) - k\theta\ \frac{dg[P]}{dP} \right] \frac{\partial P}{\partial t}$$

use NLFPE

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} dx \; \left[\phi(x) - k\theta \; \frac{dg[P]}{dP}\right]$$

$$\times \left[-\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + D\frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}\right]$$

$$u = \phi(x) - k\theta \; \frac{dg[P]}{dP} \quad \Rightarrow \quad du = \left[\frac{d\phi(x)}{dx} - k\theta \; \frac{d^2 g[P]}{dP^2}\frac{\partial P}{\partial x}\right] dx$$

$$dv = \left[-\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + D\frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}\right] dx$$

$$v = -A(x)\Psi[P(x,t)] + D\,\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}$$

Using norm conditions → $uv\big|_{-\infty}^{\infty} = 0$

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \, \left[\frac{d\phi(x)}{dx} - k\theta \, \frac{d^2 g[P]}{dP^2} \, \frac{\partial P(x,t)}{\partial x}\right]$$

$$\times \left[-A(x)\Psi[P(x,t)] + D\,\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right]$$

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \;\Psi[P(x,t)] \left[\frac{d\phi(x)}{dx} - k\theta \, \frac{d^2 g[P]}{dP^2} \, \frac{\partial P(x,t)}{\partial x}\right]$$

$$\times \left[-A(x) + D \, \frac{\Omega[P(x,t)]}{\Psi[P(x,t)]} \frac{\partial P(x,t)}{\partial x}\right]$$

Use $D = k\theta$; $\quad \dfrac{dF}{dt} \le 0$ follows if

- Condition for the H-Theorem:

$$-\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx\ \Psi[P(x,t)] \left[\frac{d\phi(x)}{dx} - k\theta\, \frac{d^2 g[P]}{dP^2}\, \frac{\partial P(x,t)}{\partial x} \right]^2 \le 0$$

- Families of FPEs:

$$-\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

$$(a)\ \text{Given } \Omega[P] \text{ and } \Psi[P] \quad \Rightarrow \quad g[P]$$

$$(b)\ \text{Given } g[P] \quad \Rightarrow \quad \text{ratio } \frac{\Omega[P]}{\Psi[P]}$$

Families of FPEs associated with the same entropy

Examples of FPEs and Associated Entropies

Use

$$-\frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}$$

a) Linear case:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + D\,\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$\Psi[P(x,t)] = P(x,t)\ ; \quad \Omega[P(x,t)] = 1$$

$$\frac{dg}{dP} = -\ln P + C \quad \Rightarrow \quad g[P] = -P\ln P + P + CP + C'$$

$$g(0) = 0 \quad \Rightarrow \quad C^{'} = 0$$

$$g(1) = 0 \quad \Rightarrow \quad C = -1$$

$$g[P] = -P \ln P \quad \Rightarrow \quad S[P] = -k \int_{-\infty}^{\infty} P(x,t) \ln[P(x,t)] \, dx$$

Linear FPE $\Longleftrightarrow$ BG entropy:

(a) Equilibrium distribution

(b) H-Theorem

(b) Nonlinear diffusion of porous medium type:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + D\ \frac{\partial^2 P^\nu (x,t)}{\partial x^2}$$

$$D\,\frac{\partial^2 P^\nu}{\partial x^2} = D\,\frac{\partial}{\partial x}\frac{\partial P^\nu}{\partial x} = D\nu\,\frac{\partial}{\partial x}\left(P^{\nu-1}\frac{\partial P}{\partial x}\right)$$

$$\Omega[P(x,t)] = \nu P^{\nu-1}\ ; \qquad \Psi[P(x,t)] = P(x,t)$$

$$\frac{d^2 g}{dP^2} = -\nu P^{\nu-2} \quad \Rightarrow \quad \frac{dg}{dP} = -\frac{\nu}{\nu-1}P^{\nu-1} + C$$

$$g[P] = -\frac{P^\nu}{\nu-1} + CP + C'$$

$$g(0) = 0 \quad \Rightarrow \quad C' = 0$$

$$g(1) = 0 \quad \Rightarrow \quad C = \frac{1}{\nu - 1}$$

$$g[P] = \frac{1}{\nu - 1}[P - P^{\nu}]$$

$$S[P] = \frac{k}{\nu - 1}\left[1 - \int_{-\infty}^{\infty} P^{\nu}(x,t)\, dx\right]$$

NLFPE $\Leftrightarrow$ Tsallis entropy:

(a) Equilibrium distribution

(b) H-Theorem

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5. Physical Application: Interacting Vortices in Type-II Superconductors

- Type-II Superconductivity

(a) Two critical magnetic fields:

$$H_{c1} \text{ and } H_{c2} \qquad (H_{c1} < H_{c2})$$

(b) $H < H_{c1}$: type-I superconductivity

(c) $H > H_{c2}$: normal state

(d) $H_{c1} < H < H_{c2}$: type-II superconductivity

➡ vortices of current generate flux tubes

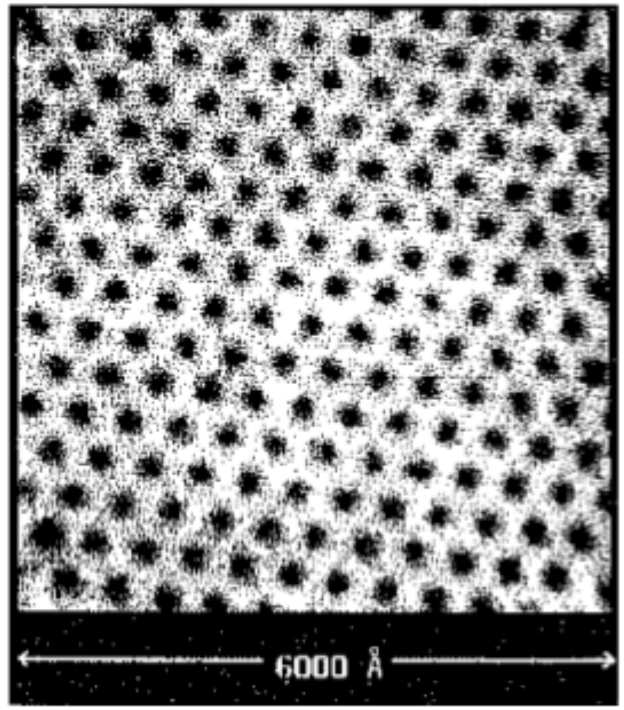
TYPE II SUPERCONDUCTORS AND THE VORTEX LATTICE

Nobel Lecture, December 8, 2003

by

ALEXEI A. ABRIKOSOV

Materials Science Division, Argonne National Laboratory, 9700 South Cass Ave., Argonne, IL 60439, USA.



