XVIII - Brazilian School of Cosmology and Gravitation

Patrick Peter

Institut d'Astrophysique de Paris

GR&CO

Inflation and Bouncing Cosmologies

Rio
Brazil
July 10, 2017

A brief history of Cosmology

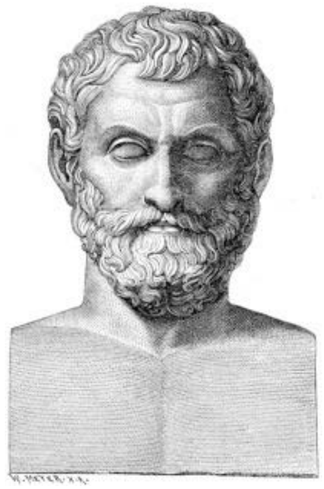
Antiquity : myths & observation

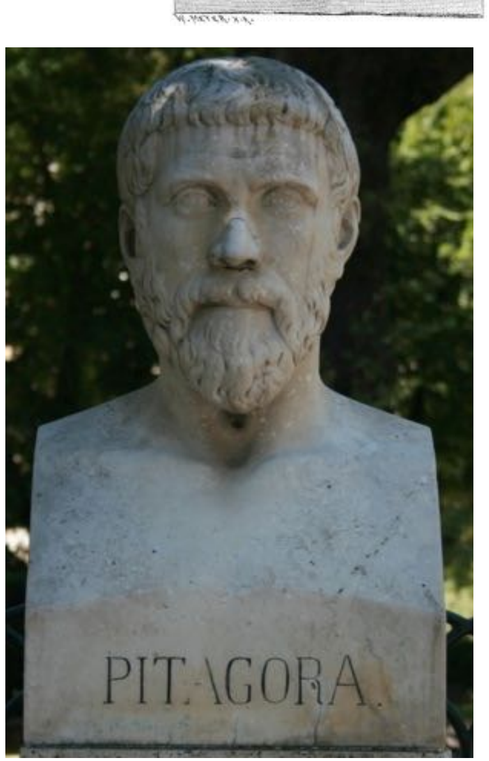


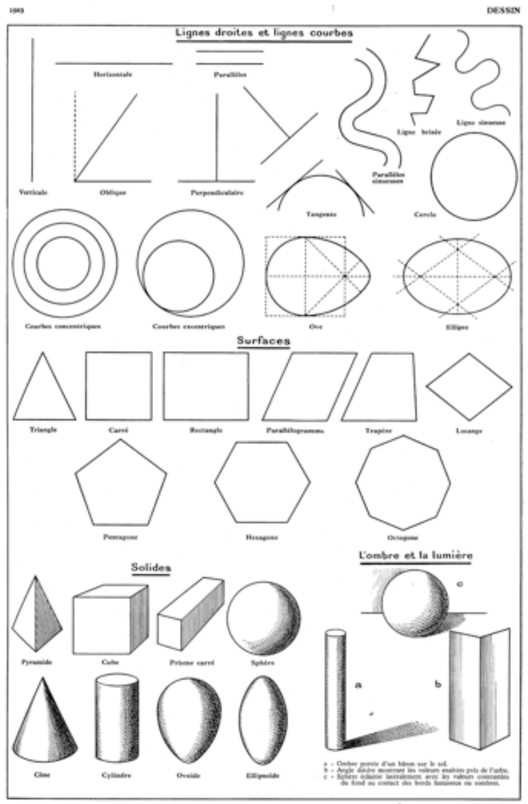
Greeks : V^th b.c. to IInd a.c.

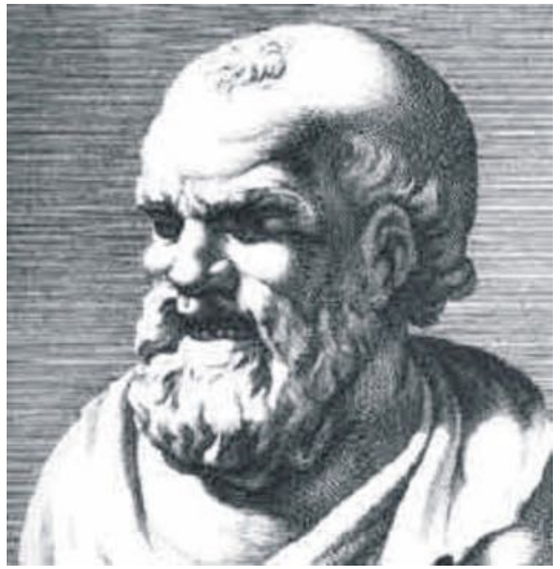
Birth of philosophy

Thales of Miletus: everything is water...









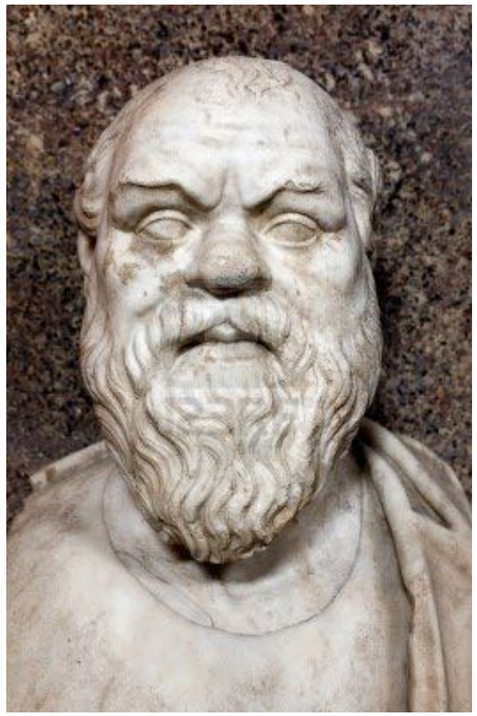
Democritus: the father of atoms

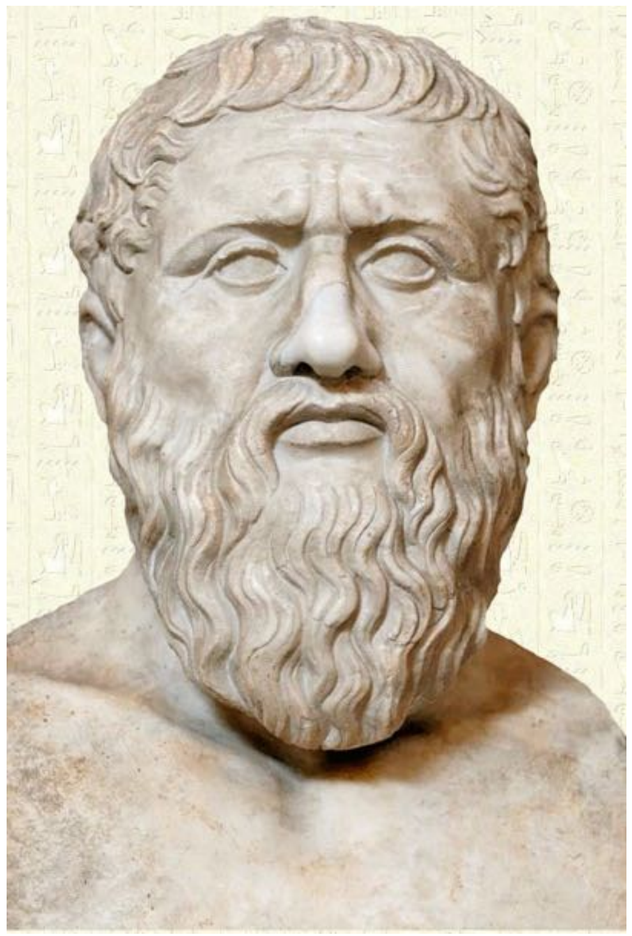
Pythagoras: mathematizing the world

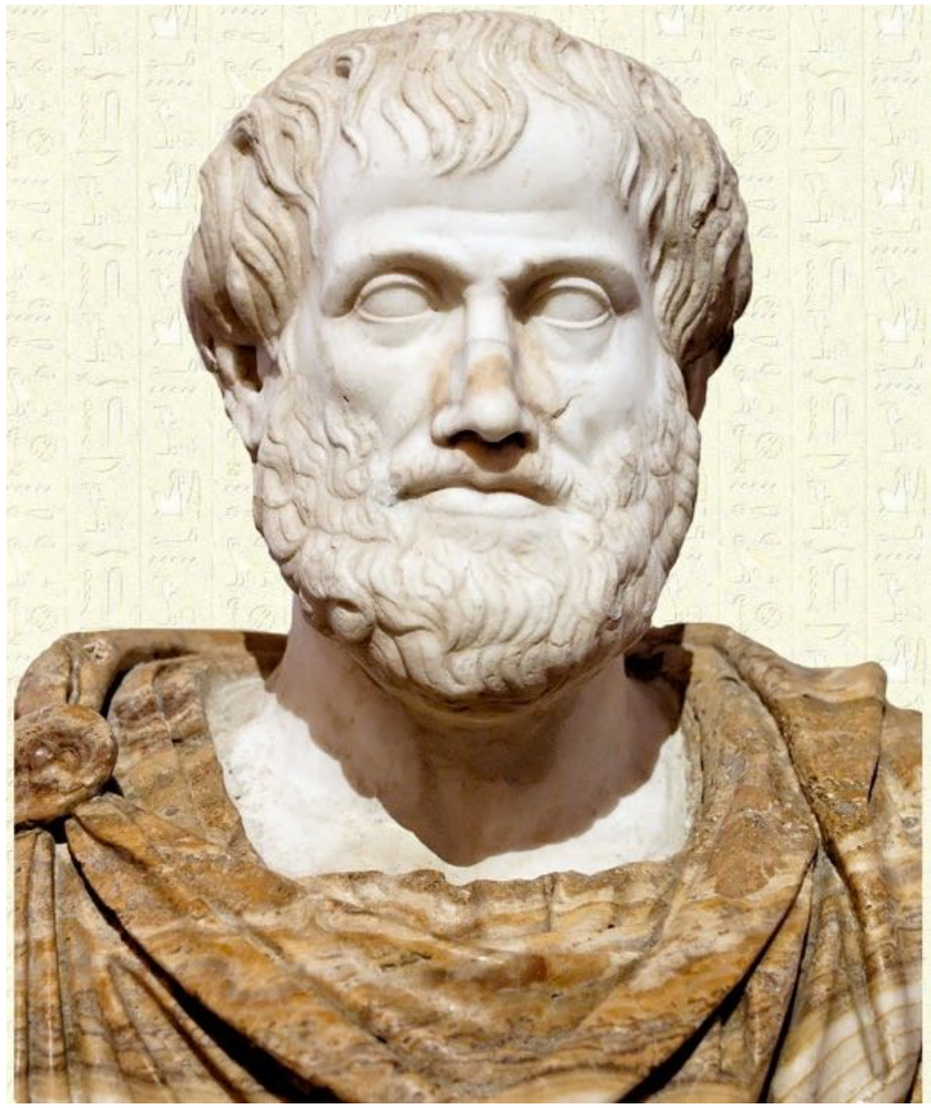
Greeks : Vth b.c. to IInd a.c.

Birth of philosophy

Socrates, Plato, Aristotle...







The world is knowledgable

Motion comes with a cause

Greeks : Vth b.c. to IInd a.c.

Retrograde motion of planets:



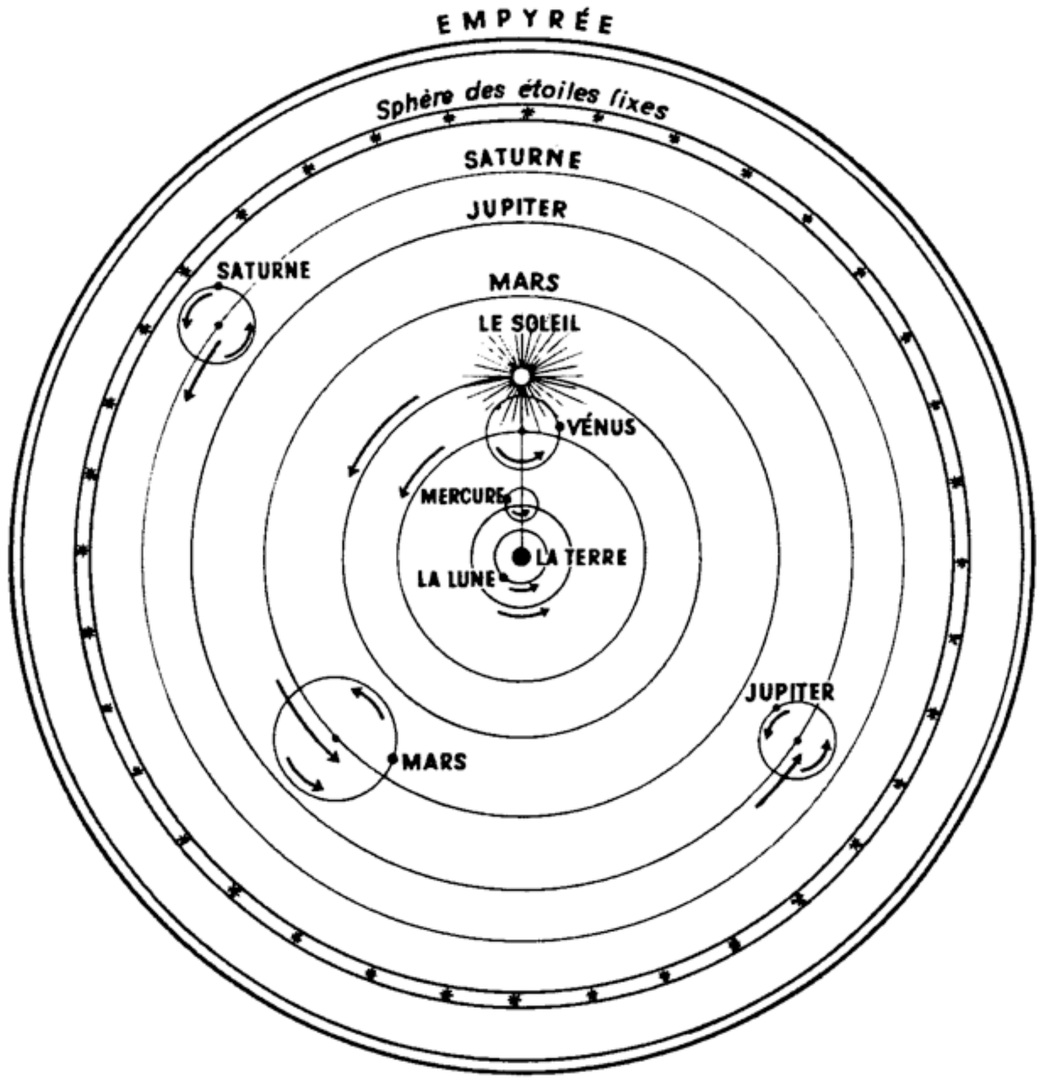
Mars trajectory (2011)

(Astronomy picture of the day / NASA)

Greeks : Vth b.c. to IInd a.c.

Ptolomeo





Cycles & epicycles

(good model!!!)

"Modern" cosmology: XVIth to XIXth

Nicolas Copernic (1473 - 1543)



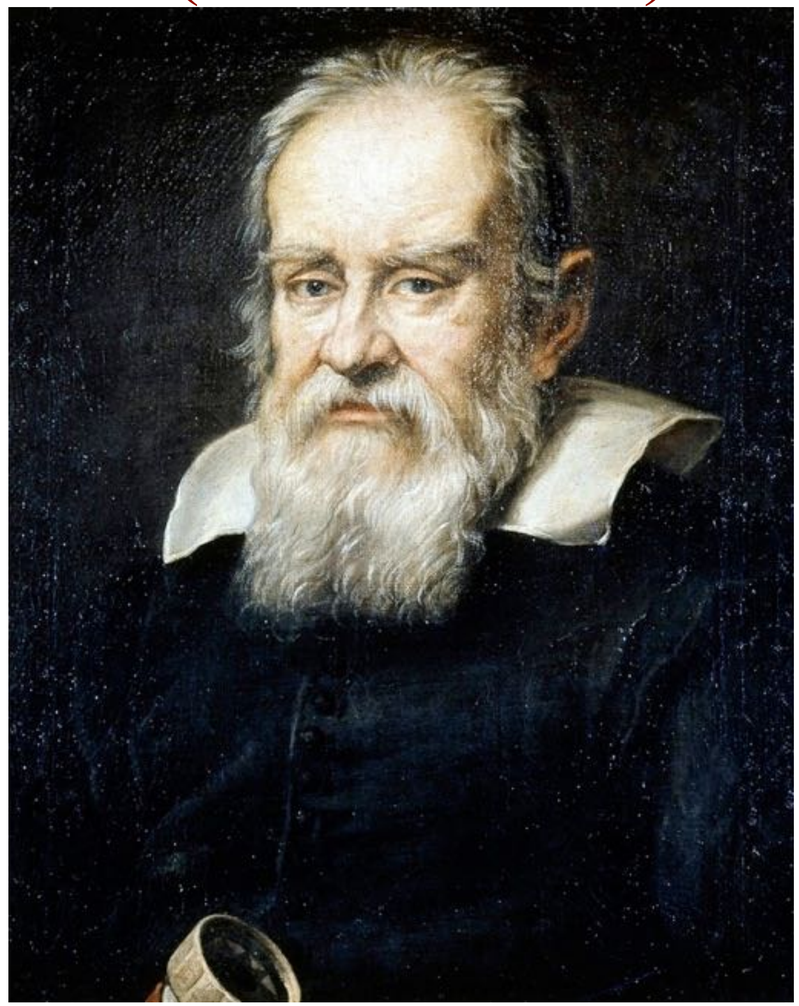
Heliocentrism

Giordano Bruno (1548 - 1600)



The universe is infinite
=> burned!

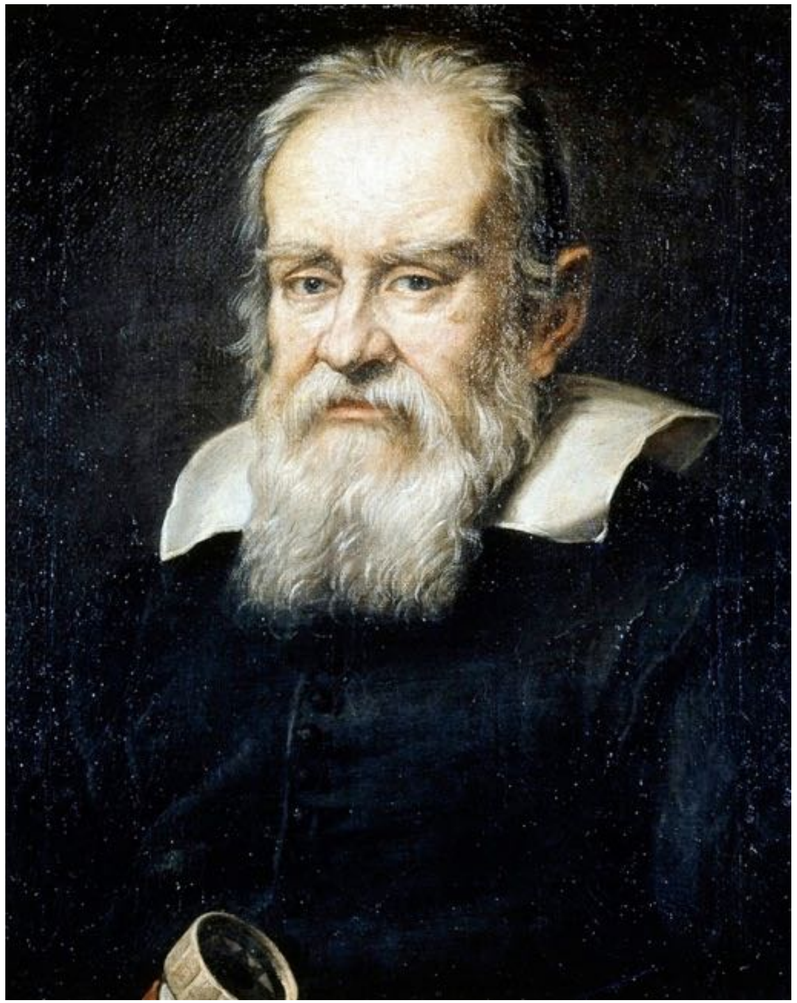
Galileo (1564 - 1642)



Other systems and trajectories

"Modern" cosmology: XVIth to XIXth

Galileo (1564 - 1642)



Birth of ***Science***

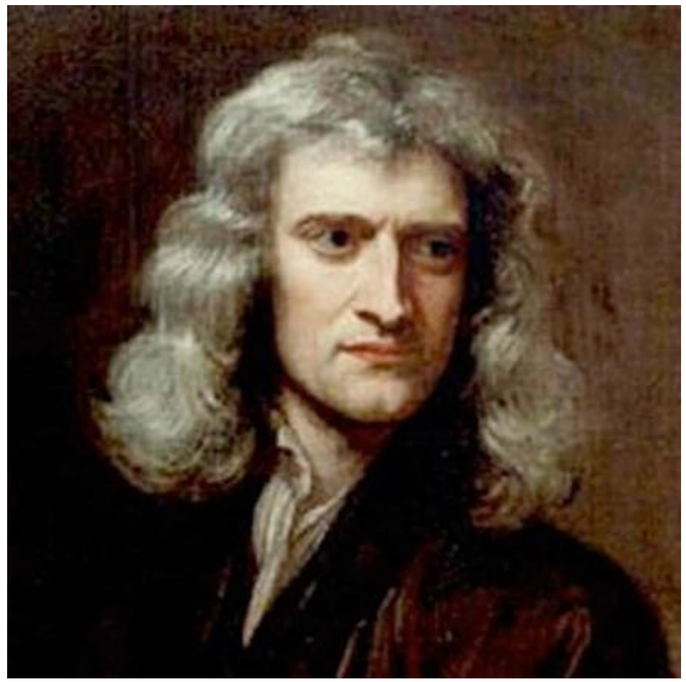


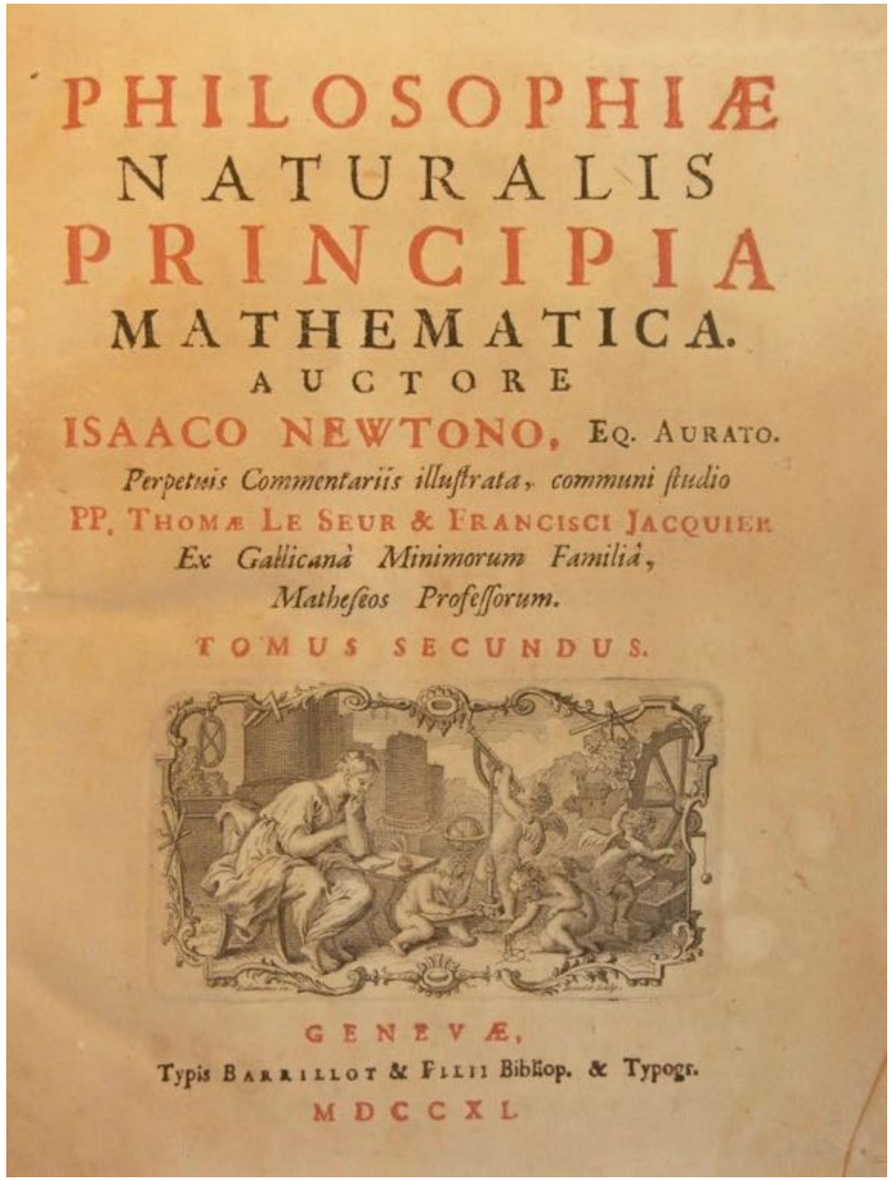
Eppur si muove!

Scientific method:
experimentation, observation & abstraction
(inductive method)

“Modern” cosmology: XVIth to XIXth

Isaac Newton (1643 - 1727)

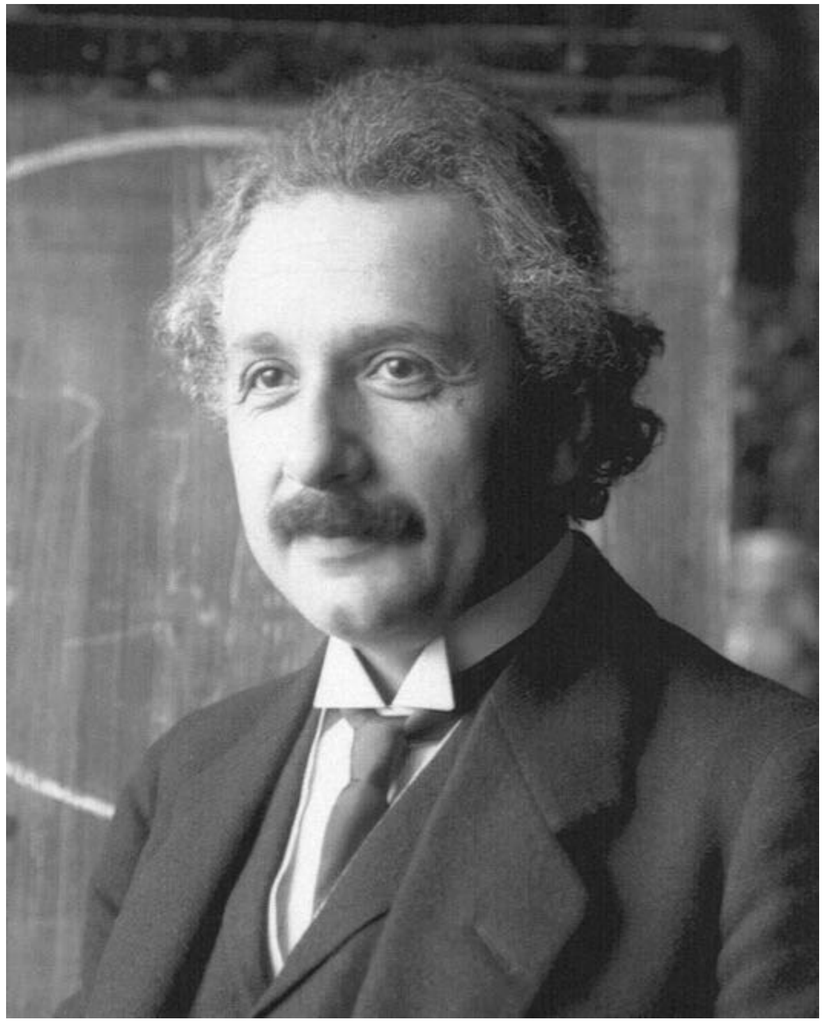



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.
AUCTORE
ISAACO NEWTONO, Eq. Aurato.
Perpetuis Commentariis illuſtrata, communi ſtudio
PP. Thomæ Le Seur & Franciſci Jacquier
Ex Gallicanâ Minimorum Familiâ,
Matheſeos Profeſſorum.
TOMUS SECUNDUS.
GENEVÆ,
Typis Barrillot & Filii Bibliop. & Typogr.
MDCCXL

Gravitation theory, differential calculus...

Contemporary cosmology: XXth, XXIst,...

Albert Einstein
(1879 - 1955)



Changing paradigms:
spacetime, quantum mechanics...

Theoretical framework 1 : Gravity = General Relativity

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

$$G_{\mu\nu}\left(\equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \Lambda g_{\mu\nu} = \frac{8\pi G_{\mathrm{N}}}{c^4}T_{\mu\nu}$$

$$R^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\nu\beta} - \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\nu\alpha}$$

$$\Gamma^\mu{}_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right)$$

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}$$

$$R \equiv R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}$$

Theoretical framework 2 : Matter = fluid

$$\mathcal{S} = \int \mathrm{d}^4 x \sqrt{-g} \left\{ \frac{1}{16\pi G_{\mathrm{N}}} (R - 2\Lambda) + \mathcal{L}_{\mathrm{matter}} \left[\phi(x), \psi(x), \cdots, g_{\mu\nu} \right] \right\}$$

$$\mathcal{L}_{\mathrm{kinetic}} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

Perfect fluid

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$$

$$g_{\mu\nu} u^\mu u^\nu = -1$$ Timelike vector → define cosmological time

$$\nabla_\mu T^{\mu\nu} = 0$$ Conservation equation = dynamics

Typical fluid?

$$p = w\rho$$

Equation of state

$$\langle v_{\rm gal}\rangle \sim 200\,{\rm km\cdot s^{-1}} \sim 10^{-3}$$

$$E_{\rm kin}/\rho \sim \tfrac{1}{2}\langle v_{\rm gal}^2\rangle \sim 10^{-6}$$

$$p/\rho \sim \tfrac{1}{3}\langle v_{\rm gal}^2\rangle \sim 10^{-6}$$

} Matter

$$w_{\rm m} \sim 0$$

Radiation

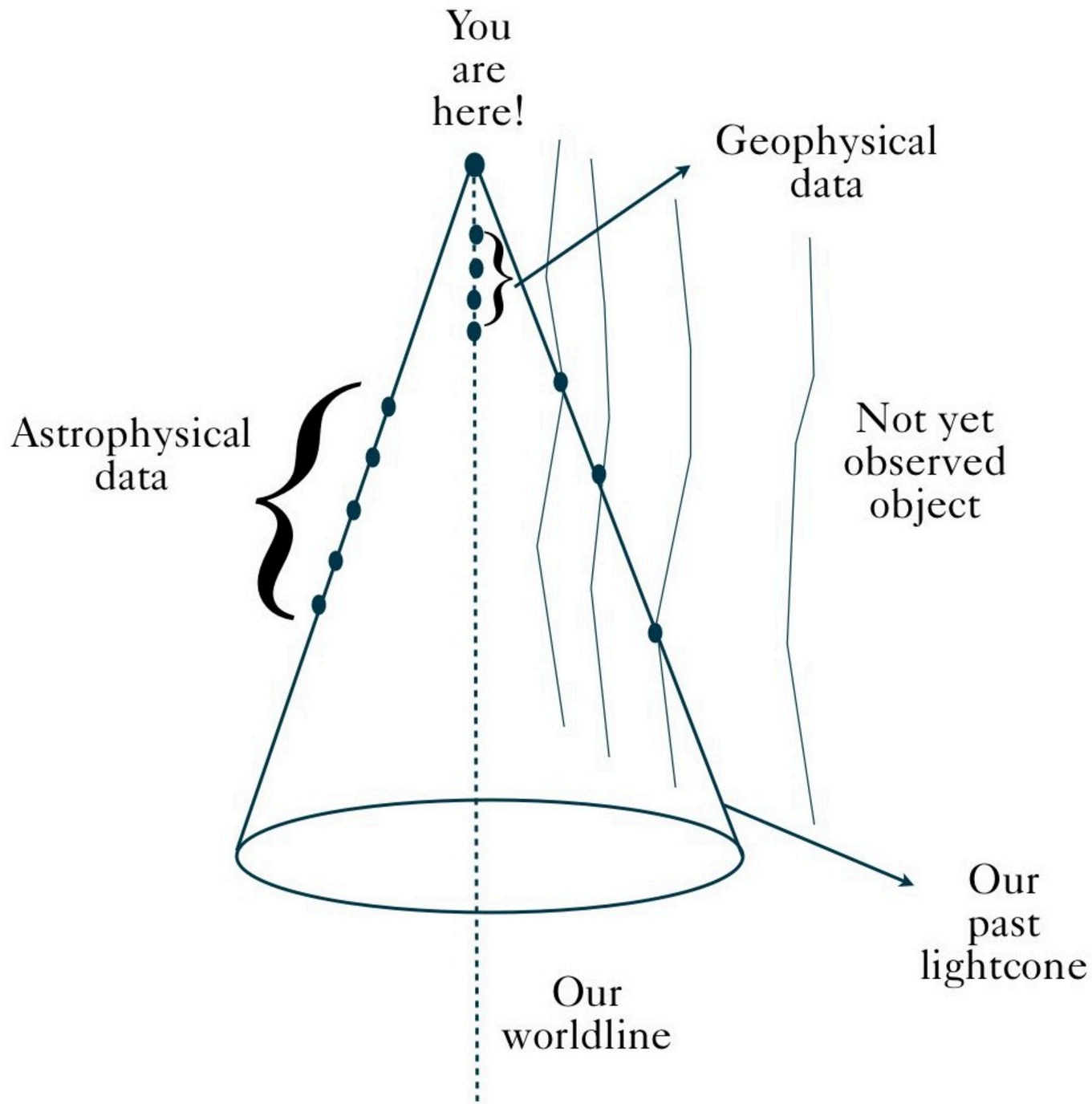
$$w_{\rm r} = \tfrac{1}{3}$$

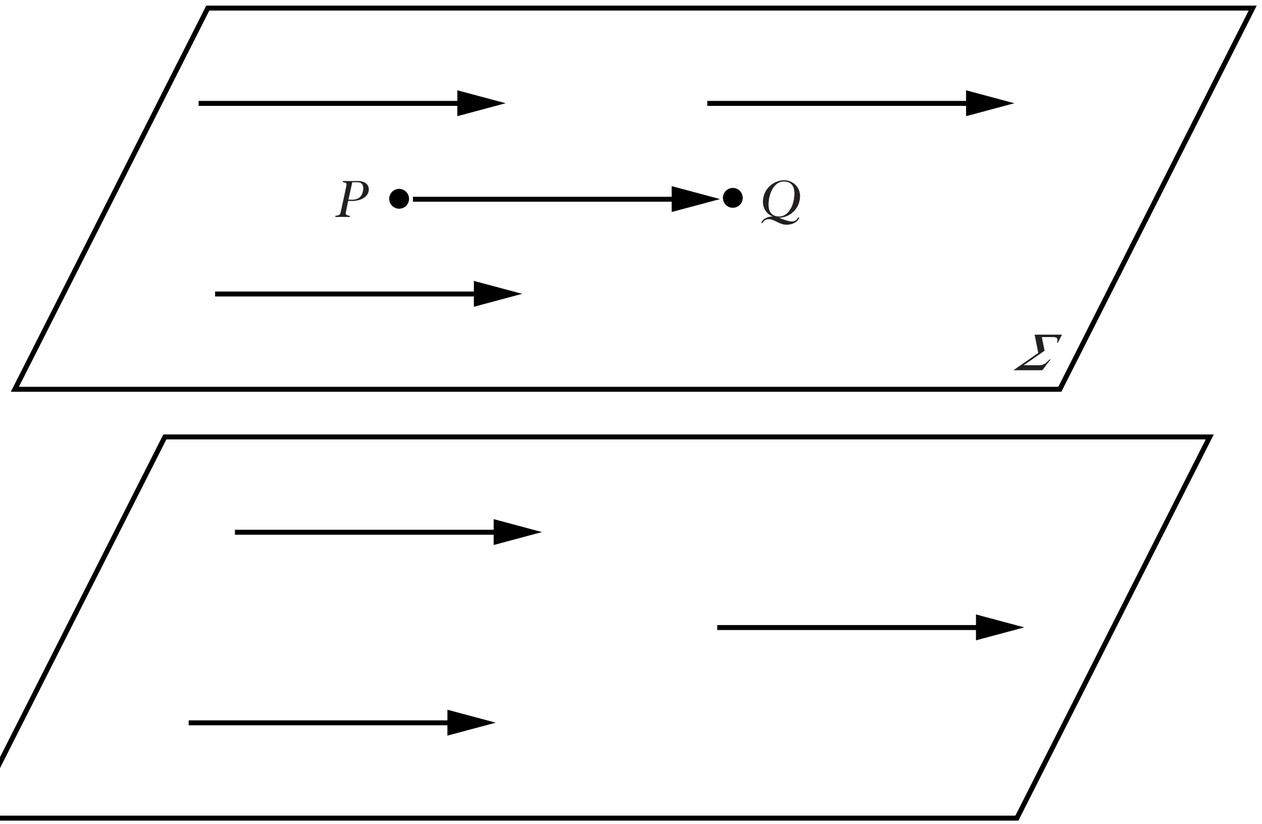
Cosmological constant (dark energy)

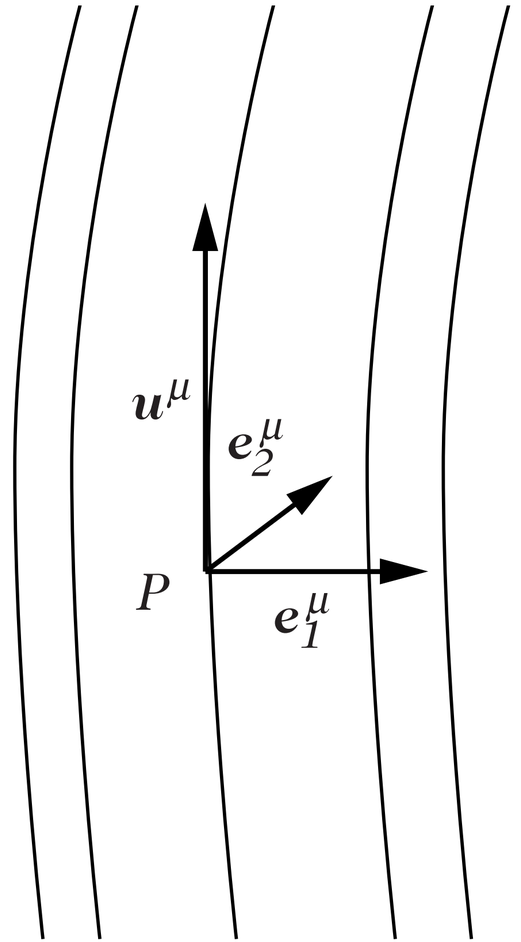
$$T^{\mu\nu}_{\Lambda} = -(\Lambda/8\pi G_{\rm N})g^{\mu\nu}$$

$$p_\Lambda = -\rho_\Lambda$$

$$w_\Lambda = -1$$


You
are
here!
Geophysical
data
Astrophysical
data
Not yet
observed
object
Our
past
lightcone
Our
worldline





Homogeneity (left) and isotropy (right) can be given exact mathematical meaning in GR: any two points P and Q can be related in an invariant way through an isometry in the hypersurfaces Σ_t (homogeneity $\rightarrow$ generalization of translations in space), and for any point P and two spacelike orthonormal vectors e_1^μ and e_2^μ such that $e^\mu u_\mu = 0$, there exists an isometry transforming $e_1 \leftrightarrow e_2$ (isotropy $\rightarrow$ generalization of rotations).

Isotropy

+

Homogeneity

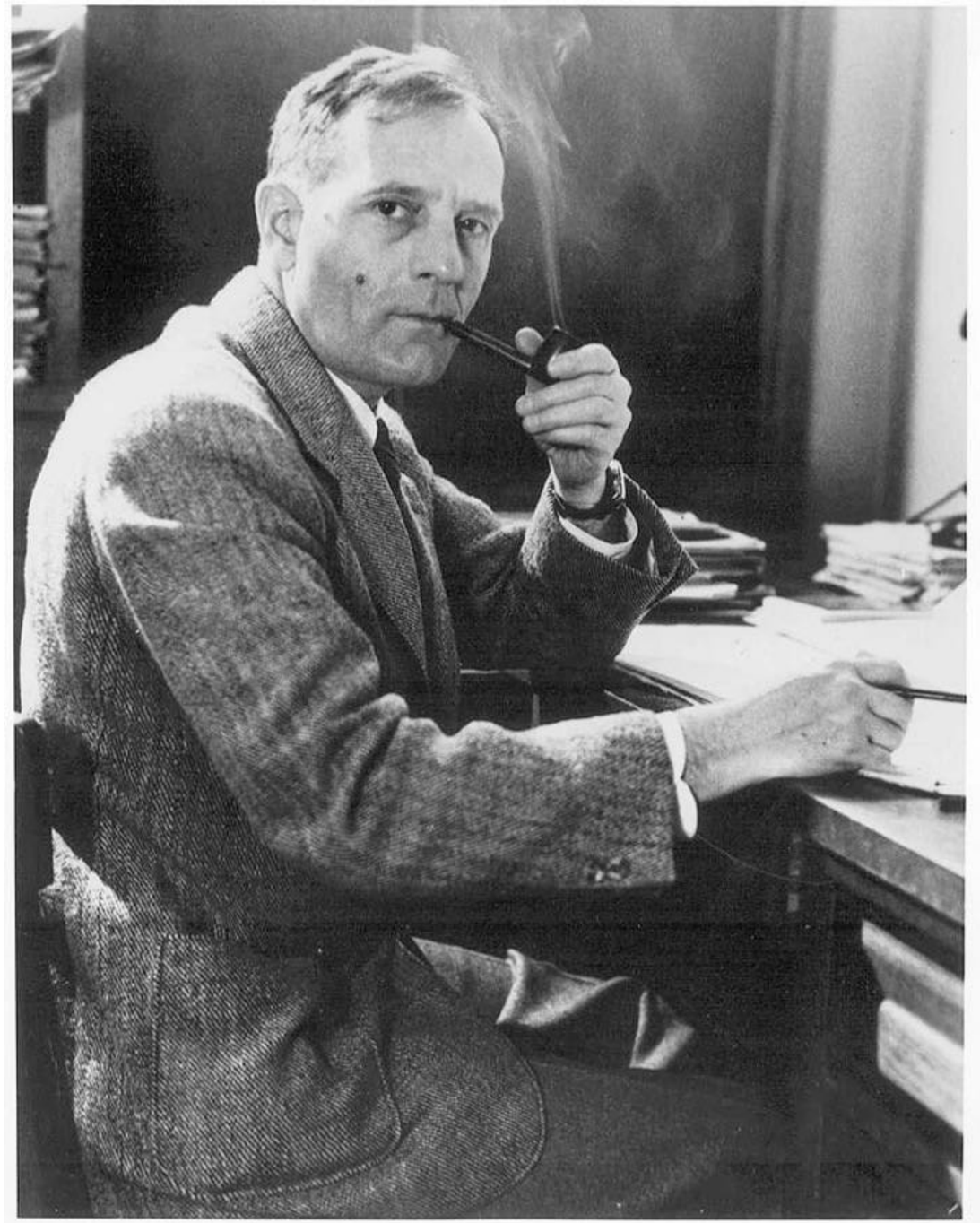
+

Einstein GR

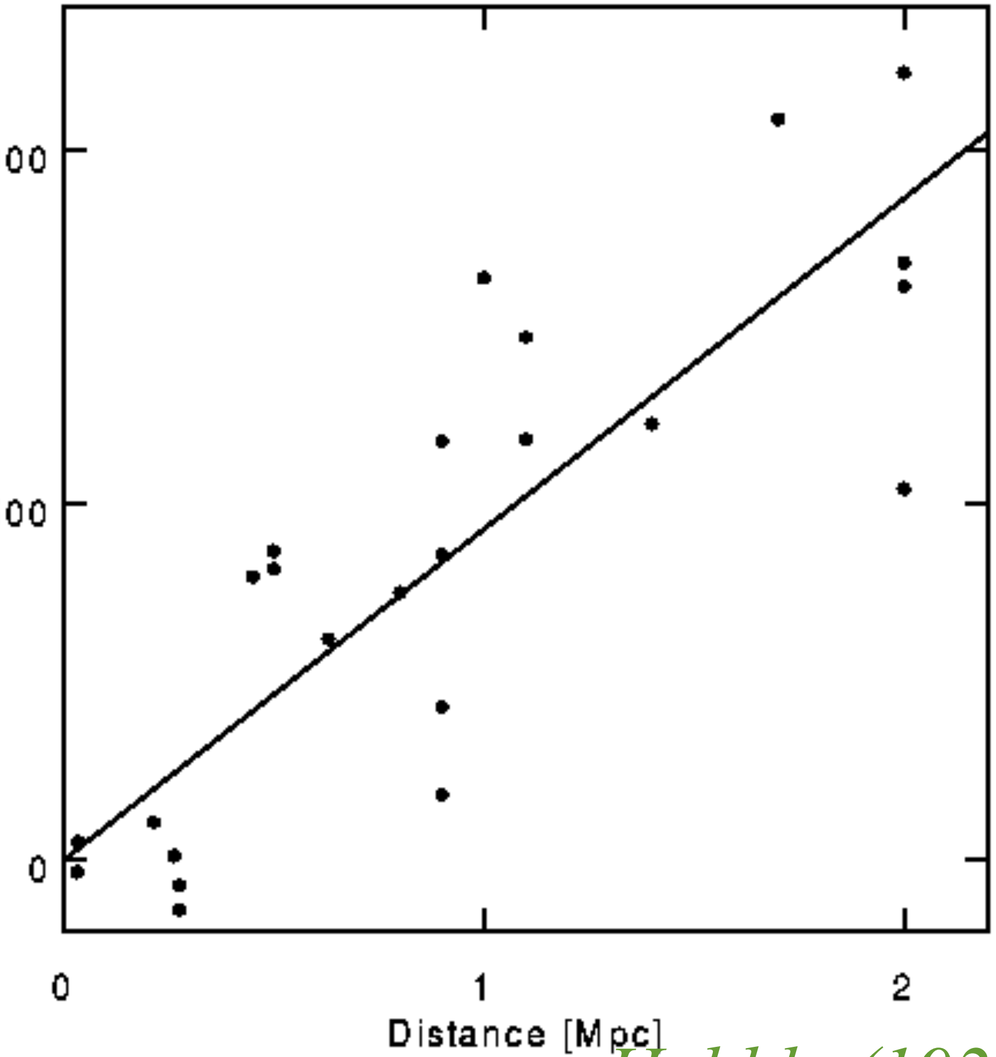
=

Expansion of the Universe

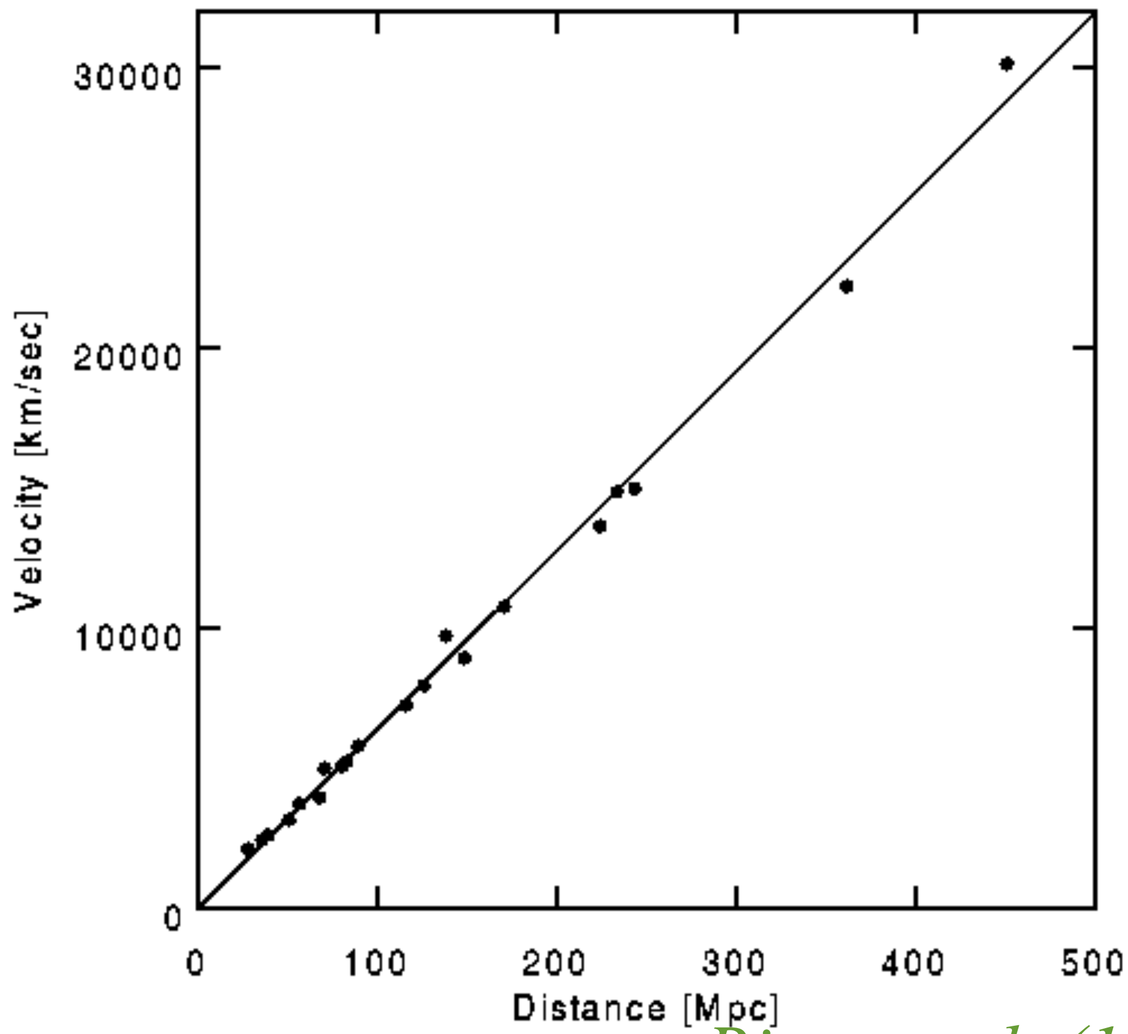
$$v = H\,d$$



Edwin Hubble



Hubble (1929)



Riess et al. (1996)

$\boldsymbol{h}(t) \equiv \boldsymbol{g}\big|_{P\in\Sigma_t}$ $\quad\Sigma_t$ = homogeneous and isotropic space with metric $\quad h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$

4D metric

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(h_{\mu\nu} - u_\mu u_\nu\right)\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(u_\mu \mathrm{d}x^\mu\right)^2 + h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

$$\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}\chi^2 + f_{\mathcal{K}}^2(\chi)\,\mathrm{d}\Omega^2$$

$$f_{\mathcal{K}} = \mathcal{K}^{-1/2}\sin\left(\sqrt{\mathcal{K}}\chi\right)$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$$

Always possible to normalize $\quad \mathcal{K} \in \{0, \pm 1\}$

Dynamics

4D metric: expansion... $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$

Conformal time

$$a\mathrm{d}\eta = \mathrm{d}t \quad \Longrightarrow \quad \mathrm{d}s^2 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \underset{\mathcal{K}\to 0}{=} a^2(\eta)\left(-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$

connections

$$\Gamma^t_{ij} = a^2 H\gamma_{ij}, \quad \Gamma^i_{tj} = H\delta^i_j \quad \text{and} \quad \Gamma^i_{jk} = \gamma^i_{jk}$$

$$\Gamma^\eta_{\eta\eta} = \mathcal{H}, \quad \Gamma^\eta_{ij} = \mathcal{H}\gamma_{ij} \quad \text{and} \quad \Gamma^i_{\eta j} = \mathcal{H}\delta^i_j$$

curvature tensors $R_{tt} = -3\dfrac{\ddot{a}}{a}, \quad R_{ti} = 0 \quad \text{and} \quad R_{ij} = a^2\gamma_{ij}\left(\dfrac{\ddot{a}}{a} + 2H^2 + 2\dfrac{\mathcal{K}}{a^2}\right)$

$$R = 6\left(H^2 + \ddot{a}/a + \mathcal{K}/a^2\right)$$

Einstein $G^t_t = -3\left(H^2 + \dfrac{\mathcal{K}}{a^2}\right), \quad G^t_i = 0 \quad \text{and} \quad G^i_j = -\delta^i_j\left(2\dfrac{\ddot{a}}{a} + H^2 + \dfrac{\mathcal{K}}{a^2}\right)$

→ Friedmann equations

Friedmann equations

$$H^2 = \frac{8\pi G_{\rm N}}{3}\rho - \frac{\mathcal{K}}{a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_{\rm N}}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

The fluid part is redundant...