Vortices in $NbSe_2$ defined by scanning tunneling microscopy (STM).

I made my derivation of the vortex lattice in 1953 but the publication was postponed since Landau first disagreed with the whole idea. Only after R. Feynman published his paper on vortices in superfluid helium [9], and Landau accepted the idea of vortices, he agreed with my derivation, and I

published my paper in 1957 [10]. Even then it did not attract attention, in spite of an English translation, and only after the discovery in the beginning of the 1960s of superconducting alloys and compounds with high critical magnetic fields there appeared an interest in my work. Nevertheless, even after that the experimentalists did not believe in the possibility of existence of a vortex lattice incommensurable with the crystalline lattice. Only after the

Reducing vortex density in superconductors using the 'ratchet effect'

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Nature 400, 337 (1999)

APPLIED PHYSICS LETTERS **96**, 192501 (2010)

Vortex manipulation in a superconducting matrix with view on applications

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We show how a single flux quantum can be effectively manipulated in a superconducting film with a matrix of *blind holes*. Such a sample can serve as a basic memory element, where the position of the vortex in a $k\times l$ matrix of pinning sites defines the desired combination of n bits of information ($2^n=k\times l$). Vortex placement is achieved by strategically applied current and the resulting position is read out via generated voltage between metallic contacts on the sample. Such a device can also act as a controllable source of a nanoengineered local magnetic field for, e.g., spintronics applications. [doi:10.1063/1.3425672]

- System: vortices interacting repulsively in 2d under overdamped motion

$$\eta \mathbf{v}_i = \mathbf{F}_i^{\mathrm{pp}} + \mathbf{F}_i^{\mathrm{ext}} \qquad (i = 1, 2, \cdots, N)$$

external force

Particle-particle interactions:

$$\mathbf{F}_i^{\mathrm{pp}} = \frac{1}{2} \sum_{j \neq i} B^{\mathrm{pp}}(r_{ij})\ \hat{\mathbf{r}}_{ij}\ ; \quad B^{\mathrm{pp}}(r_{ij}) = f_0 K_1(r_{ij}/\lambda)$$

$K_1(x)$: First-order modified Bessel function

λ : London penetration length

- Coarse graining → density $\rho(\mathbf{r},t)$

(smoothly varying around the origin)

$$\rho(\mathbf{r},t) \approx \rho(0,t) + \mathbf{r}\cdot\nabla\rho(\mathbf{r},t)$$

Average over interaction term:

$$\mathbf{F}^{\mathrm{pp}} = \frac{1}{2}\int d^2r\,\rho(\mathbf{r},t)B^{\mathrm{pp}}(r)\hat{\mathbf{r}} \approx a\nabla\rho(\mathbf{r},t)$$

$$a = \pi\int_0^\infty dr\;r^2 B^{\mathrm{pp}}(r) = 2\pi f_0\lambda^3$$

$$\eta \mathbf{v}_i = \mathbf{F}_i^{\mathrm{pp}} + \mathbf{F}_i^{\mathrm{ext}} \quad \Rightarrow \quad \eta\rho(\mathbf{r},t)\mathbf{v} = \rho(\mathbf{r},t)(\mathbf{F}^{\mathrm{pp}} + \mathbf{F}^{\mathrm{ext}})$$

$$\eta \nabla \cdot [\rho(\mathbf{r},t)\mathbf{v}] = \nabla \cdot \left\{\rho(\mathbf{r},t)\left[a\nabla\rho(\mathbf{r},t) + \mathbf{F}^{\mathrm{ext}}\right]\right\}$$

l.h.s.: $\mathbf{J} = \rho(\mathbf{r},t)\mathbf{v}$; $\nabla \cdot \mathbf{J} = \dfrac{\partial \rho(\mathbf{r},t)}{\partial t}$

$$\eta \frac{\partial \rho(\mathbf{r},t)}{\partial t} = \nabla \cdot \left\{\rho(\mathbf{r},t)\left[a\nabla\rho(\mathbf{r},t) + \mathbf{F}^{\mathrm{ext}}\right]\right\}$$

Now consider $\mathbf{F}^{\mathrm{ext}} = -A(x)\hat{\mathbf{x}}$

- From now on: $A(x) = -\alpha x \qquad (\alpha > 0)$

- Introducing uncorrelated thermal noise:

$$\eta \mathbf{v}_i = \mathbf{F}_i^{\mathrm{pp}} + \mathbf{F}_i^{\mathrm{ext}} + \mathbf{F}_i^{\mathrm{th}} \qquad (i = 1, 2, \cdots, N)$$

$$\langle \mathbf{F}_i^{\mathrm{th}}(t) \rangle = 0 \; ; \qquad \langle \mathbf{F}_i^{\mathrm{th}}(t) \cdot \mathbf{F}_i^{\mathrm{th}}(t') \rangle = 2kT\eta \; \delta(t - t')$$

- For fixed y ➡ $P(x,t) = (L_y/N)\rho(x,t)$

- Nonlinear Fokker-Planck Equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + 2D \; \frac{\partial}{\partial x} \left\{ [\lambda P(x,t)] \frac{\partial P(x,t)}{\partial x} \right\}$$

$$+kT \; \frac{\partial^2 P(x,t)}{\partial x^2}$$

- Characteristic/effective temperatures:

(i) Debye temperature (phonons T>0):

$$kT_{\mathrm{D}} = \hbar c \left(\frac{6\pi^2 N}{V} \right)^{1/3}$$

$$T \ll T_{\mathrm{D}}: \;\; C_V \sim T^3 \qquad (T \to 0)$$

$$T \gg T_{\mathrm{D}}: \;\; C_V \approx 3k$$

Solids: $T_{\mathrm{D}} \approx 200K$

(ii) Fermi temperature (for T=0):

$$kT_{\mathrm{F}} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{2s+1}\frac{N}{V}\right)^{2/3}$$

$$T \ll T_{\mathrm{F}} : \;\; C_V \sim (T/T_{\mathrm{F}}) \qquad\qquad (T \to 0)$$

$$T \gg T_{\mathrm{F}} : \;\; C_V \approx \frac{3}{2}k$$

Metals: $T_{\mathrm{F}} : \; 10^4 K \to 10^5 K$

White dwarfs: $T_{\mathrm{F}} \approx \; 10^7 K$

$$D = k\theta = \lim_{N, L_y \to \infty} \frac{N\pi f_0 \lambda^2}{L_y} = n\pi f_0 \lambda^2$$

θ ➡ Effective Temperature

n can be varied ⬌ θ can be varied !