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \Longleftarrow \quad \nabla_\mu T^{\mu\nu} = 0$$

In conformal time $\rho' + 3\mathcal{H}(\rho + p) = 0$

$$\mathcal{H}^2 + \mathcal{K} = \left(\frac{8\pi G_{\rm N}}{3}\rho + \frac{\Lambda}{3}\right)a^2$$

Define a critical density

$$\rho_{\rm c} \equiv \frac{3H^2}{8\pi G_{\rm N}} \quad \Longrightarrow \quad \Omega \equiv \frac{\rho}{\rho_{\rm c}}$$

$$\mathcal{H}' = \left[-\frac{4\pi G_{\rm N}}{3}(\rho + 3p) + \frac{\Lambda}{3}\right]a^2$$

$$\Omega_a = 8\pi G_{\rm N}\rho_a/(3H^2) = \rho_a/\rho_{\rm c}$$

$$\Omega_{\mathcal{K}} = -\mathcal{K}/(a^2H^2)$$

Friedmann = Energy budget

$$\sum_a \Omega_a + \Omega_\Lambda + \Omega_{\mathcal{K}} = 1$$

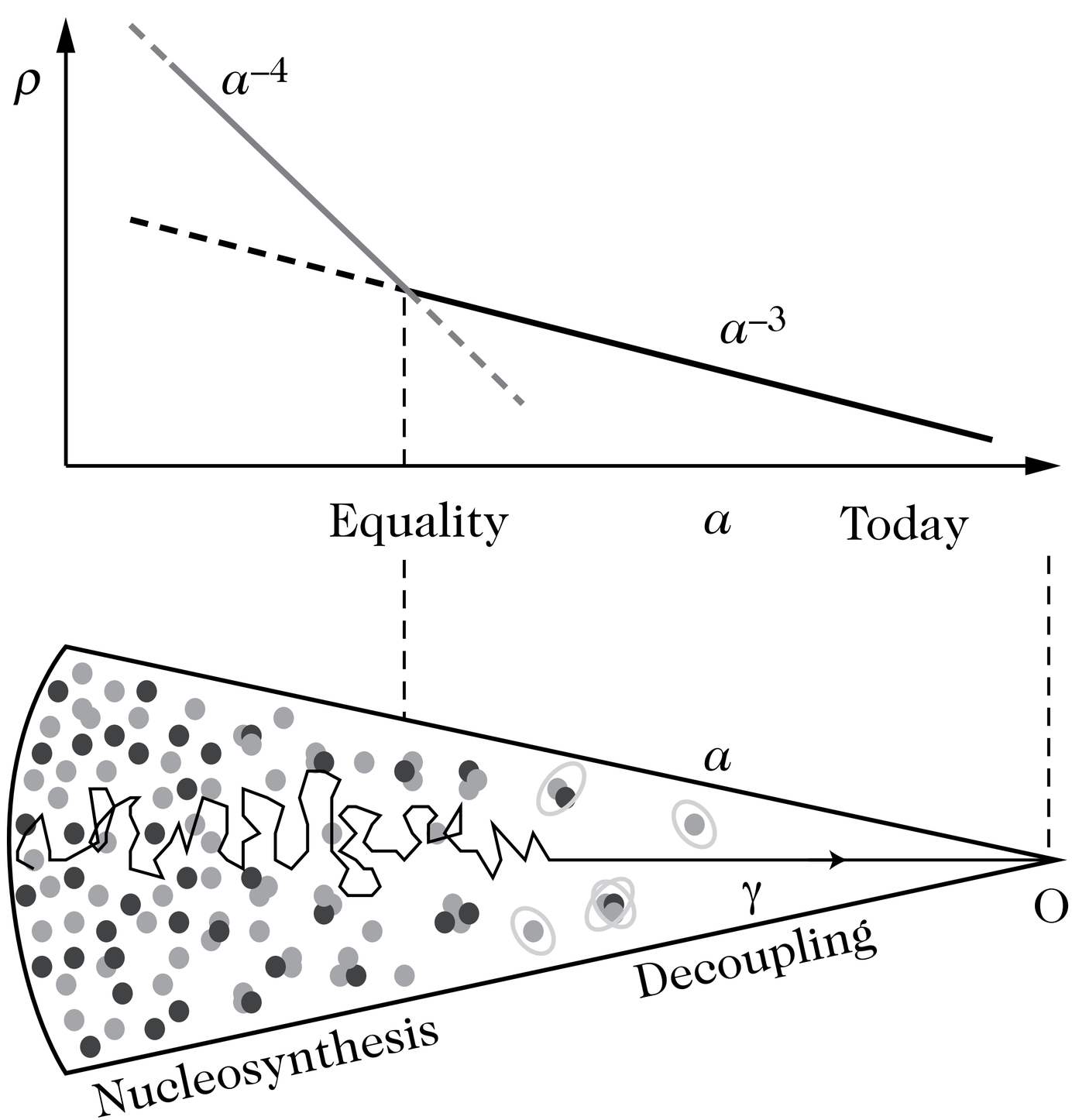
Particular solution: matter & radiation

Integrate conservation equation:

$$\rho[a(t)] = \rho_{\rm ini} \exp\left\{-3\int [1+w(a)]\,{\rm d}\ln a\right\} \underset{w\to{\rm cst}}{=} \rho_{\rm ini}\left(\frac{a}{a_{\rm ini}}\right)^{-3(1+w)}$$

$$\rightarrow \quad \rho_{\rm m} \propto a^{-3}$$

$$\rightarrow \quad \rho_{\rm r} \propto a^{-4}$$

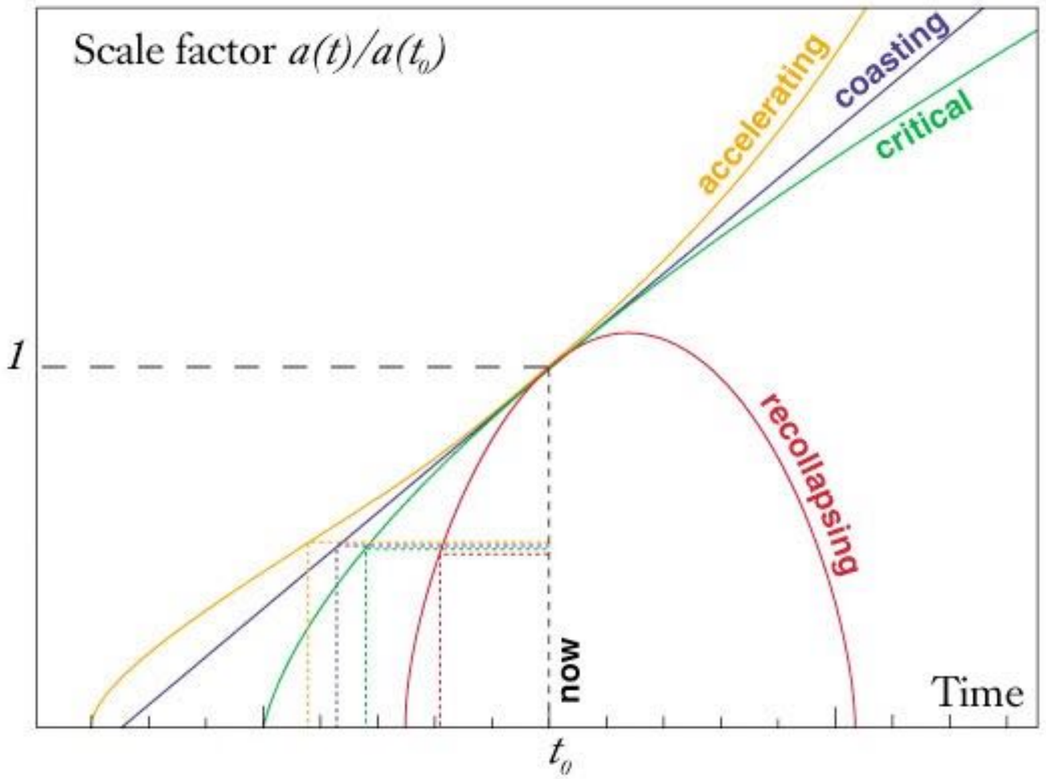

ρ
a^{-4}
a^{-3}
Equality
a
Today
a
γ
O
Decoupling
Nucleosynthesis

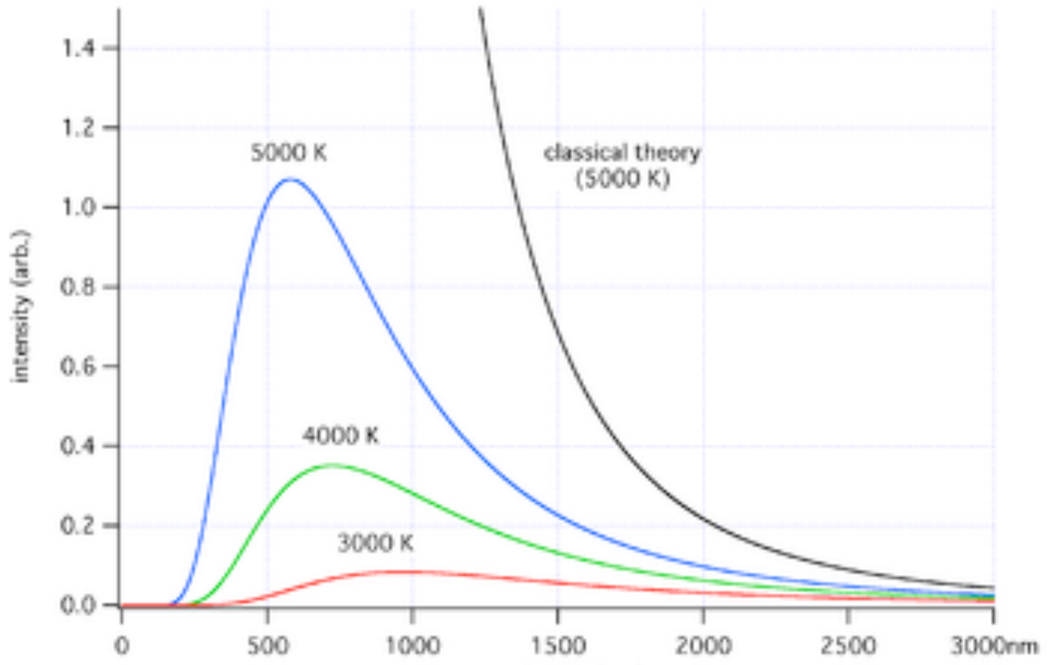
Limitations and puzzles of the standard model

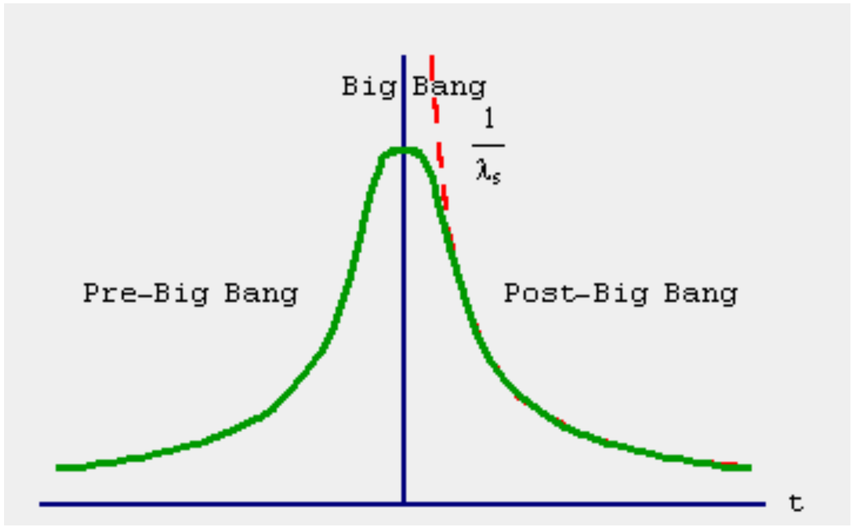
- Singularity $\quad \exists t_{(\pm\infty)}; a(t) \to 0$
- Horizon
- Flatness
- Monopoles
- **Validity of classical GR?**

Singularity problem

Purely classical effect?







Limitations and puzzles of the standard model

- Singularity $\quad \exists t_{(\pm\infty)}; a(t) \to 0$

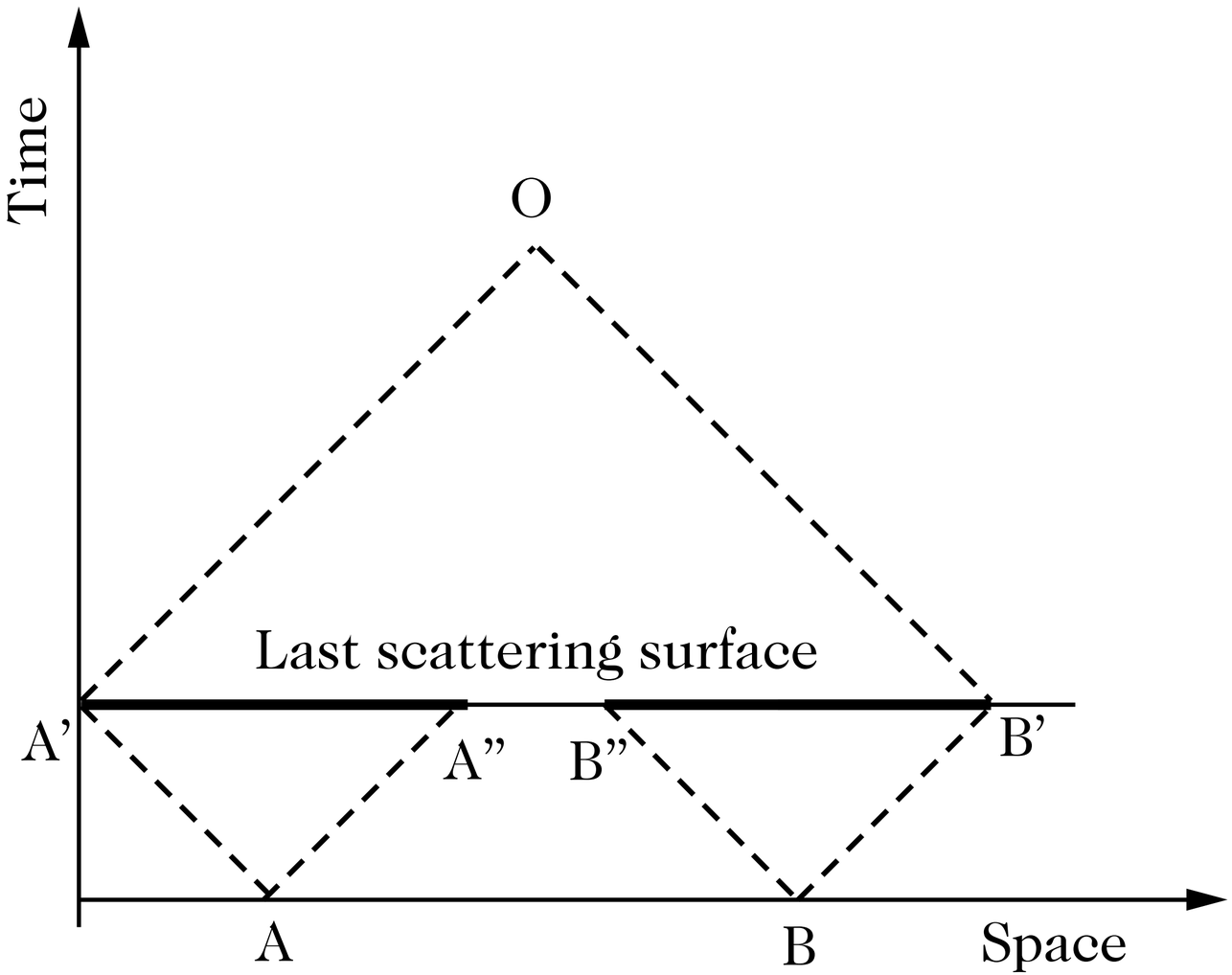
- Horizon

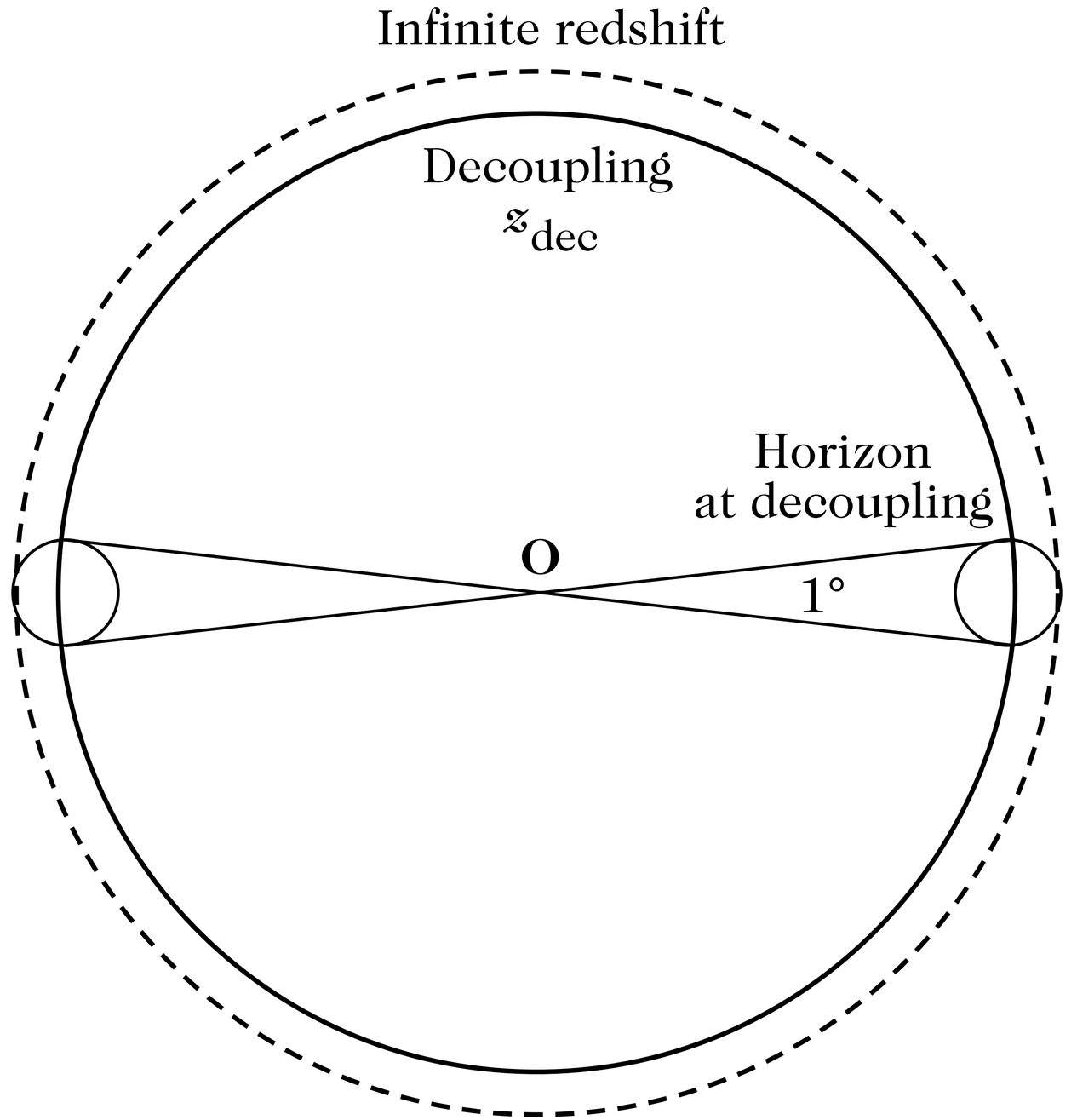
- Flatness

- Monopoles

- **Validity of classical GR?**

Horizon problem





Limitations and puzzles of the standard model

- Singularity $\quad \exists t_{(\pm\infty)}; a(t) \to 0$

- Horizon

- Flatness

- Monopoles

- **Validity of classical GR?**

Flatness problem

Observational fact: $\sum_a \Omega_a + \Omega_\Lambda = 1$, i.e. $\Omega_\mathcal{K} = 0$

Unstable fixed point!!

$$\frac{\mathrm{d}\Omega_\mathcal{K}}{\mathrm{d}\ln a} = (3w+1)(1-\Omega_\mathcal{K})\Omega_\mathcal{K}$$

constant equation of state

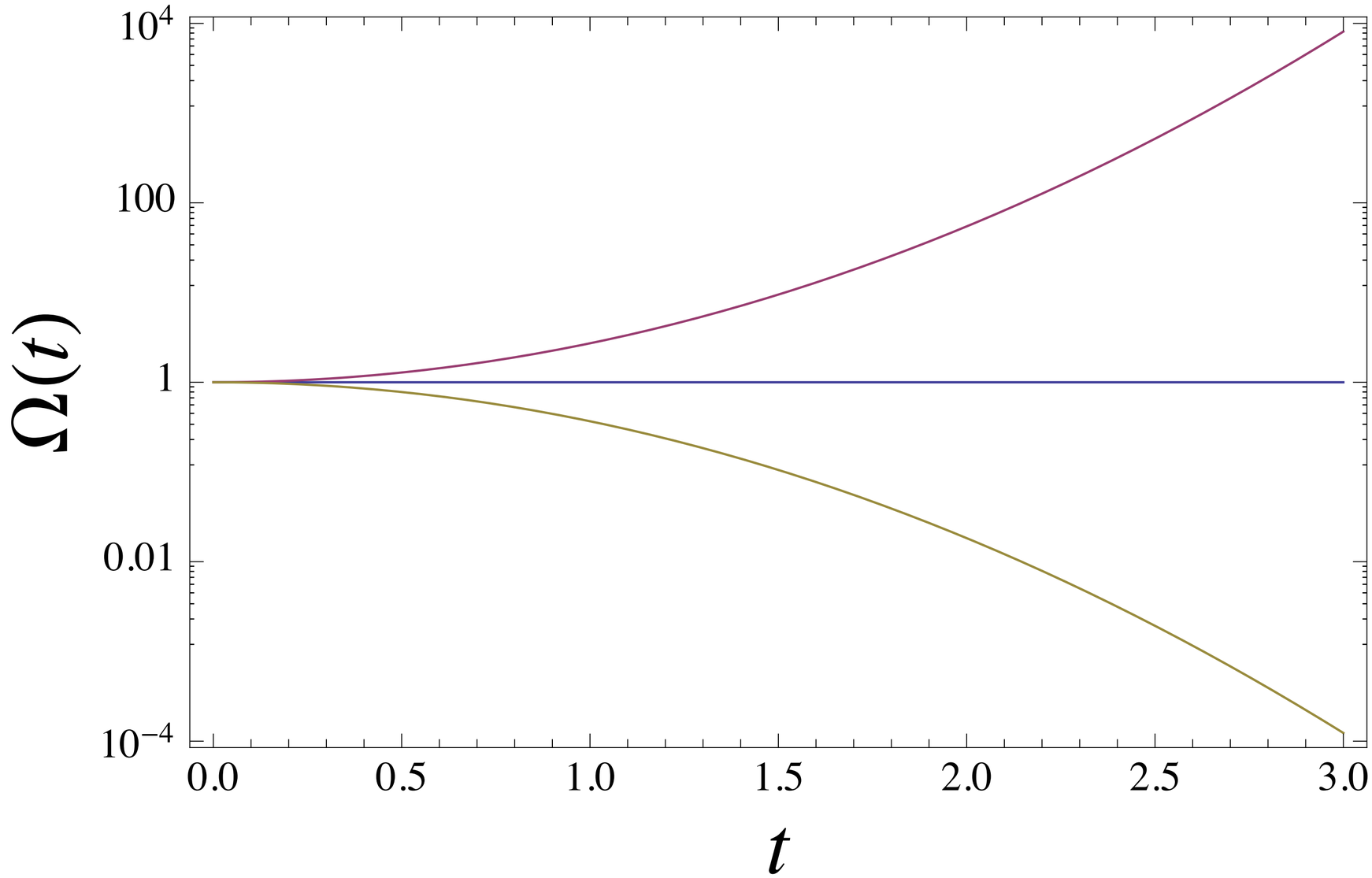
$$\Omega_\mathcal{K}(a_{\rm obs}) = \Omega_\mathcal{K}^{\rm ini}\left[\left(1-\Omega_\mathcal{K}^{\rm ini}\right)\left(\frac{a_{\rm obs}}{a_{\rm ini}}\right)^{1+3w} + \Omega_\mathcal{K}^{\rm ini}\right]^{-1}$$

$1+z$

$$|\Omega(z_{\rm eq}) - 1| < 3\times 10^{-5}$$

$$|\Omega(z_{\rm Planck}) - 1| < 10^{-60}$$

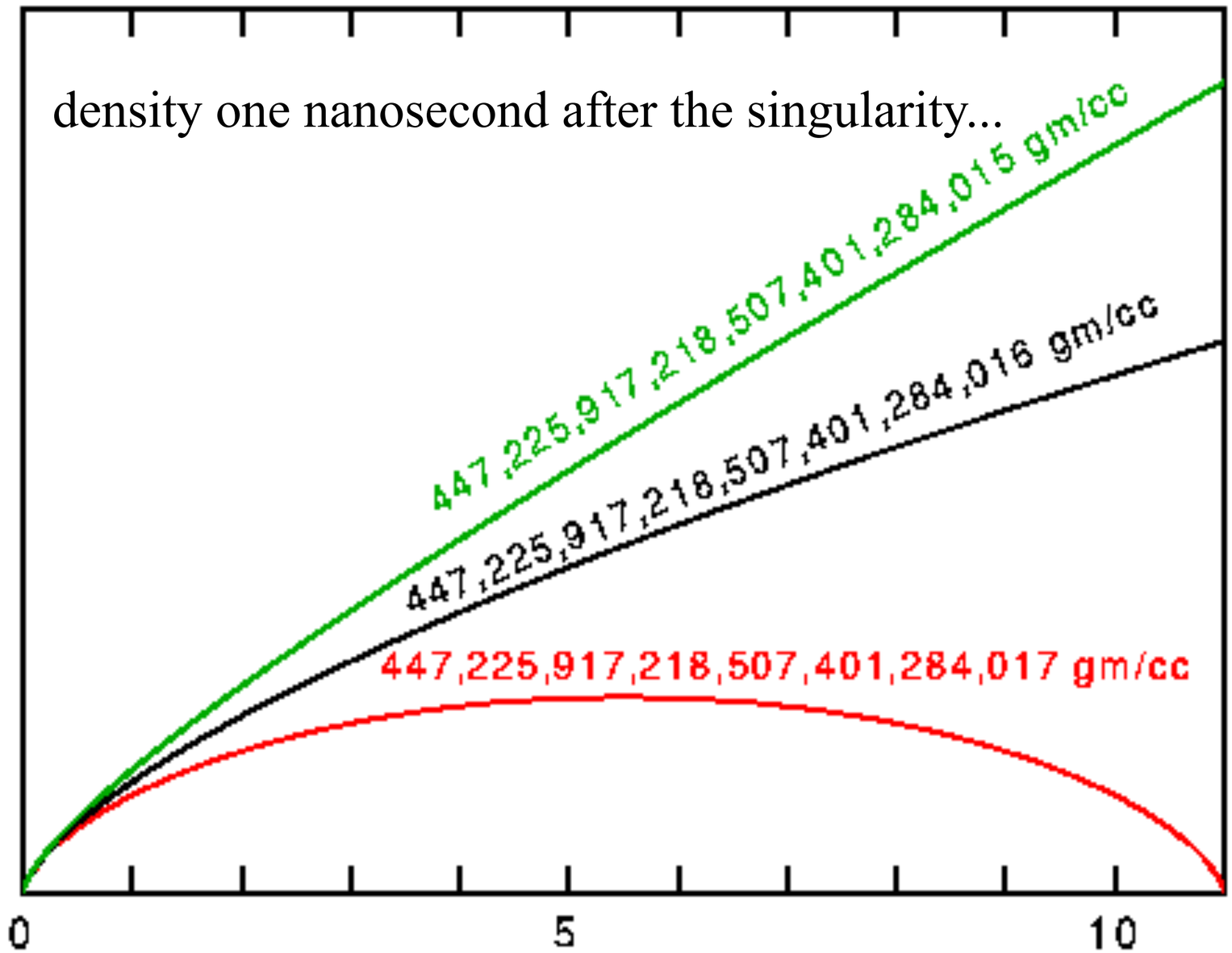
Flatness problem



Flatness problem

scale factor

$a(t)$

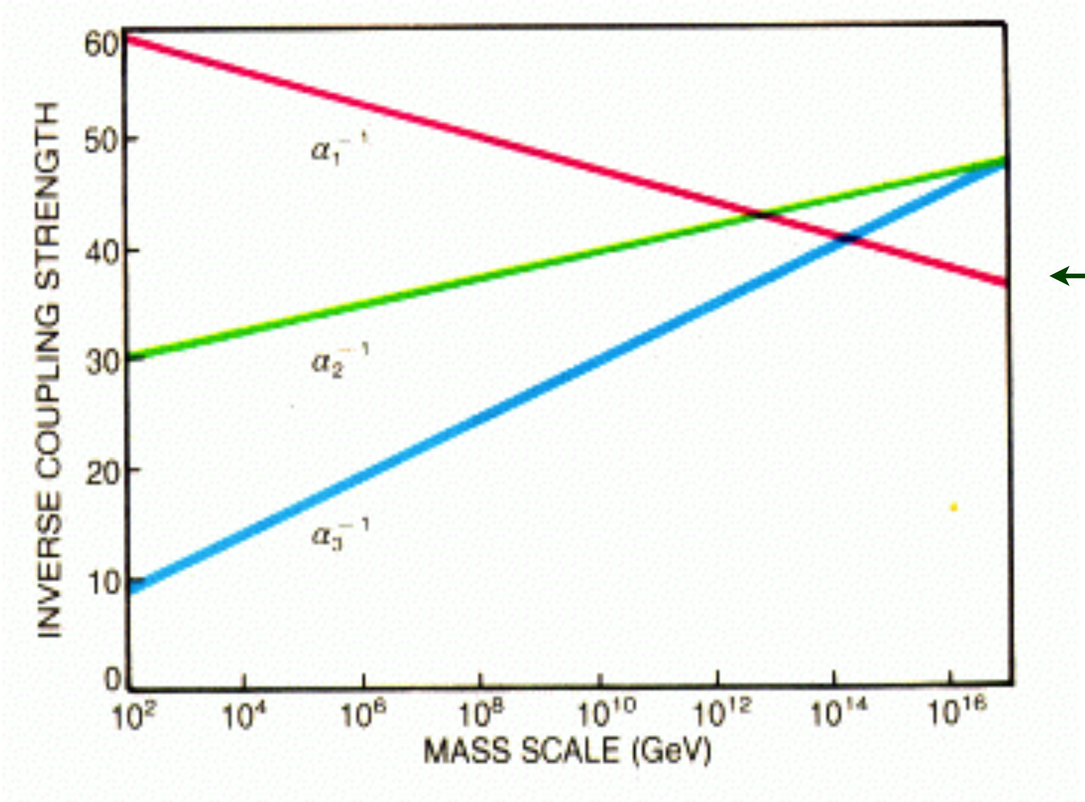


Time (Gyrs)

Limitations and puzzles of the standard model

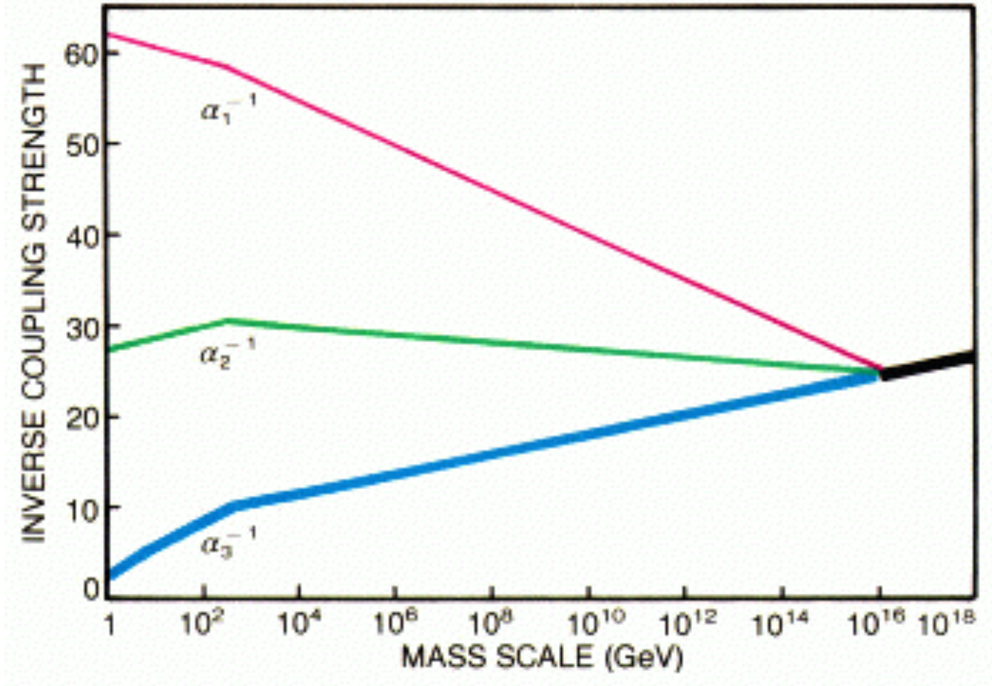
- Singularity $\quad \exists t_{(\pm\infty)}; a(t) \to 0$
- Horizon
- Flatness
- Monopoles
- **Validity of classical GR?**

GUT



SUSY

SUSY



Grand Unification

$$G \to H \to \cdots \to SU(3)_{\rm c} \otimes SU(2)_{\rm L} \otimes U(1)_{\rm Y} \to SU(3)_{\rm c} \otimes U(1)_{\rm em}$$

vacuum manifold $G/H_{\rm final} \supset U(1)$

$$\pi_2 \left[G/H_{\rm final} \supset U(1) \right] \not\sim \{\emptyset\}$$

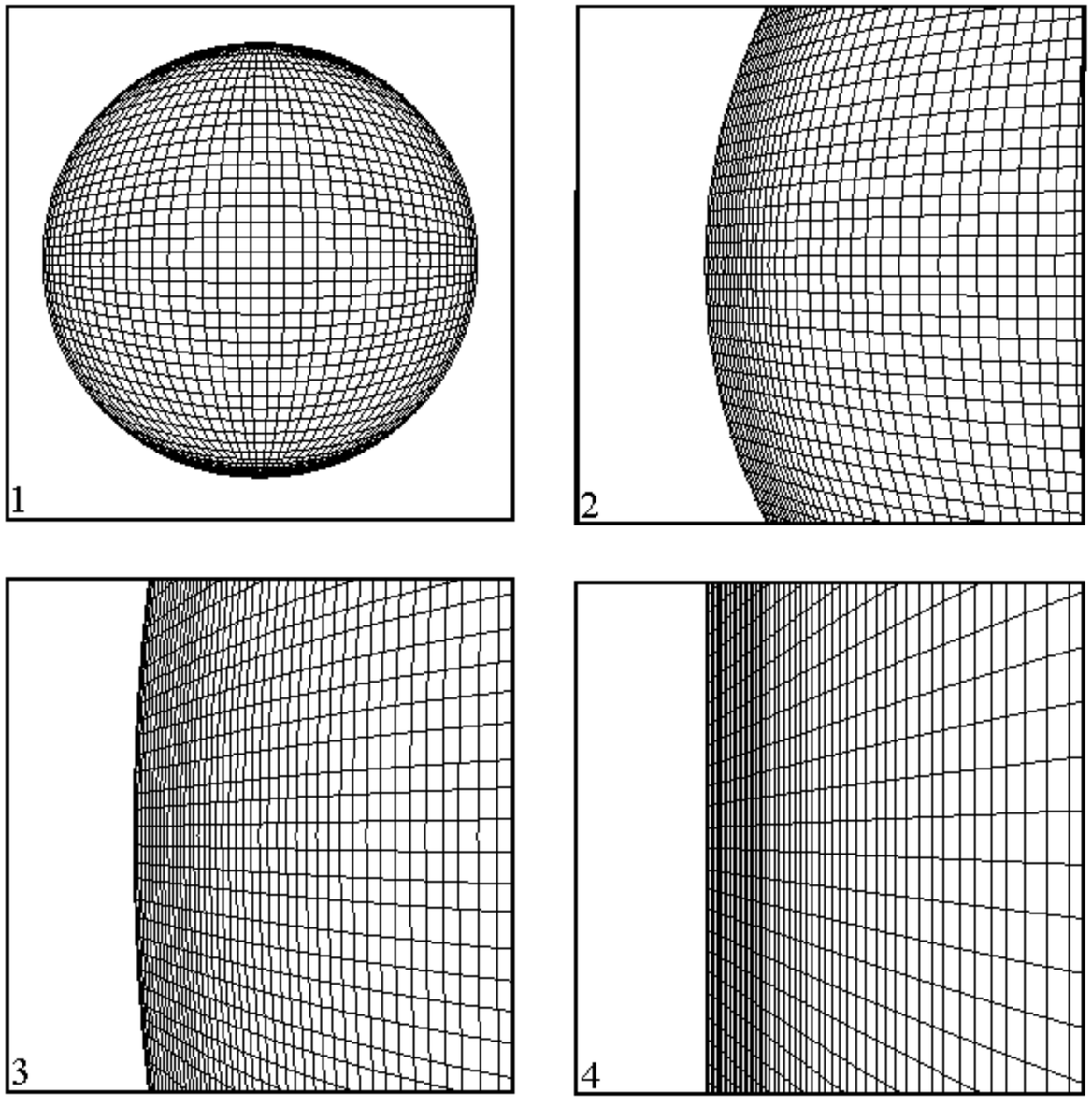
$$\Omega_{\rm monopoles} \gg 1$$

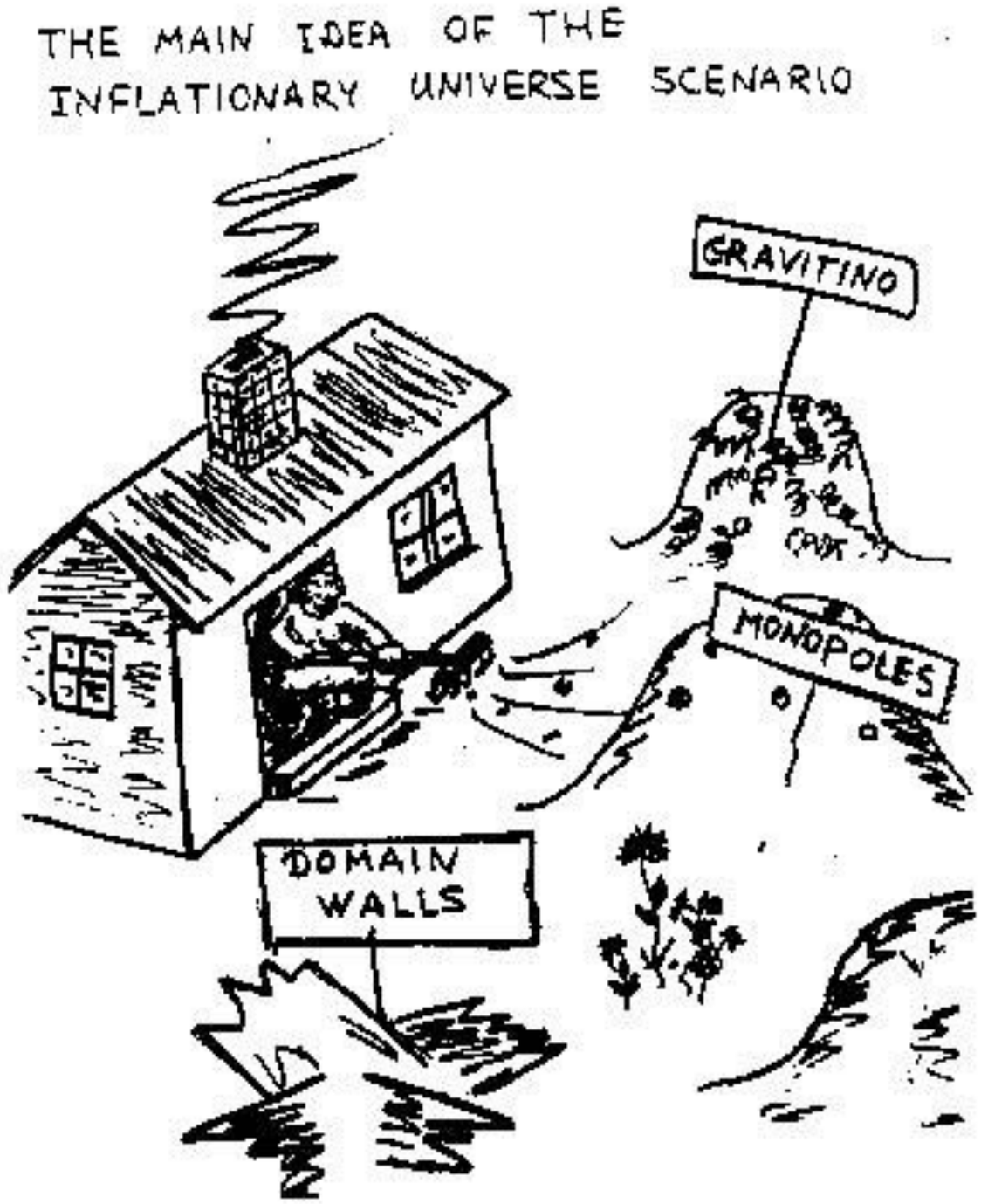
Limitations and puzzles of the standard model

- Singularity $\exists t_{(\pm\infty)}; a(t) \to 0$
- Horizon
- Flatness
- Monopoles
- **Validity of classical GR?** $E_{\rm inf} \simeq 10^{-3} M_{\rm Pl}$

Standard paradigm: inflation!

Phase of accelerated expansion





Standard Failures and inflationary solutions

Singularity Not solved... actually not addressed!

Horizon $d_{\rm H} \equiv a(t) \int_{t_{\rm i}}^{t} \frac{{\rm d}\tau}{a(\tau)}$ can be made as big as one wishes

Flatness $\frac{\rm d}{{\rm d}t} |\Omega - 1| = -2 \frac{\ddot{a}}{\dot{a}^3}$ $\qquad \ddot{a} > 0 \quad \& \quad \dot{a} > 0$

accelerated expansion (inflation)

Homogeneity & Isotropy

Initial Universe = very small patch
Accelerated expansion drives the shear to zero... $\Longrightarrow$ vacuum state!
+ attractor

Perturbations Bonus of the theory: superb predictions!!!

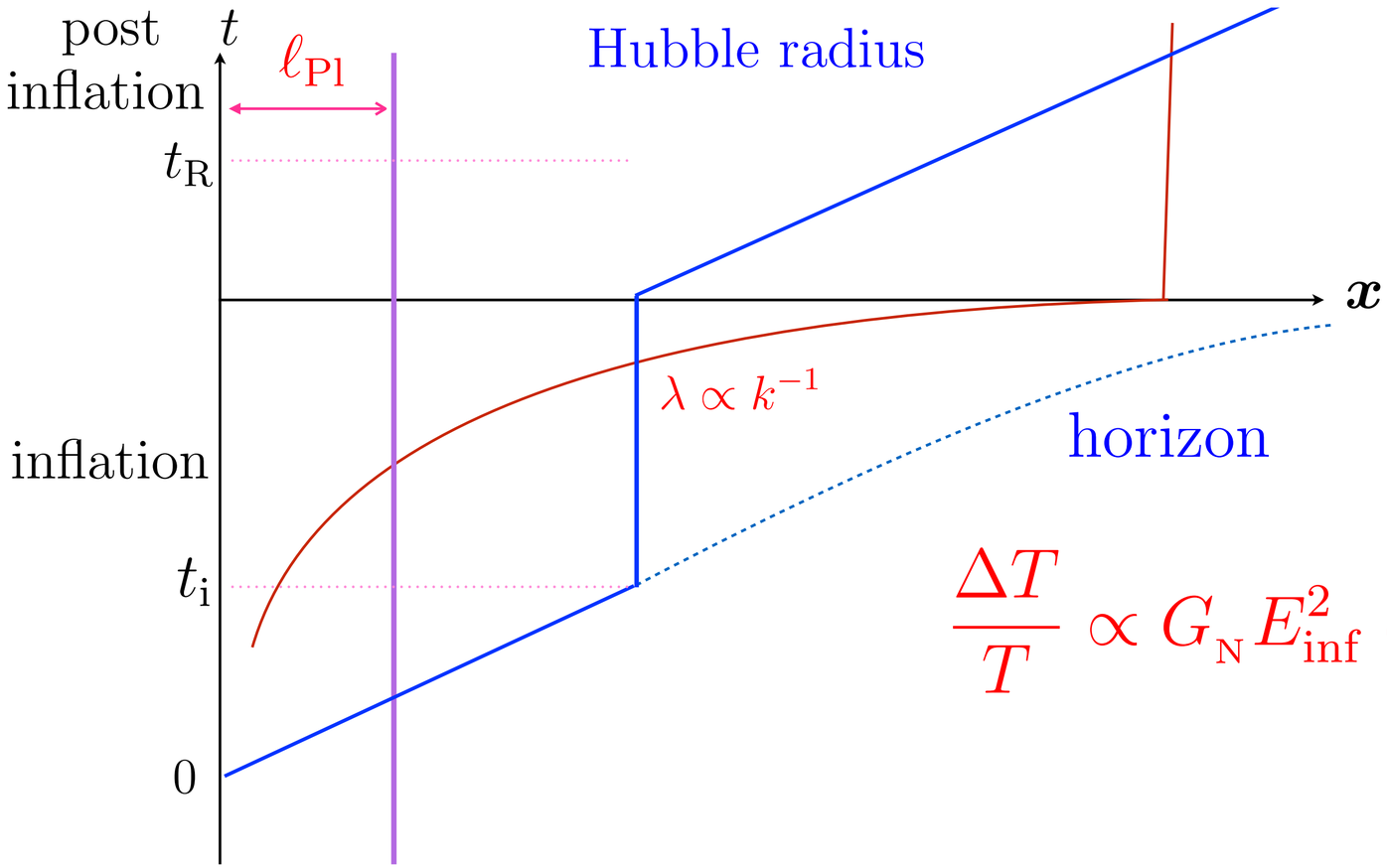
Others dark matter/energy, baryogenesis, ...

Inflation:

- ☺ **solves cosmological puzzles**
- ☺ **uses GR + scalar fields [(semi-)classical]**
- ☺ **can be implemented in high energy theories**
- ☺ **makes falsifiable predictions ...**
- ☺ **... consistent with all known observations**

Alternative model???

From R. Brandenberger, *in* M. Lemoine, J. Martin & PP (Eds.), "Inflationary cosmology", Lect. Notes Phys. **738** (Springer, Berlin, 2007).



- Singularity $\exists t_{(\pm\infty)}; a(t) \to 0$
- Trans-Planckian

$$\exists t; \ell(t) = \ell_0 \frac{a(t)}{a_0} \leq \ell_{\rm Pl}$$

- Hierarchy (amplitude)?

$$\frac{V(\varphi)}{\Delta\varphi^4} \leq 10^{-12}$$

- Classical GR?

$$\frac{\Delta T}{T} \propto G_{\rm N} E_{\rm inf}^2 \sim \left(\frac{E_{\rm inf}}{M_{\rm Pl}}\right)^2 \longrightarrow E_{\rm inf} \simeq 10^{-3} M_{\rm Pl}$$

- η problem & Lyth bound
- Initial condition & entropy
- Eternal inflation & measure (anthropic)

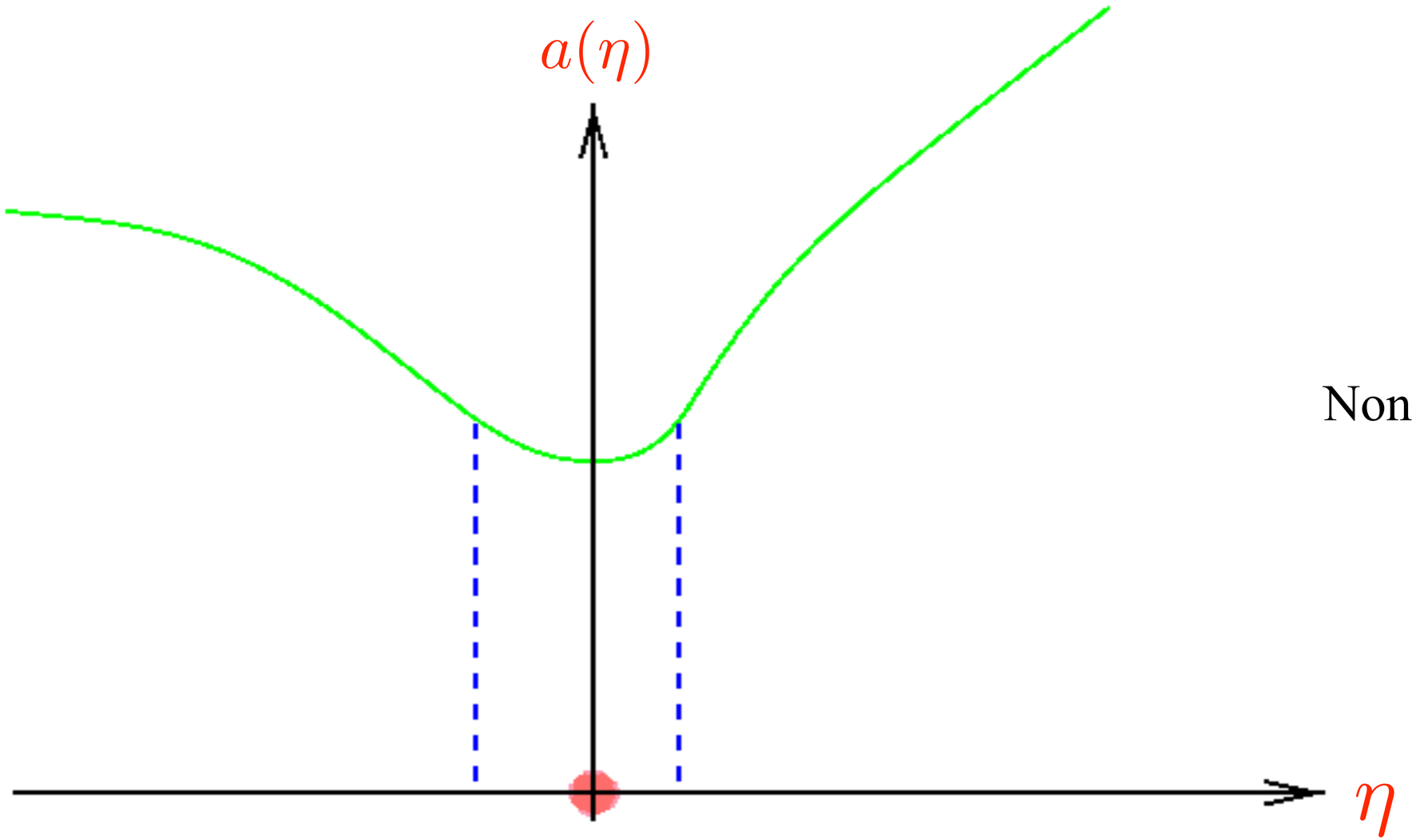
Inflation:

- ☺ **solves cosmological puzzles**
- ☺ **uses GR + scalar fields [(semi-)classical]**
- ☺ **can be implemented in high energy theories**
- ☺ **makes falsifiable predictions ...**
- ☺ **... consistent with all known observations**

Alternative model???

- **singularity, initial conditions & homogeneity**
- **string based ideas (PBB, other brane models, string gas, …)**
- **Quantum gravity / cosmology**
- **provide challengers / new ingredients!**

- **bouncing cosmology**

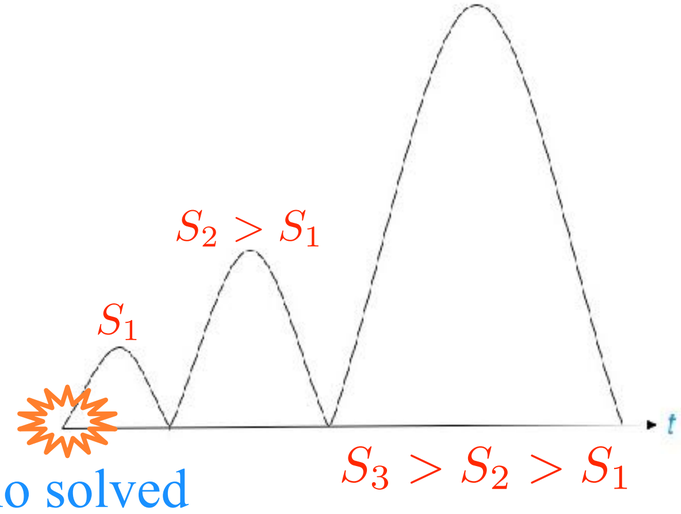


Non singular bounce

A brief history of bouncing cosmology

- R. C. Tolman, "*On the Theoretical Requirements for a Periodic Behaviour of the Universe*", PRD 38, 1758 (1931)
- G. Lemaître, "*L'Univers en expansion*", Ann. Soc. Sci. Bruxelles (1933)

…



…

Singularity pb no solved

- A. A. Starobinsky, "*On one non-singular isotropic cosmological model*", Sov. Astron. Lett. 4, 82 (1978)
- V. N. Melnikov, S.V. Orlov, Phys. Lett. A 70, 263 (1979).
- M. Novello & J. M. Salim, Phys. Rev. D20, 377 (1979).
- R. Durrer & J. Laukerman, "*The oscillating Universe: an alternative to inflation*", Class. Quantum Grav. 13, 1069 (1996)

→ Many new ideas, models…

- M. Novello & S.E. Perez Bergliaffa, "*Bouncing cosmologies*", Phys. Rep. **463**, 127 (2008)
- D. Battefeld & PP, "*A Critical Review of Classical Bouncing Cosmologies*", Phys. Rep. **571**, 1 (2015)
- R. Brandenberger & PP, "*Bouncing cosmologies: Progress and problems*", Found. Phys. (2017)

[1] V. Mukhanov, Physical foundations of cosmology (Cambridge University Press, 2005).
[2] P. Peter and J.-P. Uzan, Primordial cosmology, Oxford Graduate Texts (Oxford University Press, Oxford, 2009).
[3] P. Ade et al. (BICEP2 Collaboration), Phys.Rev.Lett. **112**, 241101 (2014), 1403.3985.
[4] A. H. Guth, Phys. Rev. **D23**, 347 (1981).
[5] A. Ijjas, P. J. Steinhardt, and A. Loeb, Phys. Lett. **B723**, 261 (2013), 1304.2785.
[6] A. Ijjas, P. J. Steinhardt, and A. Loeb (2014), 1402.6980.
[7] A. D. Linde, Lect. Notes Phys. **738**, 1 (2008), 0705.0164.
[8] M. Lemoine, J. Martin, and P. Peter, Lect. Notes Phys. **738** (2008).
[9] D. Baumann (2009), 0907.5424.
[10] J. Martin, C. Ringeval, and V. Vennin, Phys.Dark Univ. (2014), 1303.3787.
[11] J. Martin, C. Ringeval, R. Trotta, and V. Vennin, JCAP **1403**, 039 (2014), 1312.3529.
[12] D. Baumann and L. McAllister (2014), 1404.2601.
[13] A. H. Guth, D. I. Kaiser, and Y. Nomura, Phys.Lett. **B733**, 112 (2014), 1312.7619.
[14] A. Linde (2014), 1402.0526.
[15] J. D. Barrow and A. R. Liddle, Gen.Rel.Grav. **29**, 1503 (1997), gr-qc/9705048.
[16] C. Molina-Paris, M. Sakellariadou, and F. Bouchet, JCAP **0702**, 023 (2007), astro-ph/0511646.
[17] P. Peter and C. Ringeval, JCAP **1305**, 005 (2013), 1302.0953.
[18] P. Ade et al. (Planck Collaboration) (2013), 1303.5085.
[19] T. Battefeld and S. Watson, Rev. Mod. Phys. **78**, 435 (2006), hep-th/0510022.
[20] R. H. Brandenberger, A. Nayeri, and S. P. Patil, Phys.Rev. **D90**, 067301 (2014), 1403.4927.
[21] Y.-S. Piao, B. Feng, and X.-m. Zhang, Phys.Rev. **D69**, 103520 (2004), hep-th/0310206.
[22] F. T. Falciano, M. Lilley, and P. Peter, Phys. Rev. **D77**, 083513 (2008), 0802.1196.
[23] M. Lilley, L. Lorenz, and S. Clesse, JCAP **1106**, 004 (2011), 1104.3494.
(2002), gr-qc/0109038.
[48] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. **D64**, 123522 (2001), hep-th/0103239.
[49] J. Martin and P. Peter, Phys. Rev. **D68**, 103517 (2003), hep-th/0307077.
[50] B. Xue and P. J. Steinhardt, Phys. Rev. Lett. **105**, 261301 (2010), 1007.2875.
[51] N. Deruelle and V. F. Mukhanov, Phys. Rev. **D52**, 5549 (1995), gr-qc/9503050.
[52] B. Craps, T. Hertog, and N. Turok, Phys. Rev. **D86**, 043513 (2012), 0712.4180.
[53] B. Craps, T. Hertog, and N. Turok, Phys. Rev. **D80**, 086007 (2009), 0905.0709.
[54] M. Smolkin and N. Turok (2012), 1211.1322.
[55] I. Bars, S.-H. Chen, and N. Turok, Phys. Rev. **D84**, 083513 (2011), 1105.3606.
[56] I. Bars (2011), 1109.5872.
[57] I. Bars, S.-H. Chen, P. J. Steinhardt, and N. Turok, Phys. Rev. **D84**, 083513 (2011), 1105.3470.
[58] I. Bars, P. Steinhardt, and N. Turok, Phys. Rev. **D89**, 043515 (2014), 1307.1848.
[59] I. Bars, P. J. Steinhardt, and N. Turok, Phys. Lett. **B726**, 50 (2013), 1307.8106.
[60] I. Bars (2012), 1209.1068.
[61] I. Bars, P. Steinhardt, and N. Turok, Phys.Rev. **D89**, 061302 (2014), 1312.0739.
[62] P. Peter and N. Pinto-Neto, Phys. Rev. **D66**, 063509 (2002), hep-th/0203013.
[63] C. Lin, R. H. Brandenberger, and L. Perreault Levasseur, JCAP **1104**, 019 (2011), 1007.2654.
[64] T. Qiu, J. Evslin, Y.-F. Cai, M. Li, and X. Zhang, JCAP **1110**, 036 (2011), 1108.0593.
[65] Y.-F. Cai, T. Qiu, Y.-S. Piao, M. Li, and X. Zhang, JHEP **0710**, 071 (2007), 0704.1090.
[66] C. Kounnas, H. Partouche, and N. Toumbas, Nucl. Phys. **B855**, 280 (2012), 1106.0946.
[67] T. Biswas, A. Mazumdar, and W. Siegel, JCAP **0603**, 009 (2006), hep-th/0508194.
[68] T. Biswas, R. Brandenberger, A. Mazumdar, and W. Siegel, JCAP **0712**, 011 (2007), hep-th/0610274.
[69] D. Langlois and A. Naruko, Class. Quant. Grav. **30**, 205012 (2013), 1305.6346.
[70] M. Koehn, J.-L. Lehners, and B. Ovrut, Phys. Rev. **D88**, 023528 (2013), 1302.0840.
[71] M. Koehn, J.-L. Lehners, and B. A. Ovrut, Phys.Rev. **D90**, 025005 (2014), 1310.7577.
[72] L. Battarra, M. Koehn, J.-L. Lehners, and B. A. Ovrut, JCAP **1407**, 007 (2014), 1404.5067.
[73] L. Smolin, Rev. Mod. Phys. pp. 655–682 (2004), hep-th/0408048.
[74] C. Rovelli, PoS **QGQGS2011**, 003 (2011), 1102.3660.
[75] H. Nicolai, K. Peeters, and M. Zamaklar, Class. Quant. Grav. **22**, R193 (2005), hep-th/0501114.
[76] A. Henderson, A. Laddha, and C. Tomlin, Phys.Rev. **D88**, 044028 (2013), 1204.0211.
[77] A. Henderson, A. Laddha, and c. Tomlin, Phys. Rev. **D88**, 044029 (2013), 1210.3960.
[78] C. Tomlin and M. Varadarajan, Phys. Rev. **D87**, 044039 (2013), 1210.6869.
[79] W. Atwood et al. (LAT Collaboration), Astrophys. J. **697**, 1071 (2009), 0902.1089.
[80] L. Shao, Z. Xiao, and B.-Q. Ma, Astropart.Phys. **33**, 312 (2010), 0911.2276.
[81] M. Ackermann et al. (Fermi GBM/LAT Collaboration), Phys. Rev. Lett. **116**, 151105 (2016), 1602.07700.
[82] F. Aharonian et al. (H. E. S. S. Collaboration), Phys. Rev. Lett. **101**, 170402 (2008), 0810.3475.
[83] A. Abramowski et al. (HESS Collaboration), Astropart.Phys. **34**, 738 (2011), 1101.3650.
[84] G. Amelino-Camelia, J. R. Ellis, N. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature **393**, 763 (1998), astro-ph/9712103.
[85] V. Vasileiou, A. Jacholkowska, F. Piron, J. Bolmont, C. Couturier, et al., Phys. Rev. **D87**, 122001 (2013), 1305.3463.
[86] M. Bojowald, Class. Quant. Grav. **29**, 213001 (2012), 1209.3403.
[87] A. Ashtekar and P. Singh, Class. Quant. Grav. **28**, 213001 (2011), 1108.0893.
[88] G. Vereshchagin, JCAP **0407**, 013 (2004), gr-qc/0406108.
[89] P. Singh, K. Vandersloot, and G. Vereshchagin, Phys.Rev. **D74**, 043510 (2006), gr-qc/0606032.
[90] T. Cailleteau, P. Singh, and K. Vandersloot, Phys.Rev. **D80**, 124013 (2009), 0907.5591.
[91] L. Linsefors and A. Barrau, Phys. Rev. **D87**, 123509 (2013), 1301.1264.
[92] J. Amorós, J. de Haro, and S. D. Odintsov, Phys.Rev. **D87**, 104037 (2013), 1305.2344.
[93] J. Mielczarek and W. Piechocki, Phys. Rev. **D86**, 083508 (2012), 1108.0005.
[94] J. P. Gazeau, J. Mielczarek, and W. Piechocki, Phys. Rev. **D87**, 123508 (2013), 1303.1687.
[95] A. Barrau, T. Cailleteau, J. Grain, and J. Mielczarek, Class.Quant.Grav. **31**, 053001 (2014), 1309.6896.
[96] E. Wilson-Ewing, JCAP **1308**, 015 (2013), 1306.6582.
[97] B. Gupt and P. Singh, Class.Quant.Grav. **30**, 145013 (2013), 1304.7686.
[98] Y.-F. Cai and E. Wilson-Ewing, JCAP **1403**, 026 (2014), 1402.3009.
[99] A. Corichi and T. Vukasinac, Phys.Rev. **D86**, 064019 (2012), 1202.1846.
[100] M. Bojowald, G. M. Hossain, M. Kagan, and S. Shankaranarayanan, Phys. Rev. **D79**, 043505 (2009), 0811.1572.
[101] M. Bojowald, G. M. Hossain, M. Kagan, and S. Shankaranarayanan, Phys. Rev. **D78**, 063547 (2008), 0806.3929.
[102] T. Cailleteau, L. Linsefors, and A. Barrau, Class.Quant.Grav. **31**, 125011 (2014), 1307.5238.
[103] A. Ashtekar, T. Pawlowski, P. Singh, and K. Vandersloot, Phys. Rev. **D75**, 024035 (2007), gr-qc/0612104.
[104] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. **D74**, 084003 (2006), gr-qc/0607039.
[105] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. Lett. **96**, 141301 (2006), gr-qc/0602086.
[106] E. Wilson-Ewing, JCAP **1303**, 026 (2013), 1211.6269.
[107] Y. Shtanov and V. Sahni, Phys. Lett. **B557**, 1 (2003), gr-qc/0208047.
[108] M. G. Brown, K. Freese, and W. H. Kinney, JCAP **0803**, 002 (2008), astro-ph/0405353.
[109] T. Battefeld and G. Geshnizjani, Phys. Rev. **D73**, 064013 (2006), hep-th/0503160.
[110] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-ph/9905221.
[111] M. Bojowald and G. M. Paily, Phys. Rev. **D86**, 104018 (2012), 1112.1899.
[112] P. Singh, Phys.Rev. **D85**, 104011 (2012), 1112.6391.
[113] T. Thiemann, Lect. Notes Phys. **721**, 185 (2007), hep-th/0608210.
[114] P. Holland, Phys. Rept. **224**, 95 (1993).
[115] A. S. Sanz and S. Miret-Artés, Lect. Notes Phys. **850**, 187 (2012).
[116] N. Pinto-Neto and J. Fabris, Class. Quant. Grav. **30**, 143001 (2013), 1306.0820.
[117] J. Acacio de Barros, N. Pinto-Neto, and M. Sagioro-Leal, Phys. Lett. **A241**, 229 (1998), gr-qc/9710084.
[118] R. Casadio, Int.J.Mod.Phys. **D9**, 511 (2000), gr-qc/9810073.
[119] P. Peter, E. J. Pinho, and N. Pinto-Neto, Phys. Rev. **D73**, 104017 (2006), gr-qc/0605060.
[120] E. J. Pinho and N. Pinto-Neto, Phys. Rev. **D76**, 023506 (2007), hep-th/0610192.
[121] R. Y. Donagi, J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, JHEP **0111**, 041 (2001), hep-th/0105199.
[122] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok (2001), hep-th/0105212.
[123] R. Kallosh, L. Kofman, and A. D. Linde, Phys. Rev. **D64**, 123523 (2001), hep-th/0104073.
[124] R. Kallosh, L. Kofman, A. D. Linde, and A. A. Tseytlin, Phys. Rev. **D64**, 123524 (2001), hep-th/0106241.
[125] A. Lukas, B. A. Ovrut, K. Stelle, and D. Waldram, Nucl.Phys. **B552**, 246 (1999), hep-th/9806051.
[126] J. Martin, P. Peter, N. Pinto Neto, and D. J. Schwarz, Phys. Rev. **D65**, 123513 (2002), hep-th/0112128.
[127] J. Martin, P. Peter, N. Pinto-Neto, and D. J. Schwarz, Phys. Rev. **D67**, 028301 (2003), hep-th/0204222.
[128] H. Kragh (2013), 1308.0932.
[129] P. J. Steinhardt and N. Turok, Phys. Rev. **D65**, 126003 (2002), hep-th/0111098.
[130] P. J. Steinhardt and N. Turok (2001), hep-th/0111030.
[131] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok, Phys. Rev. **D65**, 086007 (2002), hep-th/0108187.
[132] G. N. Felder, A. V. Frolov, L. Kofman, and A. D. Linde, Phys.Rev. **D66**, 023507 (2002), hep-th/0202017.
[133] A. Mithani and A. Vilenkin (2012), 1204.4658.
[134] R. Brandenberger and F. Finelli, JHEP **0111**, 056 (2001), hep-th/0109004.
[135] D. H. Lyth, Phys.Lett. **B524**, 1 (2002), hep-ph/0106153.
[136] R. Durrer and F. Vernizzi, Phys. Rev. **D66**, 083503 (2002), hep-ph/0203275.
[137] P. Creminelli, A. Nicolis, and M. Zaldarriaga, Phys. Rev. **D71**, 063505 (2005), hep-th/0411270.
[138] C.-Y. Tseng, Phys. Rev. **D87**, 023518 (2013), 1210.0581.
[139] L. Battarra and J.-L. Lehners, Phys.Rev. **D89**, 063516 (2014), 1309.2281.
[140] A. Notari and A. Riotto, Nucl. Phys. **B644**, 371 (2002), hep-th/0205019.
[141] J.-L. Lehners, P. McFadden, N. Turok, and P. J. Steinhardt, Phys. Rev. **D76**, 103501 (2007), hep-th/0702153.
[142] E. I. Buchbinder, J. Khoury, and B. A. Ovrut, Phys. Rev. **D76**, 123503 (2007), hep-th/0702154.
[143] P. Creminelli and L. Senatore, JCAP **0711**, 010 (2007), hep-th/0702165.
[144] E. I. Buchbinder, J. Khoury, and B. A. Ovrut, JHEP **0711**, 076 (2007), 0706.3903.
[145] R. Kallosh, J. U. Kang, A. Linde, and V. Mukhanov, JCAP **4**, 018 (2008), 0712.2040.
[146] R. P. Woodard, Lect.Notes Phys. **720**, 403 (2007), hep-th/0601672.
[147] J. Khoury, P. J. Steinhardt, and N. Turok, Phys. Rev. Lett. **92**, 031302 (2004), hep-th/0307132.
[148] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. **D66**, 046005 (2002), hep-th/0109050.
[149] J.-L. Lehners and P. J. Steinhardt, Phys. Rev. **D79**, 063503 (2009), 0812.3388.
[150] J.-L. Lehners, Phys. Rev. **D84**, 103518 (2011), 1107.4551.
[151] J.-L. Lehners, Phys. Rept. **465**, 223 (2008), 0806.1245.
[152] J.-L. Lehners and P. J. Steinhardt, Phys. Rev. **D87**, 123533 (2013), 1304.3122.
[153] J.-L. Lehners, P. J. Steinhardt, and N. Turok, Int. J. Mod. Phys. **D18**, 2231 (2009), 0910.0834.
[154] J.-L. Lehners, Class. Quant. Grav. **28**, 204004 (2011), 1106.0172.
[155] M. C. Johnson and J.-L. Lehners, Phys. Rev. **D85**, 103509 (2012), 1112.3360.
[156] K. Enqvist and M. S. Sloth, Nucl.Phys. **B626**, 395 (2002), hep-ph/0109214.
[157] D. H. Lyth and D. Wands, Phys. Lett. **B524**, 5 (2002), hep-ph/0110002.
[158] K. Koyama, S. Mizuno, and D. Wands, Class. Quant. Grav. **24**, 3919 (2007), 0704.1152.
[159] K. Koyama and D. Wands, JCAP **0704**, 008 (2007), hep-th/0703040.
[160] K. Koyama, S. Mizuno, F. Vernizzi, and D. Wands, JCAP **0711**, 024 (2007), 0708.4321.
[161] J.-L. Lehners, P. McFadden, and N. Turok, Phys. Rev. **D75**, 103510 (2007), hep-th/0611259.
[162] J.-L. Lehners, P. McFadden, and N. Turok, Phys. Rev. **D76**, 023501 (2007), hep-th/0612026.
[163] T. Battefeld, Phys. Rev. **D77**, 063503 (2008), 0710.2540.
[164] P. Ade et al. (Planck Collaboration), Astron.Astrophys. **571**, A22 (2014), 1303.5082.
[165] P. Ade et al. (Planck Collaboration) (2013), 1303.5083.
[166] J.-L. Lehners and P. J. Steinhardt, Phys. Rev. **D77**, 063533 (2008), 0712.3779.
[167] J.-L. Lehners and P. J. Steinhardt, Phys. Rev. **D78**, 023506 (2008), 0804.1293.
[168] J.-L. Lehners, Adv. Astron. **2010**, 903907 (2010), 1001.3125.
[169] T. Qiu, X. Gao, and E. N. Saridakis, Phys. Rev. **D88**, 043525 (2013), 1303.2372.
[170] M. Li, Phys. Lett. **B724**, 192 (2013), 1306.0191.
[171] A. Fertig, J.-L. Lehners, and E. Mallwitz, Phys.Rev. **D89**, 103537 (2014), 1310.8133.
[172] A. Ijjas, J.-L. Lehners, and P. J. Steinhardt (2014), 1404.1265.
[173] B. Xue and E. Belbruno (2014), 1403.2122.
[174] E. Saridakis, Nucl. Phys. **B808**, 224 (2009), 0710.5269.
[175] T. Biswas (2008), 0801.1315.
[176] M. Novello, A. N. Araujo, and J. Salim, Int. J. Mod. Phys. C **20**, 1439 (2009), 0902.1875.
[177] L. Medeiros, Int.J.Mod.Phys. **D23**, 1250073 (2012), 1209.1124.
[178] C.-Y. Sun, Phys. Rev. **D78**, 064060 (2008), 0806.1080.
[179] E. J. Son and W. Kim, Mod.Phys. Lett. **A26**, 719 (2011), 1007.5371.
[180] C. Li, L. Wang, and Y.-K. E. Cheung, Phys.Dark Univ. **3**, 18 (2014), 1011.0202.
[181] V. Sahni and A. Toporensky, Phys. Rev. **D85**, 123542 (2012), 1203.0395.
[219] G. T. Horowitz and J. Polchinski, Phys. Rev. **D66**, 103512 (2002), hep-th/0206228.
[220] L. Cornalba and M. S. Costa, Fortsch.Phys. **52**, 145 (2004), hep-th/0310099.
[221] M. Berkooz and D. Reichmann, Nucl. Phys. Proc. Suppl. **171**, 69 (2007), 0705.2146.
[222] M. Oltean and R. Brandenberger, Phys.Rev. **D90**, 083505 (2014), 1406.7318.
[223] E. I. Buchbinder, J. Khoury, and B. A. Ovrut, Phys. Rev. Lett. **100**, 171302 (2008), 0710.5172.
[224] R. H. Brandenberger (2012), 1206.4196.
[225] Y.-F. Cai, D. A. Easson, and R. Brandenberger, JCAP **1208**, 020 (2012), 1206.2382.
[226] Y.-F. Cai, R. Brandenberger, and P. Peter, Class. Quant. Grav. **30**, 075019 (2013), 1301.4703.
[227] Y.-F. Cai, E. McDonough, F. Duplessis, and R. H. Brandenberger, JCAP **1310**, 024 (2013), 1305.5259.
[228] Y.-F. Cai, W. Xue, R. Brandenberger, and X. Zhang, JCAP **0905**, 011 (2009), 0903.0631.
[229] D. A. Easson, I. Sawicki, and A. Vikman, JCAP **1111**, 021 (2011), 1109.1047.
[230] A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. **D79**, 064036 (2009), 0811.2197.
[231] C. de Rham and G. Gabadadze, Phys. Rev. **D82**, 044020 (2010), 1007.0443.
[232] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011), 1011.1232.
[233] C. de Rham and A. J. Tolley, JCAP **1005**, 015 (2010), 1003.5917.
[234] C. Deffayet, O. Pujolas, I. Sawicki, and A. Vikman, JCAP **1010**, 026 (2010), 1008.0048.
[235] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Phys. Rev. Lett. **105**, 231302 (2010), 1008.0603.
[236] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. **D82**, 061501 (2010), 1007.5278.
[237] P. Creminelli, A. Nicolis, and E. Trincherini, JCAP **1011**, 021 (2010), 1007.0027.
[238] P. Creminelli, K. Hinterbichler, J. Khoury, A. Nicolis, and E. Trincherini, JHEP **1302**, 006 (2013), 1209.3768.
[239] G. W. Horndeski, Int.J.Theor.Phys. **10**, 363 (1974).
[240] C. Deffayet and D. A. Steer, Class. Quant. Grav. **30**, 214006 (2013), 1307.2450.
[241] M. Osipov and V. Rubakov, JCAP **1311**, 031 (2013), 1303.1221.
[242] O. Pujolas, I. Sawicki, and A. Vikman, JHEP **1111**, 156 (2011), 1103.5360.
[243] T. Qiu (2014), 1404.3060.
[244] J. D. Barrow, Class. Quant. Grav. **21**, L79 (2004), gr-qc/0403084.
[245] T. J. Battefeld and R. Brandenberger, Phys. Rev. **D70**, 121302 (2004), hep-th/0406180.
[246] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. **A173**, 211 (1939).
[247] H. van Dam and M. Veltman, Nucl. Phys. **B22**, 397 (1970).
[248] V. Zakharov, JETP Lett. **12**, 312 (1970).
[249] D. Boulware and S. Deser, Phys. Rev. **D6**, 3368 (1972).
[250] A. De Felice, A. E. Gümrükçüoğlu, and S. Mukohyama, Phys. Rev. Lett. **109**, 171101 (2012), 1206.2080.
[251] A. De Felice, A. E. Gümrükçüoğlu, C. Lin, and S. Mukohyama, JCAP **1305**, 035 (2013), 1303.4154.
[252] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, et al., Phys. Rev. **D84**, 124046 (2011), 1108.5231.
[253] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, Phys. Lett. **B717**, 295 (2012), 1206.2723.
[254] Q.-G. Huang, Y.-S. Piao, and S.-Y. Zhou, Phys. Rev. **D86**, 124014 (2012), 1206.5678.
[255] E. N. Saridakis, Class. Quant. Grav. **30**, 075003 (2013), 1207.1800.
[256] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012), 1105.3735.
[257] C. de Rham, Living Rev.Rel. **17**, 7 (2014), 1401.4173.
[258] T. Biswas, T. Koivisto, and A. Mazumdar, JCAP **1011**, 008 (2010), 1005.0590.
[259] Y.-F. Cai, C. Gao, and E. N. Saridakis, JCAP **1210**, 048 (2012), 1207.3786.
[260] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, Phys. Rev. Lett. **108**, 031101 (2012), 1110.5249.
[261] T. Biswas, A. S. Koshelev, A. Mazumdar, and S. Y. Vernov, JCAP **1208**, 024 (2012), 1206.6374.
[262] A. S. Koshelev, Class. Quant. Grav. **30**, 155001 (2013), 1302.2140.
[263] J.-L. Lehners, Phys.Rev. **D86**, 043518 (2012), 1206.1081.
[264] J. Garriga, A. Vilenkin, and J. Zhang, JCAP **1311**, 055 (2013), 1309.2847.
[265] B. Gupt and P. Singh, Phys.Rev. **D89**, 063520 (2014), 1309.2732.
[266] Y.-S. Piao, Phys. Rev. **D70**, 101302 (2004), hep-th/0407258.
[267] Y.-S. Piao, Phys. Lett. **B677**, 1 (2009), 0901.2644.
[268] Z.-G. Liu and Y.-S. Piao (2014), 1404.5748.
[269] A. Vilenkin and J. Zhang (2014), 1403.1599.
[270] J. Khoury, JHEP **0903**, 020 (2009), 0812.2285.
[271] P. Horava, Phys. Rev. **D79**, 084008 (2009), 0901.3775.
[272] P. Horava, Phys. Rev. Lett. **102**, 161301 (2009), 0902.3657.
[273] G. Calcagni, JHEP **0909**, 112 (2009), 0904.0829.
[274] E. Kiritsis and G. Kofinas, Nucl. Phys. **B821**, 467 (2009), 0904.1334.
[275] R. Brandenberger, Phys. Rev. **D80**, 043516 (2009), 0904.2835.
[276] R. Maier, Class. Quant. Grav. **30**, 115011 (2013), 1302.0139.
[277] M. Henneaux, A. Kleinschmidt, and G. Lucena Gómez, Phys.Rev. **D81**, 064002 (2010), 0912.0399.
[278] I. Kimpton and A. Padilla, JHEP **1304**, 133 (2013), 1301.6950.
[279] T. Lee and G. Wick, Nucl. Phys. **B9**, 209 (1969).
[280] T. Lee and G. Wick, Phys. Rev. **D2**, 1033 (1970).
[281] B. Grinstein, D. O'Connell, and M. B. Wise, Phys. Rev. **D77**, 025012 (2008), 0704.1845.
[282] R. Caldwell, Phys. Lett. **B545**, 23 (2002), astro-ph/9908168.
[283] J. M. Cline, S. Jeon, and G. D. Moore, Phys. Rev. **D70**, 043543 (2004), hep-ph/0311312.
[284] Y.-F. Cai, T.-t. Qiu, R. Brandenberger, and X.-m. Zhang, Phys. Rev. **D80**, 023511 (2009), 0810.4677.
[285] L. E. Allen and D. Wands, Phys. Rev. **D70**, 063515 (2004), astro-ph/0404441.
[286] K. Bhattacharya, Y.-F. Cai, and S. Das, Phys. Rev. **D87**, 083511 (2013), 1301.0661.
[287] B. Feng, X.-L. Wang, and X.-M. Zhang, Phys. Lett. **B607**, 35 (2005), astro-ph/0404224.
[288] B. Feng, M. Li, Y.-S. Piao, and X. Zhang, Phys. Lett. **B634**, 101 (2006), astro-ph/0407432.
[289] Z.-K. Guo, Y.-S. Piao, X.-M. Zhang, and Y.-Z. Zhang, (2008), 0809.2022.
[324] J. Martin and P. Peter, Phys. Rev. Lett. **92**, 061301 (2004), astro-ph/0312488.
[325] S. Weinberg (2008).
[326] D. Battefeld and T. Battefeld, Phys. Rev. **D80**, 063518 (2009), 0907.2443.
[327] J. J. Heckman, A. Tavanfar, and C. Vafa, JHEP **1004**, 054 (2010), 0812.3155.
[328] B. S. Acharya, P. Kumar, K. Bobkov, G. Kane, J. Shao, et al., JHEP **0806**, 064 (2008), 0804.0863.
[329] E. Huguet and P. Peter, Astropart.Phys. **12**, 277 (2000), hep-ph/9901370.
[330] Y.-F. Cai, R. Brandenberger, and X. Zhang, Phys. Lett. **B703**, 25 (2011), 1105.4286.
[331] J. Polchinski (1998).
[332] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, JHEP **0605**, 043 (2006), hep-th/0512177.
[333] V. Braun, Y.-H. He, and B. A. Ovrut, JHEP **1309**, 008 (2013), 1301.6767.
[334] R. Maier, S. Pereira, N. Pinto-Neto, and B. B. Siffert, Phys. Rev. **D85**, 023508 (2012), 1111.0946.
[335] J. K. Erickson, D. H. Wesley, P. J. Steinhardt, and N. Turok, Phys. Rev. **D69**, 063514 (2004), hep-th/0312009.
[336] D. Garfinkle, W. C. Lim, F. Pretorius, and P. J. Steinhardt, Phys. Rev. **D78**, 083537 (2008), 0808.0542.
[337] J.-L. Lehners and P. J. Steinhardt, Phys. Rev. Lett. **106**, 081301 (2011), 1008.4567.
[338] P. Ade et al. (Planck Collaboration) (2013), 1303.5076.
[339] A. D. Linde, Newton Centenary Volume (Cambridge U. Press) (1986).
[340] J. Garriga, A. D. Linde, and A. Vilenkin, Phys.Rev. **D69**, 063521 (2004), hep-th/0310034.
[341] L. Abbott, Sci.Am. **258**, 106 (1988).
[342] P. J. Steinhardt and N. Turok, Science **312**, 1180 (2006), astro-ph/0605173.
[343] M. G. Romania, N. Tsamis, and R. Woodard, Lect.Notes Phys. **863**, 375 (2013), 1204.6558.
[344] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[345] R. H. Brandenberger and J. Martin, Class. Quant. Grav. **30**, 113001 (2013), 1211.6753.
[346] S. D. Vitenti and N. Pinto-Neto, Phys.Rev. **D85**, 023524 (2012), 1111.0888.
[347] N. Pinto-Neto and S. Vitenti, Phys.Rev. **D89**, 028301 (2014), 1312.7790.
[348] I. P. Heard and D. Wands, Class.Quant.Grav. **19**, 5435 (2002), gr-qc/0206085.
[349] B. Xue, PhD dissertation, Princeton University (2014).
[350] V. Belinsky, I. Khalatnikov, and E. Lifshitz, Adv. Phys. **19**, 525 (1970).
[351] E. Lifshitz and I. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[352] C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).
[353] J. D. Barrow and K. Yamamoto, Phys. Rev. **D82**, 063516 (2010), 1004.4767.
[354] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, JHEP **0603**, 025 (2006), hep-th/0512260.
[355] V. Rubakov (2014), 1401.4024.
[356] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty, and S. Mukohyama, JHEP **0405**, 074 (2004), hep-th/0312099.
[357] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, JHEP **0612**, 080 (2006), hep-th/0606090.
[358] K. Hinterbichler, A. Joyce, J. Khoury, and G. E. Miller, Phys. Lett. **B608**, 177 (2005), astro-ph/0410654.
[290] Y.-F. Cai, T. Qiu, R. Brandenberger, Y.-S. Piao, and X. Zhang, JCAP **0803**, 013 (2008), 0711.2187.
[291] A. De Felice and S. Tsujikawa, Living Rev.Rel. **13**, 3 (2010), 1002.4928.
[292] K. Bamba, A. N. Makarenko, A. N. Myagky, S. Nojiri, and S. D. Odintsov (2013), 1309.3748.
[293] A. V. Astashenok, Astrophys.Space Sci. **351**, 377 (2014), 1308.0581.
[294] I. Brevik, V. Obukhov, and A. Timoshkin (2014), 1404.1887.
[295] M. Bouhmadi-Lopez, J. Morais, and A. B. Henriques, Phys. Rev. **D87**, 103528 (2013), 1210.1761.
[296] K. Bamba, A. N. Makarenko, A. N. Myagky, and S. D. Odintsov (2014), 1403.3242.
[297] A. S. Koshelev, Class. Quant. Grav. **30**, 155001 (2013), 1302.2140.
[298] K. Hayashi and T. Shirafuji, Phys. Rev. **D19**, 3524 (1979).
[299] E. V. Linder, Phys. Rev. **D81**, 127301 (2010), 1005.3039.
[300] Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, Class. Quant. Grav. **28**, 215011 (2011), 1104.4349.
[301] H. Maeda, Phys. Rev. **D85**, 124012 (2012), 1112.5178.
[302] S. Mukherji and M. Peloso, Phys.Lett. **B547**, 297 (2002), hep-th/0205180.
[303] N. Turok, M. Perry, and P. J. Steinhardt, Phys.Rev. **D70**, 106004 (2004), hep-th/0408083.
[304] G. Geshnizjani, T. J. Battefeld, and G. Geshnizjani, Phys. Rev. **D74**, 084501 (2006), hep-th/0506139.
[305] V. Bozza and G. Veneziano, JCAP **0509**, 007 (2005), gr-qc/0506040.
[306] R. Maier, I. D. Soares, and E. Tonini, Phys. Rev. **D79**, 023522 (2009), 1212.3766.
[307] R. Maier, N. Pinto-Neto, and I. D. Soares, Phys. Rev. **D87**, 043528 (2013), 1301.5250.
[308] R. Maier, F. Pace, and I. d. a. Soares, Phys. Rev. D **88**, 106003 (2013), 1310.5036.
[309] Y.-F. Cai and E. N. Saridakis, JCAP **0910**, 020 (2009), 0906.1789.
[310] A. H. Chamseddine and V. Mukhanov, JHEP **1311**, 135 (2013), 1308.5410.
[311] A. H. Chamseddine, V. Mukhanov, and A. Vikman (2014), 1403.3961.
[312] M. Novello and J. Salim, Phys. Rev. **D20**, 377 (1979).
[313] W. Heisenberg and H. Euler, Z.Phys. **98**, 714 (1936), physics/0605038.
[314] M. Artymowski and Z. Lalak, Phys. Lett. **B707**, 203 (2012), 1109.5901.
[315] T. Kibble, J.Math.Phys. **2**, 212 (1961).
[316] D. W. Sciama, Rev. Mod. Phys. **36**, 463 (1964).
[317] S. Capozziello, M. De Laurentis, A. Lasenby, Class.Quant.Grav. **24**, 6329 (2007), 0706.2367.
[318] S. Brechet, M. Hobson, and A. Lasenby (2008), 0807.2523.
[319] N. J. Poplawski, Phys. Rev. **D85**, 107502 (2012), 1111.4595.
[320] S. Alexander, C. Bambi, A. Marciano, and L. Modesto (2014), 1402.5880.
[321] A. Borde, A. H. Guth, and A. Vilenkin, Phys.Rev.Lett. **90**, 151301 (2003), gr-qc/0110012.
[322] P. Peter (2009), 0912.1293.
[323] P. Peter and N. Pinto-Neto, Phys. Rev. **D78**, 063506 (2008), 0809.2022.
[359] B. Elder, A. Joyce, and J. Khoury, Phys. Rev. **D89**, 044027 (2014), 1311.5889.
[360] A. Nicolis, R. Rattazzi, and E. Trincherini, JHEP **1005**, 095 (2010), 0912.4258.
[361] P. Creminelli, M. Serone, and E. Trincherini, JHEP **1310**, 040 (2013), 1306.2946.
[362] C. de Rham, M. Fasiello, and A. J. Tolley (2013), 1308.2702.
[363] V. Rubakov, Phys. Rev. **D88**, 044015 (2013), 1305.2614.
[364] L. R. Abramo and N. Pinto-Neto, Phys. Rev. **D73**, 063522 (2006), astro-ph/0511562.
[365] L. R. Abramo, I. Yasuda, and P. Peter, Phys. Rev. **D81**, 023511 (2010), 0910.3422.
[366] L. P. Chimento and M. I. Forte, Phys.Rev. **D73**, 063502 (2006), astro-ph/0510706.
[367] J. De-Santiago, J. L. Cervantes-Cota, and D. Wands, Phys.Rev. **D87**, 023502 (2013), 1204.3631.
[368] B. Xue, D. Garfinkle, F. Pretorius, and P. J. Steinhardt, Phys. Rev. **D88**, 083509 (2013), 1308.3044.
[369] R. V. Buniy, S. D. Hsu, and B. M. Murray, Phys. Rev. **D74**, 063518 (2006), hep-th/0606091.
[370] M. Koehn, J.-L. Lehners, and B. Ovrut, Phys. Rev. **D87**, 065022 (2013), 1212.2185.
[371] B. Xue and P. J. Steinhardt, Phys. Rev. **D84**, 083520 (2011), 1106.1416.
[372] D. A. Easson, I. Sawicki, and A. Vikman, JCAP **1307**, 014 (2013), 1304.3903.
[373] D. Wands, Phys. Rev. **D60**, 023507 (1999), gr-qc/9809062.
[374] E. Kiritsis and C. Kounnas, Phys. Lett. **B331**, 51 (1994), hep-th/9404092.
[375] J. J. Atick and E. Witten, Nucl. Phys. **B310**, 291 (1988).
[376] I. Antoniadis and C. Kounnas, Phys. Lett. **B261**, 369 (1991).
[377] J. Barbon and E. Rabinovici, JHEP **0203**, 057 (2002), hep-th/0112173.
[378] J. Barbon and E. Rabinovici (2004), hep-th/0407

Standard Failures and bouncing solutions

Singularity Merely a non issue in the bounce case!

Horizon $d_{\rm H} \equiv a(t) \int_{t_{\rm i}}^{t} \frac{{\rm d}\tau}{a(\tau)}$ can be made divergent easily if $t_{\rm i} \to -\infty$

Flatness $\frac{\rm d}{{\rm d}t} |\Omega - 1| = -2 \frac{\ddot{a}}{\dot{a}^3}$ $\ddot{a} < 0$ & $\dot{a} < 0$

accelerated expansion (inflation) or decelerated contraction (bounce)

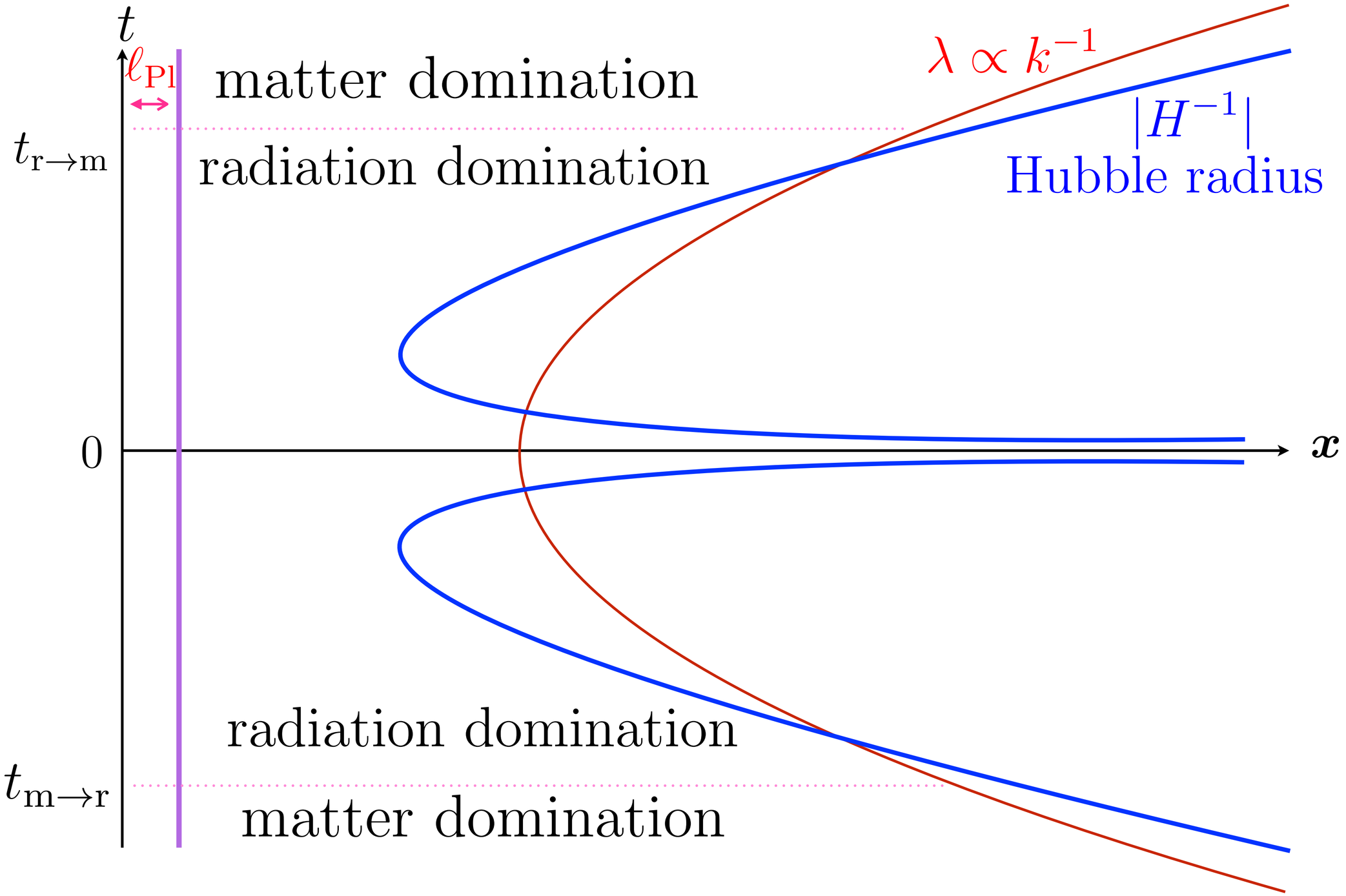
Homogeneity Large & flat Universe + low initial density + diffusion

$\frac{t_{\rm dissipation}}{t_{\rm Hubble}} \propto \frac{\lambda}{R_{\rm H}^{1/3}} \left(1 + \frac{\lambda}{A R_{\rm H}^2} \right)$ enough time to dissipate any wavelength

$\Longrightarrow$ quantum vacuum fluctuations...