- Type-II Superconductors: $\theta \gg T$
- Neglecting thermal effects:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + 2D\,\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}$$

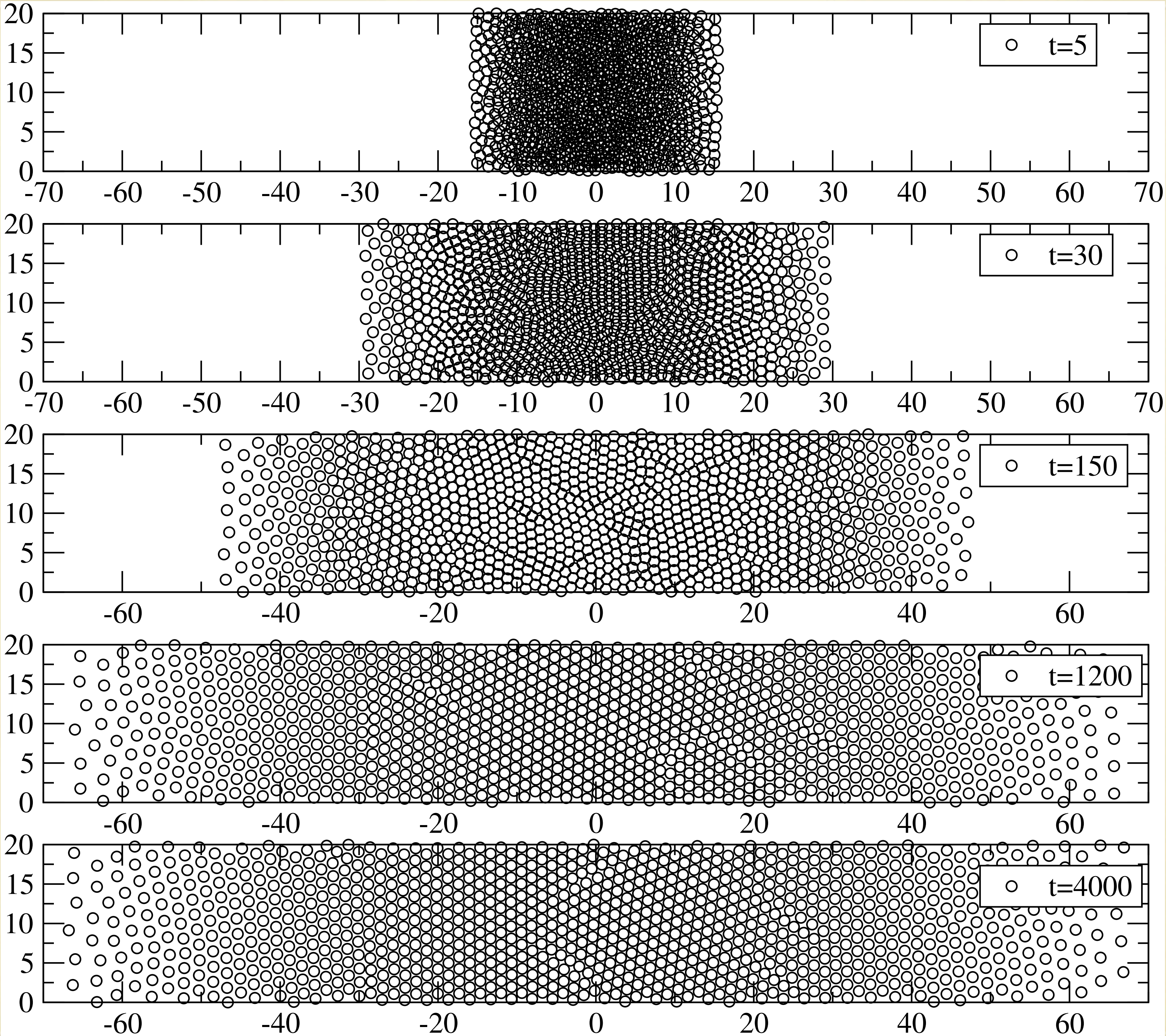
- Entropy: S_q

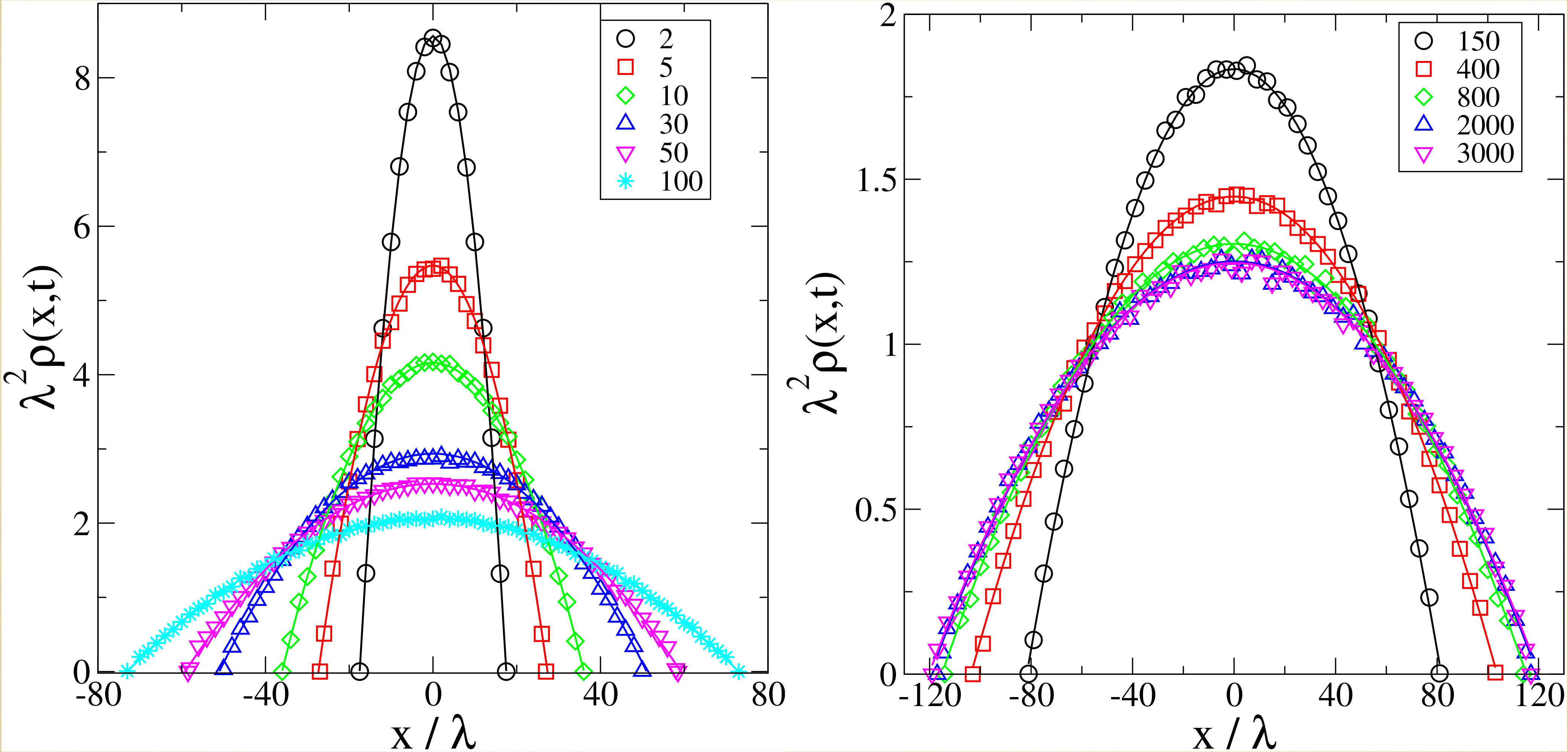
$$s[P] = k\{1 - \lambda \int_{-\infty}^{\infty} dx\ [P(x,t)]^2\ \} \qquad (q = 2)$$