Isotropy Potentially problematic: model dependent

Others dark matter/energy, baryogenesis, ...


t
$\ell_{\rm Pl}$
matter domination
$t_{\rm r\to m}$
radiation domination
$\lambda \propto k^{-1}$
$|H^{-1}|$
Hubble radius
0
x
radiation domination
$t_{\rm m\to r}$
matter domination

Standard Failures and bouncing solutions

Singularity Merely a non issue in the bounce case!

Horizon $d_{\rm H} \equiv a(t)\int_{t_{\rm i}}^{t}\frac{{\rm d}\tau}{a(\tau)}$ can be made divergent easily if $t_{\rm i}\to-\infty$

Flatness $\frac{\rm d}{{\rm d}t}\left|\Omega-1\right| = -2\frac{\ddot{a}}{\dot{a}^3}$ $\ddot{a}<0$ & $\dot{a}<0$

accelerated expansion (inflation) or decelerated contraction (bounce)

Homogeneity Large & flat Universe + low initial density + diffusion

$\frac{t_{\rm dissipation}}{t_{\rm Hubble}} \propto \frac{\lambda}{R_{\rm H}^{1/3}}\left(1+\frac{\lambda}{AR_{\rm H}^2}\right)$ enough time to dissipate any wavelength

$\Longrightarrow$ quantum vacuum fluctuations...

Isotropy Potentially problematic: model dependent

Others dark matter/energy, baryogenesis, ...

$$d_{\rm H}^{\rm cont} = \frac{3(1+w)}{1+3w} t_{\rm end} \left[1 - \left(\frac{t_{\rm ini}}{t_{\rm end}} \right)^{(1+3w)/[3(1+w)]} \right]$$

$$t_{\rm ini} \to -\infty$$

Standard Failures and bouncing solutions

Singularity Merely a non issue in the bounce case!

Horizon $d_{\rm H} \equiv a(t)\int_{t_{\rm i}}^{t}\frac{{\rm d}\tau}{a(\tau)}$ can be made divergent easily if $t_{\rm i}\to-\infty$

Flatness $\frac{\rm d}{{\rm d}t}\left|\Omega-1\right| = -2\frac{\ddot{a}}{\dot{a}^3}$ $\qquad \ddot{a}<0\ \&\ \dot{a}<0$

accelerated expansion (inflation) or decelerated contraction (bounce)

Homogeneity Large & flat Universe + low initial density + diffusion

$\frac{t_{\rm dissipation}}{t_{\rm Hubble}} \propto \frac{\lambda}{R_{\rm H}^{1/3}}\left(1+\frac{\lambda}{AR_{\rm H}^2}\right)$ enough time to dissipate any wavelength

$\Longrightarrow$ quantum vacuum fluctuations...

Isotropy Potentially problematic: model dependent

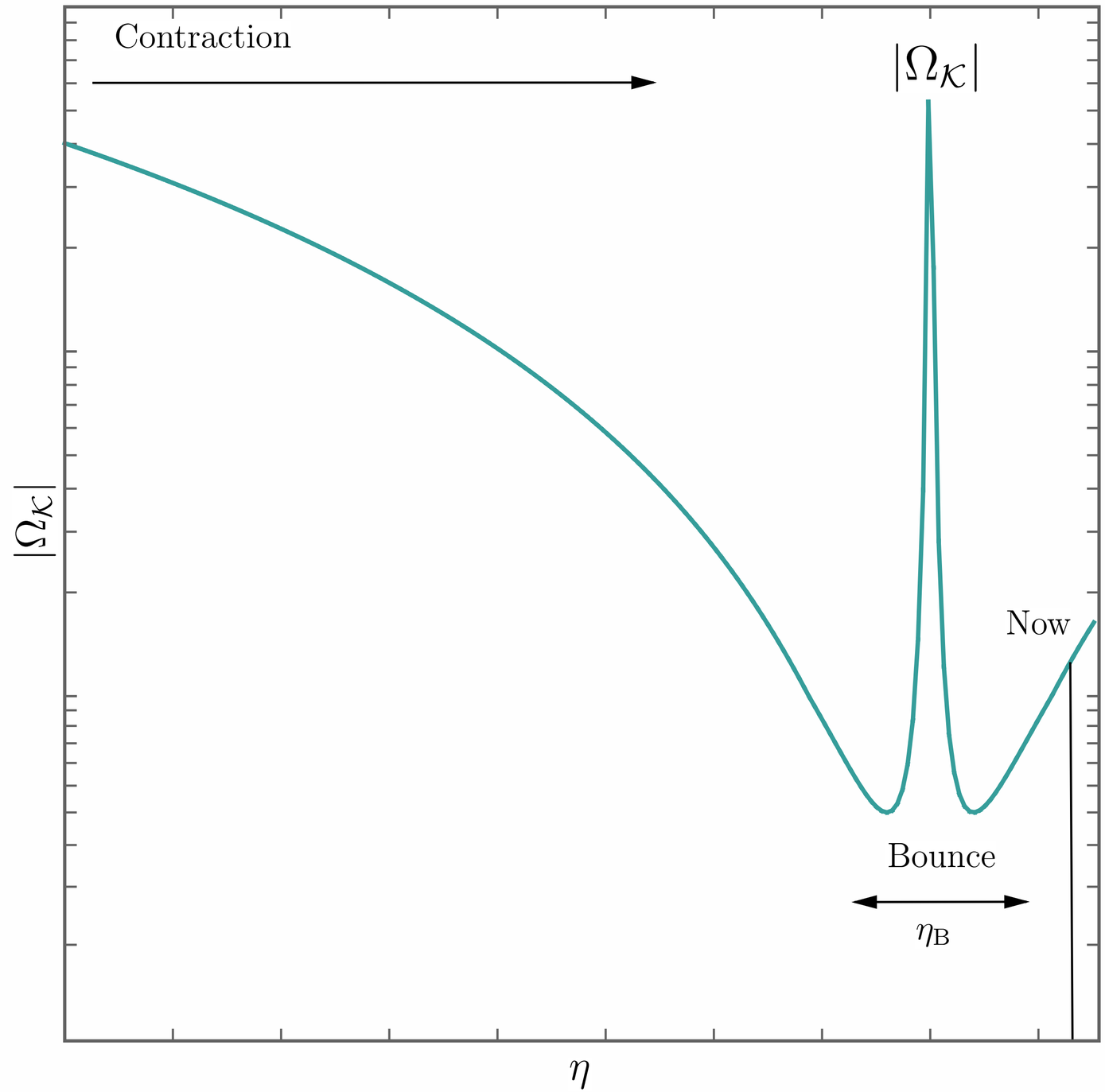
Others dark matter/energy, baryogenesis, ...

$$H^2 = \frac{1}{3}\left[-\frac{3\mathcal{K}}{a^2} + \frac{\rho_{\rm m0}}{a^3} + \frac{\rho_{\rm r0}}{a^4} + \frac{\rho_{\theta 0}}{a^6} + \ldots + \frac{\rho_{\phi 0}}{a^{3(1+w_\phi)}}\right]$$

Critical density

$$\rho_{\rm c} \equiv \frac{3H^2}{8\pi G_{\rm N}} \quad \Longrightarrow \quad \Omega \equiv \frac{\rho}{\rho_{\rm c}}$$

Density parameter



Standard Failures and bouncing solutions

Singularity Merely a non issue in the bounce case!

Horizon $d_{\rm H} \equiv a(t)\int_{t_{\rm i}}^{t}\frac{{\rm d}\tau}{a(\tau)}$ can be made divergent easily if $t_{\rm i}\to-\infty$

Flatness $\frac{\rm d}{{\rm d}t}\left|\Omega-1\right| = -2\frac{\ddot{a}}{\dot{a}^3}$ $\qquad \ddot{a}<0\ \&\ \dot{a}<0$

accelerated expansion (inflation) or decelerated contraction (bounce)

Homogeneity Large & flat Universe + low initial density + diffusion

$\frac{t_{\rm dissipation}}{t_{\rm Hubble}} \propto \frac{\lambda}{R_{\rm H}^{1/3}}\left(1+\frac{\lambda}{AR_{\rm H}^2}\right)$ enough time to dissipate any wavelength

$\Longrightarrow$ quantum vacuum fluctuations...

Shear Potentially problematic: model dependent

Others dark matter/energy, baryogenesis, ...

Standard Failures and bouncing solutions

Singularity Merely a non issue in the bounce case!

Horizon $d_{\rm H} \equiv a(t)\int_{t_{\rm i}}^{t}\frac{{\rm d}\tau}{a(\tau)}$ can be made divergent easily if $t_{\rm i}\to-\infty$

Flatness $\frac{\rm d}{{\rm d}t}\left|\Omega-1\right| = -2\frac{\ddot{a}}{\dot{a}^3}$ $\qquad \ddot{a}<0\ \&\ \dot{a}<0$

accelerated expansion (inflation) or decelerated contraction (bounce)

Homogeneity Large & flat Universe + low initial density + diffusion

$\frac{t_{\rm dissipation}}{t_{\rm Hubble}} \propto \frac{\lambda}{R_{\rm H}^{1/3}}\left(1+\frac{\lambda}{AR_{\rm H}^2}\right)$ enough time to dissipate any wavelength

$\Longrightarrow$ quantum vacuum fluctuations...

Shear Potentially problematic: model dependent

Others dark matter/energy, baryogenesis, ...

The problem with contraction: BKL/shear instability

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\sum_i \mathrm{e}^{2\theta_i(t)}\sigma^i\sigma^i$$

Ricci flat:

$$\sigma^i = \mathrm{d}x^i$$

$$\sum_i \theta_i = 0$$

Average scale factor

$$\frac{\dot{a}}{a}$$ Mean Hubble parameter

$$H_i \equiv \frac{1}{a\mathrm{e}^{\theta_i}}\frac{\mathrm{d}}{\mathrm{d}t}\left(a\mathrm{e}^{\theta_i}\right) = H + \dot{\theta}_i$$

Friedman equations

$$H^2 = \frac{\rho_{\mathrm{T}}}{3M_{\mathrm{Pl}}^2} + \frac{1}{6}\sum_i \dot{\theta}_i^2$$

$$\dot{H} = -\frac{\rho_{\mathrm{T}} + p_{\mathrm{T}}}{2M_{\mathrm{Pl}}^2} - \frac{1}{2}\sum_i \dot{\theta}_i^2$$

$$\ddot{\theta}_i + 3H\dot{\theta}_i = 0$$

$$\rho_{\mathrm{shear}} \propto a^{-6}$$

Slow contraction solution:

$$w_{\rm ekp} \gg 1 \quad \Longrightarrow \rho_{\rm ekp} \propto a^{-3\left(1+w_{\rm ekp}\right)} \gg a^{-6} \text{ when } \quad a \to 0$$

Standard Failures and bouncing solutions

Singularity Merely a non issue in the bounce case!

Horizon $d_{\rm H} \equiv a(t)\int_{t_{\rm i}}^{t}\frac{{\rm d}\tau}{a(\tau)}$ can be made divergent easily if $t_{\rm i}\to-\infty$

Flatness $\frac{\rm d}{{\rm d}t}\left|\Omega-1\right| = -2\frac{\ddot{a}}{\dot{a}^3}$ $\qquad \ddot{a}<0\ \&\ \dot{a}<0$

accelerated expansion (inflation) or decelerated contraction (bounce)

Homogeneity Large & flat Universe + low initial density + diffusion

$\frac{t_{\rm dissipation}}{t_{\rm Hubble}} \propto \frac{\lambda}{R_{\rm H}^{1/3}}\left(1+\frac{\lambda}{AR_{\rm H}^2}\right)$ enough time to dissipate any wavelength

$\Longrightarrow$ quantum vacuum fluctuations...

Shear Potentially problematic: model dependent

Others dark matter/energy, baryogenesis, ...

The **Jean's length**: newtonian perturbation theory

Actual distance between two objects $\boldsymbol{r} = a(t)\boldsymbol{x}$

$$\longrightarrow \quad \frac{\partial \boldsymbol{r}}{\partial t} = aH\boldsymbol{x} + a\frac{\partial \boldsymbol{x}}{\partial t} \quad \Longrightarrow \quad \boldsymbol{v} = aH\boldsymbol{x} + u$$

cosmic flow *peculiar velocity*

Total density $\rho(\boldsymbol{x}, t) = \bar{\rho}(t)\left[1 + \delta(\boldsymbol{x}, t)\right]$

background *perturbation*

continuity equation: $\dot{\rho} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla}_{\boldsymbol{x}}\left(\rho \boldsymbol{u}\right) = 0$

$\longrightarrow$ *background:* $\dot{\bar{\rho}} + 3H\bar{\rho} = 0 \quad \Longrightarrow \quad \rho \propto a^{-3}$

$\longrightarrow$ *perturbation:* $\dot{\delta} + \frac{1}{a}\boldsymbol{\nabla}\left[\left(1+\delta\right)\boldsymbol{u}\right] = 0$ *derive w.r.t time*

Euler equation: $\dot{\boldsymbol{v}} + \left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi$

$$\longrightarrow \dot{\boldsymbol{u}} + H\boldsymbol{u} + \frac{1}{a}\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} = -\frac{1}{a}\left(\frac{1}{\rho}\boldsymbol{\nabla}P + \boldsymbol{\nabla}\Phi\right)$$

$$\Delta\Phi = 4\pi G_{\mathrm{N}}\bar{\rho}a^2(t)\delta$$

Matter fluid perturbation equation

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_{\rm s}^2}{a^2}\Delta\delta = 4\pi G_{\rm N}\bar{\rho}\delta$$

expansion damping — *sound-wave* — *gravitation*

$$c_{\rm s}^2 = \left(\frac{\partial P}{\partial \rho}\right)_S$$

+ Fourier & no expansion: $\delta_k(t) \propto \exp\left[\sqrt{4\pi G_{\rm N}\bar{\rho}\left(1 - \frac{\lambda_{\rm J}^2}{\lambda^2}\right)}\,t\right]$

$\lambda > \lambda_{\rm J}$ → *exponential collapse*

$\lambda < \lambda_{\rm J}$ → *soundlike oscillation*

$$\lambda_{\rm J} \equiv c_{\rm s}\sqrt{\frac{\pi}{G_{\rm N}\bar{\rho}}}$$

expansion: power-law rather than exponential collapse

Measurements: statistics

correlation function

$$\xi(\boldsymbol{x},\boldsymbol{r}) = \langle \delta(\boldsymbol{x})\delta(\boldsymbol{x}+\boldsymbol{r})\rangle$$

statistical isotropy and homogeneity $\quad \xi(\boldsymbol{x},\boldsymbol{r}) = \xi(r)$

Fourier $\quad \delta(\boldsymbol{r}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \delta_{\boldsymbol{k}} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ $\qquad\qquad$ $\delta^*_{\boldsymbol{k}} = \delta_{-\boldsymbol{k}}$

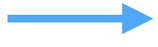
$$\longrightarrow \langle \delta_{\boldsymbol{k}}\delta_{\boldsymbol{k}'}\rangle = \delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}')P_\delta(k) \longrightarrow P_\delta(k) = \int \mathrm{d}^3\boldsymbol{r}\, \xi(r)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

power spectrum

$$\boxed{\xi(r) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} P_\delta(k)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = \int_0^\infty \frac{\mathrm{d}k}{2\pi^2} k^2 P_\delta(k) \frac{\sin kr}{kr}}$$

Link with observable quantities and smoothing

finite part
single realization
stochastic process



spatial average $\xi_{\rm obs}(r) = \dfrac{1}{V}\displaystyle\int_V \mathrm{d}^3\boldsymbol{r}'\delta_{\rm obs}(\boldsymbol{r}')\delta_{\rm obs}(\boldsymbol{r}+\boldsymbol{r}')$

window function $\delta_{{\rm obs},R}(\boldsymbol{x}) = \displaystyle\int_V \mathrm{d}^3\boldsymbol{y}\delta_{\rm obs}(\boldsymbol{y})W_R(\boldsymbol{x}-\boldsymbol{y})$

'top-hat' and Gaussian filters.

$$\int W_R(\boldsymbol{x})\mathrm{d}^3\boldsymbol{x} = 1$$

$$\int W_R(\boldsymbol{x})x^2\mathrm{d}^3\boldsymbol{x} = R^2$$

GR **perturbative expansion**: metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

background FLRW *perturbation*

inverse metric $g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu}$

check: $\delta g^{\mu\nu} = -\bar{g}^{\mu\sigma}\bar{g}^{\nu\lambda}\delta g_{\sigma\lambda}$

explicitly

$$\mathrm{d}s^2 = a^2(\eta)\left[-(1+2A)\mathrm{d}\eta^2 + 2B_i\mathrm{d}x^i\mathrm{d}\eta + (\gamma_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right]$$

trace $h \equiv h_{ij}\gamma^{ij}$

SVT decomposition

scalar-vector-tensor

Vector = gradient of scalar + divergenceless $\quad B^i = D^i B + \overline{B}^i \quad$ with $\quad D^i \overline{B}_i = 0$

(id. velocity, potential & vorticity)

rank 2 symmetric tensor

$$h_{ij} = 2C\gamma_{ij} + 2D_i D_j E + 2D_{(i}\overline{E}_{j)} + 2\overline{E}_{ij} \quad \text{with} \quad D_i \overline{E}^{ij} = 0,\ \overline{E}^i_i = 0$$

full metric

$$\mathrm{d}s^2 = a^2(\eta)\left[-(1+2A)\mathrm{d}\eta^2 + 2B_i \mathrm{d}x^i \mathrm{d}\eta + (\gamma_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j\right]$$

The 10 degrees of freedom of the metric have thus been decomposed as

- 4 *scalars*: A, B, C and E, corresponding to 4 degrees of freedom,
- 2 *vectors*: $\overline{B}^i$ and $\overline{E}^i$ corresponding to $2 \times (3-1) = 4$ degrees of freedom,
- 1 *tensor*: $\overline{E}^{ij}$, corresponding to $3 \times 2 - 1 - 3 = 2$ degrees of freedom.

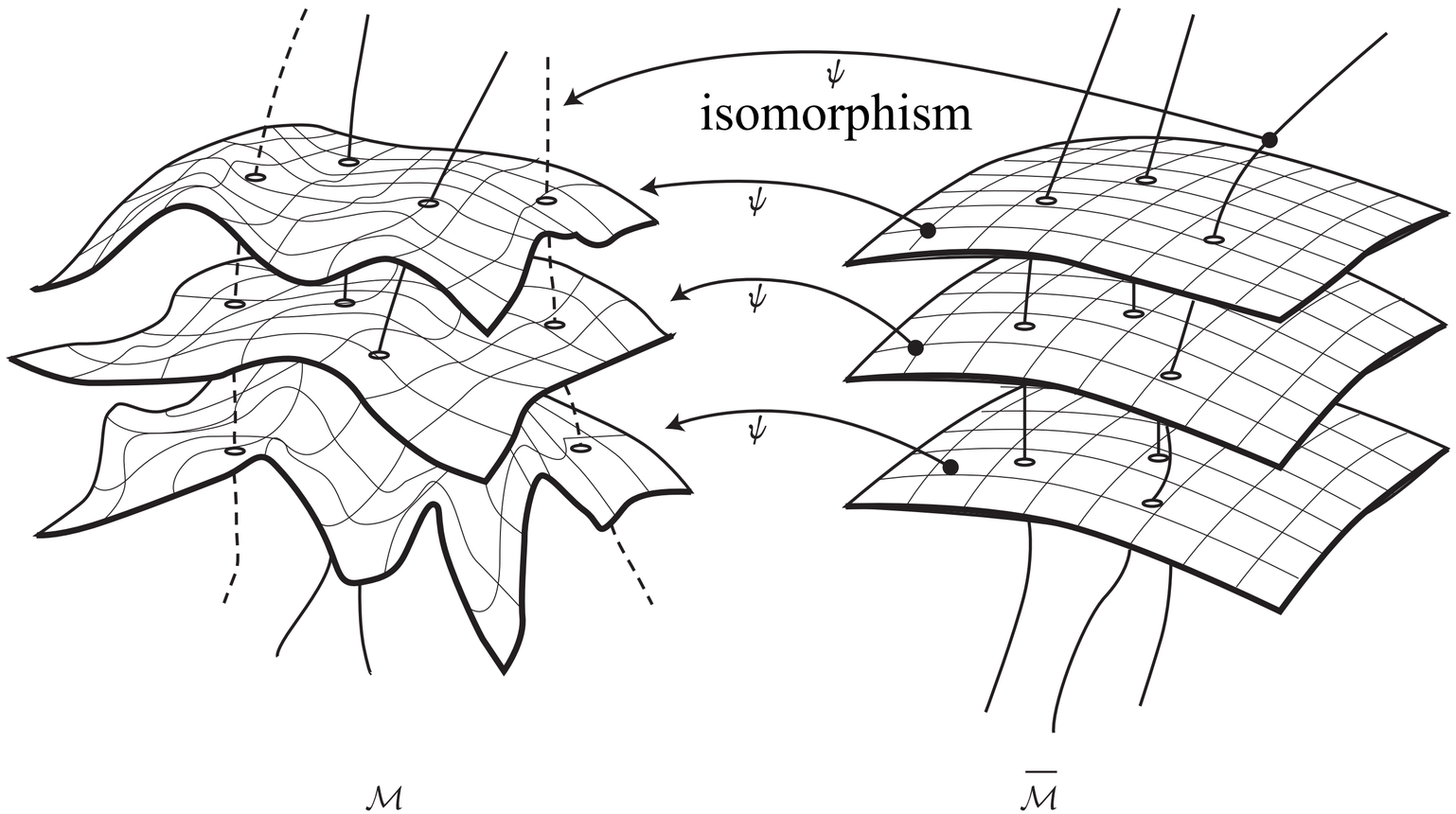
→ Don't mix at linear order

The gauge issue

Field theory $\delta Q(\boldsymbol{x},t) = Q(\boldsymbol{x},t) - \overline{Q}(\boldsymbol{x},t)$ fixed spacetime & varying fields at same point

GR perturbed space close to FLRW background… ???

$(\mathcal{M}, g_{\mu\nu})$ $(\overline{\mathcal{M}}, \bar{g}_{\mu\nu})$



Change of coordinate (ex): $x^i \to y^i = x^i - \xi^i\left(x^j, \eta\right)$

$$\longrightarrow \quad \mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + 2\xi_i'\mathrm{d}y^i\mathrm{d}\eta + (\gamma_{ij} + 2D_{(i}\xi_{j)})\mathrm{d}y^i\mathrm{d}y^j\right]$$

not perturbation terms!
(choice of coordinates)

general change of coordinate
(SVT decomposition… SV!)

$$x^\mu \to x^\mu - \xi^\mu \longrightarrow \begin{cases} \xi^0 = T \\ \& \\ \xi^i = L^i = D^i L + \bar{L}^i \end{cases}$$

metric transforms as $g_{\mu\nu} \to g_{\mu\nu} + \pounds_\xi g_{\mu\nu}$

first order in the perturbations $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \pounds_\xi \bar{g}_{\mu\nu} = \delta g_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$

$$A \to A + T' + \mathcal{H}T, \qquad \overline{B}^i \to \overline{B}^i + \bar{L}^{i\prime}, \qquad \overline{E}_{ij} \to \overline{E}_{ij},$$
$$B \to B - T + L', \qquad \overline{E}^i \to \overline{E}^i + \bar{L}^i,$$
$$C \to C + \mathcal{H}T,$$
$$E \to E + L,$$

Gauge invariant variables

10 original degrees of freedom - 4 possible gauge changes = 6 actual d.o.f.

$$\Psi \equiv -C - \mathcal{H}(B - E'), \qquad 1$$

$$\Phi \equiv A + \mathcal{H}(B - E') + (B - E')', \qquad 1 \quad \Big\} \text{ 2 scalar}$$

$$\bar{\Phi}^i \equiv \overline{E}^{i\prime} - \overline{B}^i, \qquad 2 \quad \text{vector}$$

$$\overline{E}^{ij}. \qquad 2 \quad \text{tensor}$$

Other gauge invariant quantities could have been constructed.

$$X = A - C - \left(\frac{C}{\mathcal{H}}\right)' \longrightarrow X = \Psi + \Phi + \left(\frac{\Psi}{\mathcal{H}}\right)'$$

Description of matter

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$$ normalization $\bar{u}^\mu = a^{-1}\delta^\mu_0$

$$\delta T_{\mu\nu} = (\delta\rho + \delta P)\bar{u}_\mu \bar{u}_\nu + \delta P \bar{g}_{\mu\nu} + 2(\rho + P)\bar{u}_{(\mu}\delta u_{\nu)} + P\delta g_{\mu\nu} + a^2 P \pi_{\mu\nu}$$

$$\delta u^\mu = a^{-1}(-A, v^i) \quad \delta u_\mu = a(-A, v_k + B_k)$$

$$v_i = D_i v + \overline{v}_i$$

anisotropic stress

$\pi_{00} = \pi_{0i} = 0$

$$\pi_{ij} = \Delta_{ij}\overline{\pi} + D_{(i}\overline{\pi}_{j)} + \overline{\pi}_{ij}$$

$$\Delta_{ij} \equiv D_i D_j - \frac{1}{3}\gamma_{ij}\Delta$$

entropy perturbation

$$\delta P = c_{\rm S}^2 \delta\rho + P\Gamma$$

$$\delta P_{\rm nad} = P\Gamma$$

explicitly

$$\delta T_{00} = \rho a^2 \left(\delta + 2A\right),$$

$$\delta T_{0i} = -\rho a^2 \left[(1+w)\left(D_i v + \overline{v}_i\right) + D_i B + \overline{B}_i\right],$$

$$\delta T_{ij} = P a^2 \left(h_{ij} + \frac{\delta P}{P}\gamma_{ij} + \pi_{ij}\right),$$

Gauge invariant quantities

$$\begin{aligned} &\delta\rho \to \delta\rho + \rho' T, && v \to v - L', \\ &\delta P \to \delta P + P' T, && \overline{v}_i \to \overline{v}_i - \overline{L}_i'. \end{aligned}$$

$$\Rightarrow \begin{cases} \delta^{\mathrm{N}} = \delta + \dfrac{\rho'}{\rho}(B - E'), \\ \delta^{\mathrm{F}} = \delta - \dfrac{\rho'}{\rho}\dfrac{C}{\mathcal{H}}, \\ \delta^{\mathrm{C}} = \delta + \dfrac{\rho'}{\rho}(v + B), \\ V = v + E', \\ \overline{V}_i = \overline{v}_i + \overline{B}_i, \\ \overline{W}_i = \overline{v}_i + \overline{E}_i'. \end{cases}$$

not independent: $\delta^{\mathrm{F}} = \delta^{\mathrm{N}} + \dfrac{\rho'}{\rho}\dfrac{\Psi}{\mathcal{H}}, \qquad \delta^{\mathrm{C}} = \delta^{\mathrm{N}} + \dfrac{\rho'}{\rho}V, \qquad \overline{W}_i = \overline{V}_i + \overline{\Phi}_i.$

Gauge choices

Newtonian or longitudinal gauge the scalar part of the perturbed metric is diagonal

$$B = 0 \qquad E = 0 \qquad \overline{B}_i = 0$$

fixes the gauge completely →

$$\begin{cases} T = B - E' \\ L = -E \\ \overline{L}_i' = -\overline{B}_i \end{cases}$$

other quantities: $A = \Phi \quad C = -\Psi \quad \delta = \delta^{\mathrm{N}} \quad \delta P = \delta P^{\mathrm{N}} \quad v = V \qquad \overline{v}_i = \overline{V}_i$

Flat-slicing gauge the scalar part of the curvature perturbation of the spatial sections vanishes

$$C = 0 \qquad E = 0 \qquad \overline{E}_i = 0$$

also completely fixed →

$$\begin{cases} T = -\dfrac{C}{\mathcal{H}} \\ L = -E \\ \overline{L}_i = -\overline{E}_i \end{cases}$$

$$A = X \qquad E' = -\frac{\Psi}{\mathcal{H}} \qquad \delta = \delta^{\mathrm{F}} \qquad \delta P = \delta P^{\mathrm{F}} \qquad v = V \qquad \overline{v}_i = \overline{W}_i$$

Synchronous or Gauss gauges only the spatial sections are perturbed $A = 0$

$x^i = \text{const.}$ are geodesics orthogonal to constant-time hypersurfaces $B_i = 0$

proper time of a comoving observer corresponds to the cosmic time

→ every point corresponds to a free-falling observer

not completely fixed

$$\begin{cases} t \to t' = f(t) \\ \text{or} \\ x^i \to y^i = f^i(x^j) \end{cases}$$

transformation to pass from an arbitrary gauge to a synchronous gauge $C(x^i)$ integration constants

$$T = -\frac{1}{a}\int A a \, \mathrm{d}\eta + \frac{C_T}{a} \qquad L = \int (T - B)\, \mathrm{d}\eta + C_L \qquad \overline{L}^i = -\int \overline{B}^i \, \mathrm{d}\eta + \overline{C}^i_L$$

Comoving gauge moving with the fluid (vanishing fluid velocity) $\delta T^0_i = 0$

clear meaning with only one fluid, less obvious with many…

cannot always be satisfied, as it implies that $v_i + B_i = 0$

must add other conditions $E = 0$ and $\overline{E}_i = 0$

Description of matter

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$$ normalization $\bar{u}^\mu = a^{-1}\delta^\mu_0$

$$\delta T_{\mu\nu} = (\delta\rho + \delta P)\bar{u}_\mu \bar{u}_\nu + \delta P \bar{g}_{\mu\nu} + 2(\rho + P)\bar{u}_{(\mu}\delta u_{\nu)} + P\delta g_{\mu\nu} + a^2 P \pi_{\mu\nu}$$

$$\delta u^\mu = a^{-1}(-A, v^i) \quad \delta u_\mu = a(-A, v_k + B_k)$$

$$v_i = D_i v + \overline{v}_i$$

anisotropic stress

$\pi_{00} = \pi_{0i} = 0$

$$\pi_{ij} = \Delta_{ij}\overline{\pi} + D_{(i}\overline{\pi}_{j)} + \overline{\pi}_{ij}$$

$$\Delta_{ij} \equiv D_i D_j - \frac{1}{3}\gamma_{ij}\Delta$$

entropy perturbation

$$\delta P = c_{\rm S}^2 \delta\rho + P\Gamma$$

$$\delta P_{\rm nad} = P\Gamma$$

explicitly

$$\delta T_{00} = \rho a^2 \left(\delta + 2A\right),$$

$$\delta T_{0i} = -\rho a^2 \left[(1+w)\left(D_i v + \overline{v}_i\right) + D_i B + \overline{B}_i\right],$$

$$\delta T_{ij} = P a^2 \left(h_{ij} + \frac{\delta P}{P}\gamma_{ij} + \pi_{ij}\right),$$

Einstein equations $\delta G^\mu_\nu = \kappa \delta T^\mu_\nu$ fluid conservation $\delta\left(\nabla_\mu T^\mu_\nu\right) = 0$ Fourier space

Vectors $V_i = k_i V + \overline{V}_i$

$k^i \overline{V}_i = 0$ $\overline{V}^i$ lives in the subspace perpendicular to k^i $\{e^1, e^2\}$

transverse modes

$e^a_i k_j \gamma^{ij} = 0$

$e^a_i e^b_j \gamma^{ij} = \delta^{ab}$

→ decomposition $\overline{V}_i(k_i, \eta) = \sum_{a=1,2} V_a(\hat{k}_i, \eta)\, e^a_i(\hat{k}_i)$

projection operator onto the subspace perpendicular to k^i $P_{ij} \equiv e^1_i e^1_j + e^2_i e^2_j = \gamma_{ij} - \hat{k}_i \hat{k}_j$

→ $V_i = (\hat{k}^j V_j)\hat{k}_i + P^j_i V_j$

$$(\Delta + 2K)\,\overline{\Phi}_i = -6\mathcal{H}^2\Omega(1+w)\overline{V}_i$$

$$\overline{\Phi}'_i + 2\mathcal{H}\overline{\Phi}_i = 3\mathcal{H}^2\Omega w \overline{\pi}_i \quad (\overline{\pi}_i \to 0)$$

$\overline{\Phi}_i \propto a^{-2}$

$\overline{V}^i \propto a^{-(1-3c_{\rm S}^2)}$

washed out

$$\overline{V}'_i + \mathcal{H}\left(1 - 3c_{\rm S}^2\right)\overline{V}_i = -\frac{1}{2}\frac{w}{1+w}(\Delta + 2K)\overline{\pi}_i \quad (\overline{\pi}_i \to 0)$$

Tensors $V_{ij} = T\gamma_{ij} + \Delta_{ij}S + 2D_{(i}\overline{V}_{j)} + 2\overline{V}_{ij}$

$$\Delta_{ij} \equiv D_i D_j - \tfrac{1}{3}\Delta\gamma_{ij}$$

polarization tensors $\varepsilon_{ij}^{\lambda} = \dfrac{e_i^1 e_j^1 - e_i^2 e_j^2}{\sqrt{2}}\delta_+^{\lambda} + \dfrac{e_i^1 e_j^2 + e_i^2 e_j^1}{\sqrt{2}}\delta_\times^{\lambda}$

→ decomposition $\overline{V}_{ij}(k_i,\eta) = \sum_{\lambda=+,\times} V_\lambda(k^i,\eta)\,\varepsilon_{ij}^{\lambda}(\hat{k}_i)$

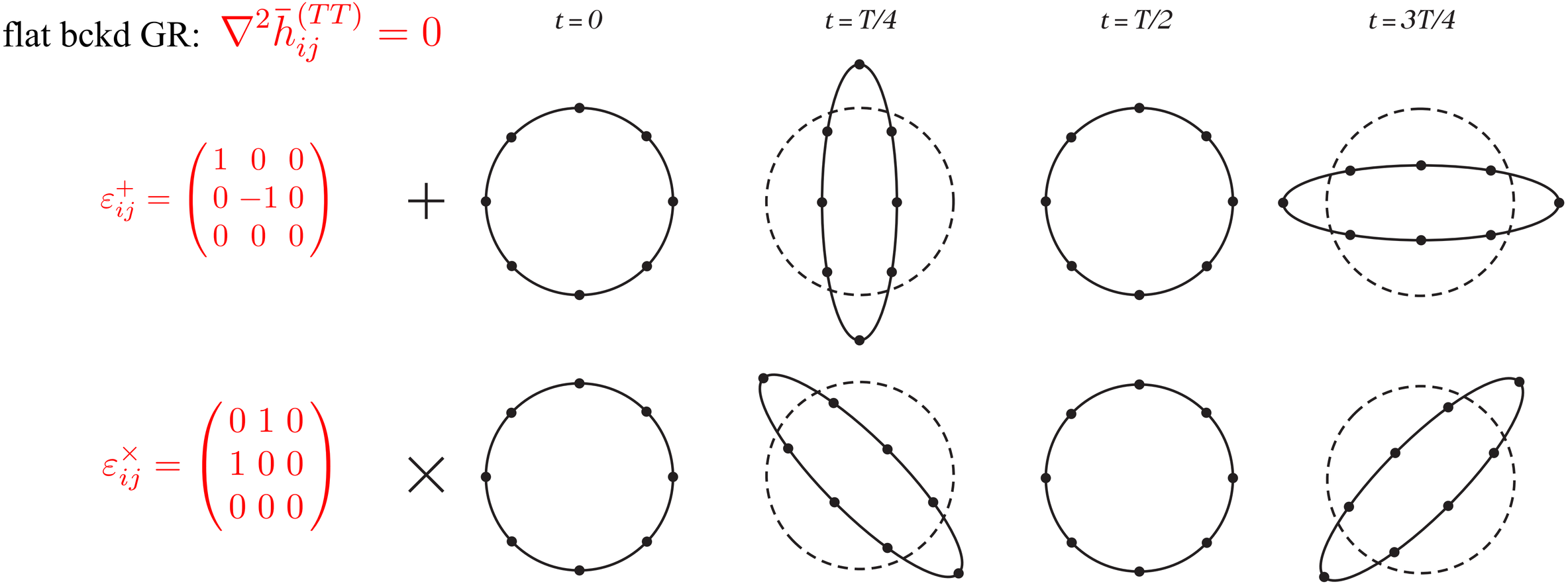
$$\boxed{\overline{E}_{kl}'' + 2\mathcal{H}\overline{E}_{kl}' + (2K-\Delta)\,\overline{E}_{kl} = \kappa a^2 P\overline{\pi}_{kl}.}$$

recall $\mathrm{d}s^2 = a^2(\eta)\left[-(1+2A)\mathrm{d}\eta^2 + 2B_i\mathrm{d}x^i\mathrm{d}\eta + (\gamma_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right]$

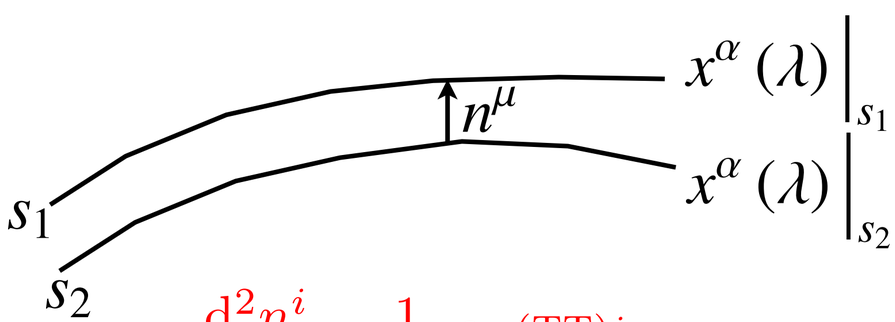
$h_{ij} = 2C\gamma_{ij} + 2D_iD_jE + 2D_{(i}\overline{E}_{j)} + 2\overline{E}_{ij}$ with $D_i\overline{E}^{ij} = 0,\ \overline{E}_i^i = 0$

→ $\overline{E}_{kl}'' + 2\mathcal{H}\overline{E}_{kl}' + (2K-\Delta)\,\overline{E}_{kl} = 3\mathcal{H}^2\Omega w\overline{\pi}_{kl}$

flat bckd GR: $\nabla^2 \bar{h}_{ij}^{(TT)} = 0$



$$h_{ij}^{(\mathrm{TT})} = E_{+}(ct - z)\varepsilon_{ij}^{+} + E_{\times}(ct - z)\varepsilon_{ij}^{\times}$$



$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}t^2} = \frac{1}{2}\partial_t^2 h_j^{(\mathrm{TT})i} n^j$$

$$\longrightarrow \quad \overline{E}''_{kl} + 2\mathcal{H}\overline{E}'_{kl} + (2K - \Delta)\,\overline{E}_{kl} = 3\mathcal{H}^2\Omega w \overline{\pi}_{kl}^{\,0}$$

$$\longrightarrow \text{decomposition} \qquad \overline{E}_{ij} = \sum_{\lambda} \overline{E}_{\lambda}\varepsilon^{\lambda}_{ij} \qquad\qquad u_{\mathrm{T}} = a\overline{E}_{\lambda}$$

$$u''_{\mathrm{T}} + \left(2K - \Delta - \frac{a''}{a}\right) u_{\mathrm{T}} = \kappa a^3 P \overline{\pi}_{\lambda}^{\,0}$$

$$a \propto \eta^{\nu} \longrightarrow \frac{\mathrm{d}^2\overline{E}_{ij}}{\mathrm{d}x^2} + \frac{2\nu}{x}\frac{\mathrm{d}\overline{E}_{ij}}{\mathrm{d}x} + \overline{E}_{ij} = 0$$

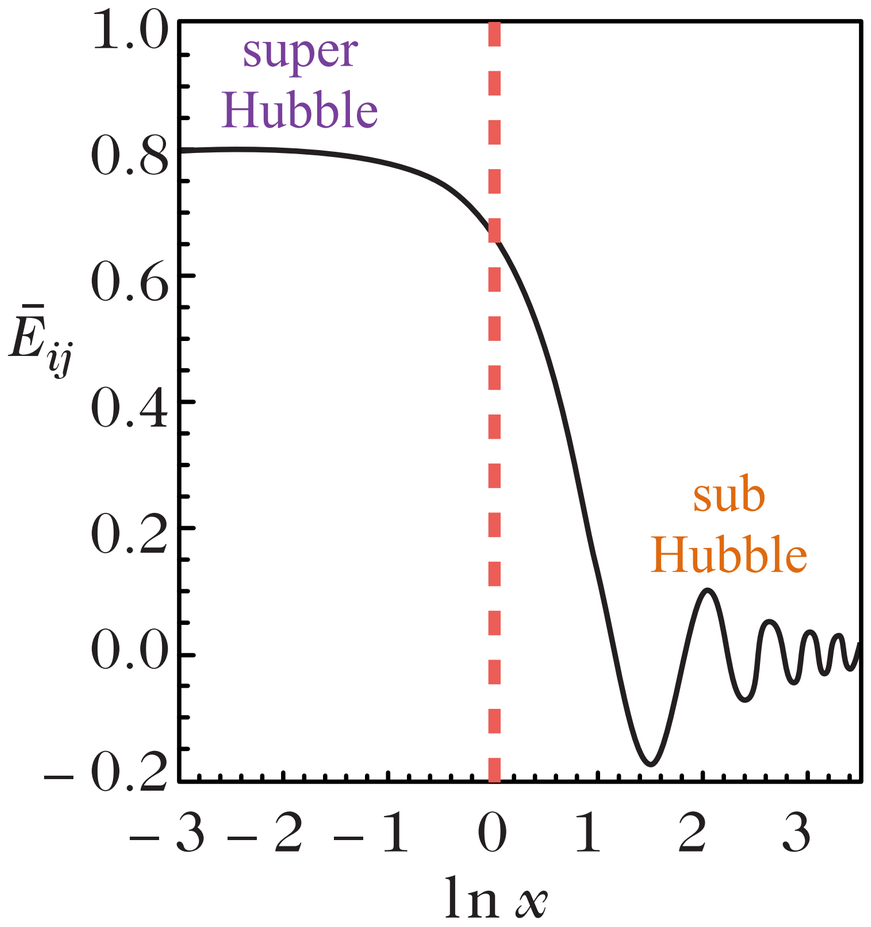
$$x \equiv k\eta$$

$$\longrightarrow \overline{E}_{ij} = x^{1/2-\nu}\left[A_{ij}\, J_{\nu-\frac{1}{2}}(x) + B_{ij}\, N_{\nu-\frac{1}{2}}(x)\right]$$

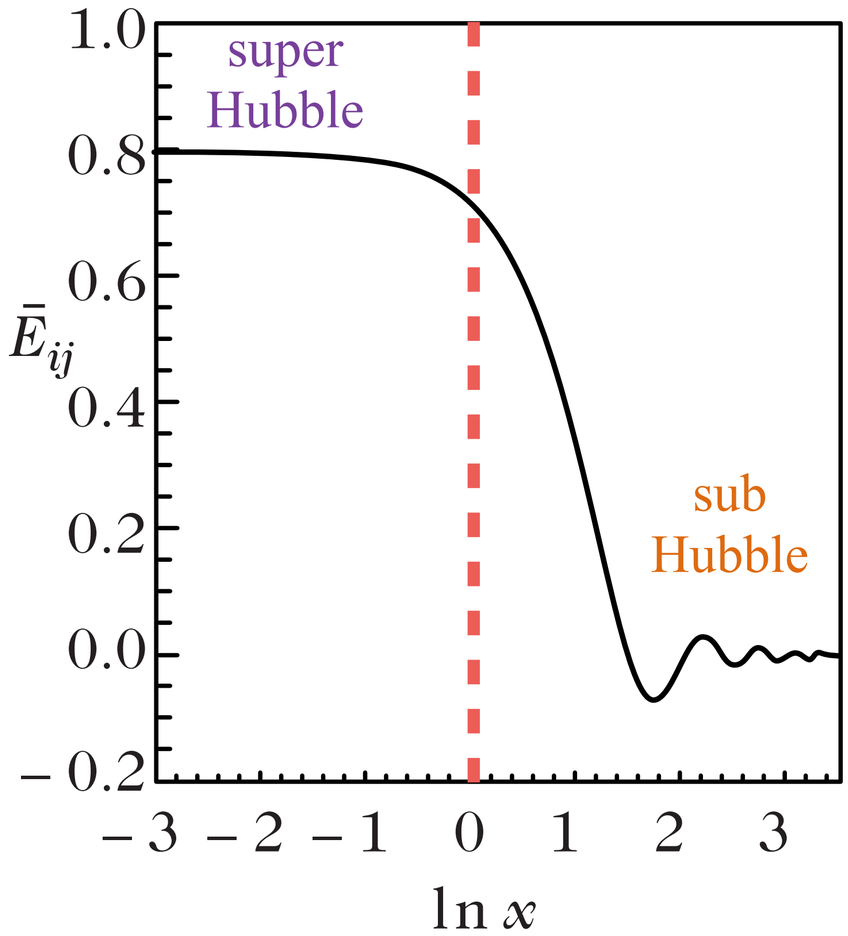
$$\longrightarrow \overline{E}_{ij} = x^{1/2-\nu} J_{\nu-\frac{1}{2}}(x)\, A_{ij} = x^{1-\nu} j_{\nu-1}(x)\, A_{ij}$$

$x \equiv k\eta$

solutions…

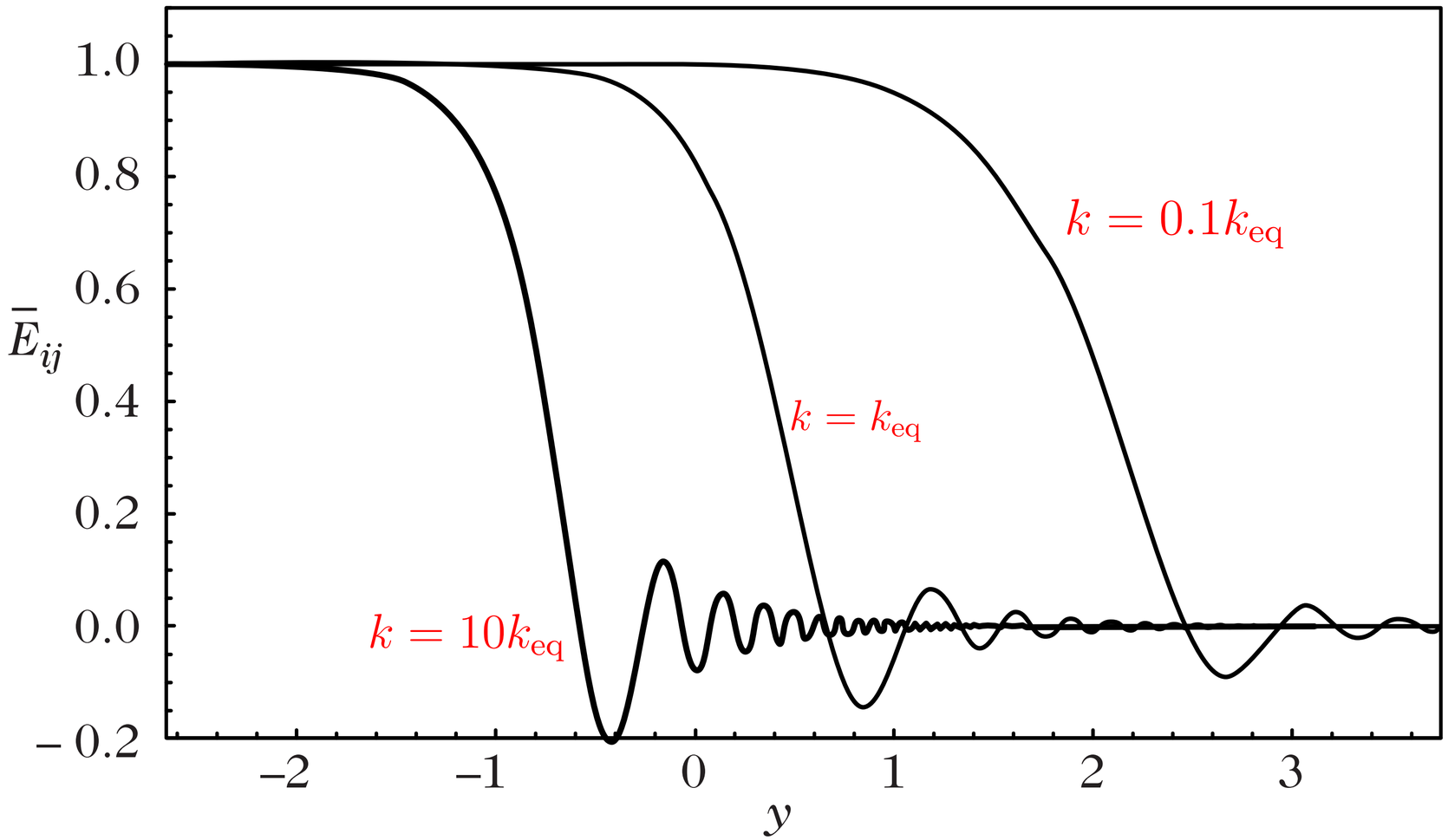


radiation
domination
$(w = \frac{1}{3}, \nu = 1)$



matter
domination
$(w = 0, \nu = 2)$

radiation-to-matter transition



$$y \equiv \frac{a}{a_{\rm eq}}$$

- there is a characteristic scale!

General solutions for long-wavelength modes

$$u_{_{\rm T}}(\eta,k) = a(\eta)\left[A_1(k) + A_2(k)\int^{\eta}\frac{{\rm d}\tilde{\eta}}{a^2(\tilde{\eta})} + k^2\int^{\eta}\frac{{\rm d}\tilde{\eta}}{a^2(\tilde{\eta})}\int^{\tilde{\eta}}{\rm d}\bar{\eta}\,a^2(\bar{\eta})u_{_{\rm T}}(\bar{\eta},k)\right]$$

$$k \ll a''/a \quad\longrightarrow\quad u_{_{\rm T}}(\eta,k) = a(\eta)\,A_1(k) + A_2(k)\,a(\eta)\int^{\eta}\frac{{\rm d}\tilde{\eta}}{a^2(\tilde{\eta})} + \mathcal{O}\left(\frac{k^2 a}{a''}\right)$$

Equation of motion $\mu_{\mathrm{T}}'' + \left(k^2 + 2K - \dfrac{a''}{a}\right)\mu_{\mathrm{T}} = 0$

$$\mathrm{d}^2\psi(x)/\mathrm{d}x + [E - V(x)]\,\psi(x) = 0$$

time-independent one-dimensional Schrödinger equation

derivable from $\mathcal{S}^{(2)} = \dfrac{1}{2}\sum_{+,\times}\int \sqrt{\gamma}\mathrm{d}^3x\mathrm{d}\eta\left[\left(\mu_{\mathrm{T}}'\right)^2 - \gamma^{ij}\partial_i\mu_{\mathrm{T}}\partial_j\mu_{\mathrm{T}} - 2K\mu_{\mathrm{T}}^2 + \dfrac{a''}{a}\mu_{\mathrm{T}}^2\right]$

scalar field in constant spatial curvature spacetime…
easy to handle from the point of view of QFT!

Scalar modes

constraint equations

$$\begin{cases} (\Delta + 3K)\,\Psi = \dfrac{\kappa}{2} a^2 \rho \delta^{\mathrm{C}} \\ \Psi - \Phi = \kappa a^2 P \cancel{\overline{\pi}} \end{cases} \longrightarrow \quad \Phi = \Psi$$

One scalar d.o.f.

evolution equations: $\Psi' + \mathcal{H}\Phi = -\dfrac{\kappa}{2} a^2 \rho (1+w) V$

$$\Psi'' + 3\mathcal{H}(1+c_{\mathrm{S}}^2)\Psi' + \left[2\mathcal{H}' + (\mathcal{H}^2 - K)(1+3c_{\mathrm{S}}^2)\right]\Psi - c_{\mathrm{S}}^2 \Delta\Psi = 0$$

$$- \left(\mathcal{H}^2 + 2\mathcal{H}' + K\right)\left[\frac{1}{2}\Gamma + (3\mathcal{H}^2 + 2\mathcal{H}')\cancel{\overline{\pi}} + \mathcal{H}\cancel{\overline{\pi}}' + \frac{1}{3}\Delta\cancel{\overline{\pi}}\right]$$

$$-9c_{\mathrm{S}}^2 \mathcal{H}^2 \left(\mathcal{H}^2 + K\right)\cancel{\overline{\pi}}$$

matter

$$\boxed{\delta^{\mathrm{N}\prime} + 3\mathcal{H}(c_{\mathrm{S}}^2 - w)\delta^{\mathrm{N}} = -(1+w)\left(\Delta V - 3\Psi'\right) - 3\mathcal{H} w \Gamma,}$$

$$\boxed{V' + \mathcal{H}(1 - 3c_{\mathrm{S}}^2)V = -\Phi - \frac{c_{\mathrm{S}}^2}{1+w}\delta^{\mathrm{N}} - \frac{w}{1+w}\left[\Gamma + \frac{2}{3}(\Delta + 3K)\overline{\pi}\right].}$$

Evolution of the gravitational potential $(\overline{\pi} = 0)$

$$\Phi'' + 3\mathcal{H}\left(1 + c_{\rm S}^2\right)\Phi' + \left[2\mathcal{H}' + \left(\mathcal{H}^2 - K\right)\left(1 + 3c_{\rm S}^2\right)\right]\Phi - c_{\rm S}^2\Delta\Phi = \frac{\kappa}{2}a^2 P\Gamma$$

reduced form

$$\theta = \frac{1}{a}\left(\frac{\rho}{\rho + P}\right)^{1/2}\left(1 - \frac{3K}{\kappa\rho a^2}\right)^{1/2} \longrightarrow u_{\rm S} = \frac{2}{3}\frac{a^2\theta}{\mathcal{H}}\Phi$$

after a tedious, but straightforward, computation...

$$u_{\rm S}'' - \left(\frac{\theta''}{\theta} + c_{\rm S}^2\Delta\right)u_{\rm S} = \frac{\kappa}{3}\frac{\theta}{\mathcal{H}}a^4 P\Gamma$$

→ *Integral solution for long-wavelength modes*

$$u_{\rm S}(k,\eta) = A(k)\theta(\eta) + B(k)\theta(\eta)\int^{\eta}\frac{{\rm d}\eta'}{\theta^2(\eta')} + \mathcal{O}\left(\frac{k^2 c_{\rm S}^2\theta}{\theta''}\right)$$

First integral $\zeta \equiv \Phi + \dfrac{2}{3\mathcal{H}} \dfrac{\Phi' + \mathcal{H}\Phi}{1+w}$

$$\longrightarrow \zeta' = \frac{2}{3}\frac{\mathcal{H}}{1+w}\left\{ \frac{K}{\mathcal{H}^2}\left[\frac{1+3w}{2\mathcal{H}}\Phi' + 3\left(c_{\rm S}^2 - w\right)\Phi\right] + \frac{\kappa}{2}\frac{a^2}{\mathcal{H}^2}P\Gamma - c_{\rm S}^2\left(\frac{k}{\mathcal{H}}\right)^2\Phi \right\} = 0$$

$K/\mathcal{H}^2 \ll 1$ $\qquad$ $\Gamma = 0$ $\qquad$ $k/\mathcal{H} \ll 1$

adiabatic super-Hubble

If curvature not negligible

$$\zeta_{\rm BST} = \Phi - \frac{2}{3}\frac{\mathcal{H}}{(1+w)(\mathcal{H}^2+K)}\left\{\Phi' + \left[1 - \frac{K}{\mathcal{H}^2} + \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\right]\mathcal{H}\Phi\right\}$$

$$\longrightarrow \zeta'_{\rm BST} = -\frac{2}{3}\frac{\mathcal{H}^2}{(1+w)(\mathcal{H}^2+K)}\left[\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi' + \mathcal{H}\Phi) + \frac{\kappa}{2}\frac{a^2}{\mathcal{H}^2}P\Gamma\right] = 0$$

$k/\mathcal{H} \ll 1$ $\qquad$ $\Gamma = 0$

super-Hubble $\qquad$ adiabatic

radiation-to-matter transition

$$u_{\rm S}'' - \left(\frac{\theta''}{\theta} + c_{\rm S}^2\Delta\right) u_{\rm S} = \frac{\kappa}{3}\frac{\theta}{\mathcal{H}} a^4 P\Gamma \quad \longrightarrow \quad u'' + \left(c_{\rm s}^2 k^2 - \frac{\theta''}{\theta}\right) = 0$$

$$\longrightarrow \quad c_{\rm s}^2 k^2 \ll \frac{\theta''}{\theta} \qquad u(\eta) \simeq \theta(\eta) + \theta(\eta)\int^{\eta} \frac{{\rm d}\tau}{\theta(\tau)^2} \qquad \& \qquad \Phi = \frac{3\mathcal{H}u}{2a^2\theta}$$

$$\theta^{-2} = a^2(1+w)$$

$$\longrightarrow \quad \Phi = \frac{\mathcal{H}}{a^2}\left[D + G\int a^2(1+w){\rm d}\tau\right] \quad \text{valid at all times}$$

decaying — growing

$$\Longrightarrow \quad \Phi = \frac{2(1+w)}{5+3w} G$$

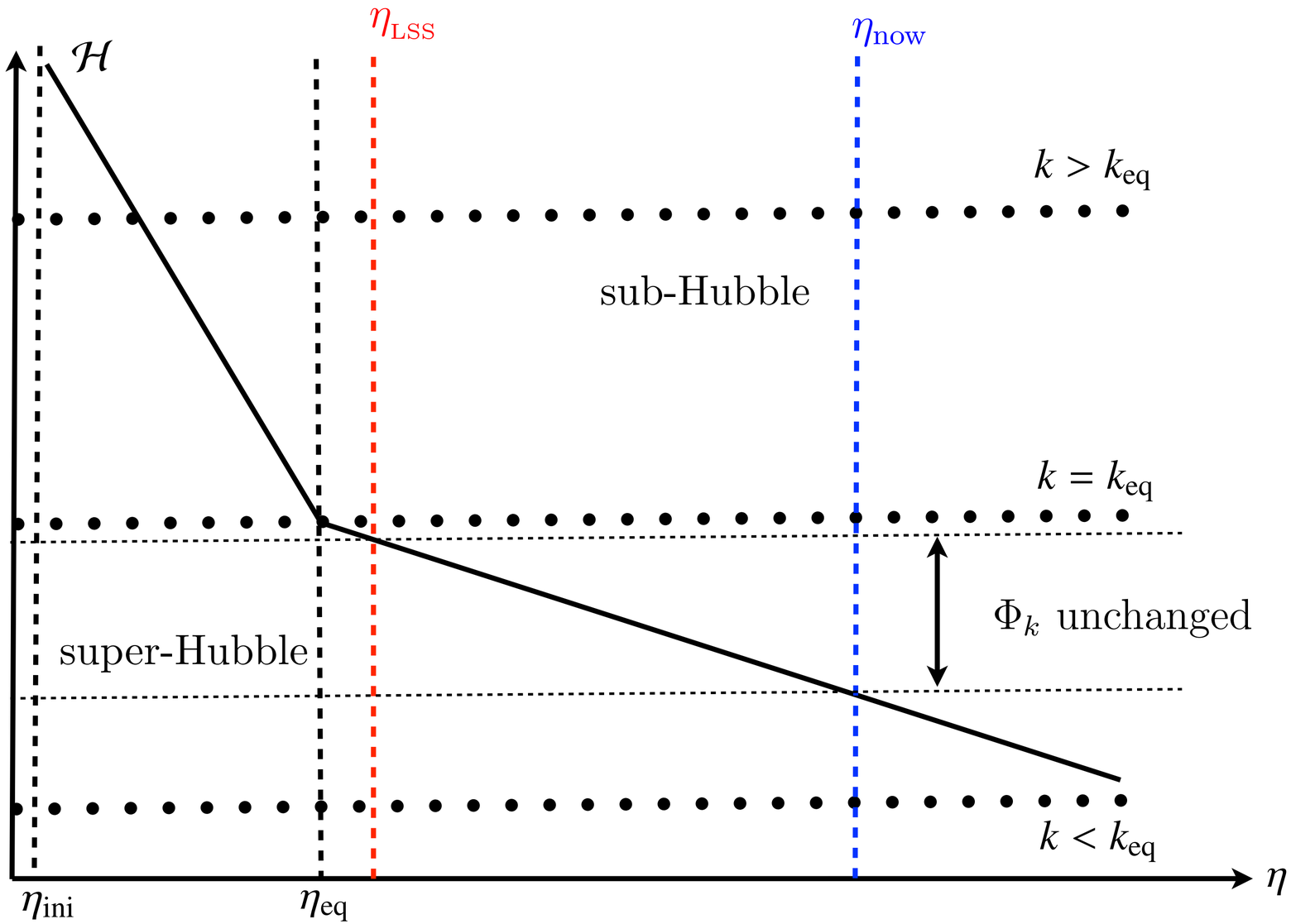
$$\frac{\Phi(\eta \gg \eta_{\rm eq})}{\Phi(\eta \ll \eta_{\rm eq})} = \frac{1+w_+}{5+3w_+} \cdot \frac{5+3w_-}{1+w_-} \quad \to \frac{9}{10}$$

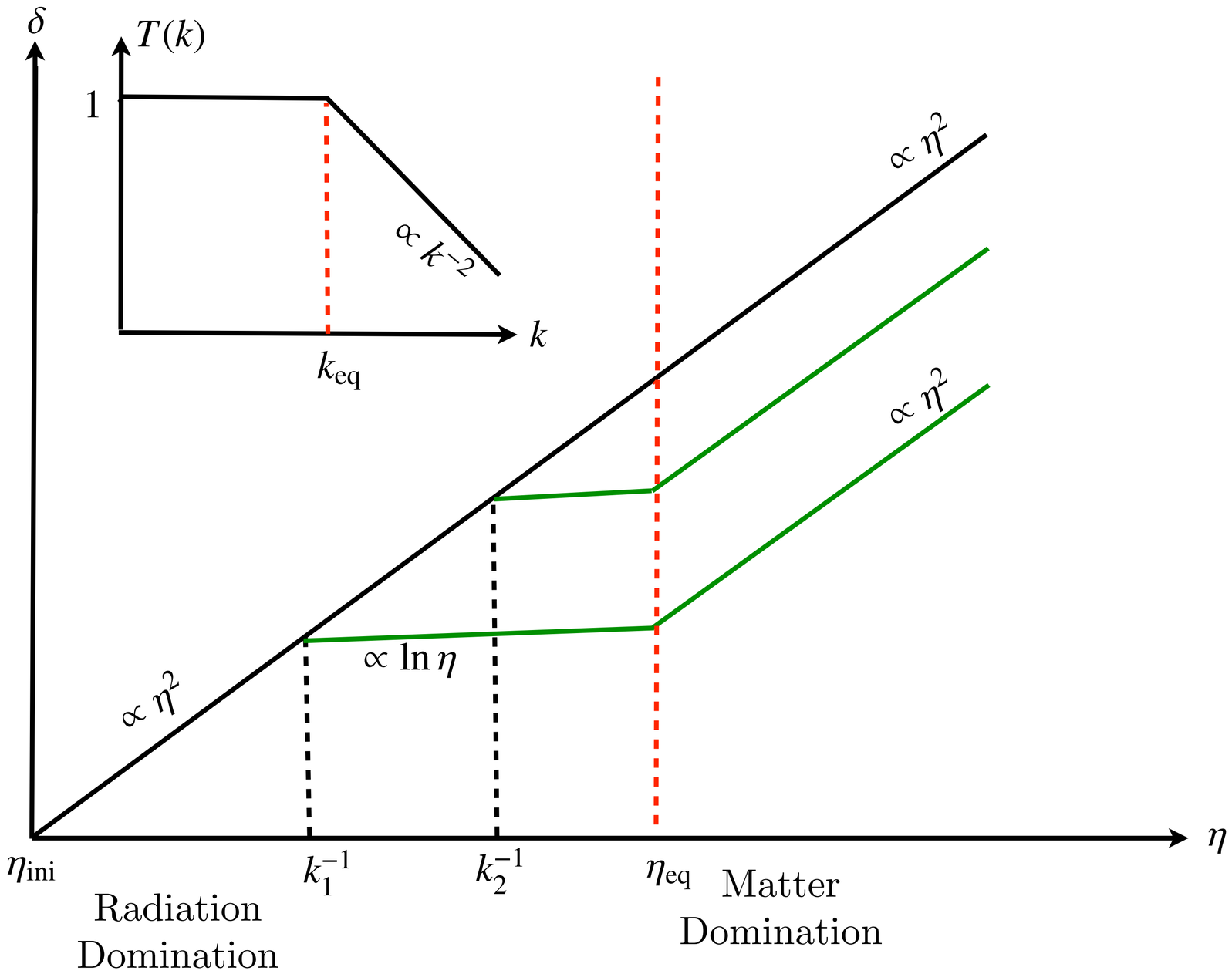
($w_+ \to 0$, $w_- \to 1/3$)

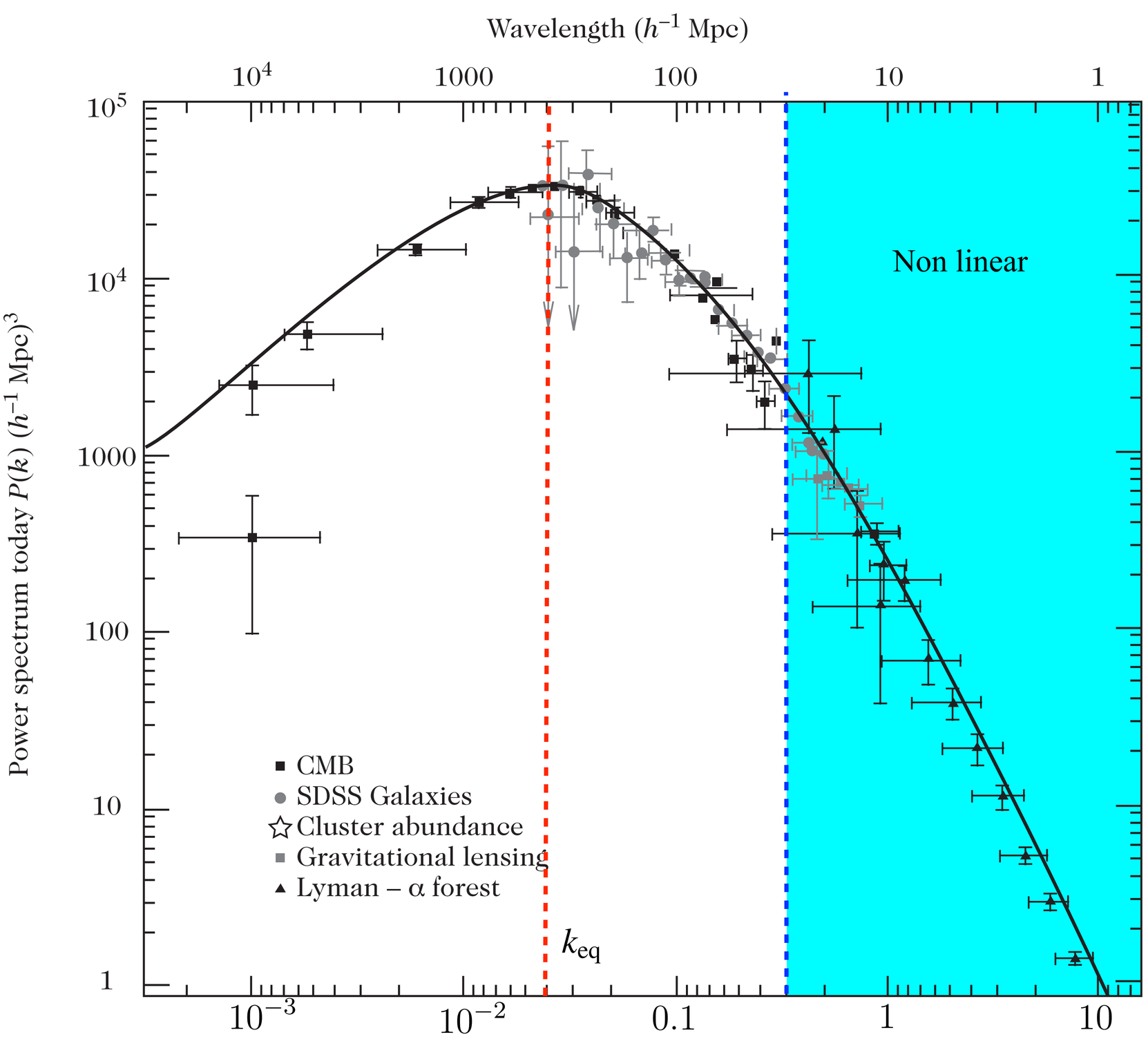
same by simply applying

$$[\zeta]_{\pm} = 0$$

Transfer function and power spectrum




δ
$T(k)$
1
$\propto k^{-2}$
k
$k_{\rm eq}$
$\propto \eta^2$
$\propto \eta^2$
$\propto \eta^2$
$\propto \ln \eta$
η
$\eta_{\rm ini}$
k_1^{-1}
k_2^{-1}
$\eta_{\rm eq}$
Radiation Domination
Matter Domination


Wavelength (h^{-1} Mpc)
10^4
1000
100
10
1
10^5
10^4
1000
100
10
1
Power spectrum today $P(k)$ (h^{-1} Mpc)^3
Non linear
CMB
SDSS Galaxies
Cluster abundance
Gravitational lensing
Lyman – α forest
k_{eq}
10^{-3}
10^{-2}
0.1
1
10
Wave number k (h / Mpc)

Quantum vacuum fluctuations

scale factor

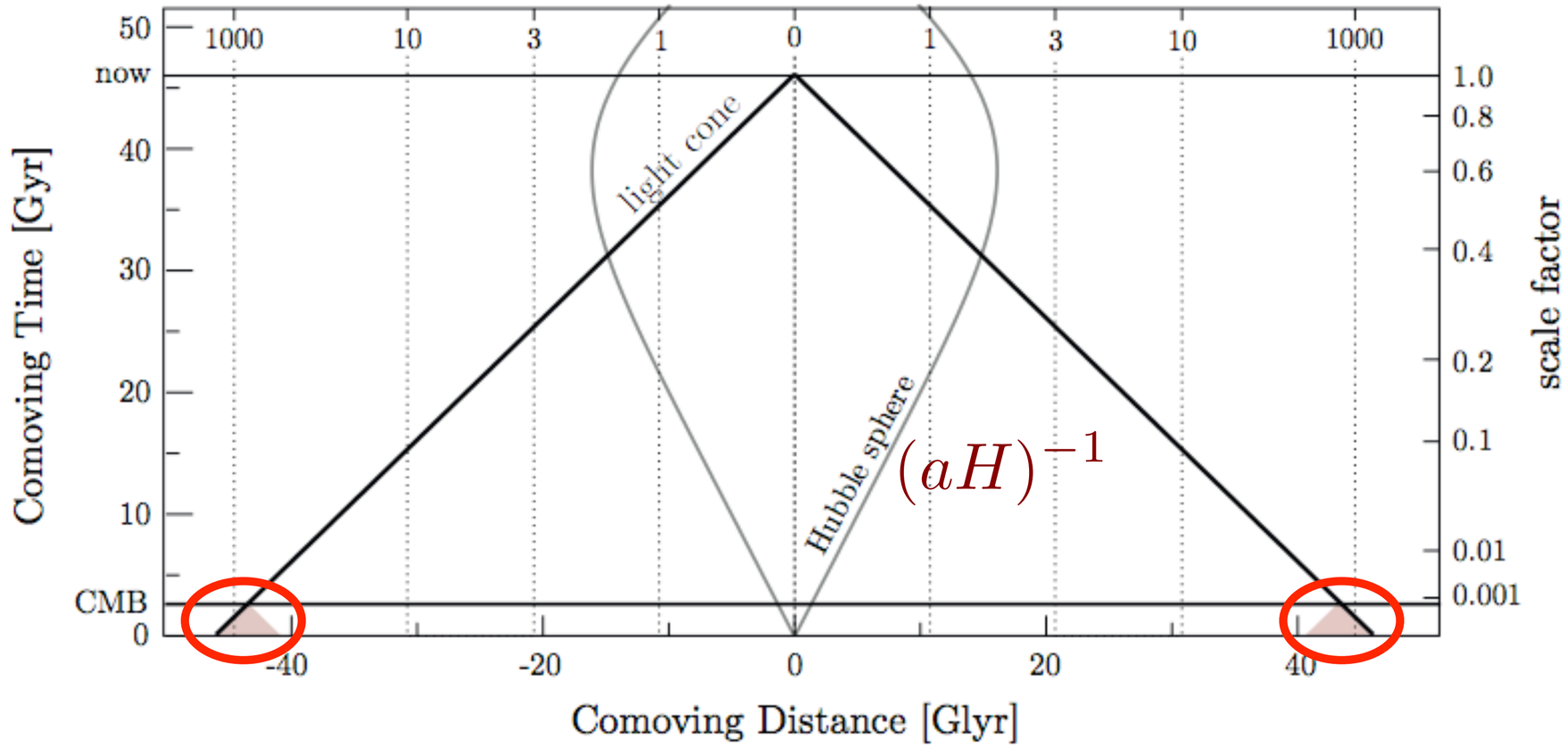
comoving (conformal) time

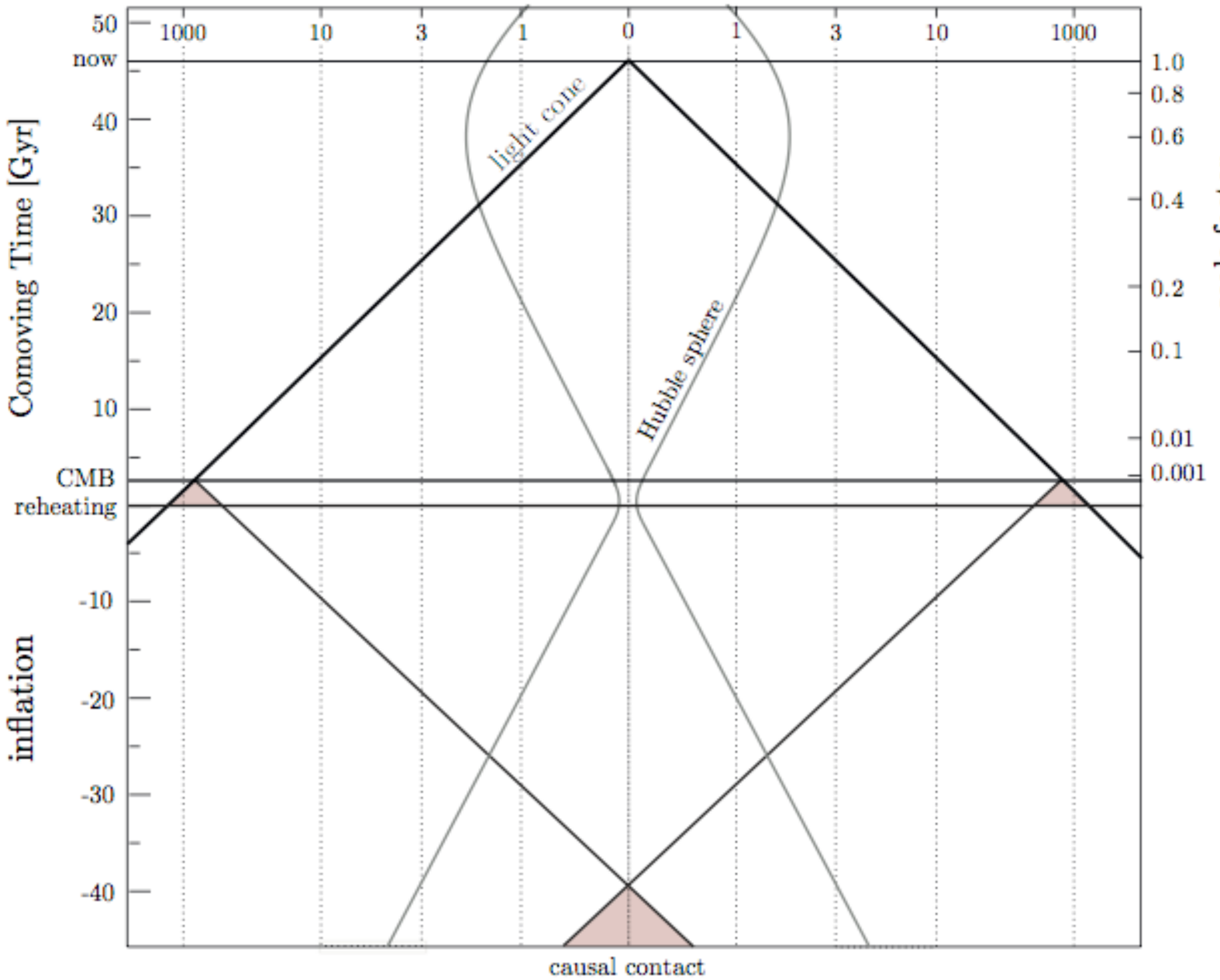
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^{\,2} = a(\tau)^2 \left(-d\tau^2 + \mathrm{d}\vec{x}^{\,2}\right)$$

cosmic time

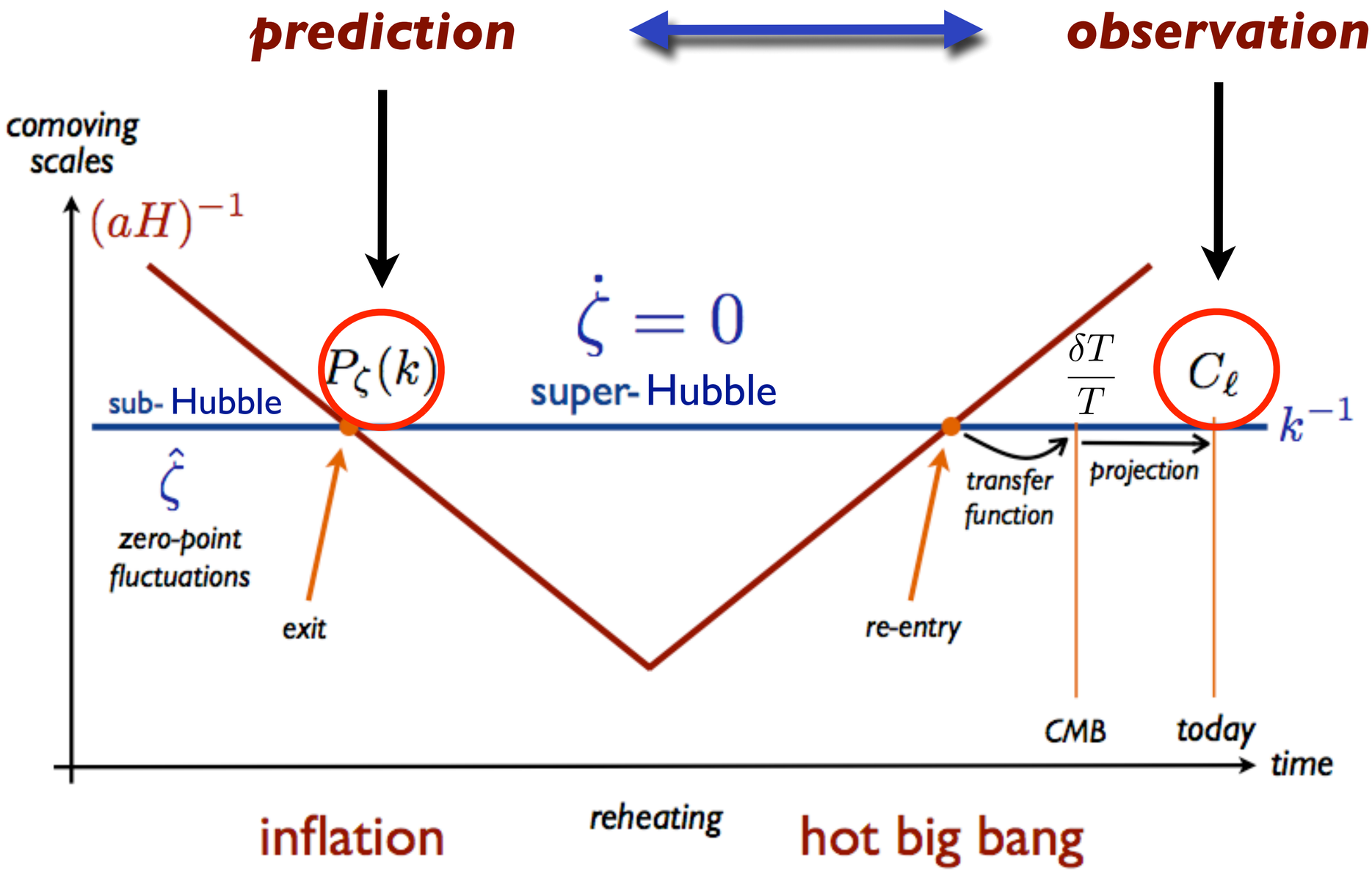
Hubble scale $H = \dfrac{\dot{a}}{a}$

comoving distance

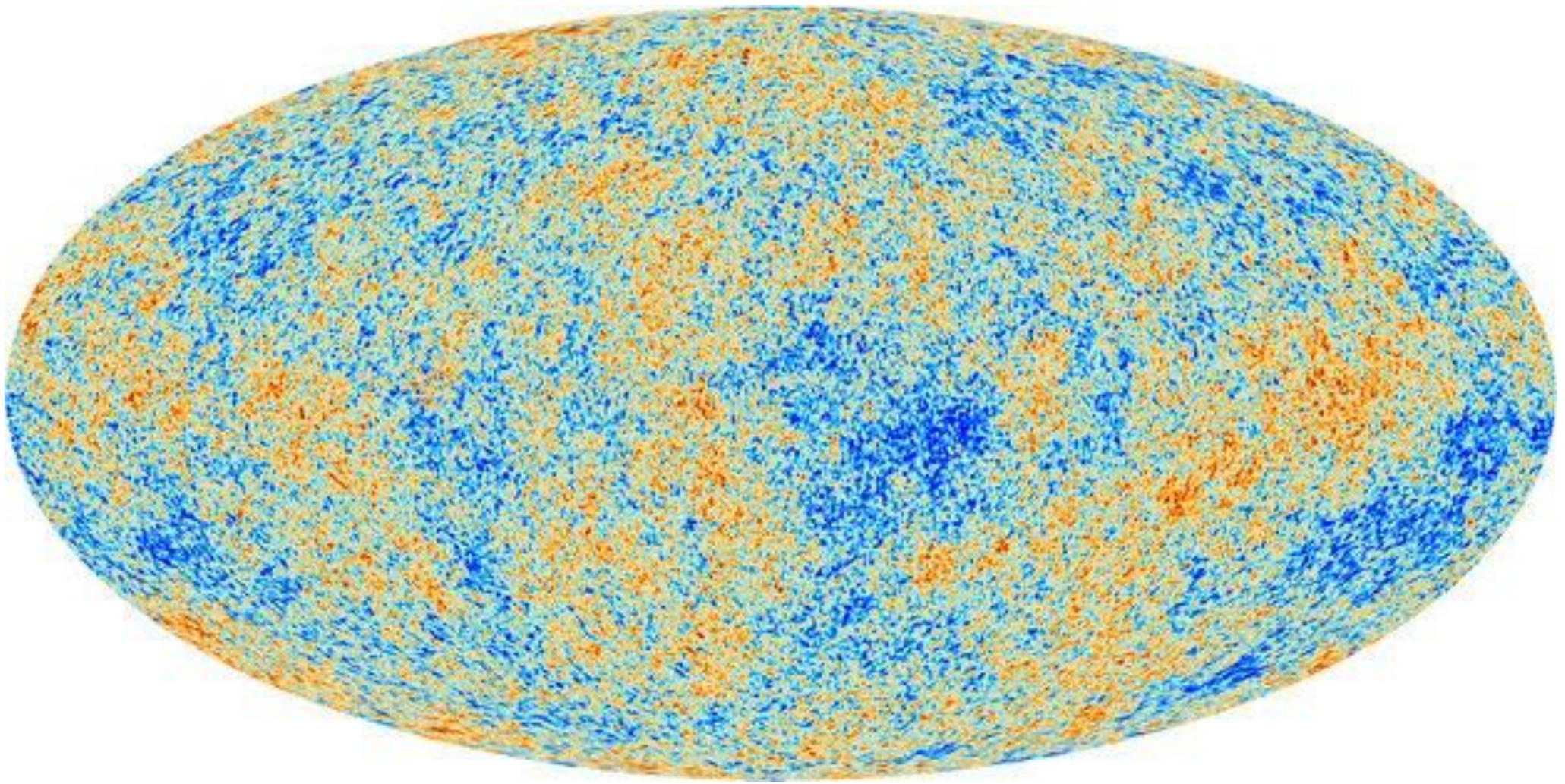



50
now
40
30
20
10
CMB
reheating
-10
-20
-30
-40
Comoving Time [Gyr]
inflation
1000
10
3
1
0
1
3
10
1000
light cone
Hubble sphere
1.0
0.8
0.6
0.4
0.2
0.1
0.01
0.001
scale factor
causal contact

$$\langle \zeta_{\boldsymbol{k}} \zeta_{\boldsymbol{k'}} \rangle = (2\pi)^3 P_\zeta(k) \delta(\boldsymbol{k} + \boldsymbol{k'})$$



Planck map



homogeneous

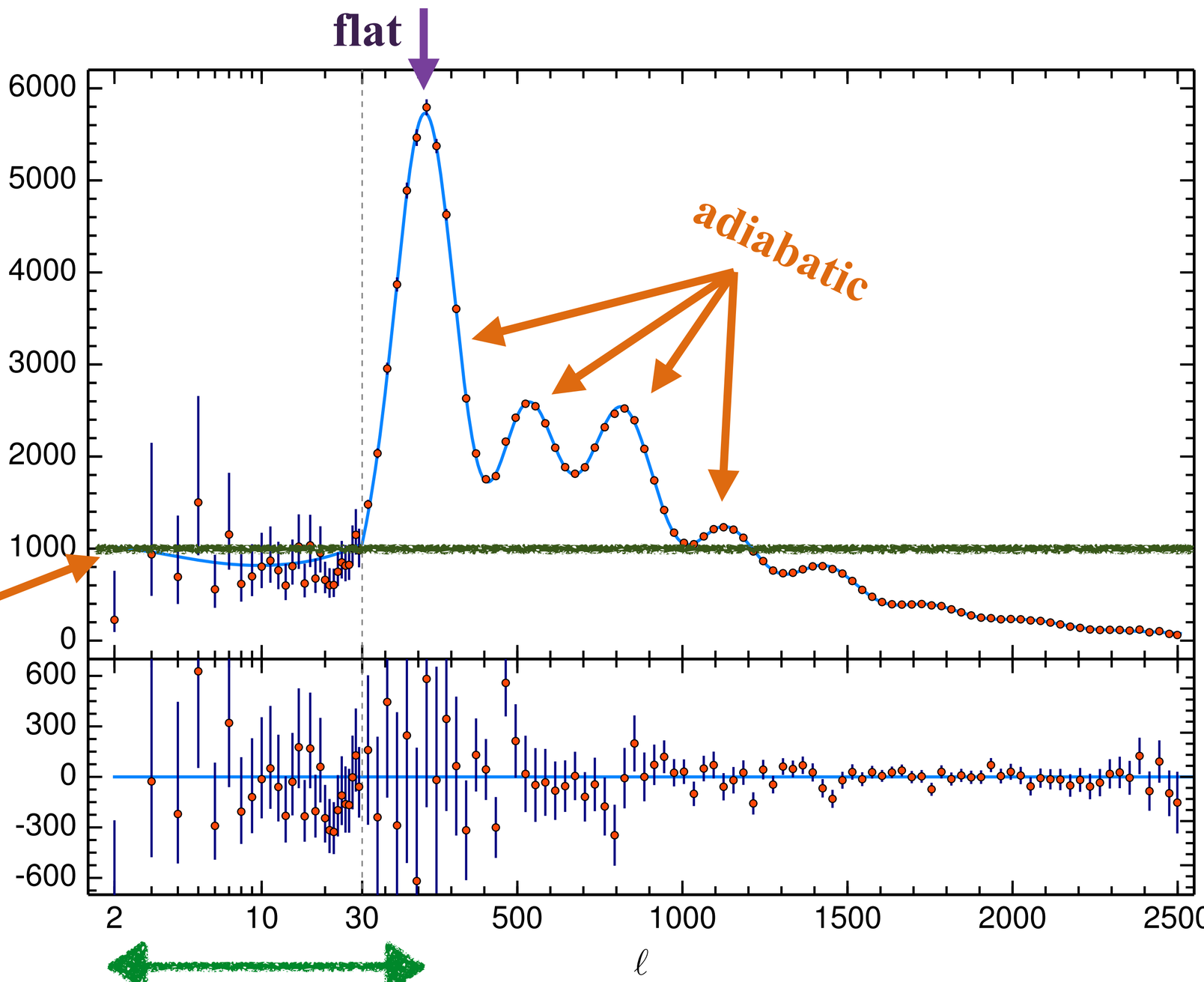
gaussian

scale invariant

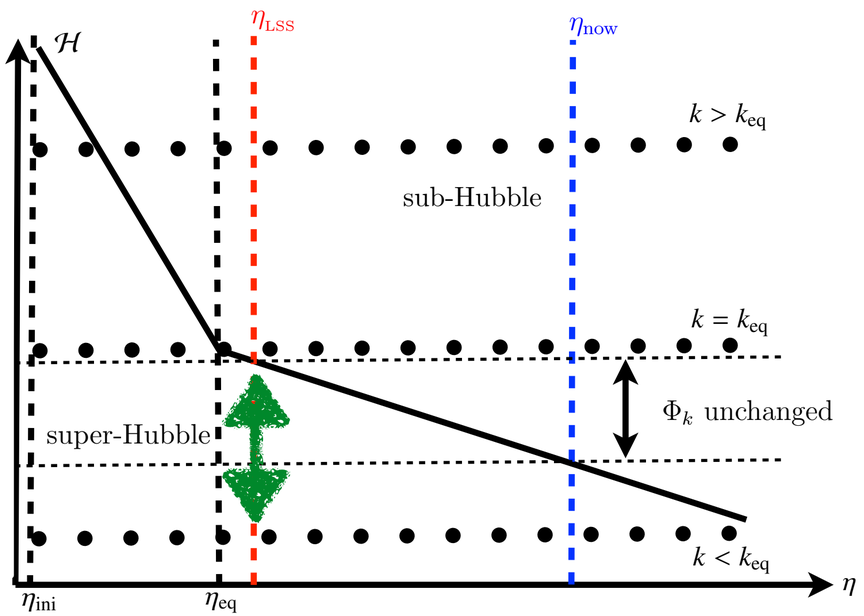
flat

adiabatic

$\ell(\ell+1)C_\ell/2\pi[\mu K^2]$



super Hubble @LSS



Implementing inflation

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \quad \Longrightarrow \quad T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] \longrightarrow \begin{cases} \rho = \frac{1}{2}\dot{\phi}^2 + V \\ P = \frac{1}{2}\dot{\phi}^2 - V \end{cases}$$

inflation $\Longleftrightarrow \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0$

$$\rho + 3P = 2\left(\dot{\phi}^2 - V\right)$$

Friedmann $\quad \frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{\kappa}{3}\left(\dot{\phi}^2 - V\right)$

Potential energy must dominate over kinetic energy

$V > \dot{\phi}^2$ … in practice: $V \gg \dot{\phi}^2$

Limit $\dot{\phi}^2 \to 0 \quad \Longrightarrow \quad V \to V_0 \quad \Longrightarrow \quad H^2 = \frac{\kappa V_0}{3} \quad \& \quad \frac{\ddot{a}}{a} = \frac{\kappa V_0}{3}$

$$a \propto \exp\left(\sqrt{\frac{\kappa V_0}{3}}t\right)$$

Friedmann $$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{\kappa}{3}\left(\dot{\phi}^2 - V\right)$$

Potential energy must dominate over kinetic energy

formally define

$$\epsilon := -\frac{\dot{H}}{H^2} = 3\frac{\frac{1}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} \ll 1$$

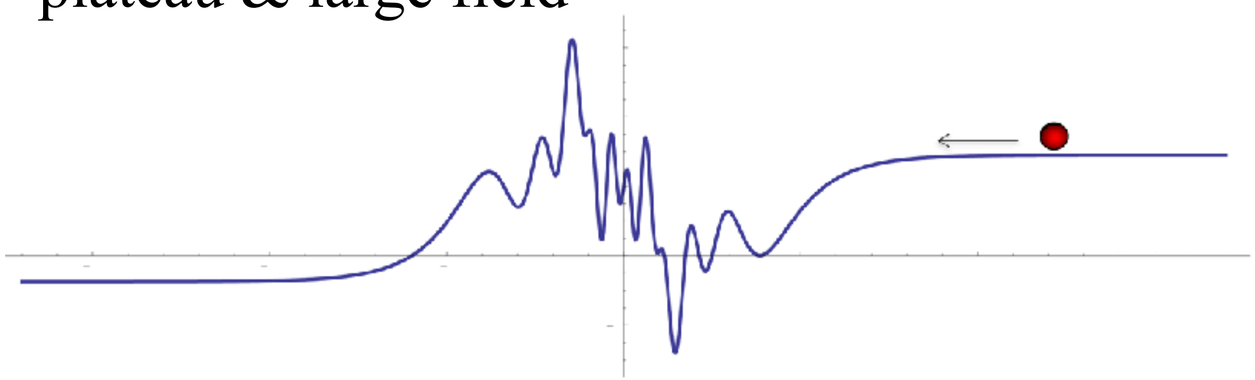
Klein-Gordon $$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{dV}}{\mathrm{d}\phi} = 0$$

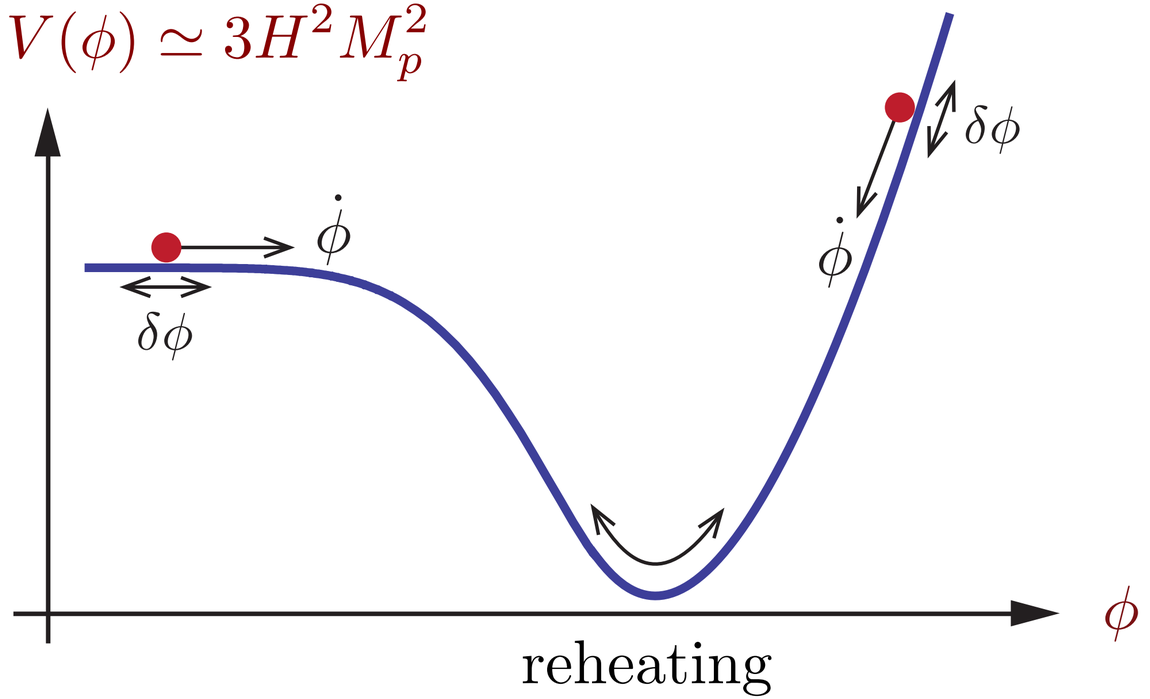
slow roll parameters $$\delta := -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\dot{\epsilon}}{2H\epsilon} + \epsilon$$

$$\epsilon \approx \frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2$$

$$\delta \approx -\frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 + \frac{1}{8\pi G}\frac{V_{,\phi\phi}}{V}$$

plateau & large field





Other slow-roll parameter hierarchy/definition:

$$\epsilon_0 = \frac{H_{\rm ini}}{H(t)}$$

$$\epsilon_{n+1} = \frac{{\rm d}\ln|\epsilon_{\rm n}|}{{\rm dN}} = \frac{\dot{\epsilon}_n}{H\epsilon_n}$$

$${\rm dN} = {\rm d}\ln\left(\frac{\rm a}{\rm a_{ini}}\right) = \frac{\rm da}{\rm a} = \frac{\dot{\rm a}{\rm dt}}{\rm a} = {\rm Hdt}$$