$$P(x,t) = B(t)[1 - \beta(t)x^2]_+$$

$$t \ \text{ large}: \ P(x,t) \to P_{\text{st}}(x)$$

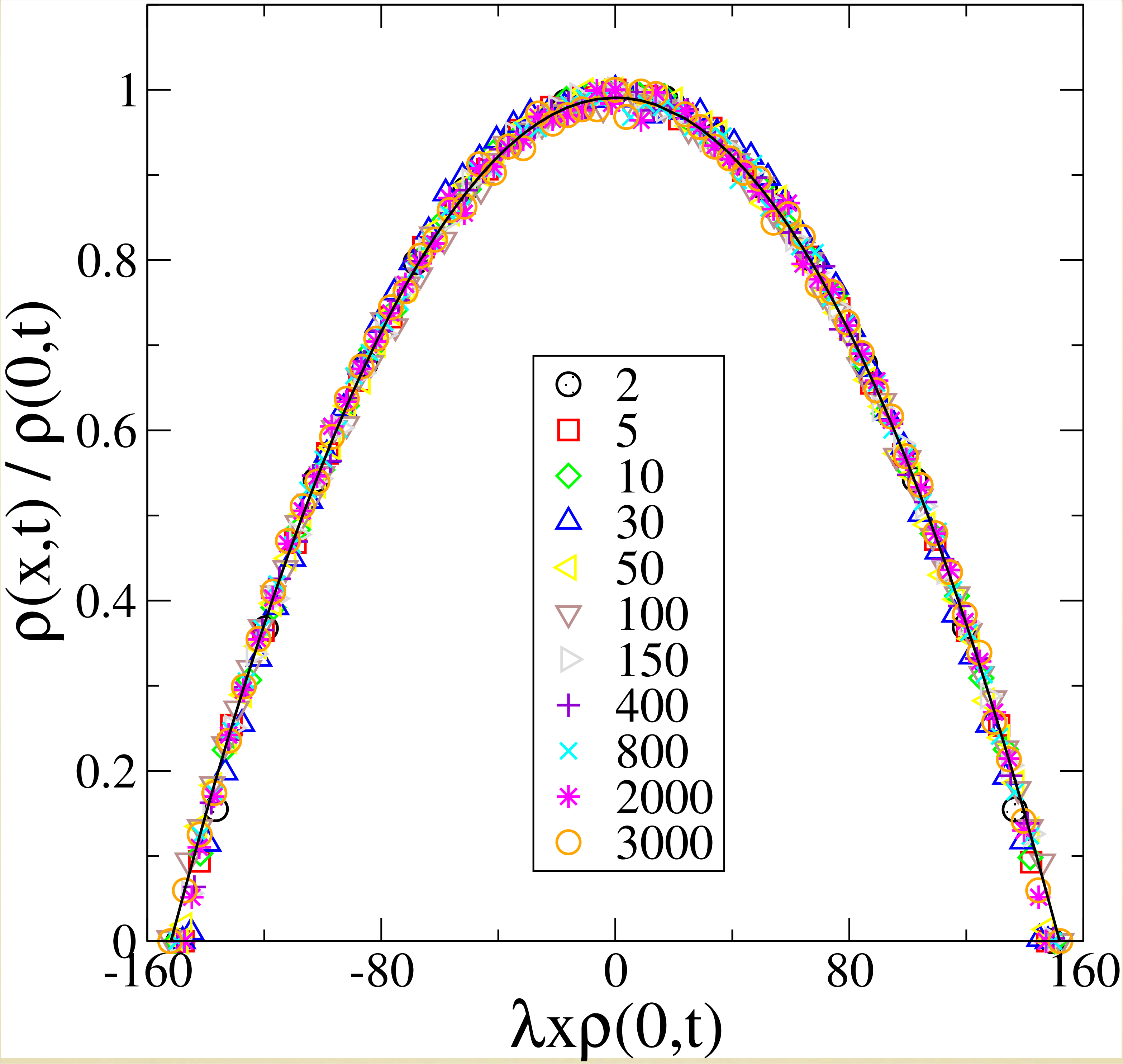
- Thermodynamics: stationary/equilibrium state


t=5
t=30
t=150
t=1200
t=4000
20
15
10
5
0
-70
-60
-50
-40
-30
-20
-10
0
10
20
30
40
50
60
70