Background equations $H^2 = \dfrac{\kappa V}{3-\epsilon} \approx \dfrac{\kappa}{3} V(\phi) + \mathcal{O}(\epsilon)$ & $\dfrac{\mathrm{d}\phi}{\mathrm{dt}} = -\dfrac{V_{,\phi}}{(3-\delta)H} \approx -\dfrac{V_{,\phi}}{3H} + \mathcal{O}(\delta)$

concrete example $V(\phi) = \alpha \left(\dfrac{\phi}{M_\mathrm{p}} \right)^n$

$$\Longrightarrow \quad \epsilon = \frac{n^2 M_\mathrm{P}^2}{16\pi\phi^2} \quad \text{slow-roll} \quad \epsilon \ll 1 \quad \text{demands} \quad \phi > \phi_\mathrm{end} = \frac{n}{4\sqrt{\pi}} M_\mathrm{P}$$

solve eom: $\phi(t) \propto (A + Bt)^{2/(4-n)}$

and $a(t) \propto \exp\left\{ \dfrac{4\pi}{n M_\mathrm{P}^2} \left[\phi_\mathrm{ini}^2 - \phi^2(t) \right] \right\}$

Implementing a bounce

Quantized scalar field effect model:

Parker & Fulling ’73: massive scalar field, if $\langle a^\dagger a \rangle \gg 1$, then $\exists$ solution ($\mathcal{K} > 0$)

$$a(t) = \left(\frac{|B_2|^2 - |B_1|^2}{4m^2 |B_2|^2} + \frac{8\pi G m^2 |B_2|^2 t^2}{3} \right)^{1/2};$$

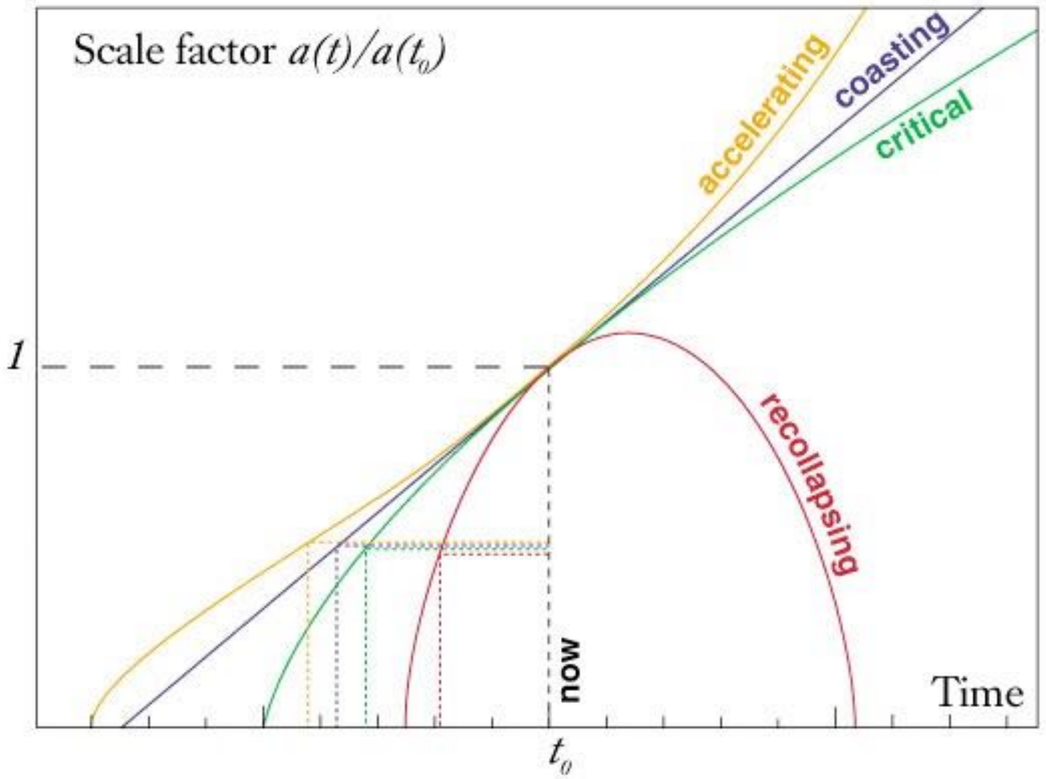
Probability that it occurs: $\mathcal{P} \sim 10^{-43}$

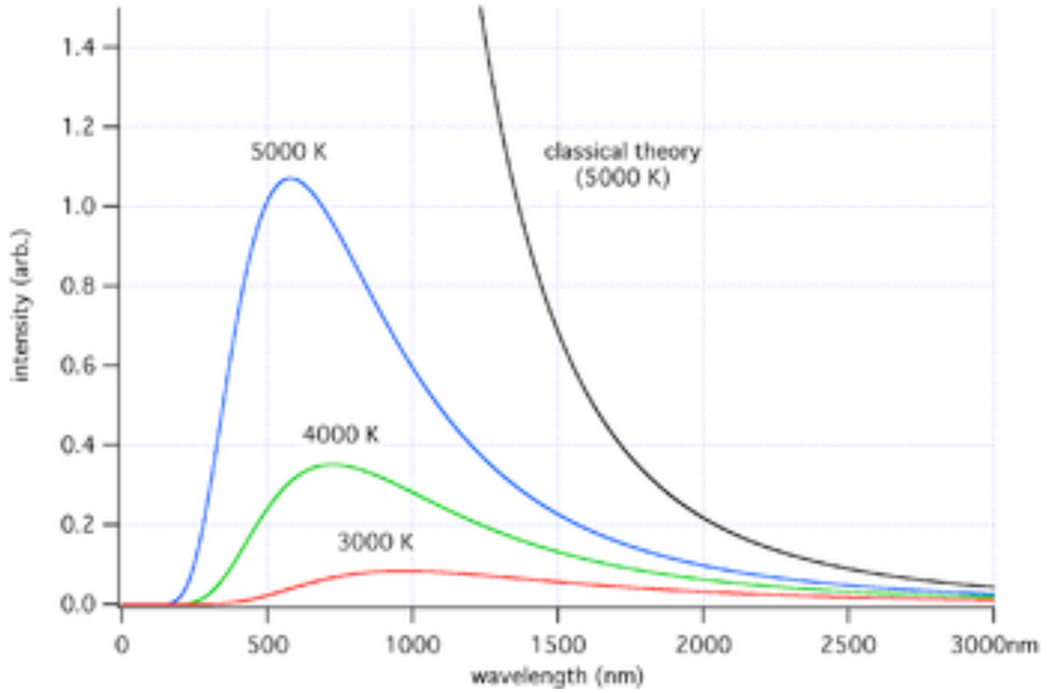
Implementing a bounce

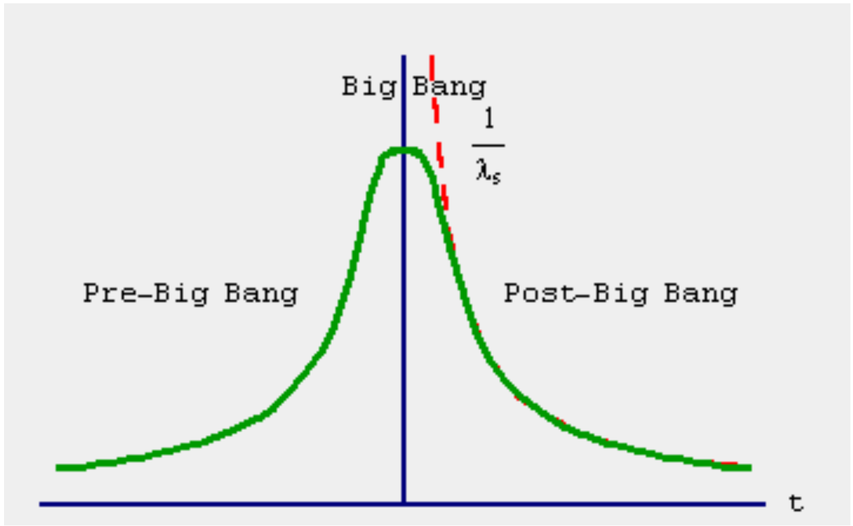
Model listing:

Quantum gravity	LQG & LQC	Non relativistic quantum gravity
	Canonical quantum gravity (WdW)	
	String theory	

Singularity problem Quantum effect?

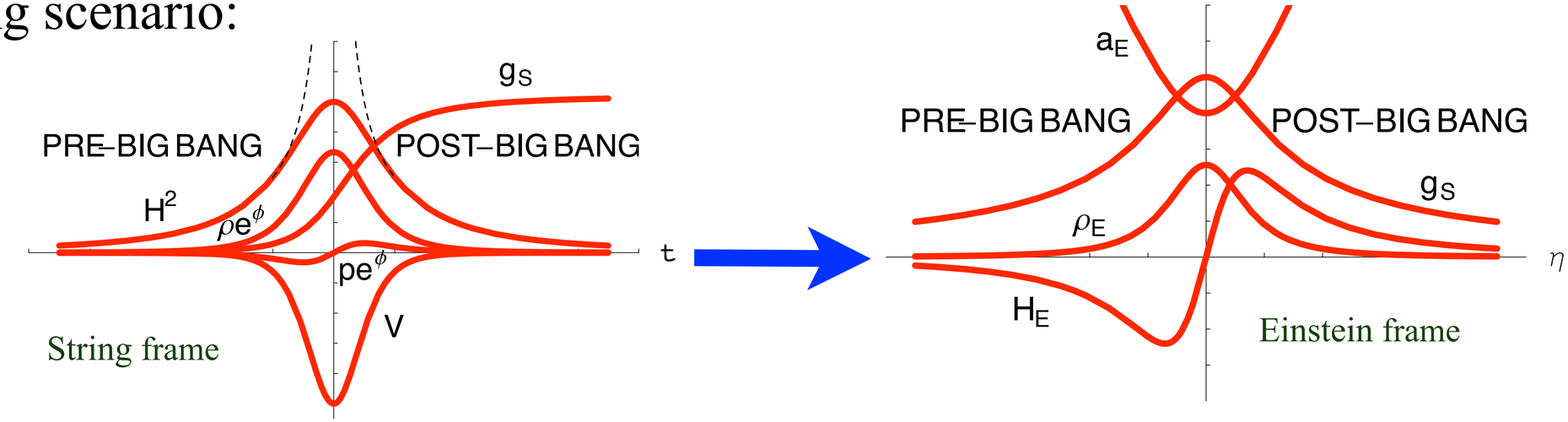




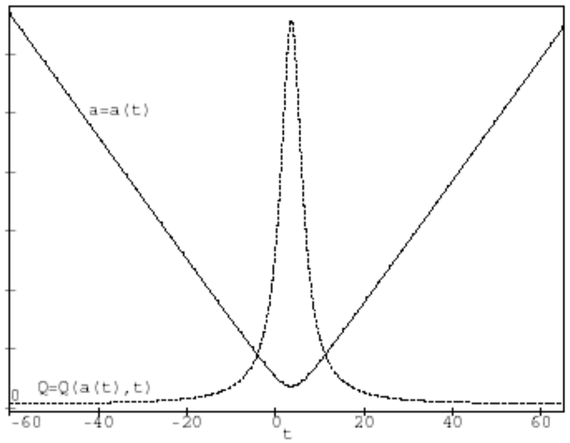


M.Gasperini & G. Veneziano, *Phys. Rep*. **373**, 1 (2003), hep-th/0207130 & hep-th/0703055

Pre Big Bang scenario:

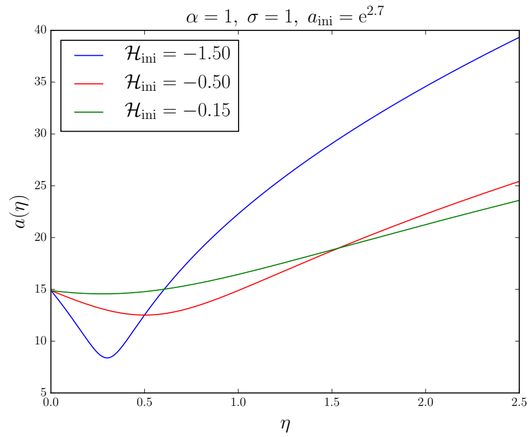


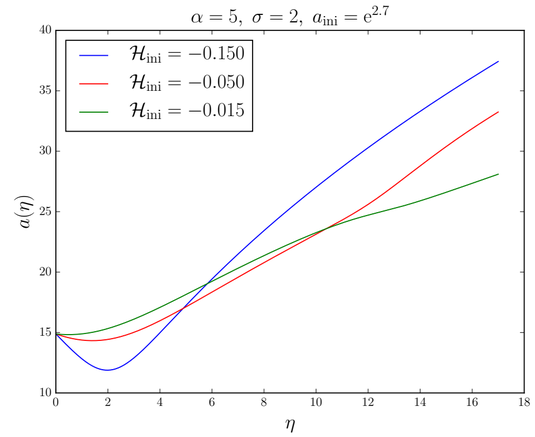
trajectory quantum cosmology:

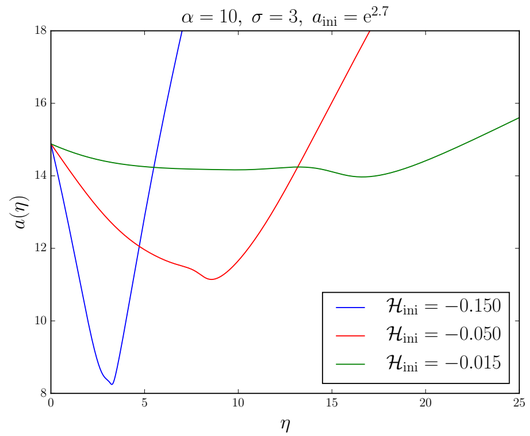


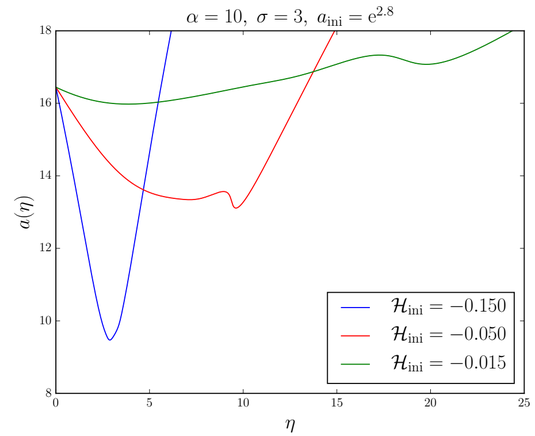
J. Acacio de Barros, N. Pinto-Neto & M. Sagorio-Leal
Phys. Lett. A**241**, 229 (1998)

S. Vitenti & PP
Mod.Phys.Lett. A**31**, 1640006 (2016).









Model listing:

Quantum gravity

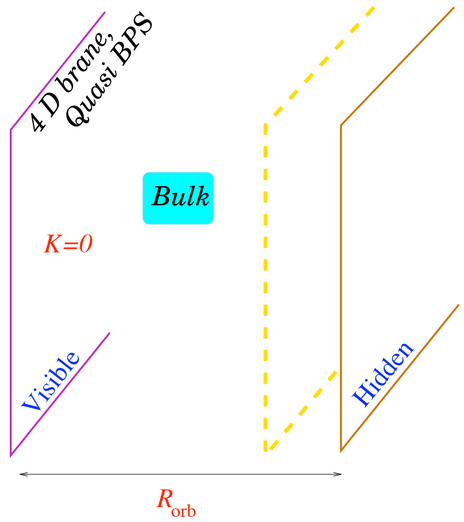
LQG & LQC

Canonical quantum gravity (WdW)

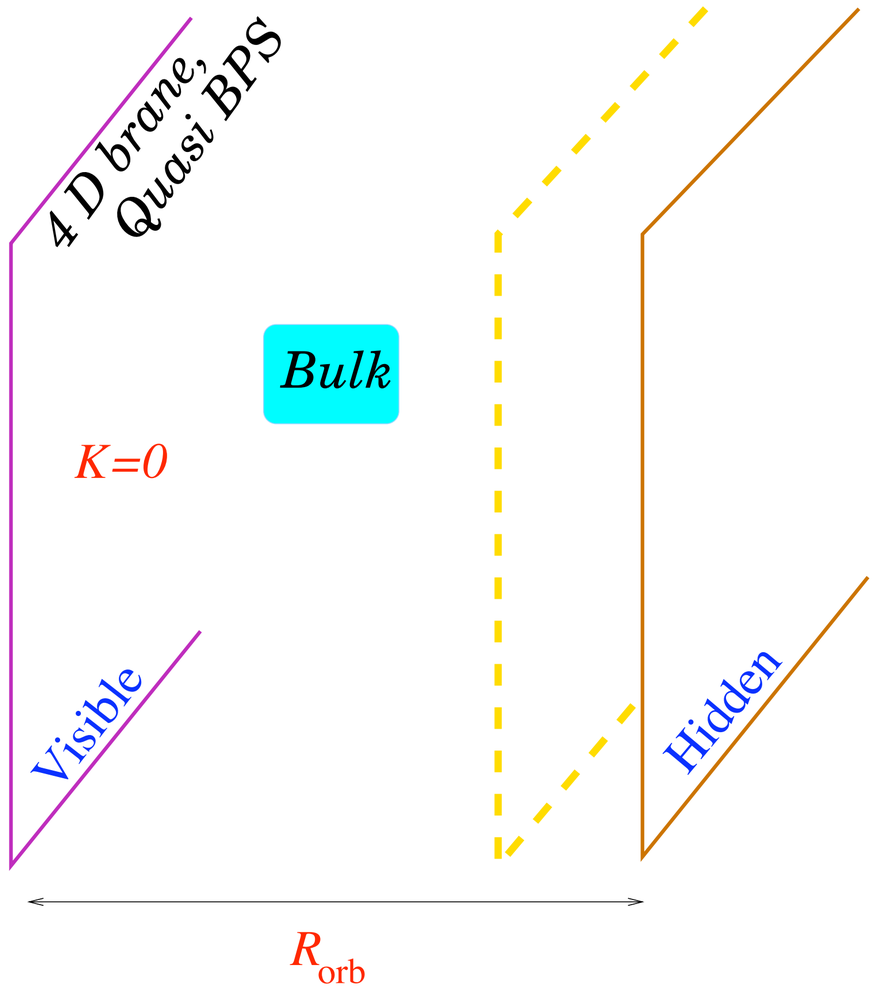
String theory

Non relativistic quantum gravity

Ekpyrotic & cyclic Branes



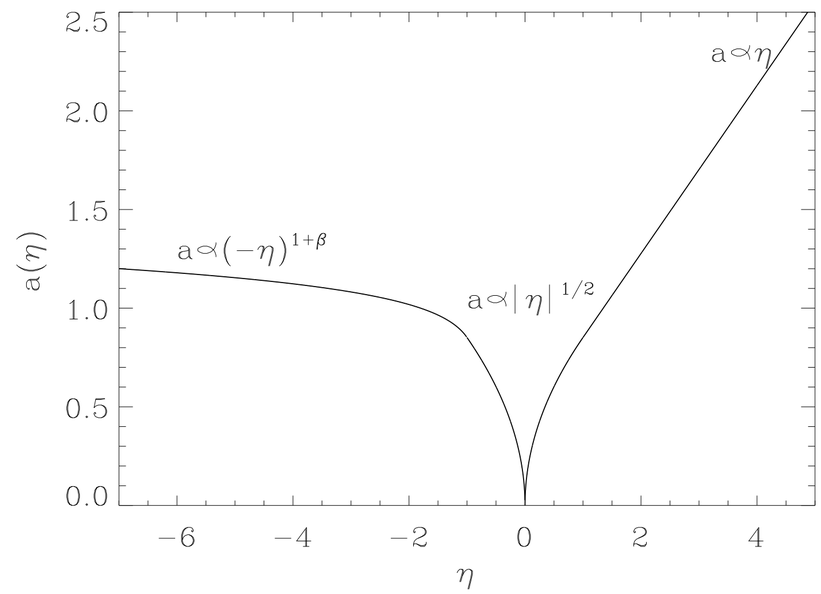
Ekpyrotic scenario:



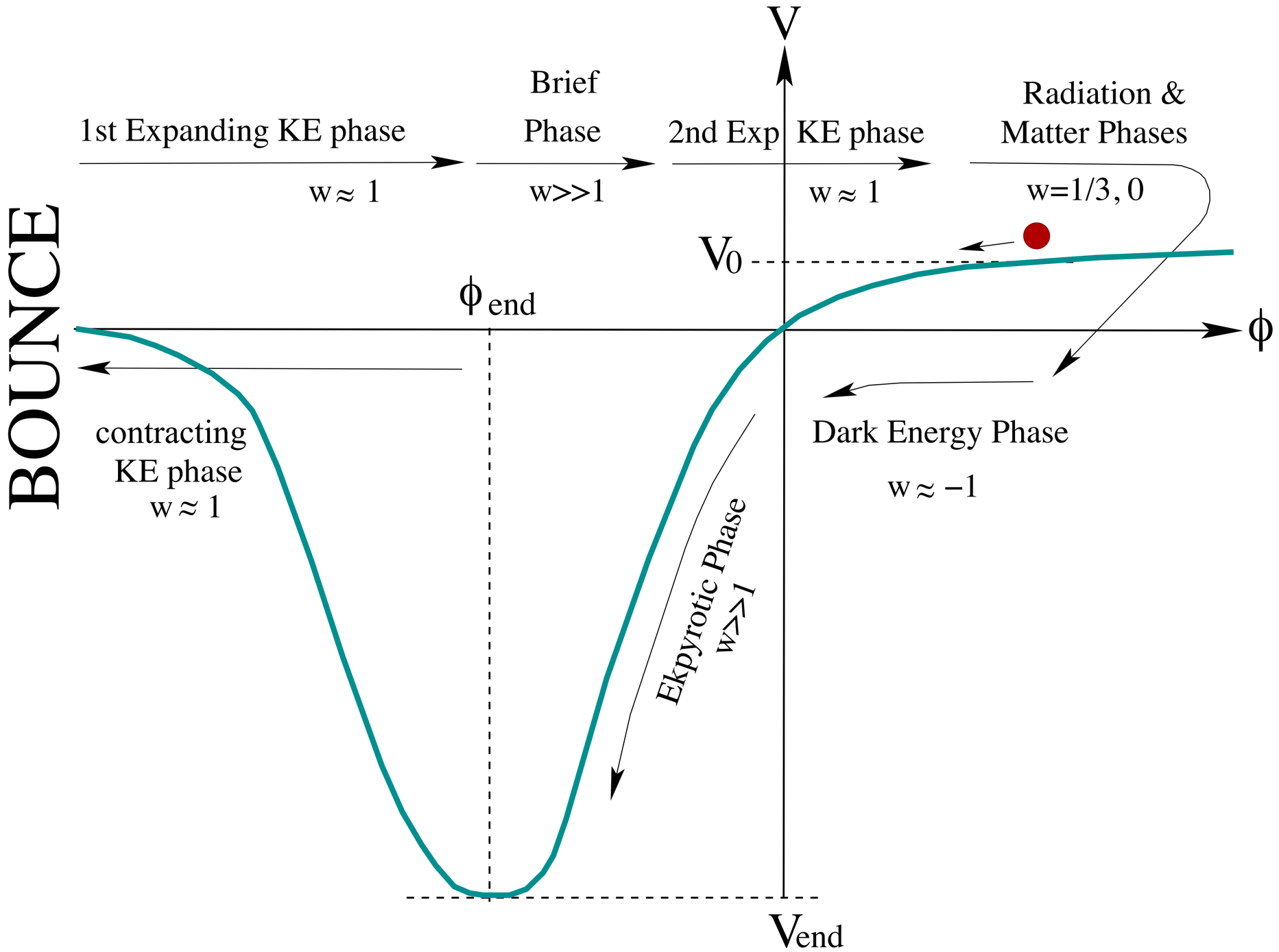
$$\mathcal{S}_5 \propto \int_{\mathcal{M}_5} \mathrm{d}^5x\sqrt{-g_5}\left[R_{(5)} - \frac{1}{2}\left(\partial\varphi\right)^2 - \frac{3}{2}\frac{\mathrm{e}^{2\varphi}\mathcal{F}^2}{5\,!}\right],$$

$$\mathcal{S}_4 = \int_{\mathcal{M}_4} \mathrm{d}^4x\sqrt{-g_4}\left[\frac{R_{(4)}}{2\kappa} - \frac{1}{2}\left(\partial\phi\right)^2 - V(\phi)\right];$$

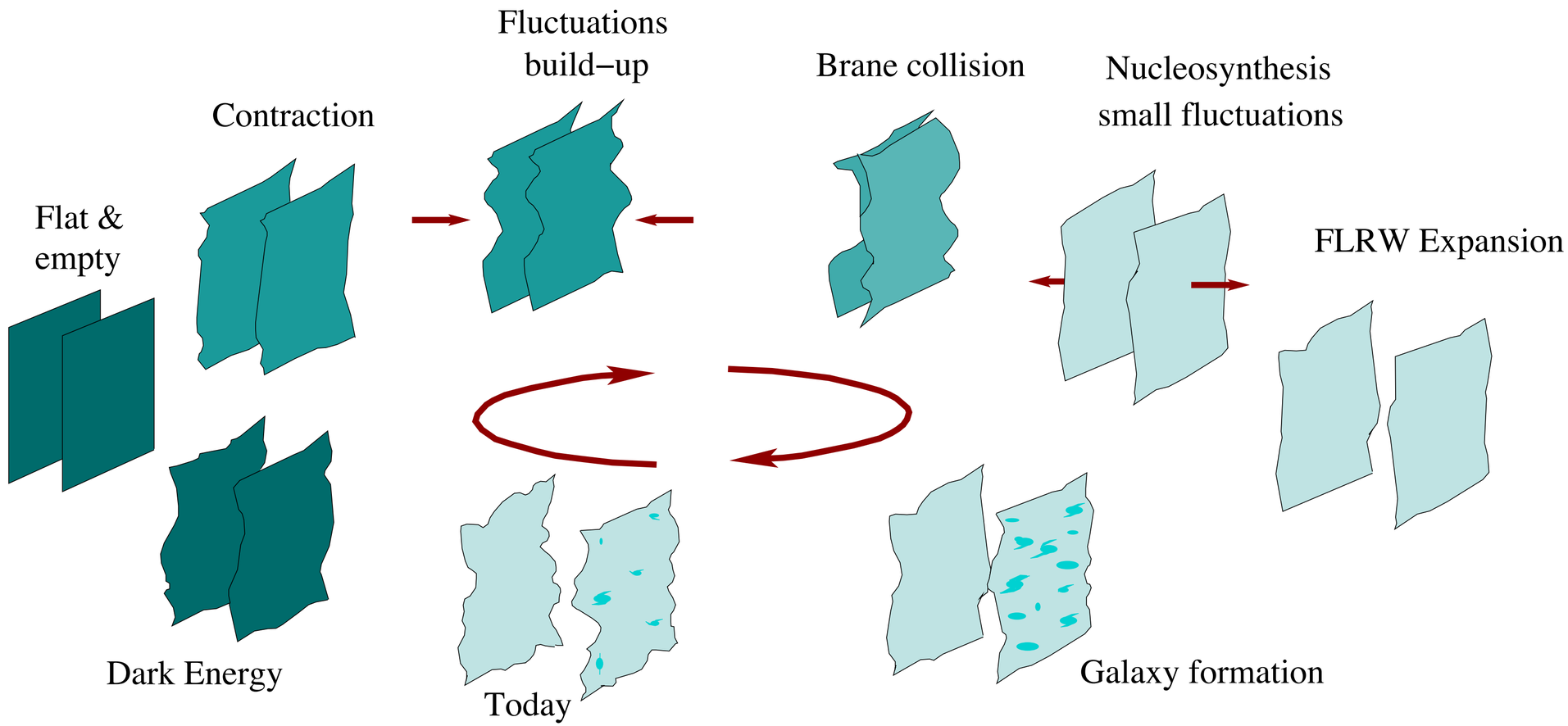
$$V(\varphi) = -V_{\rm i}\exp\left[-\frac{4\sqrt{\pi\gamma}}{m_{\rm Pl}}(\varphi-\varphi_{\rm i})\right];$$

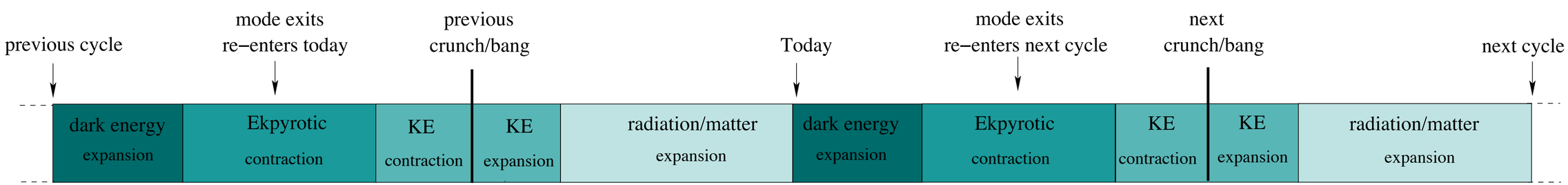


Singular…


V
Brief
Phase
1st Expanding KE phase
w ≈ 1
w>>1
2nd Exp KE phase
w ≈ 1
Radiation &
Matter Phases
w=1/3, 0
BOUNCE
V0
ϕ end
ϕ
contracting
KE phase
w ≈ 1
Dark Energy Phase
w ≈ −1
Ekpyrotic Phase
w>>1
Vend

Cyclic extension





Model listing:

Quantum gravity

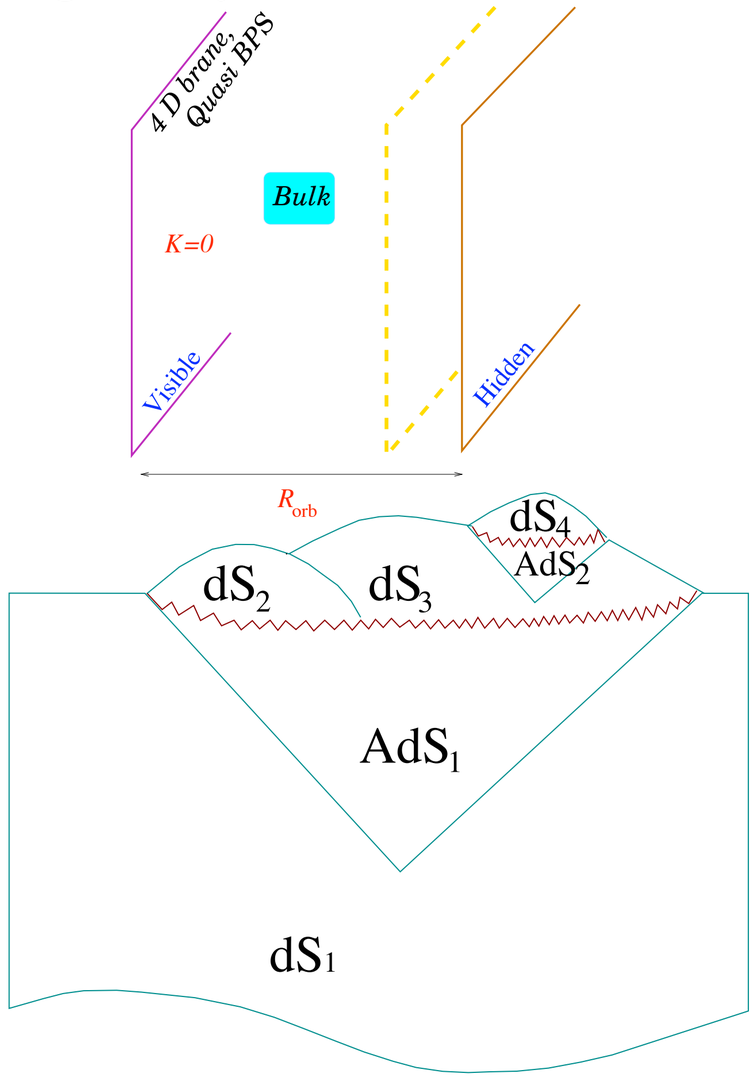
LQG & LQC

Canonical quantum gravity (WdW)

String theory

Ekpyrotic & cyclic

Branes



String gas cosmology

Antigravity

Galileon

Massive gravity

Multiverse models

Strings & AdS/CFT

Non relativistic quantum gravity

Horava-Lifshitz

Lee-Wick & Quintom

F(R), *f(T)*, Gauss-Bonnet

Mimetic matter

Non-linear electromagnetic action

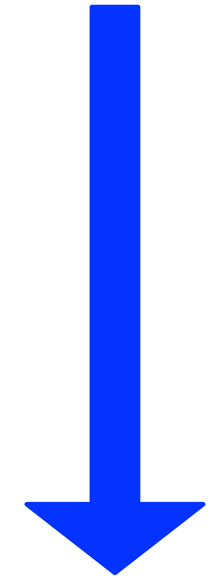
Spinors & torsion

Implementing a bounce = problem with GR!

$$\dot{H} = \frac{\mathcal{K}}{a^2} - \frac{1}{2}(\rho + P)$$

Violation of Null Energy Condition (NEC)

$$\rho + P \leq 0$$



Instabilities for perfect fluids

Implementing a bounce = problem with GR!

$$\dot{H} = \frac{\mathcal{K}}{a^2} - \frac{1}{2}(\rho + P)$$

Violation of Null Energy Condition (NEC)

$$\rho + P \leq 0$$

Positive spatial curvature + scalar field

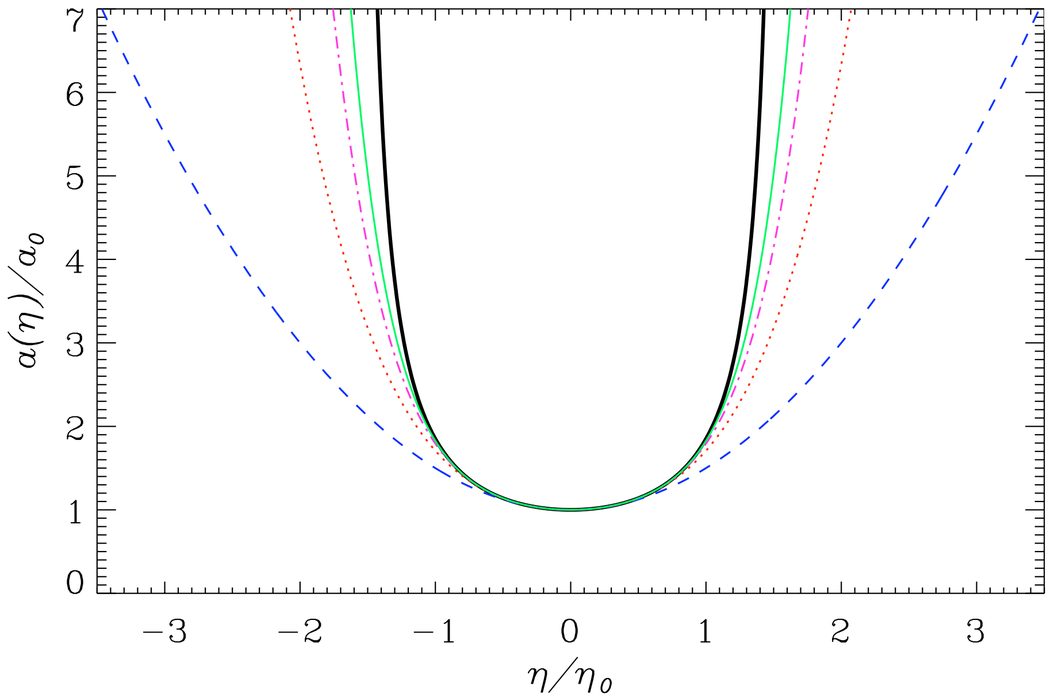
Self consistent bounce:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2\mathrm{d}\Omega^2\right)$$

⟶ One d.o.f. + 4 dimensions G.R.

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{6\ell_{\mathrm{Pl}}^2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V\right) - \frac{\mathcal{K}}{a^2}$$ Positive spatial curvature



J. Martin & PP., *Phys. Rev.* **D68**, 103517 (2003)

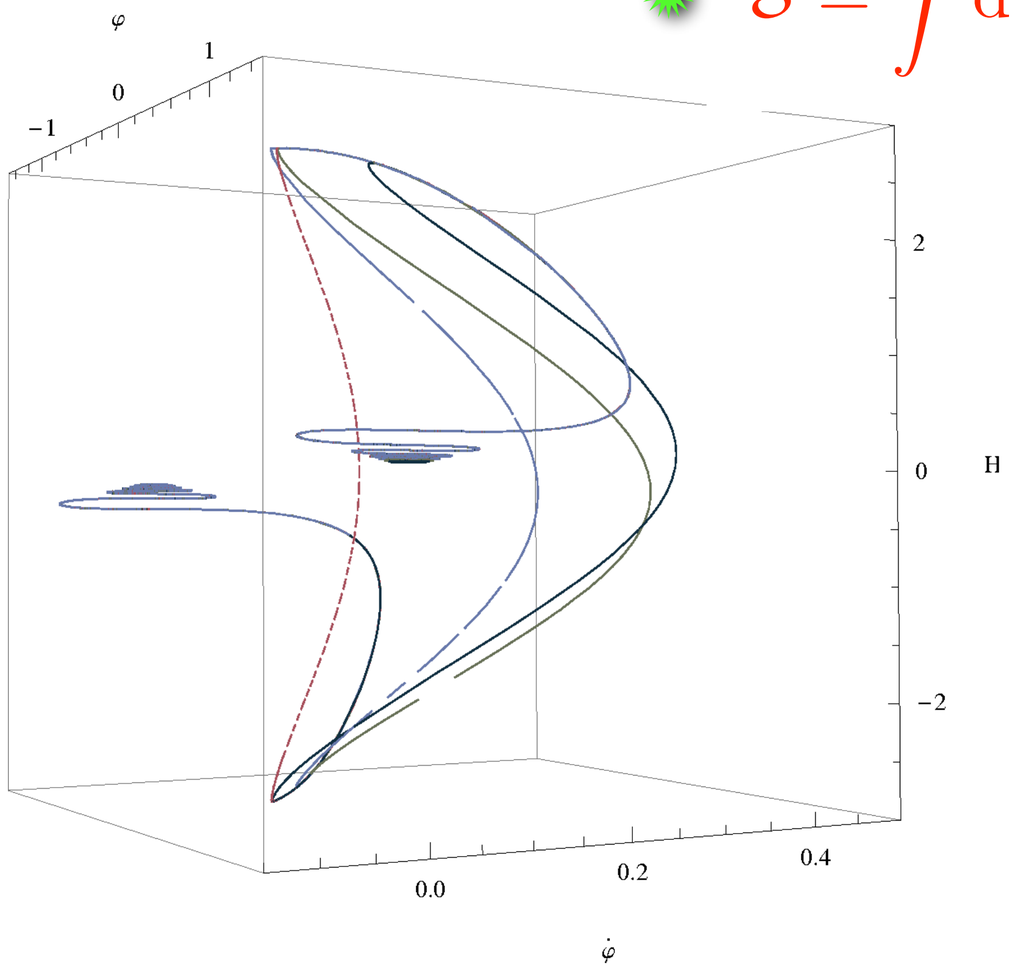
Self consistent bounce:

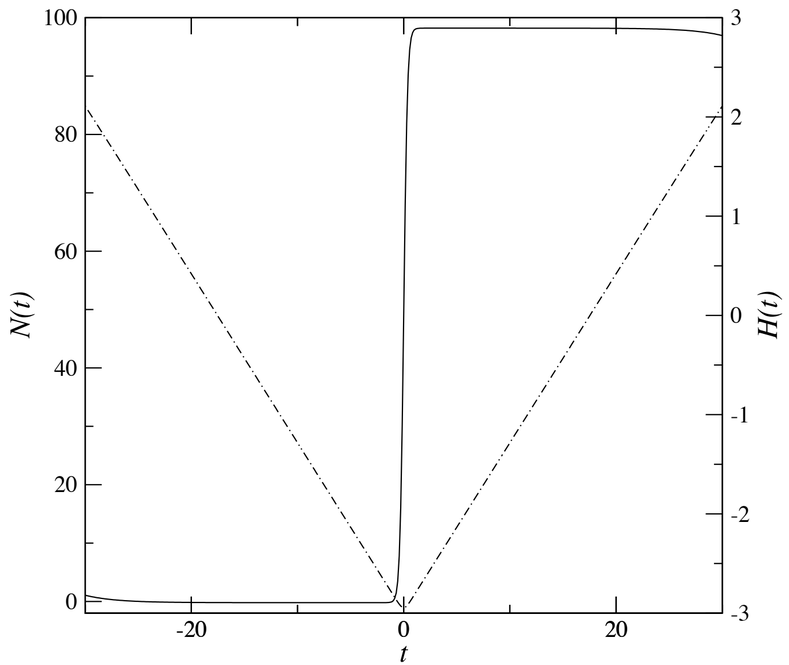
$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2\mathrm{d}\Omega^2\right)$$

→ One d.o.f. + 4 dimensions G.R.

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{6\ell_{\mathrm{Pl}}^2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V\right) - \frac{\mathcal{K}}{a^2}$$ Positive spatial curvature





F. Falciano, M. Lilley & P. P., *Phys. Rev.* **D77**, 083513 (2008)

Implementing a bounce = problem with GR!

$$\dot{H} = \frac{\mathcal{K}}{a^2} - \frac{1}{2}(\rho + P)$$

Violation of Null Energy Condition (NEC)

$$\rho + P \leq 0$$

Positive spatial curvature + scalar field

Modify GR?

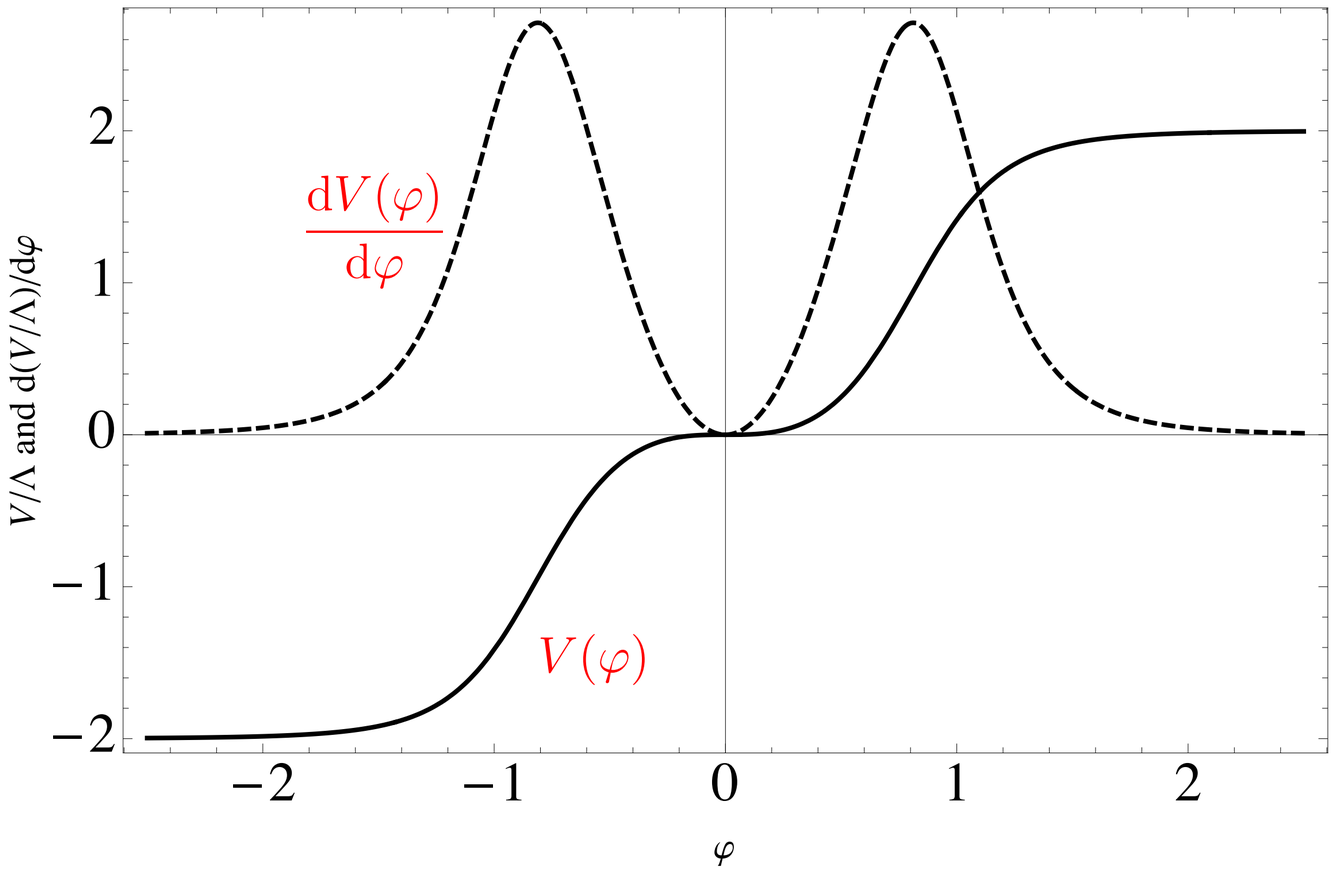
Add new terms?

K-bounce, Ghost condensates, Galileons...?