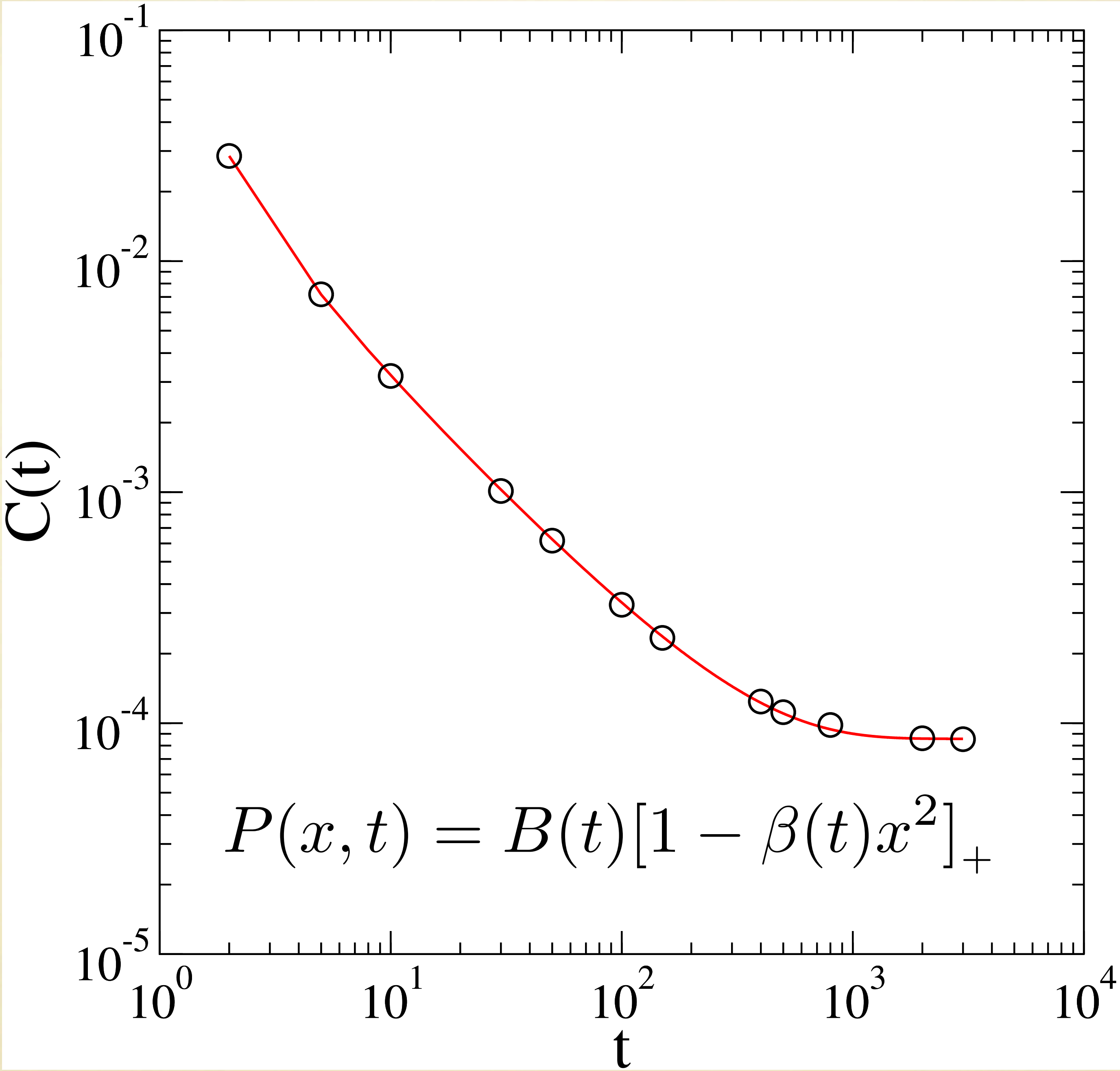


$$a = \pi \int_0^\infty dr\; r^2 B^{\mathrm{pp}}(r) = 2\pi f_0 \lambda^3$$

$$a = (5.87 \pm 0.02) f_0 \lambda^3$$



ρ(x,t) / ρ(0,t)
1
0.8
0.6
0.4
0.2
0
-160
-80
0
80
160
λxρ(0,t)
2
5
10
30
50
100
150
400
800
2000
3000



$$C(t) = \lambda^4 N B(t) \beta(t) / L_y$$

- Analysis of Stationary State:

$$D = k\theta = \lim_{N, L_y \to \infty} \frac{N\pi f_0 \lambda^2}{L_y} = n\pi f_0 \lambda^2$$

Dimensionless variable: $\tau = \dfrac{k\theta}{\alpha\lambda^2} = \dfrac{n\pi f_0}{\alpha}$

$$P_{\rm st}(x) = \frac{\alpha}{4D\lambda}(x_e^2 - x^2) = \frac{1}{4\tau\lambda}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right]$$

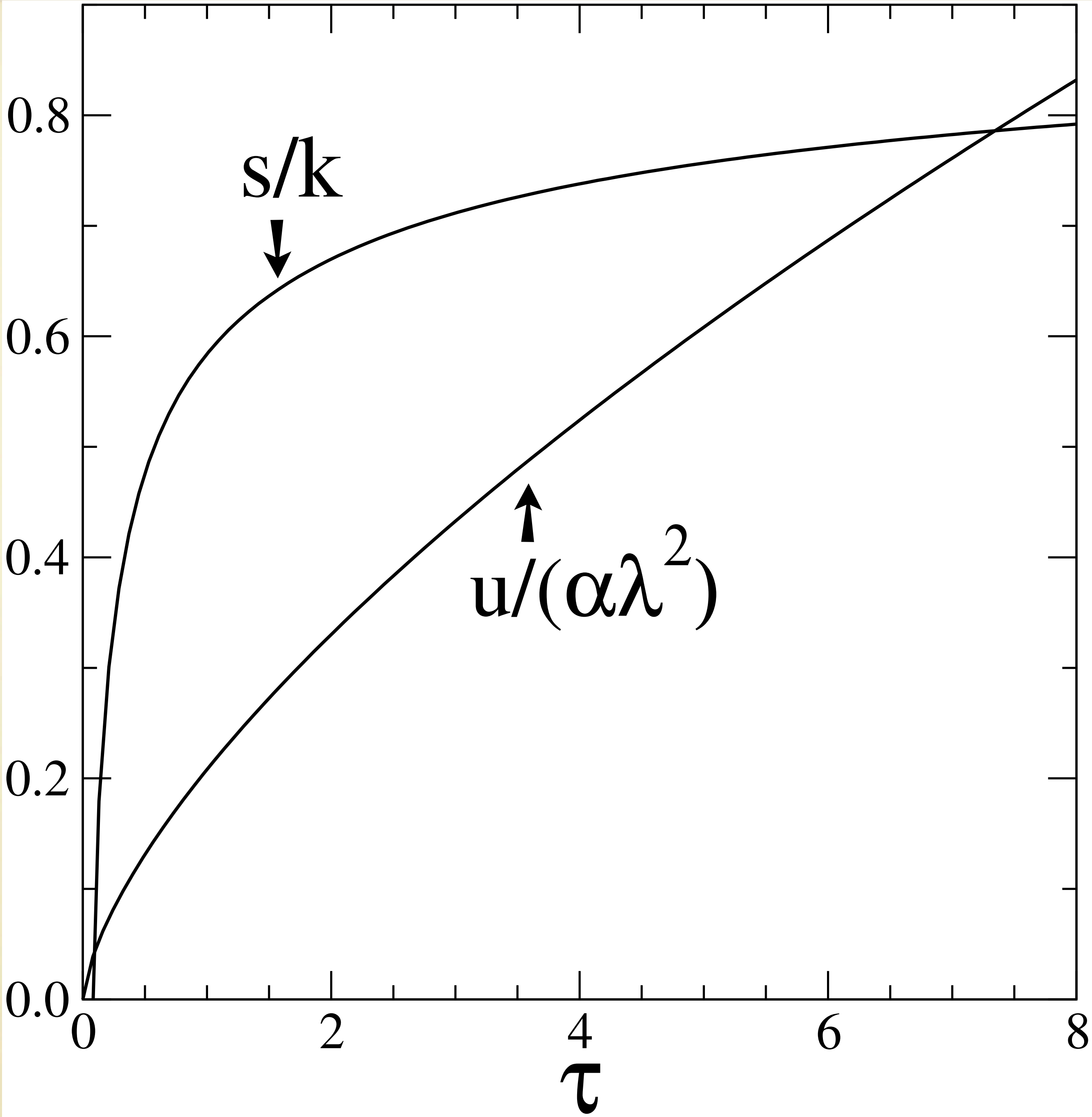
$$(|x| < x_e) \qquad x_e = (3D\lambda/\alpha)^{1/3} = (3\tau)^{1/3}\lambda$$