→ Modify GR to non singular theories (curvature invariants)

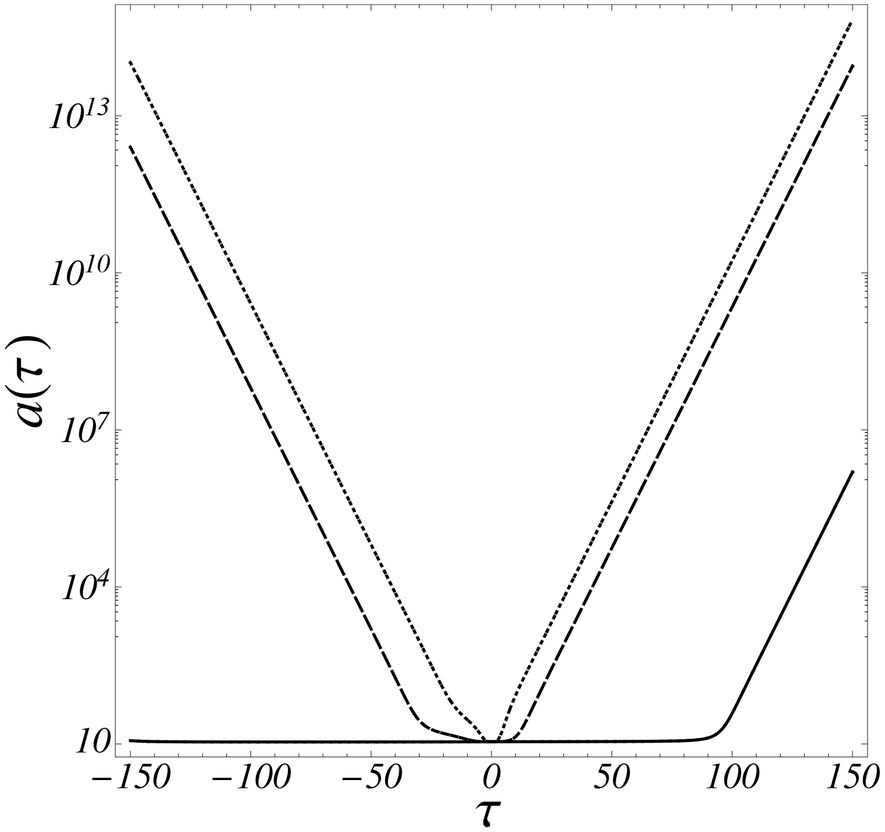
$$\mathcal{S} = \frac{1}{16\pi G_{\rm N}} \int \mathrm{d}^4x\sqrt{-g}\left[R + \sum_{i=1}^{N} \varphi_i I^{(i)} - V\left(\boldsymbol{\varphi}\right)\right] \qquad \Longrightarrow \frac{\mathrm{d}V}{\mathrm{d}\varphi} = I$$

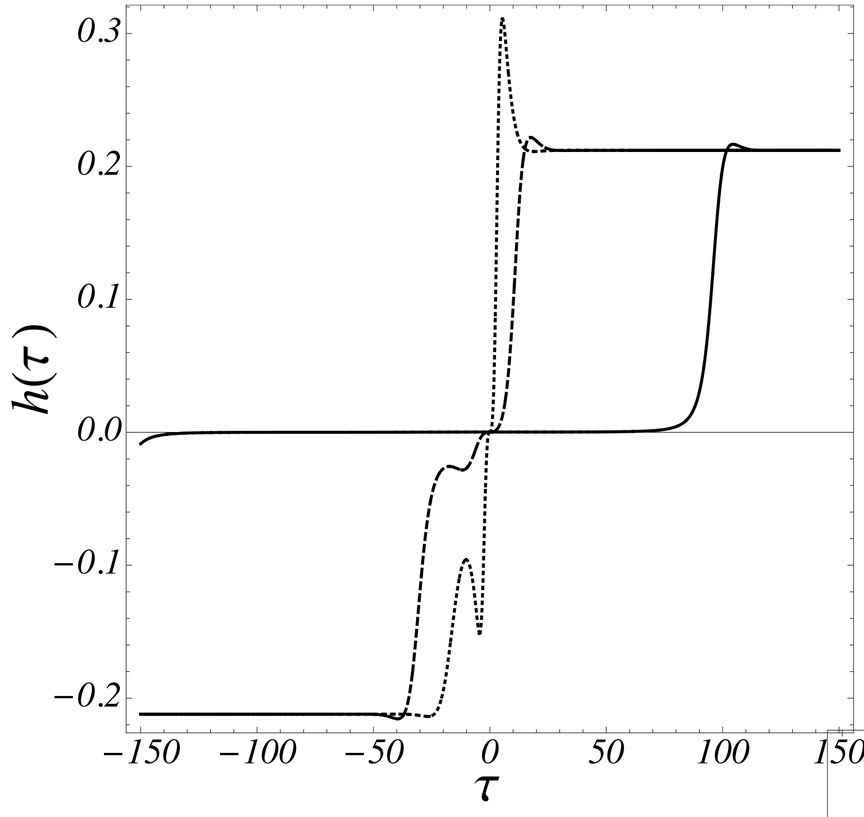
R. Brandenberger, V. F. Mukhanov and A. Sornborger, *Phys. Rev.* **D48**, 1629 (1993)

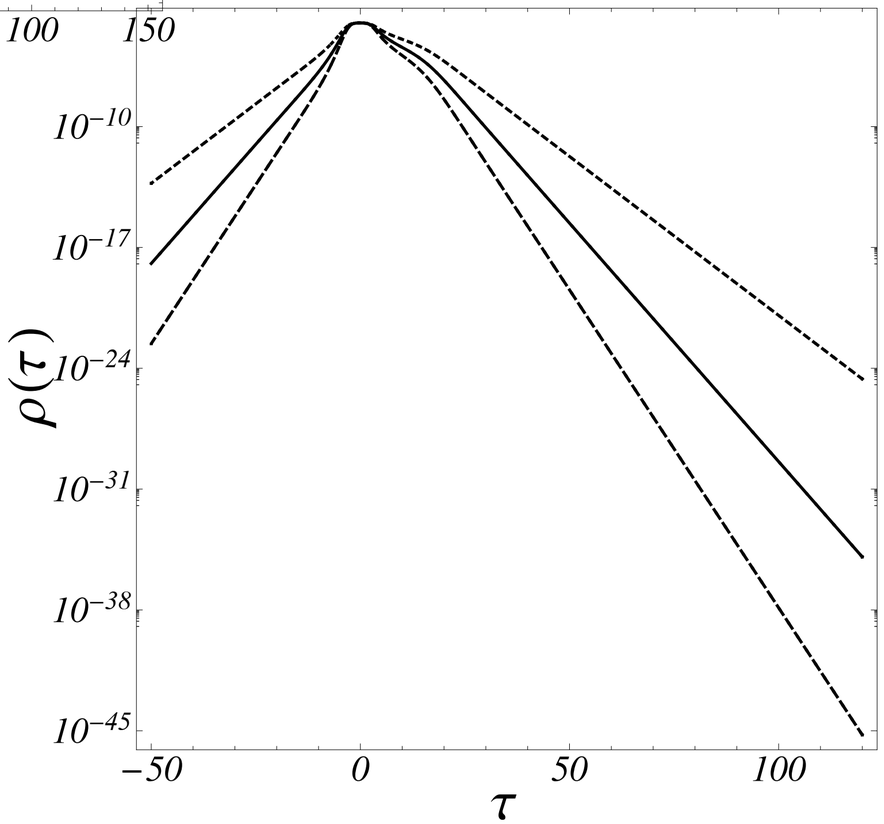


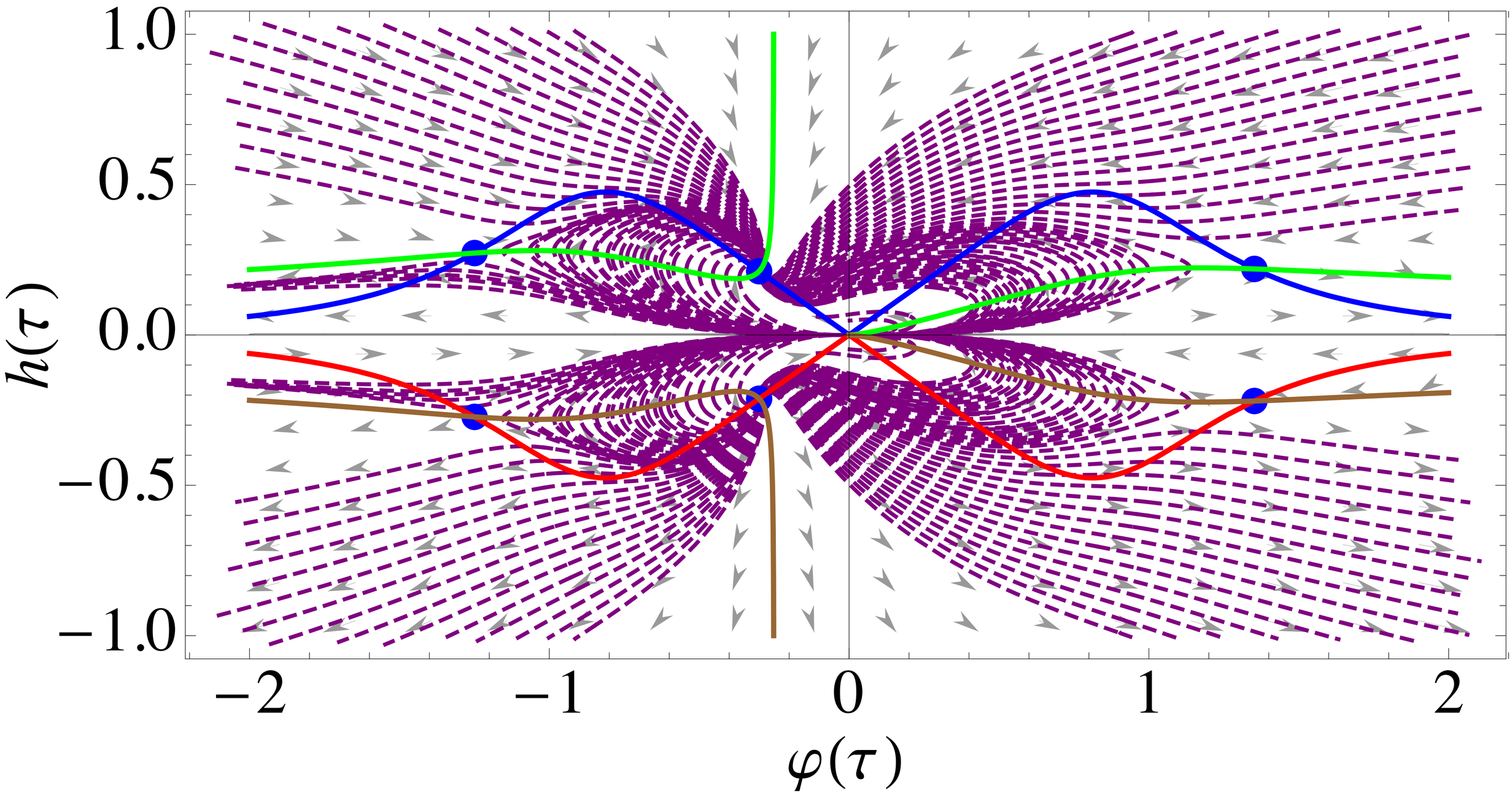
$$I = R - \sqrt{3\left(4R_{\mu\nu}R^{\mu\nu} - R^2\right)}$$

R. Abramo, P. P. & I. Yasuda, *Phys. Rev.* **D81**, 023511 (2010)


10^{13}
10^{10}
10^{7}
10^{4}
10
$a(\tau)$
−150 −100 −50 0 50 100 150
τ


0.3
0.2
0.1
0.0
−0.1
−0.2
$h(\tau)$
−150 −100 −50 0 50 100 150
τ


10^{-10}
10^{-17}
10^{-24}
10^{-31}
10^{-38}
10^{-45}
$\rho(\tau)$
−50 0 50 100
τ


1.0
0.5
0.0
−0.5
−1.0
h(τ)
−2
−1
0
1
2
φ(τ)

- *K*-bounce: $\mathcal{L} = p(X, \varphi)$

$$X \equiv \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

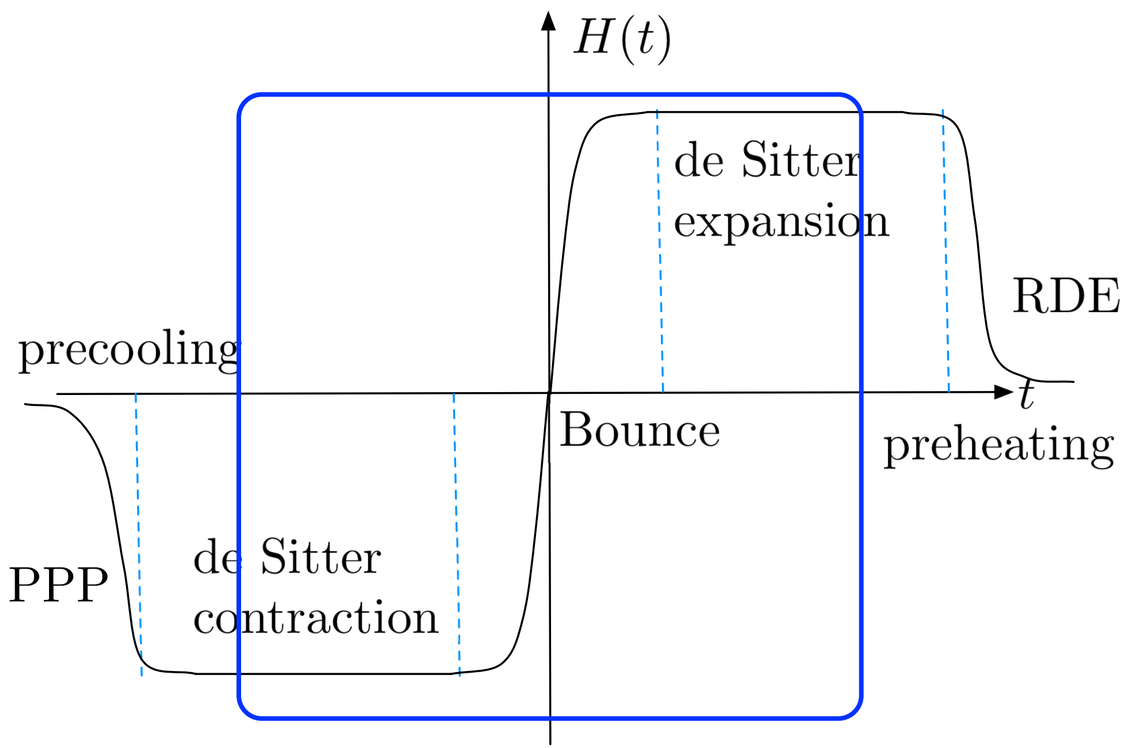
$$\Rightarrow \quad T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu - p g^{\mu\nu}$$

$$\rho \equiv 2X \frac{\partial p}{\partial X} - p$$

vanishing spatial curvature possible in 4 dimensions G.R.?

$$u_\mu \equiv \frac{\partial_\mu \varphi}{\sqrt{2X}}$$

$$\rho\left(t_{\text{bounce}}\right) = 0 \Longrightarrow p\left(t_{\text{bounce}}\right) < 0$$



Implementing a bounce = problem with GR!

$$\dot{H} = \frac{\mathcal{K}}{a^2} - \frac{1}{2}(\rho + P)$$

Violation of Null Energy Condition (NEC)

$$\rho + P \leq 0$$

Positive spatial curvature + scalar field

Modify GR?

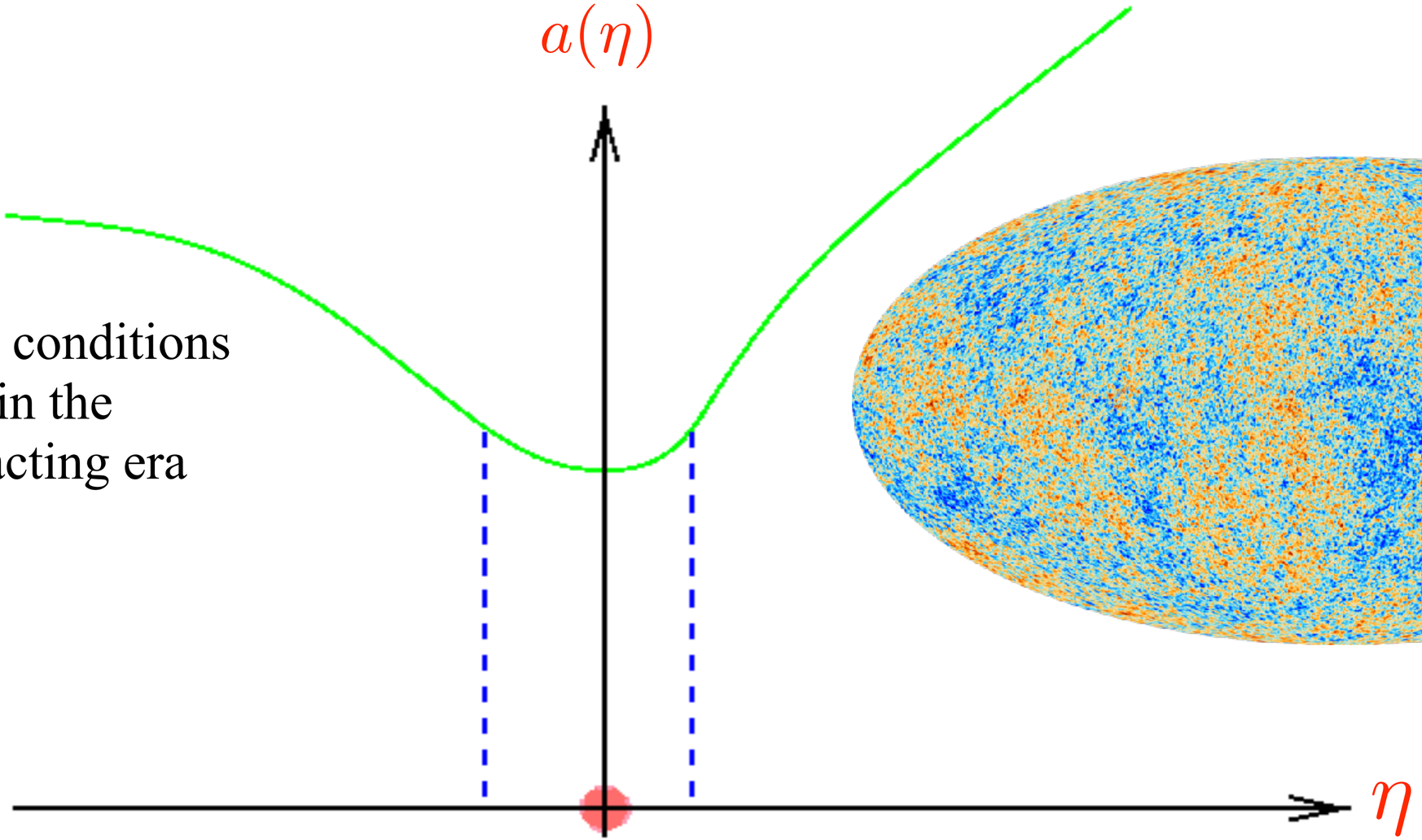
Add new terms?

K-bounce, Ghost condensates, Galileons...?

Various instabilities may arise!
(e.g. radiation for matter bounce or curvature perturbations)

Perturbations: $\mathrm{d}s^2 = a^2(\eta)\left\{(1+2\Phi)\,\mathrm{d}\eta^2 - [(1-2\Phi)\,\gamma_{ij} + h_{ij}]\,\mathrm{d}x^i\mathrm{d}x^j\right\}$

Initial conditions fixed in the contracting era

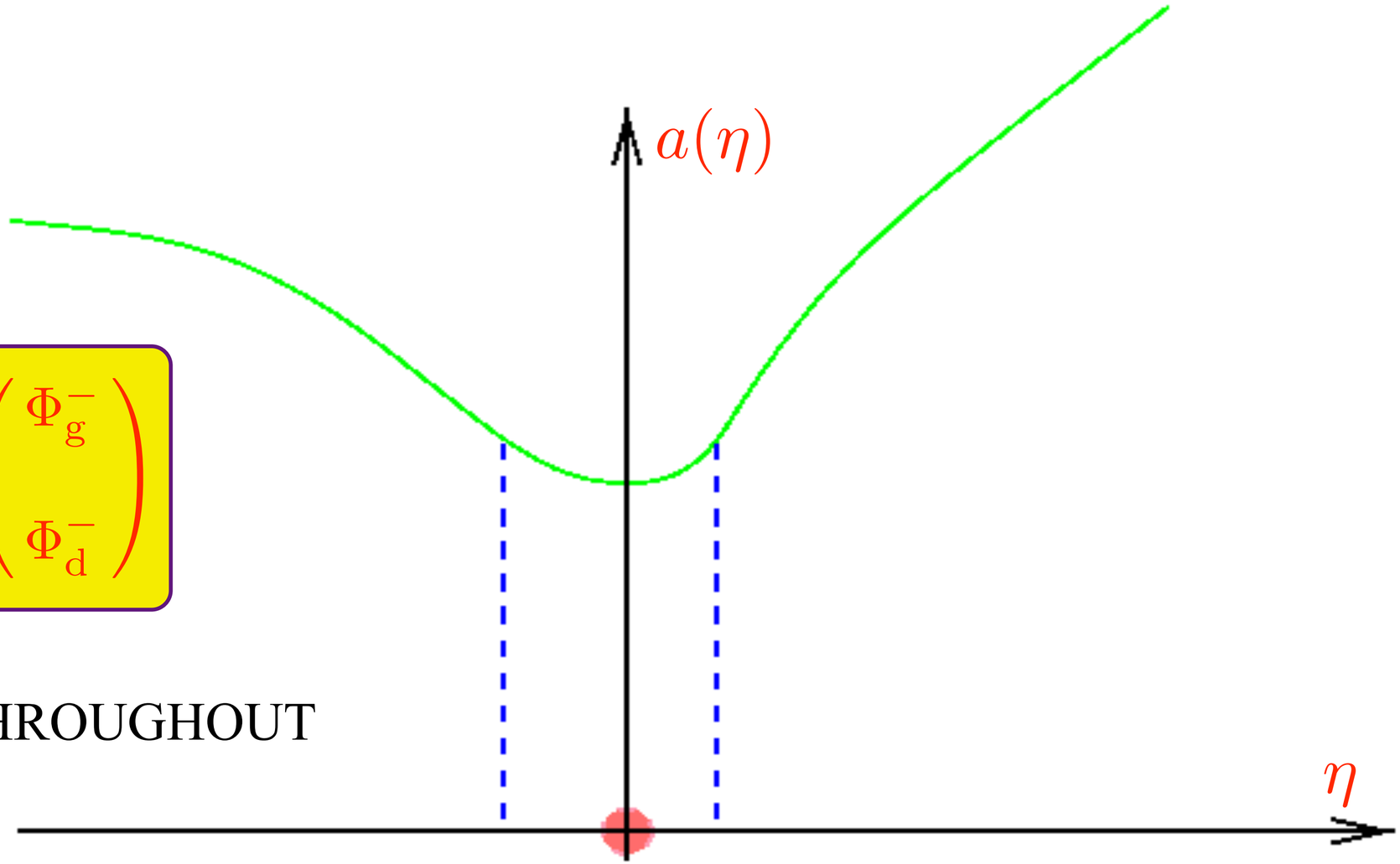




??????

Perturbations: $\mathrm{d}s^2 = a^2(\eta)\left\{(1+2\Phi)\,\mathrm{d}\eta^2 - [(1-2\Phi)\,\gamma_{ij} + h_{ij}]\,\mathrm{d}x^i\mathrm{d}x^j\right\}$

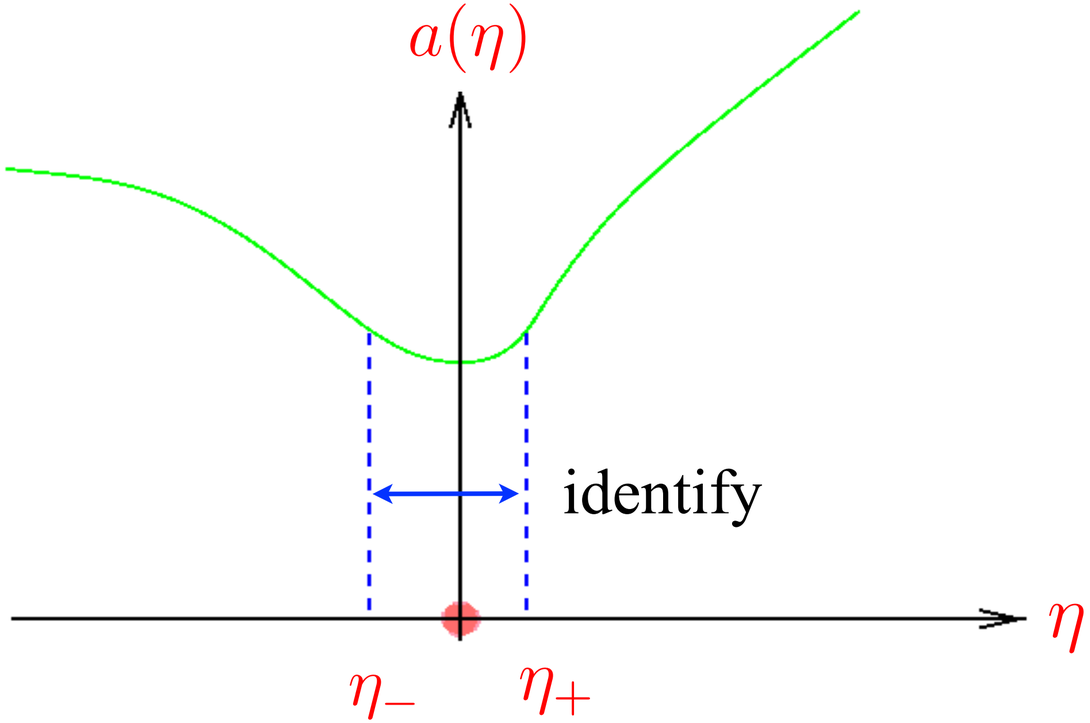
$$\begin{pmatrix} \Phi_{\rm g}^{+} \\ \\ \Phi_{\rm d}^{+} \end{pmatrix} = \mathbf{T}_{ij}(k) \begin{pmatrix} \Phi_{\rm g}^{-} \\ \\ \Phi_{\rm d}^{-} \end{pmatrix}$$



ASSUME LINEARITY THROUGHOUT

``central feature of bouncing cosmology = the bounce''…

A generic model-independent treatment of the bounce phase?

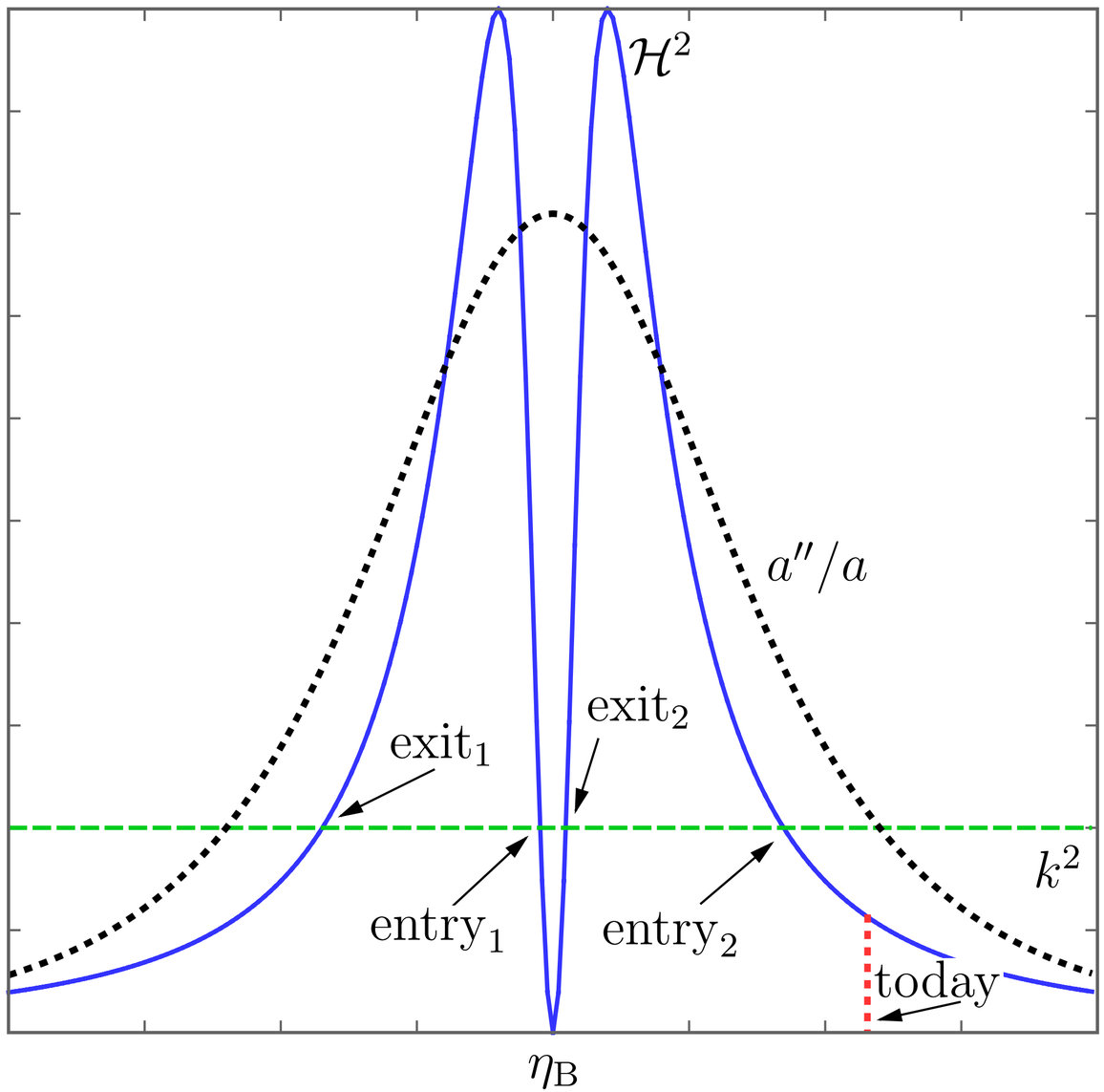


Geometric matching conditions?

Continuity of metric	$[a]_\pm = 0$	OK
Continuity of extrinsic curvature	$[H]_\pm = 0$	???
Perturbations?	$[\zeta]_\pm = 0$	???

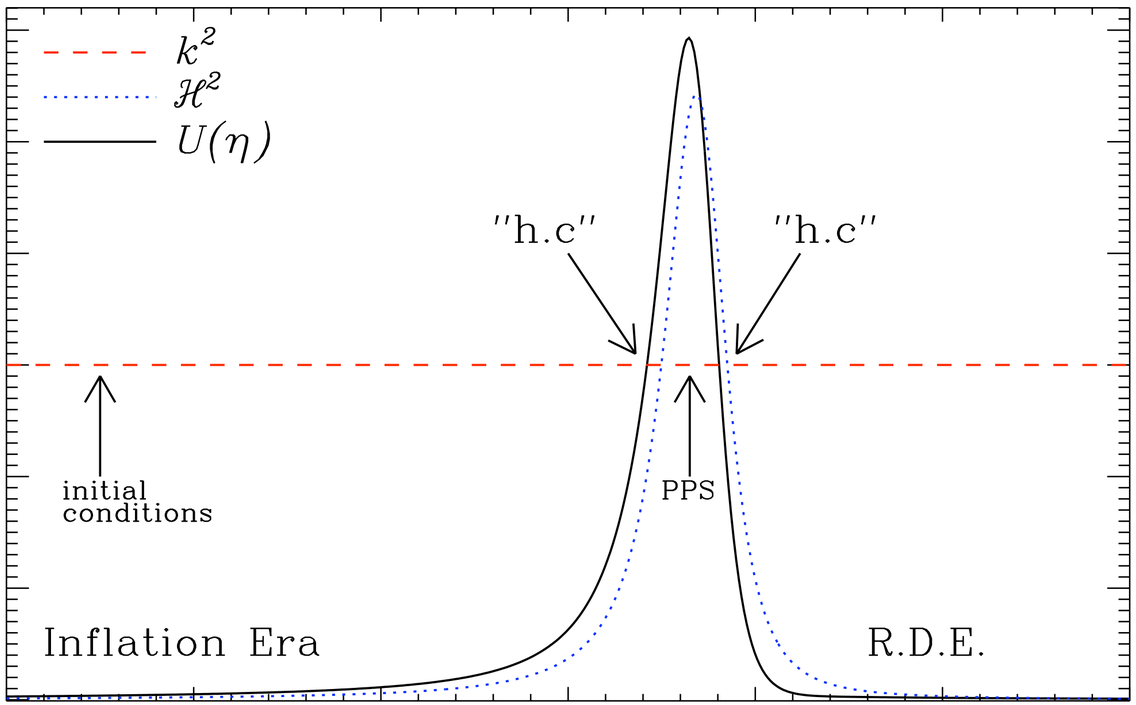
Hubble & potential: tensor example

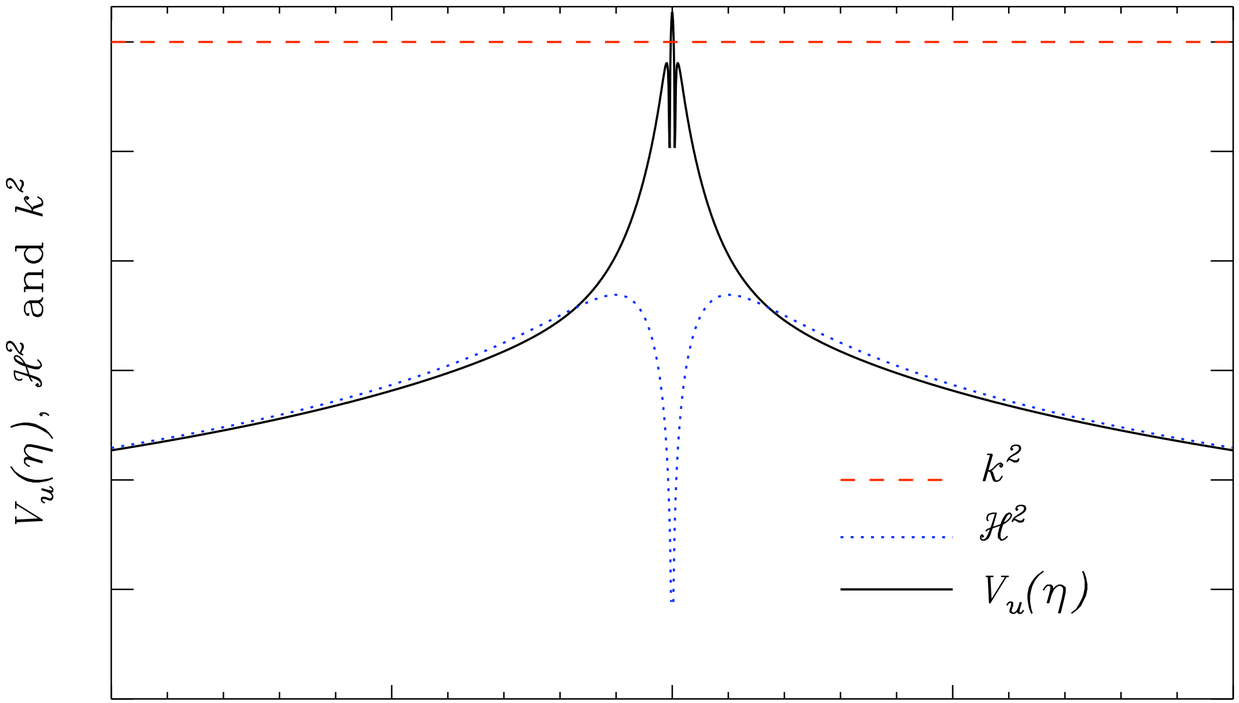
$$\mu_{\rm T}'' + \left(k^2 + \frac{a''}{a}\right)\mu_{\rm T} = 0$$



Hubble & potential: scalar example

J. Martin& PP, *PR***D68,** 103517 (2003)





$$u_{\boldsymbol{k}}'' + \left[k^2 - V_u(\eta)\right] u_{\boldsymbol{k}} = 0$$

Perturbations: $$\mathrm{d}s^2 = a^2(\eta)\left\{(1+2\Phi)\,\mathrm{d}\eta^2 - \left[(1-2\Phi)\,\gamma_{ij} + h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}$$

$$\Longleftrightarrow \qquad \Phi = \frac{3\mathcal{H}u}{2a^2\theta} \qquad \theta \equiv \frac{1}{a}\sqrt{\frac{\rho_\varphi}{\rho_\varphi + p_\varphi}\left(1 - \frac{3\mathcal{K}}{\rho_\varphi a^2}\right)}$$

$$u'' + \left[k^2 - \frac{\theta''}{\theta} - 3\mathcal{K}\left(1 - c_{\mathrm{S}}^2\right)\right]u = 0$$

$$V_u(\eta) \equiv \frac{\theta''}{\theta} + 3\mathcal{K}(1 - c_{\mathrm{S}}^2) = \frac{P_{24}(\eta)}{Q_{24}(\eta)}\,,$$

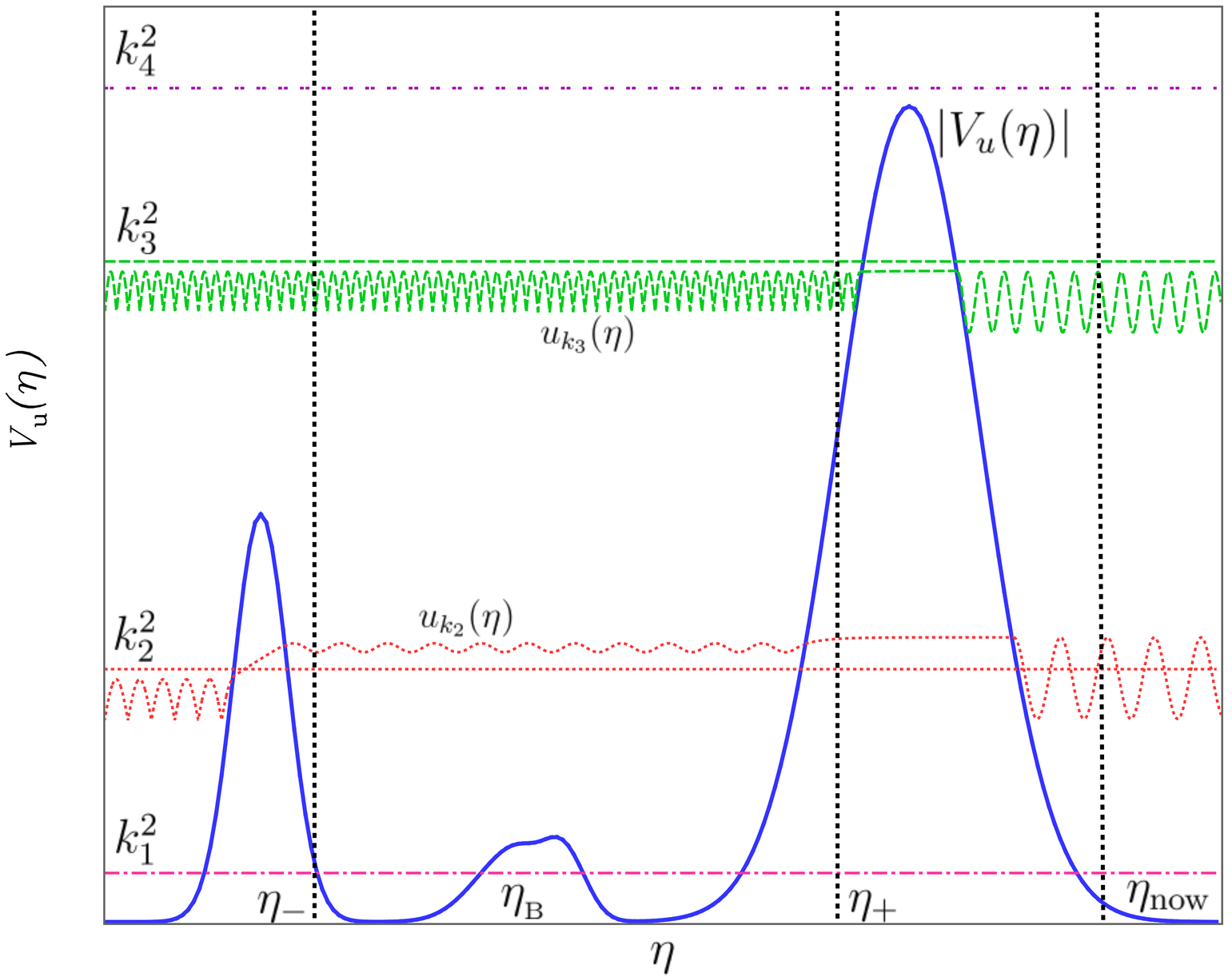
Non trivial transfer matrix

$$\boldsymbol{T}_{ij}(k) = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix}$$

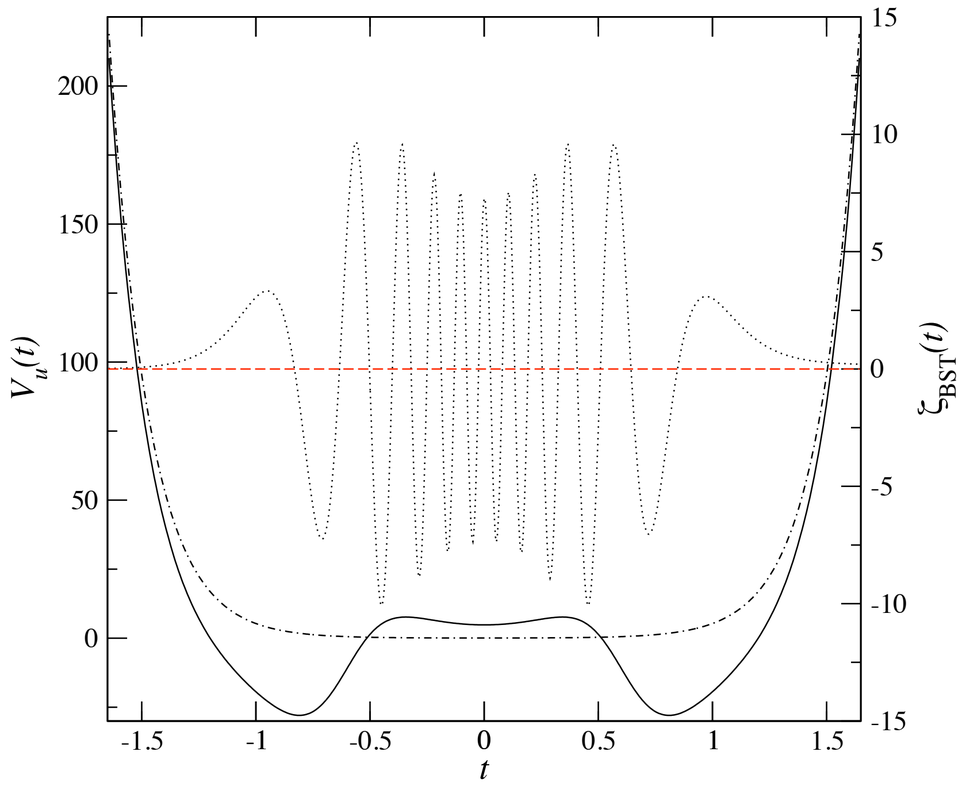
“Causality” argument... J. Martin & PP, *Phys. Rev. Lett.* **92,** 061301 (2004)

Resulting spectrum: very much model dependent…

J. Martin& PP, *PR***D68,** 103517 (2003)

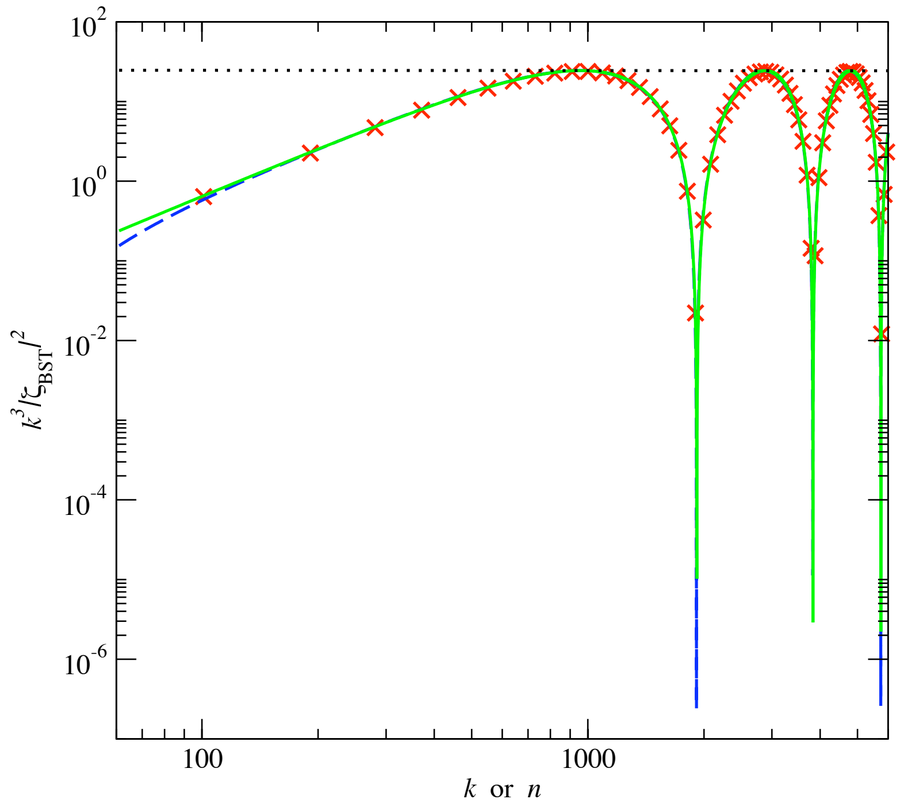


$$\mathcal{P}_\zeta = \mathcal{A} k^{n_{\rm S}-1} \cos^2\left(\omega \frac{k_{\rm ph}}{k_\star} + \psi\right)$$



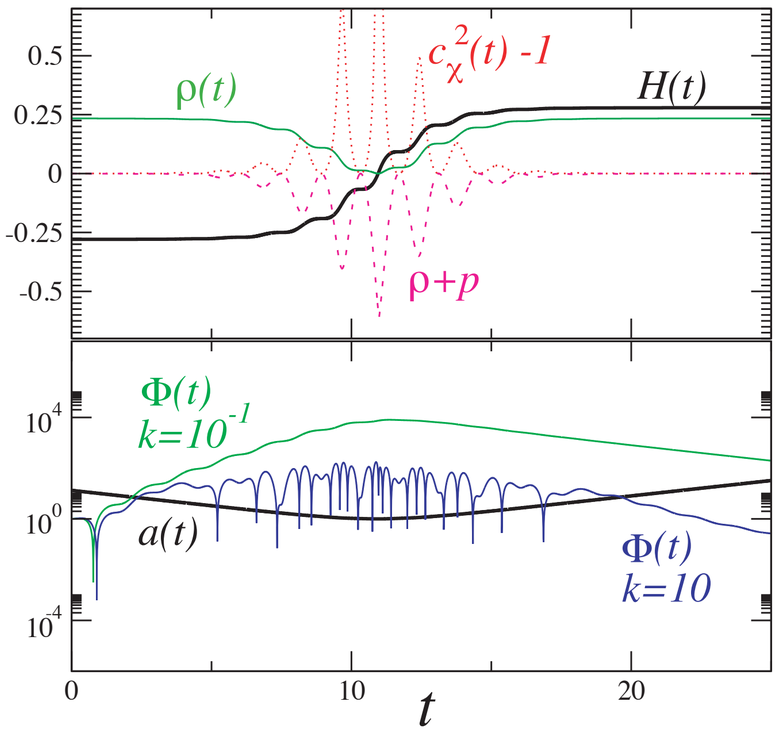
Different parameters

primordial spectrum

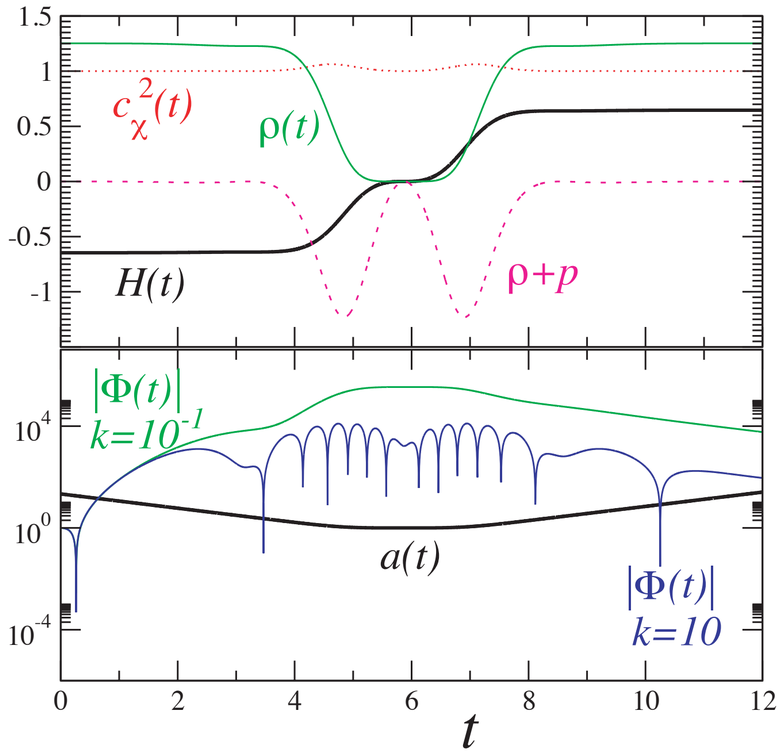


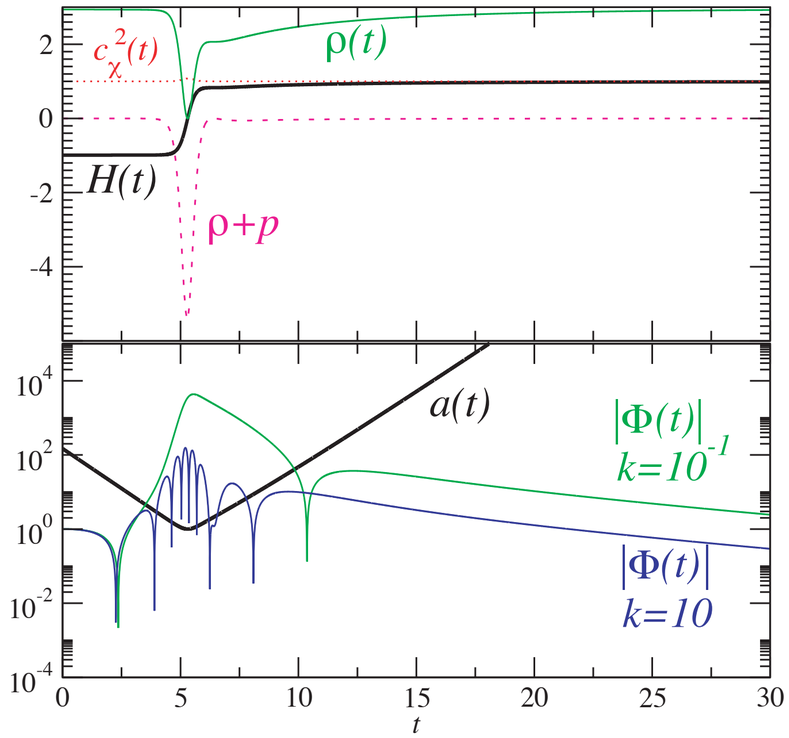
Perturbations in the *K*-bounce

$$p = p_0 + p_X (X - X_0) + p_\varphi \varphi + p_{X\varphi} \varphi (X - X0) + \frac{1}{2} p_{XX} (X - X_0)^2 + \frac{1}{2} p_{\varphi\varphi} \varphi^2 + \cdots$$



Slow

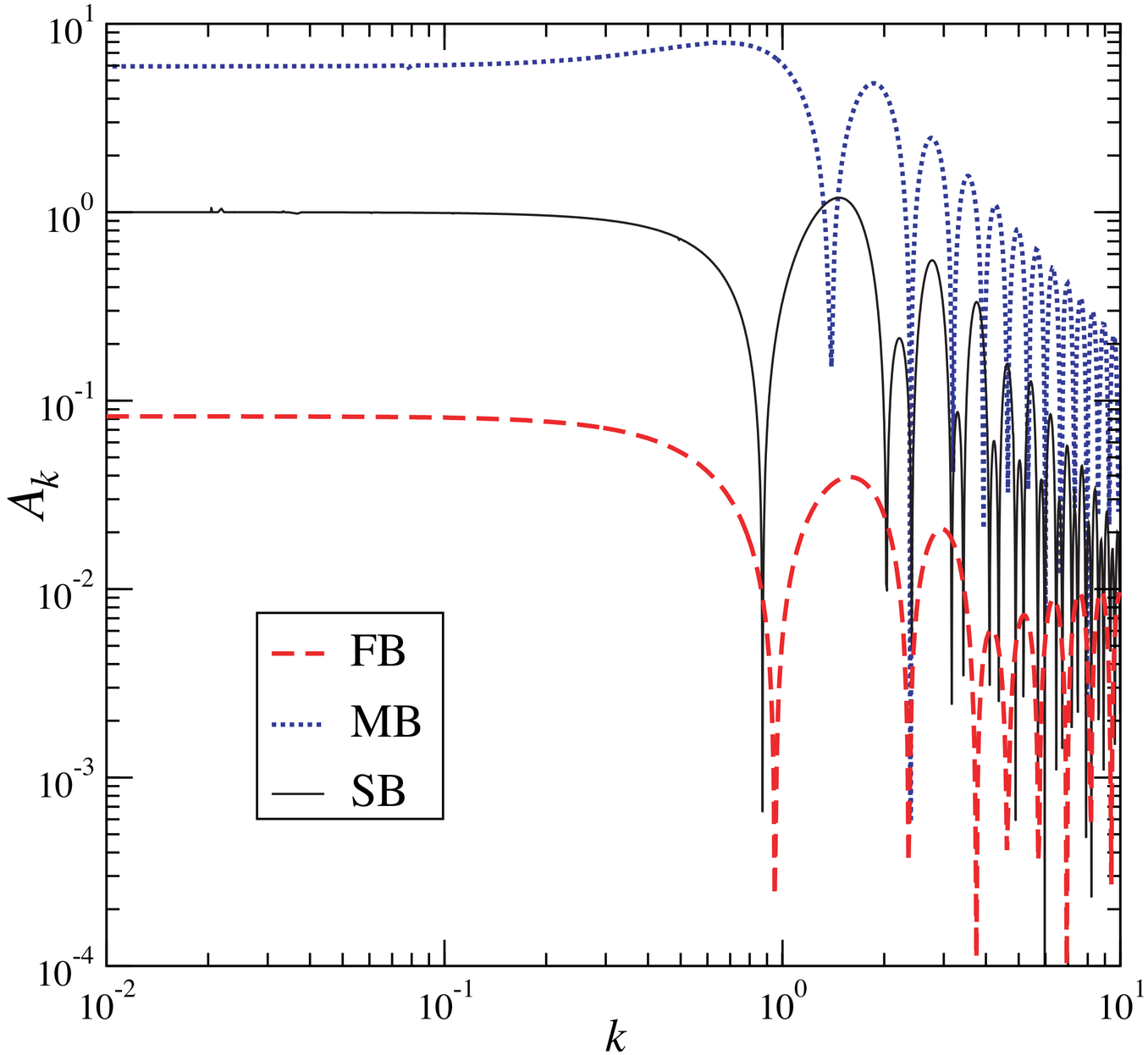




Fast

Oscillations + ζ conserved

R. Abramo & P. P., *JCAP* **09**, 001 (2007)



k-mode mixing ...