$$u = \int_{-x_e}^{x_e} dx \ \frac{\alpha x^2}{2} \ P_{\rm st}(x) = \frac{3^{2/3}}{10} \ \alpha\lambda^2 \ \tau^{2/3}$$

$$s = k\{1 - \lambda \int_{-x_e}^{x_e} dx \ [P_{\rm st}(x)]^2 \ \}$$

$$s = k\left[1 - \frac{1}{5}\left(\frac{9}{\tau}\right)^{1/3}\right]$$

$$s(u) = k\left[1 - \frac{3}{5}\left(\frac{\alpha\lambda^2}{10u}\right)^{1/2}\right] \Rightarrow \frac{\partial s(u)}{\partial u} = \frac{1}{\theta}$$



0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
τ
s/k
u/(αλ2)

$$f = \alpha\lambda^2 \left[\frac{3^{5/3}}{10}\,\tau^{2/3} - \tau\right] \;\Rightarrow\; \frac{\partial f}{\partial \theta} = -s$$

$$c = \frac{\partial u}{\partial \theta} = \theta\frac{\partial s}{\partial \theta} = -\theta\frac{\partial^2 f}{\partial^2 \theta} = \frac{3^{2/3}}{15}\,k\tau^{-1/3}$$

Kinetic temperature: $\langle v^2 \rangle \propto kT$

Present system: $\boxed{\langle x^2 \rangle \propto (k\theta)^{2/3}}$

- First Law:

$$du = \delta Q + \delta W = \theta ds + \sigma d\alpha$$

$$u = u(s, \alpha) \ ; \qquad u(s,\alpha) = \frac{9}{250} \, \frac{\alpha \lambda^2}{(1 - s/k)^2}$$

$$\left(\frac{\partial u}{\partial s}\right)_\alpha = \theta \ ; \qquad \left(\frac{\partial u}{\partial \alpha}\right)_s = \sigma$$

$$\frac{\partial^2 u}{\partial \alpha \, \partial s} = \frac{\partial^2 u}{\partial s \, \partial \alpha} \quad \Rightarrow \quad \left(\frac{\partial \sigma}{\partial s}\right)_\alpha = \left(\frac{\partial \theta}{\partial \alpha}\right)_s$$

- Equation of state:

$$\sigma = \frac{3^{2/3}}{10}\,\lambda^2\left(\frac{k\theta}{\alpha\lambda^2}\right)^{2/3} \quad ; \qquad \sigma = \frac{u}{\alpha}$$

$$s(u,\alpha) = k\left[1 - \frac{3}{5}\left(\frac{\alpha\lambda^2}{10u}\right)^{1/2}\right]$$

$\Rightarrow$ $$s = \text{const} \quad \Rightarrow \quad \sigma = \frac{u}{\alpha} = \text{const}$$

- Isothermal process:

$$Q = \int_{s_i}^{s_f} \theta ds = \frac{3^{2/3}}{5} (k\theta)^{2/3} \left[(\alpha_i \lambda^2)^{1/3} - (\alpha_f \lambda^2)^{1/3} \right]$$

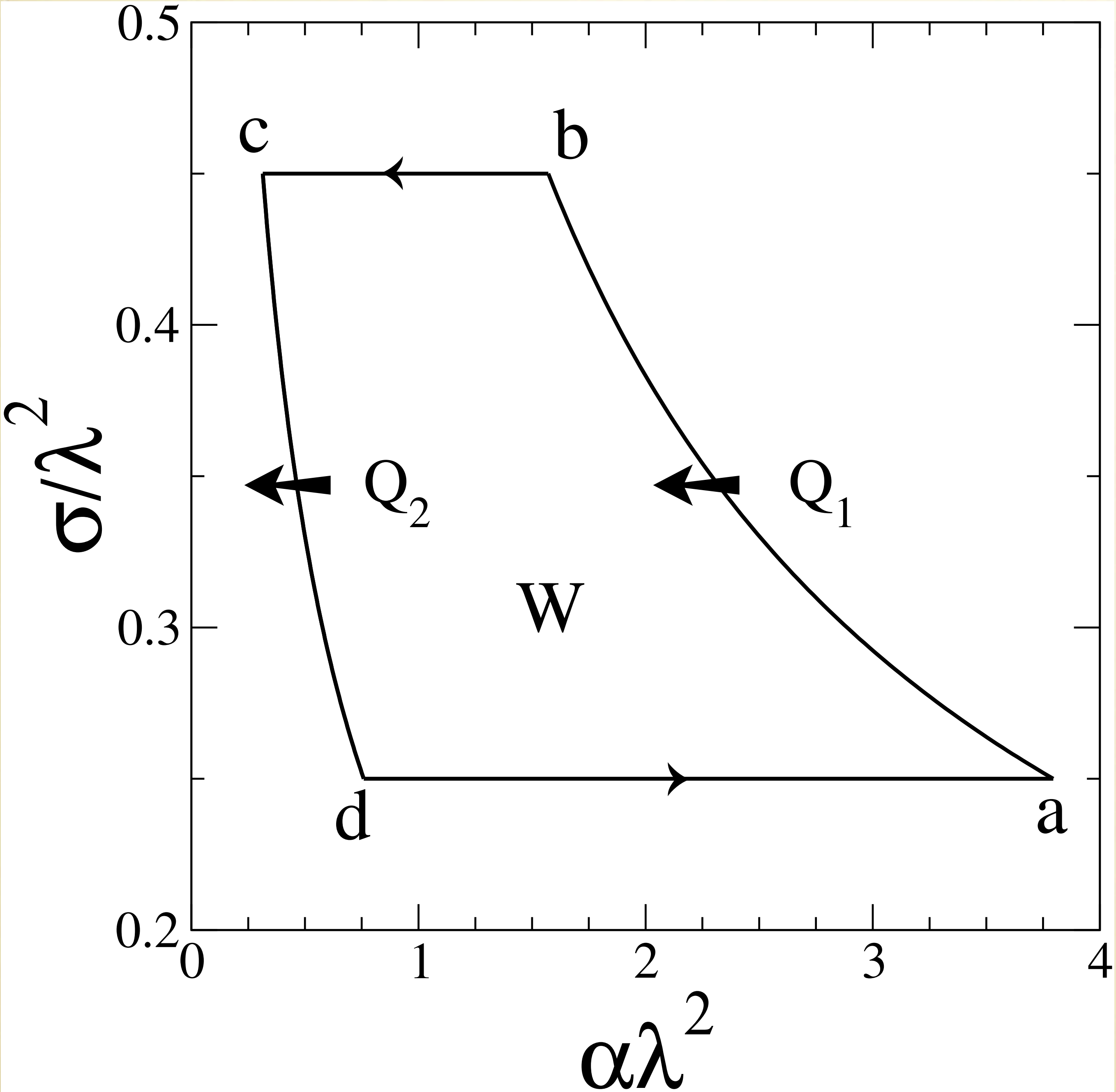
$$W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \frac{3^{5/3}}{10} (k\theta)^{2/3} \left[(\alpha_f \lambda^2)^{1/3} - (\alpha_i \lambda^2)^{1/3} \right]$$

$$u_f - u_i = Q + W = \frac{3^{2/3}}{10} (k\theta)^{2/3} \left[(\alpha_f \lambda^2)^{1/3} - (\alpha_i \lambda^2)^{1/3} \right]$$

- Adiabatic process:

$$u_f - u_i = W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \sigma(\alpha_f - \alpha_i)$$

Carnot Cycle:



$$\theta_1 > \theta_2$$

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2}$$

$$e = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$e = 1 - \frac{\theta_2}{\theta_1}$$

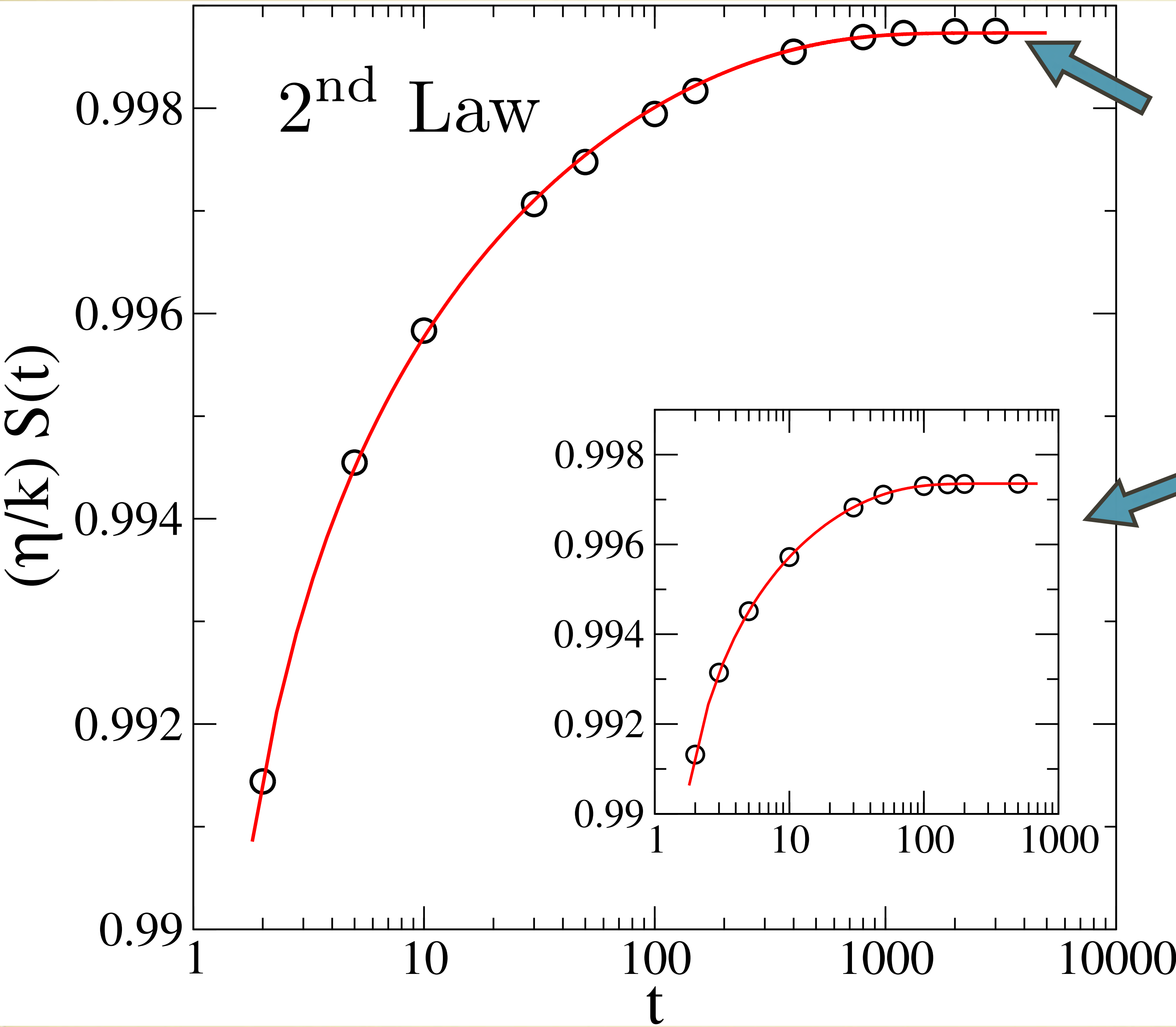
- Entropy: $s = k\{1 - \lambda \int_{-x_e(t)}^{x_e(t)} dx \, [P(x,t)]^2 \}$

- Second Law of Thermodynamics:

"The total entropy of a thermally isolated system never decreases"

$\Delta S = 0$: reversible processes

$\Delta S > 0$: irreversible processes



$$\alpha = 10^{-3}\ \frac{f_0}{\lambda}$$

$$\alpha = 10^{-2}\ \frac{f_0}{\lambda}$$

$$n = 200$$

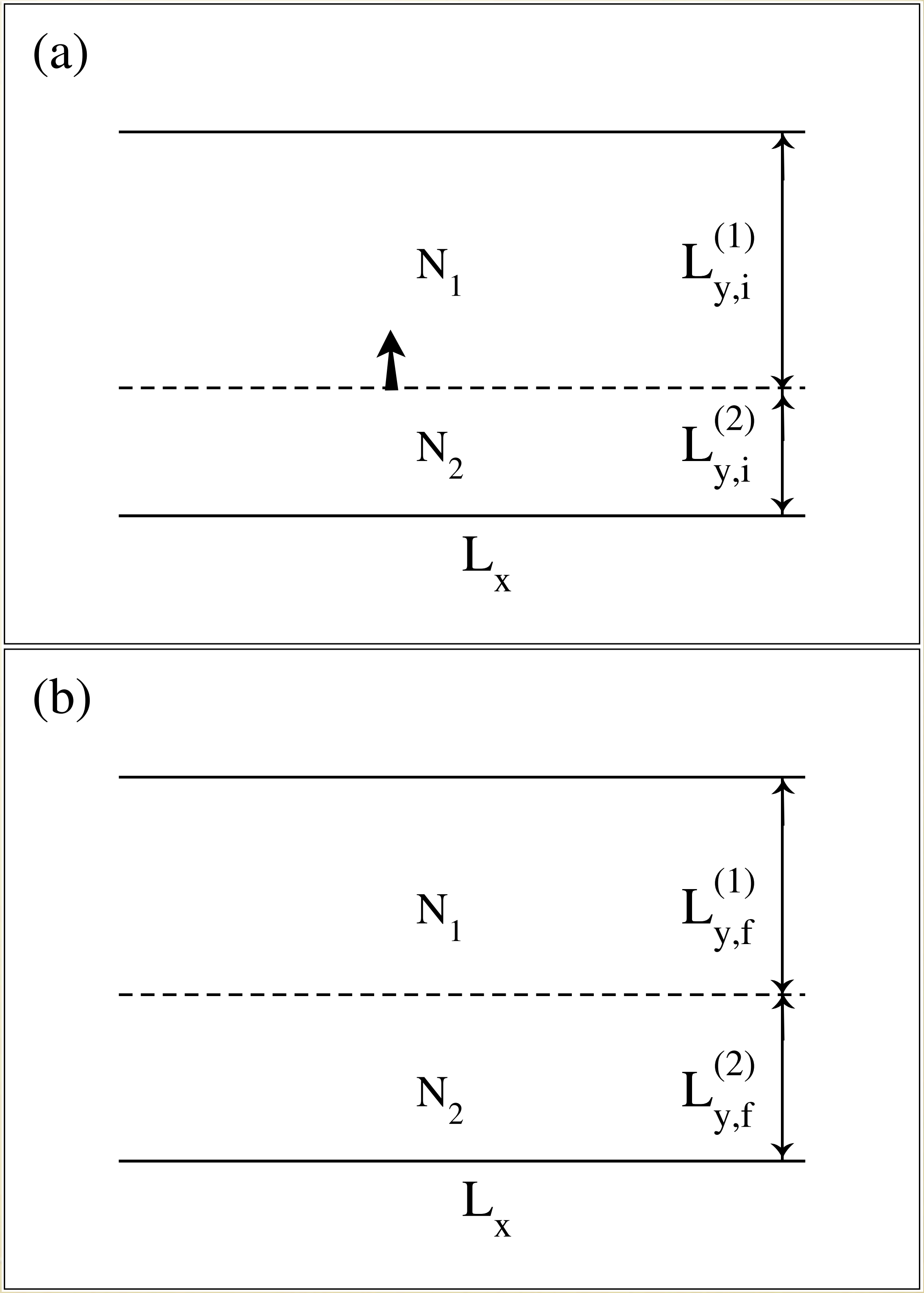
Contact between two systems: change $\theta = n\pi f_0 \lambda^2$

$$(n = N/L_y)$$

$$a) \quad \frac{N_1}{L_{y,i}^{(1)}} < \frac{N_2}{L_{y,i}^{(2)}}$$

$$b) \quad \frac{N_1}{L_{y,f}^{(1)}} = \frac{N_2}{L_{y,f}^{(2)}}$$

$$\Rightarrow \quad \theta_f^{(1)} = \theta_f^{(2)} = \theta_f$$



Entropy Variations:

$$\frac{\delta s^{(1)}}{k} = \frac{1}{k}\left[s_f^{(1)} - s_i^{(1)}\right] = \frac{3^{2/3}}{5}\left[\left(\frac{\alpha\lambda^2}{k\theta_i^{(1)}}\right)^{1/3} - \left(\frac{\alpha\lambda^2}{k\theta_f}\right)^{1/3}\right]$$

$$\frac{\delta s^{(2)}}{k} = \frac{1}{k}\left[s_f^{(2)} - s_i^{(2)}\right] = \frac{3^{2/3}}{5}\left[\left(\frac{\alpha\lambda^2}{k\theta_i^{(2)}}\right)^{1/3} - \left(\frac{\alpha\lambda^2}{k\theta_f}\right)^{1/3}\right]$$

$$\theta_i^{(1)} < \theta_i^{(2)} : \qquad \delta s^{(1)} > \delta s^{(2)}$$

$$\Rightarrow \quad \theta_i^{(1)} > \theta_i^{(2)} : \qquad \delta s^{(1)} < \delta s^{(2)}$$

- Use conservation: $L_{y,f}^{(1)} + L_{y,f}^{(2)} = L_{y,i}^{(1)} + L_{y,i}^{(2)}$

$$\frac{1}{\theta_f} = \frac{N_1}{N_1 + N_2}\,\frac{1}{\theta_i^{(1)}} + \frac{N_2}{N_1 + N_2}\,\frac{1}{\theta_i^{(2)}}$$

or equivalently,

$$\theta_f = \frac{L_{y,i}^{(1)}}{L_{y,i}^{(1)} + L_{y,i}^{(2)}}\,\theta_i^{(1)} + \frac{L_{y,i}^{(2)}}{L_{y,i}^{(1)} + L_{y,i}^{(2)}}\,\theta_i^{(2)}$$

$\Rightarrow$ θ_f : between $\theta_i^{(1)}$ and $\theta_i^{(2)}$

- Zeroth Law:

$$\frac{N_1}{L_{y,1}} = \frac{N_2}{L_{y,2}} \qquad (\theta_1 = \theta_2)$$

$$\frac{N_1}{L_{y,1}} = \frac{N_3}{L_{y,3}} \qquad (\theta_1 = \theta_3)$$

$$\Rightarrow \quad \frac{N_2}{L_{y,2}} = \frac{N_3}{L_{y,3}} \qquad (\theta_2 = \theta_3)$$

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