A few problems…

spectral index $n_s < 1$

Non gaussianities: *phenomenological description* $S = -\int \mathrm{d}^4x\sqrt{-g}\left[R + (\partial\phi)^2 + V(\phi)\right]$

$$a(\eta) = a_0\left[1 + \frac{1}{2}\left(\frac{\eta}{\eta_c}\right)^2 + \lambda_3\left(\frac{\eta}{\eta_c}\right)^3 + \frac{5}{24}\left(1+\lambda_4\right)\left(\frac{\eta}{\eta_c}\right)^4\right] \quad + \textit{scalar field}$$

$$\begin{cases} \dfrac{\phi'^2}{a^2} = \dfrac{2}{a^2}\left(\mathcal{H}^2 - \mathcal{H}' + \mathcal{K}\right) \\ -\dfrac{6}{a^2}\mathcal{H}' = -2V(\phi)\left[1 - \dfrac{\phi'^2}{a^2V(\phi)}\right] \end{cases} \qquad \phi'' + 2\mathcal{H}\phi' + a^2V_{,\phi} = 0$$

$$\varepsilon_V = \frac{V_0'}{V_0} \qquad \textit{"slow-roll"}$$

$$\eta_V = \frac{V_0''}{V_0}$$

$$\Upsilon \equiv \phi_0'^2/2 \longrightarrow \eta_c^2 = \frac{1}{1-\Upsilon}$$

complete set of parameters

perturbed metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2\left(-\mathrm{e}^{2\Phi}\mathrm{d}\eta^2 + \mathrm{e}^{-2\Psi}\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)$

X. Gao, M. Lilley & P. P., *JCAP* **07**, 010 (2014)

perturbations up to 2nd order $X\left(\boldsymbol{x},\eta\right)=X_{(1)}\left(\boldsymbol{x},\eta\right)+\frac{1}{2}X_{(2)}\left(\boldsymbol{x},\eta\right)+\cdots$ $\qquad\mathcal{D}\,\Psi_{(i)}=\mathcal{S}\left[\Psi_{(i-1)}\right]$

first order $\Psi_{(1)}''+F\left(\eta\right)\Psi_{(1)}'-\bar{\nabla}^{2}\Psi_{(1)}+W\left(\eta\right)\Psi_{(1)}=0$

$2\left(\mathcal{H}-\frac{\bar{\phi}''}{\bar{\phi}'}\right)$ $\qquad 2\left(\mathcal{H}'-\mathcal{H}\frac{\bar{\phi}''}{\bar{\phi}'}-2\mathcal{K}\right)$

positive spatial curvature: decomposition on the 3-sphere

$$\Psi_{(1)}(\boldsymbol{x},\eta)=\sum_{\ell mn}\Psi_{\ell mn}(\eta)Q_{\ell mn}(\chi,\theta,\varphi)$$

$Q_{\ell mn}(\chi,\theta,\varphi)=R_{\ell n}(\chi)Y_{\ell m}(\theta,\varphi)$ *hyperspherical harmonics*

Legendre

$$R_{n\ell}\left(\chi\right)=\sqrt{\frac{\left(n+1\right)\left(n+\ell+1\right)!}{\left(n-\ell\right)!}}\sqrt{\frac{\mathcal{K}}{f_{\mathcal{K}}\left(\chi\right)}}P_{n+\frac{1}{2}}^{-\ell-\frac{1}{2}}\left[\cos\left(\sqrt{\mathcal{K}}\chi\right)\right]$$

effect of the bounce itself: initial conditions = classical gaussian fields

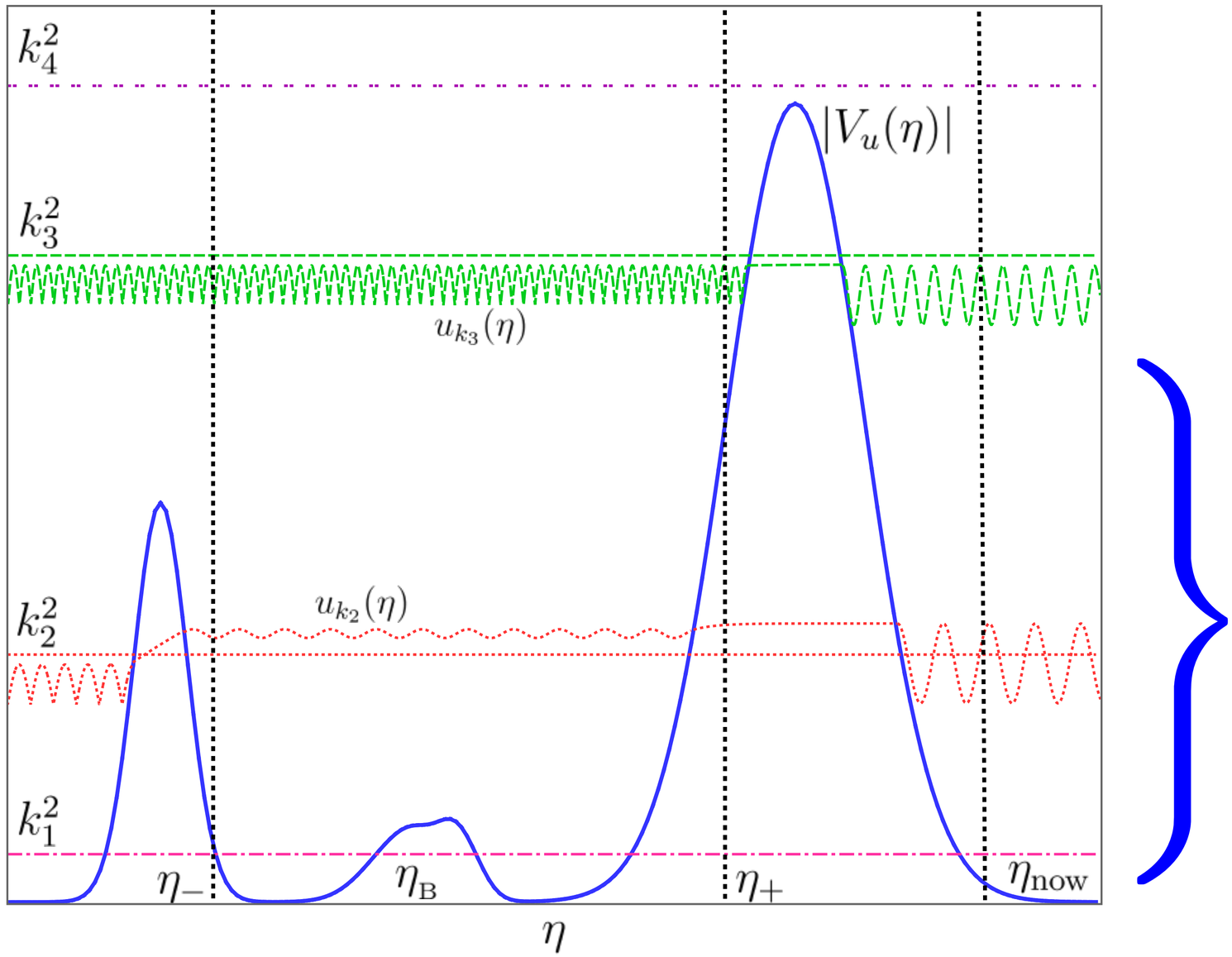
$$\begin{bmatrix}\Psi_{(1)}\left(\boldsymbol{k},\eta_{-}\right)\\ \Psi_{(1)}'\left(\boldsymbol{k},\eta_{-}\right)\end{bmatrix}\equiv\begin{bmatrix}\hat{x}_{1}\left(\boldsymbol{k}\right)\\ \hat{x}_{2}\left(\boldsymbol{k}\right)\end{bmatrix}$$

spectra

$$\left\langle \hat{x}_{i}\left(\boldsymbol{k}\right)\hat{x}_{j}\left(\boldsymbol{k}'\right)\right\rangle \equiv\delta_{\boldsymbol{k},\boldsymbol{k}'}P_{ij}\left(k\right)$$

$\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$

$u \propto a\Psi_{(1)}/\phi' \longrightarrow u_{\boldsymbol{k}}'' + \left[k^2 - V_u(\eta)\right] u_{\boldsymbol{k}} = 0$



2nd order

$$\Psi''_{(2)} + 2\left(\mathcal{H} - \frac{\bar{\phi}''}{\bar{\phi}'}\right)\Psi'_{(2)} - \bar{\nabla}^2\Psi_{(2)} + 2\left(\mathcal{H}' - 2\mathcal{K} - \mathcal{H}\frac{\bar{\phi}''}{\bar{\phi}'}\right)\Psi_{(2)} = \mathcal{S}_{(2)}$$

$$\mathcal{S}_{(2)} = 4\left(2\mathcal{H}^2 - \mathcal{H}' + 2\mathcal{H}\frac{\bar{\phi}''}{\bar{\phi}'} + 6\mathcal{K}\right)\Psi^2_{(1)} + 8\Psi'^2_{(1)} + 8\left(2\mathcal{H} + \frac{\bar{\phi}''}{\bar{\phi}'}\right)\Psi_{(1)}\Psi'_{(1)} + 8\Psi_{(1)}\bar{\nabla}^2\Psi_{(1)} - \frac{4}{3}\left(\bar{\nabla}_i\Psi_{(1)}\right)^2$$
$$- \left[2\left(2\mathcal{H}^2 - \mathcal{H}'\right) - \frac{\bar{\phi}'''}{\bar{\phi}'}\right]\phi^2_{(1)} - \frac{2}{3}\left(\bar{\nabla}_i\phi_{(1)}\right)^2 - 2\left(\frac{\bar{\phi}''}{\bar{\phi}'} + 2\mathcal{H}\right)\bar{\nabla}^{-2}\bar{\nabla}^i\left(2\Psi'_{(1)}\bar{\nabla}_i\Psi_{(1)} + \phi'_{(1)}\bar{\nabla}_i\phi_{(1)}\right)$$
$$+ \left[2\left(\mathcal{H}' - \mathcal{H}\frac{\bar{\phi}''}{\bar{\phi}'}\right) + \frac{1}{3}\bar{\nabla}^2\right]\left[2F\left(\Psi_{(1)}\right) + F\left(\phi_{(1)}\right)\right] + \mathcal{H}\left[2F\left(\Psi_{(1)}\right) + F\left(\phi_{(1)}\right)\right]'$$

$$F(X) = \left(\bar{\nabla}^2\bar{\nabla}^2 + 3\mathcal{K}\bar{\nabla}^2\right)^{-1}\left[\bar{\nabla}_i\bar{\nabla}^j\left(3\bar{\nabla}^i X\bar{\nabla}_j X - \delta^i_j\left(\bar{\nabla}_k X\right)^2\right)\right]$$

general solution

$$\mathcal{S}_{(2)}(\boldsymbol{k},\eta) = \sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}\mathcal{G}_{\boldsymbol{k},\boldsymbol{p}_1,\boldsymbol{p}_2}\,\tilde{\Sigma}_{ij}(k,p_1,p_2;\eta)\,\hat{a}_i(\boldsymbol{p}_1)\,\hat{a}_j(\boldsymbol{p}_2)$$

$$\Psi_{(2)}(\boldsymbol{k},\eta) = \Psi^{(0)}_{(2)}(\boldsymbol{k},\eta) + \sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}\mathcal{G}_{\boldsymbol{k},\boldsymbol{p}_1,\boldsymbol{p}_2}\,\Pi_{ij}(k,p_1,p_2;\eta)\,\hat{x}_i(\boldsymbol{p}_1)\,\hat{x}_j(\boldsymbol{p}_2)$$

$$\Pi_{ij}(k,p_1,p_2;\eta) \equiv \int_{\eta_-}^{\eta}\mathrm{d}\eta' G\left(k,\eta,\eta'\right)\Sigma_{ij}\left(k,p_1,p_2;\eta'\right)$$

Green

Bispectrum

$$\langle \Psi(\boldsymbol{k}_1,\eta)\,\Psi(\boldsymbol{k}_2,\eta)\,\Psi(\boldsymbol{k}_3,\eta)\rangle \equiv \frac{1}{2}\mathcal{G}_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{B}_\Psi(k_1,k_2,k_3;\eta)$$

$$\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)$$

$$\mathcal{B}_\Psi(k_1,k_2,k_3) = \frac{6}{5} f_{\rm NL}\left[P_{\Psi\Psi}(k_1)P_{\Psi\Psi}(k_2) + P_{\Psi\Psi}(k_2)P_{\Psi\Psi}(k_3) + P_{\Psi\Psi}(k_3)P_{\Psi\Psi}(k_1)\right]$$

$$7P_{\Psi\Psi}(k) + 11P_{\Psi\Psi'}(k) + 4P_{\Psi'\Psi'}(k)$$

$$f_{\rm NL} = -\frac{5(k_1+k_2+k_3)}{3\Upsilon K_3(k_1,k_2,k_3)}\left(\left[\prod_{\sigma(i,j,\ell)}(k_i+k_j-k_\ell)\right]\left\{\sum_{\sigma(i,j,\ell)}\frac{K_1(k_i)K_1(k_j)}{k_\ell^2} - 4\left[\frac{K_1(k_i)K_2(k_j)}{k_j^2k_\ell^2}+\frac{K_1(k_j)K_2(k_i)}{k_i^2k_\ell^2}\right]\right\}\right.$$

$$\left. - \sum_{\sigma(i,j,\ell)}\left[\frac{7}{3}+\frac{2}{3}\left(\frac{k_i^2+k_j^2}{k_\ell^2}\right) - 3\left(\frac{k_i^2-k_j^2}{k_\ell^2}\right)^2\right]K_1(k_i)K_1(k_j)\right) + \cdots,$$

$$6P_{\Psi\Psi}(k) + 7P_{\Psi\Psi'}(k) + 2P_{\Psi'\Psi'}(k)$$

$$81\sum_{\sigma(i,j)}P_{\Psi\Psi}(k_i)P_{\Psi\Psi}(k_j) + 108\sum_{\sigma(i,j)}P_{\Psi\Psi}(k_i)P_{\Psi\Psi'}(k_j) + 36\sum_{\sigma(i,j)}P_{\Psi\Psi}(k_i)P_{\Psi'\Psi'}(k_j) +$$

$$144\sum_{\sigma(i,j)}P_{\Psi\Psi'}(k_i)P_{\Psi\Psi'}(k_j) + 48\sum_{\sigma(i,j)}P_{\Psi\Psi'}(k_i)P_{\Psi'\Psi'}(k_j) + 16\sum_{\sigma(i,j)}P_{\Psi'\Psi'}(k_i)P_{\Psi'\Psi'}(k_j)$$

equilateral $k_1 = k_2 = k_3 = k$

squeezed $k_i = k_j = k \;\; \& \;\; k_\ell = p \ll k$

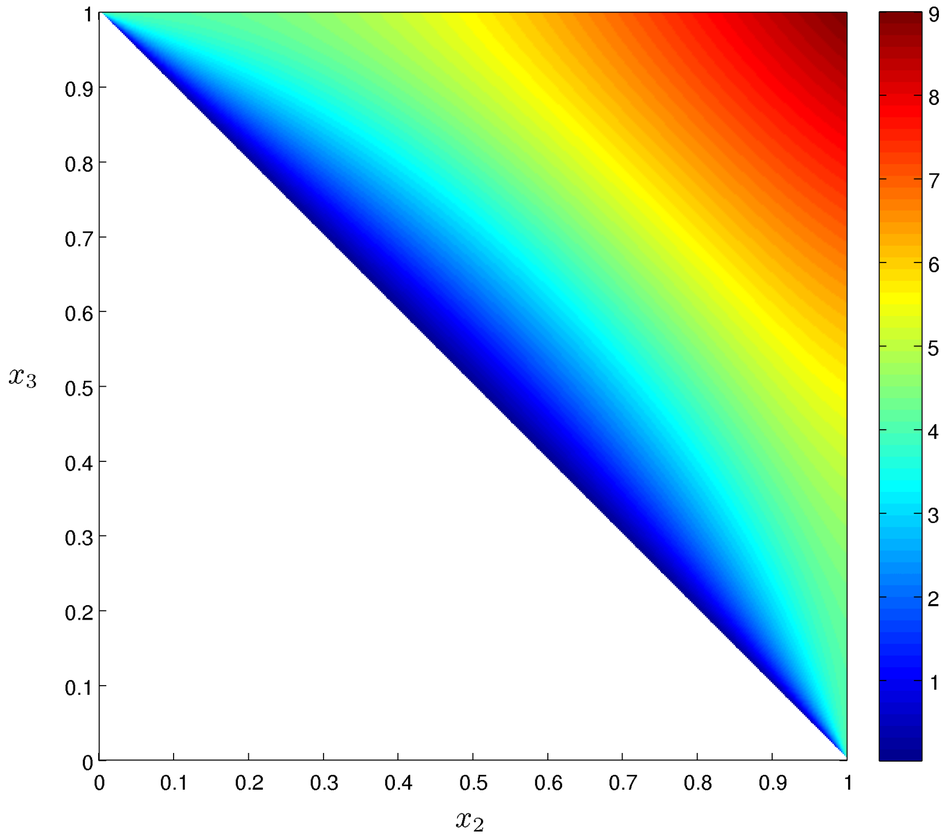
folded $k_2 = k_3 = \frac{1}{2} k_1$

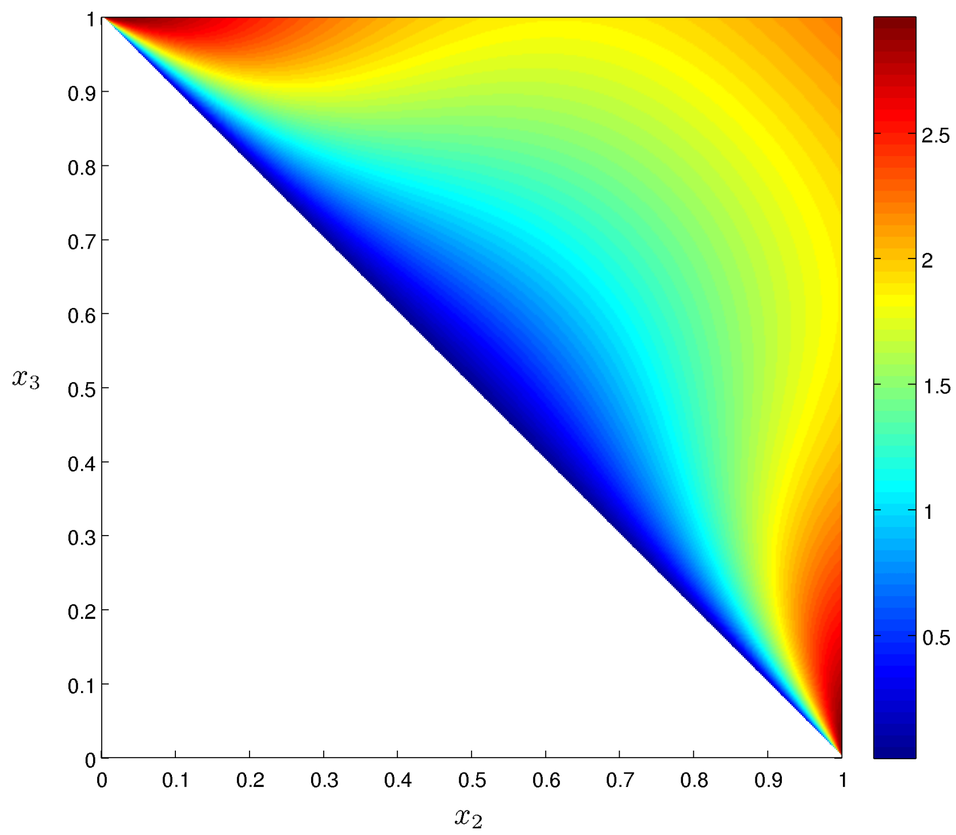
$$f_{\rm NL}^{\rm equi} = -\frac{15k^2}{\Upsilon} \frac{K_1^2(k)}{K_3(k,k,k)}$$

$$f_{\rm NL}^{\rm sq} = -\frac{20k^2}{3\Upsilon} \frac{K_1^2(k) + K_1(k)K_1(p)}{K_3(k,k,p)}$$

$$f_{\rm NL}^{\rm fold} = \frac{40}{9\Upsilon} \frac{K_1(k)\,[K_1(k) - 16K_1(2k)]}{K_3(k,k,2k)}$$

$$f_{\rm NL} \;\propto\; (k_1^2/\Upsilon) \times \mathcal{S}(x_2, x_3)$$





$10^{-2} h^{-1}\,{\rm Mpc} \leq k_{\rm phys}^{-1} \leq 10^{3} h^{-1}\,{\rm Mpc}$ → $10^2 \lesssim k \lesssim 10^8$ *with* $\Omega_{\mathcal{K}} \leq 10^{-2}$

X. Gao, M. Lilley & P. P., *1406.4119*

The problem with contraction: BKL/shear instability

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\sum_i \mathrm{e}^{2\theta_i(t)}\sigma^i\sigma^i$$

Ricci flat:

$$\sigma^i = \mathrm{d}x^i$$

$$\sum_i \theta_i = 0$$

Average scale factor

$$\frac{\dot{a}}{a}$$ Mean Hubble parameter

$$H_i \equiv \frac{1}{a\mathrm{e}^{\theta_i}}\frac{\mathrm{d}}{\mathrm{d}t}\left(a\mathrm{e}^{\theta_i}\right) = H + \dot{\theta}_i$$

Friedman equations

$$H^2 = \frac{\rho_{\mathrm{T}}}{3M_{\mathrm{Pl}}^2} + \frac{1}{6}\sum_i \dot{\theta}_i^2$$

$$\dot{H} = -\frac{\rho_{\mathrm{T}} + p_{\mathrm{T}}}{2M_{\mathrm{Pl}}^2} - \frac{1}{2}\sum_i \dot{\theta}_i^2$$

$$\ddot{\theta}_i + 3H\dot{\theta}_i = 0$$

$$\rho_{\mathrm{shear}} \propto a^{-6}$$

slow contraction solution:

$$w_{\rm ekp} \gg 1 \quad \Longrightarrow \rho_{\rm ekp} \propto a^{-3\left(1+w_{\rm ekp}\right)} \gg a^{-6} \text{ when } \quad a \to 0$$

Problem: regular bounce ➡ $\exists$ phase with $w_{\rm bounce} < -1$

So finally...

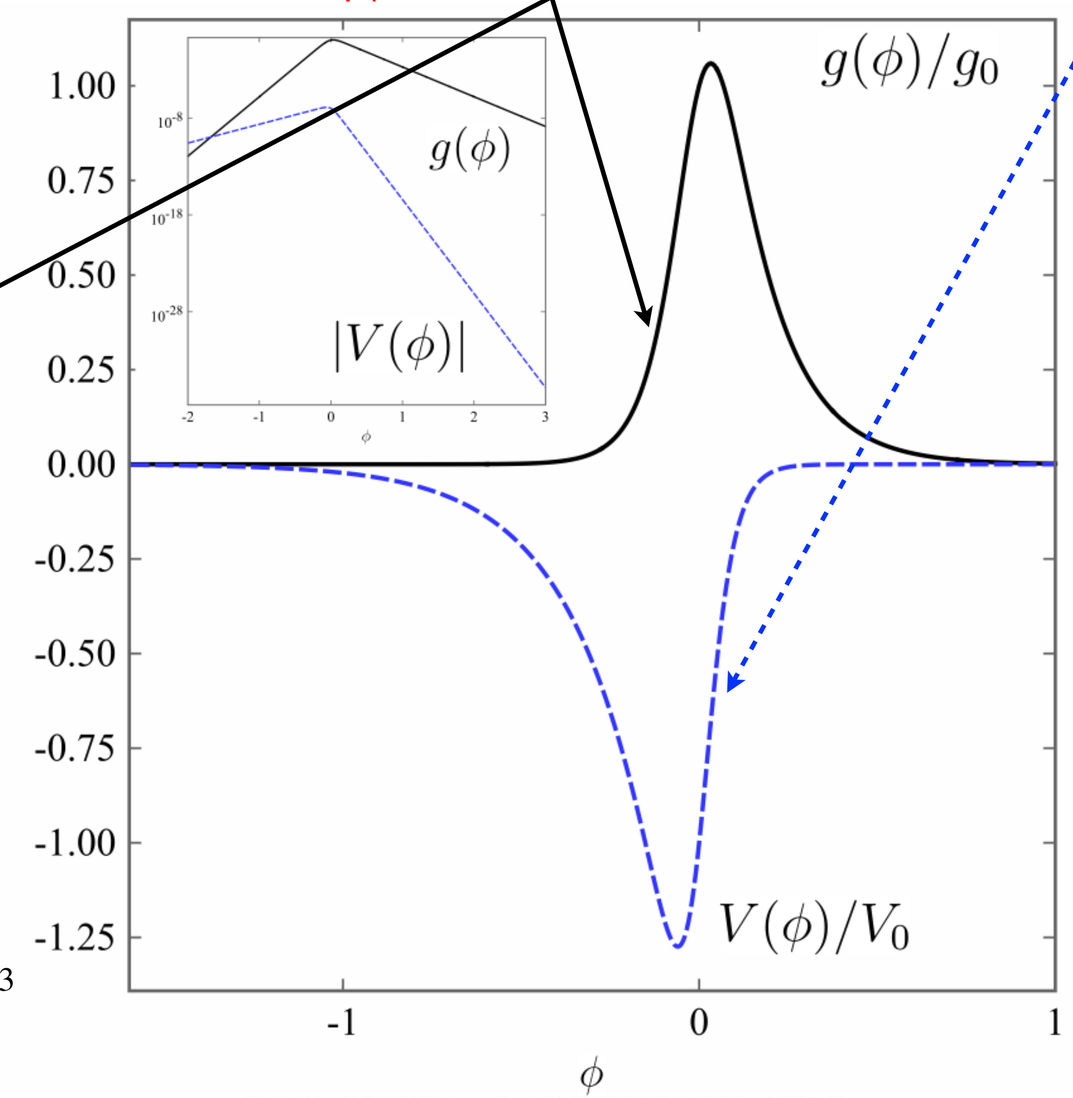
$$\rho_{\rm Shear} \equiv \frac{M_{\rm Pl}^2}{2} \sum_i \dot{\theta}_i^2 \propto a^{-6} \gg \rho_{\rm Fluid}$$

➡ Singularity!

A nonsingular bounce model: ghost condensate & Galileon

$$\mathcal{L}\left[\phi\left(x\right)\right] = K(\phi, X) + G(\phi, X)\Box\phi \quad \text{with kinetic term} \quad X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \qquad + \text{Fluid}$$

Specific choices: $K(\phi, X) = M_{\rm Pl}^2\left[1 - g(\phi)\right]X + \beta X^2 - V(\phi)$ $\qquad G(X) = \gamma X$



$$g(\phi) = \frac{2g_0}{\mathrm{e}^{-\sqrt{\frac{2}{p}}\phi} + \mathrm{e}^{b_g\sqrt{\frac{2}{p}}\phi}}$$

$$V(\phi) = -\frac{2V_0}{\mathrm{e}^{-\sqrt{\frac{2}{q}}\phi} + \mathrm{e}^{b_V\sqrt{\frac{2}{q}}\phi}}$$

+Bianchi

$V_0 = 10^{-7}, g_0 = 1.1, \beta = 5, \gamma = 10^{-3}$

$b_V = 5, b_g = 0.5, p = 0.01, q = 0.1$

Stress-energy tensor

$$T^{\phi}_{\mu\nu} = \left(-K + 2XG_{,\phi} + G_{,X}\nabla_\sigma X \nabla^\sigma \phi\right) g_{\mu\nu} + \left(K_{,X} + G_{,X}\Box\phi - 2G_{,\phi}\right)\nabla_\mu\phi\nabla_\nu\phi - G_{,X}\left(\nabla_\mu X \nabla_\nu \phi + \nabla_\nu X \nabla_\mu \phi\right)$$

→ Energy density & Pressure

$$\rho_\phi = \frac{1}{2}M_{\rm Pl}^2(1-g)\dot{\phi}^2 + \frac{3}{4}\beta\dot{\phi}^4 + 3\gamma H\dot{\phi}^3 + V(\phi)$$

$$p_\phi = \frac{1}{2}M_{\rm Pl}^2(1-g)\dot{\phi}^2 + \frac{1}{4}\beta\dot{\phi}^4 - \gamma\dot{\phi}^2\ddot{\phi} - V(\phi)$$

+ Fluid $\quad p = w\rho$

↓

Einstein equation + $\nabla_\mu T^{\mu\nu}_{\rm Fluid} = 0$

+ modified Klein-Gordon $\quad \mathcal{P}\ddot{\phi} + \mathcal{D}\dot{\phi} + V_{,\phi} = 0$

with...

$$\mathcal{P} = (1-g)M_{\rm Pl}^2 + 6\gamma H\dot{\phi} + 3\beta\dot{\phi}^2 + \frac{3\gamma^2}{2M_{\rm Pl}^2}\dot{\phi}^4$$

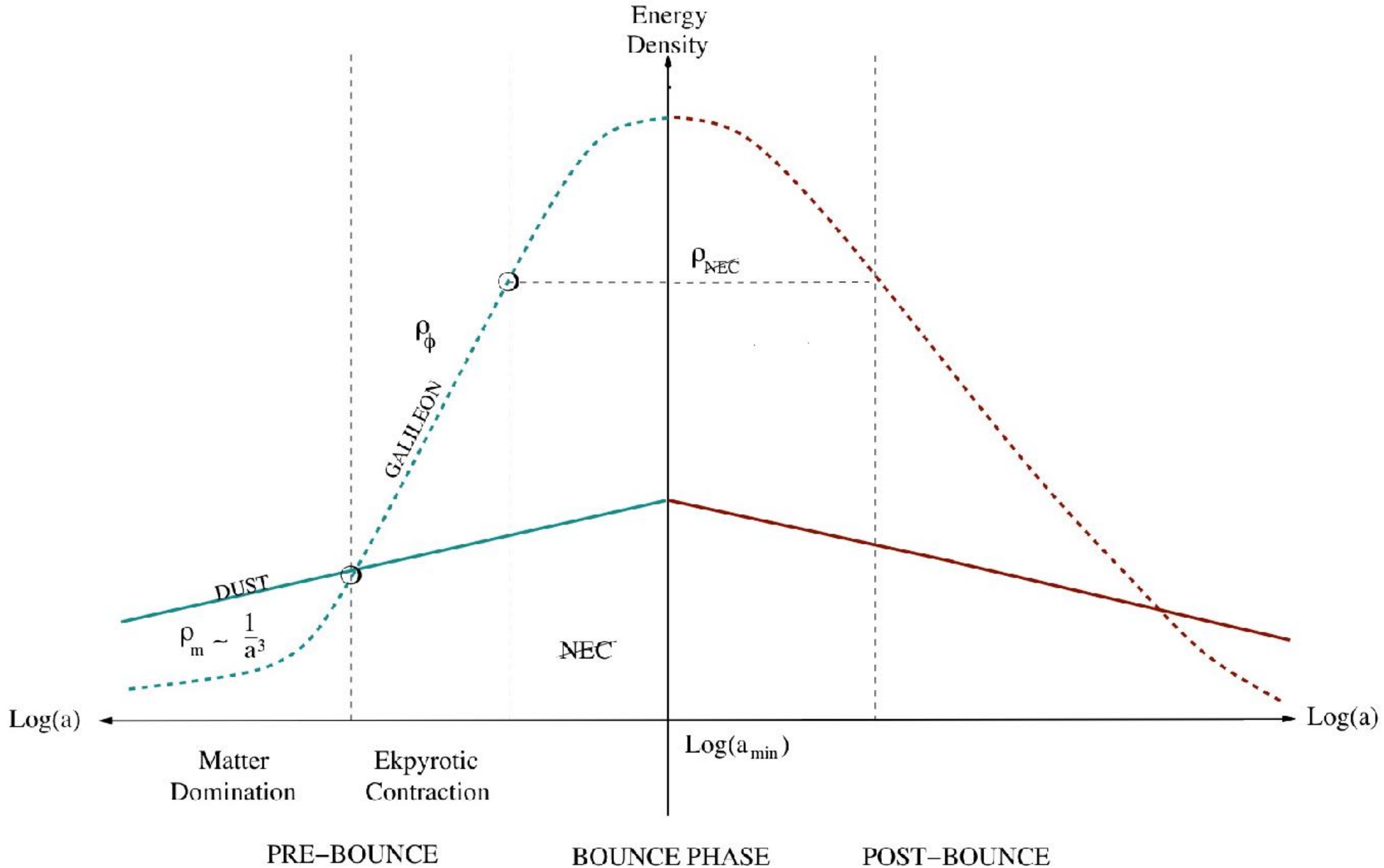
$$\mathcal{D} = 3(1-g)M_{\rm Pl}^2 H + \left(9\gamma H^2 - \frac{1}{2}M_{\rm Pl}^2 g_{,\phi}\right)\dot{\phi} + 3\beta H\dot{\phi}^2$$

$$-\frac{3}{2}(1-g)\gamma\dot{\phi}^3 - \frac{9\gamma^2 H\dot{\phi}^4}{2M_{\rm Pl}^2} - \frac{3\beta\gamma\dot{\phi}^5}{2M_{\rm Pl}^2}$$

$$-\frac{3}{2}G_{,X}\sum_i \dot{\theta}_i^2\dot{\phi} - \frac{3G_{,X}}{2M_{\rm Pl}^2}(\rho_{\rm m}+p_{\rm m})\dot{\phi}$$

5 phases:

A.	→	Matter contraction	Produces scale invariant perturbations
B.	→	Ekpyrotic contraction	Removes anisotropies
C.	→	The bounce itself	Leads to expansion
D.	→	Fast-roll expansion	Connects to standard model!!
E.	→	Radiation + Matter + ...	BB cosmology

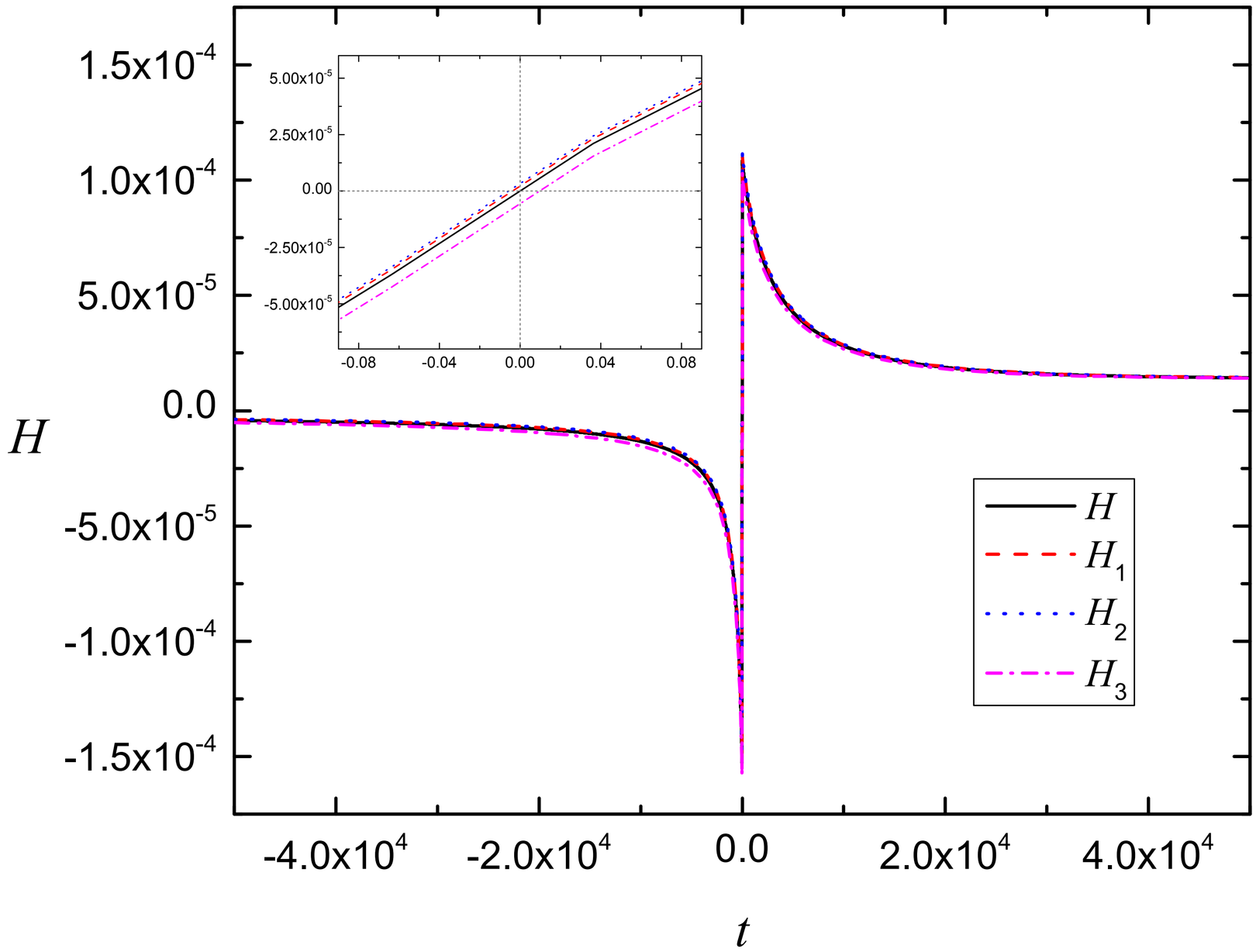

Energy
Density
ρ_{NEC}
ρ_ϕ
GALILEON
DUST
$\rho_m \sim \frac{1}{a^3}$
NEC
Log(a)
Log(a)
Log(a_{min})
Matter
Domination
Ekpyrotic
Contraction
PRE–BOUNCE
BOUNCE PHASE
POST–BOUNCE

Anisotropies can remain small all throughout!!!!

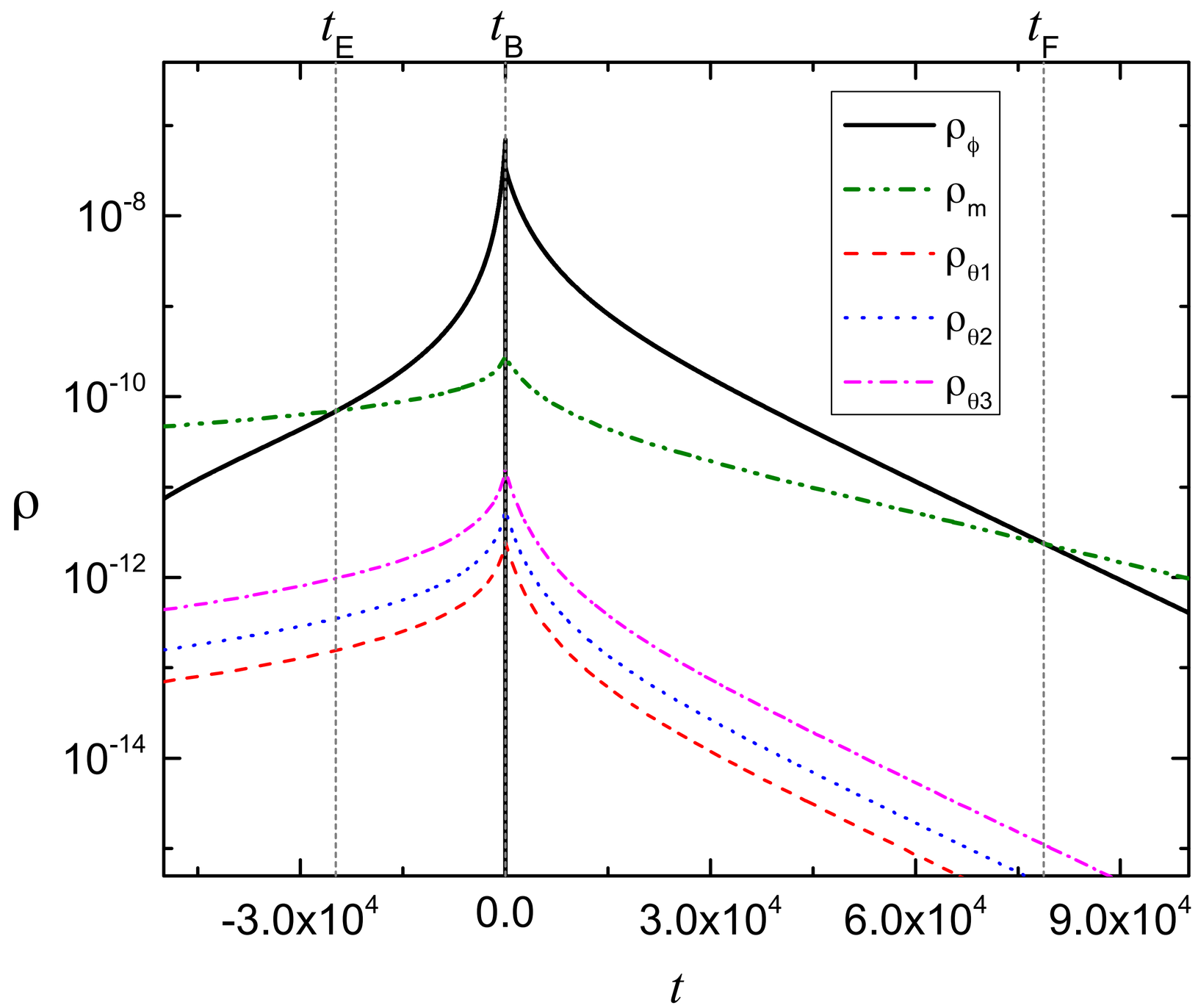
explicit example...

$$V_0 = 10^{-7}, \quad g_0 = 1.1, \quad \beta = 5, \quad \gamma = 10^{-3}$$
$$b_V = 5, \quad b_g = 0.5, \quad p = 0.01, \quad q = 0.1$$
$$\rho_{\mathrm{m,B}} = 2.8 \times 10^{-10}, \quad M_{\theta,1} = 2.2 \times 10^{-6}$$
$$M_{\theta,2} = 3.4 \times 10^{-6}, \quad M_{\theta,3} = -5.6 \times 10^{-6}$$
$$\phi_{\mathrm{ini}} = -2, \quad \dot{\phi}_{\mathrm{ini}} = 7.8 \times 10^{-6}$$

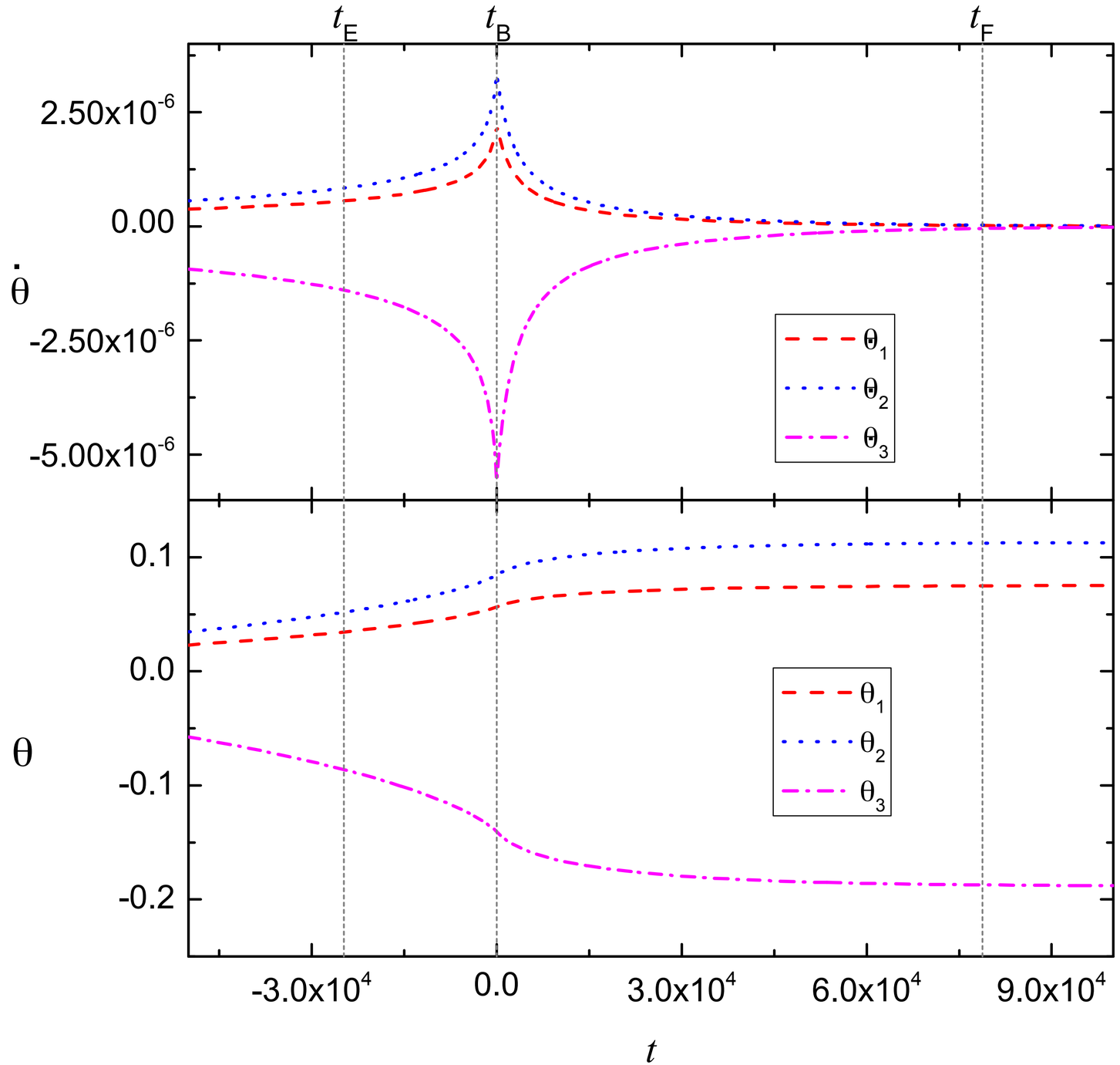
Hubble parameters



Energy densities



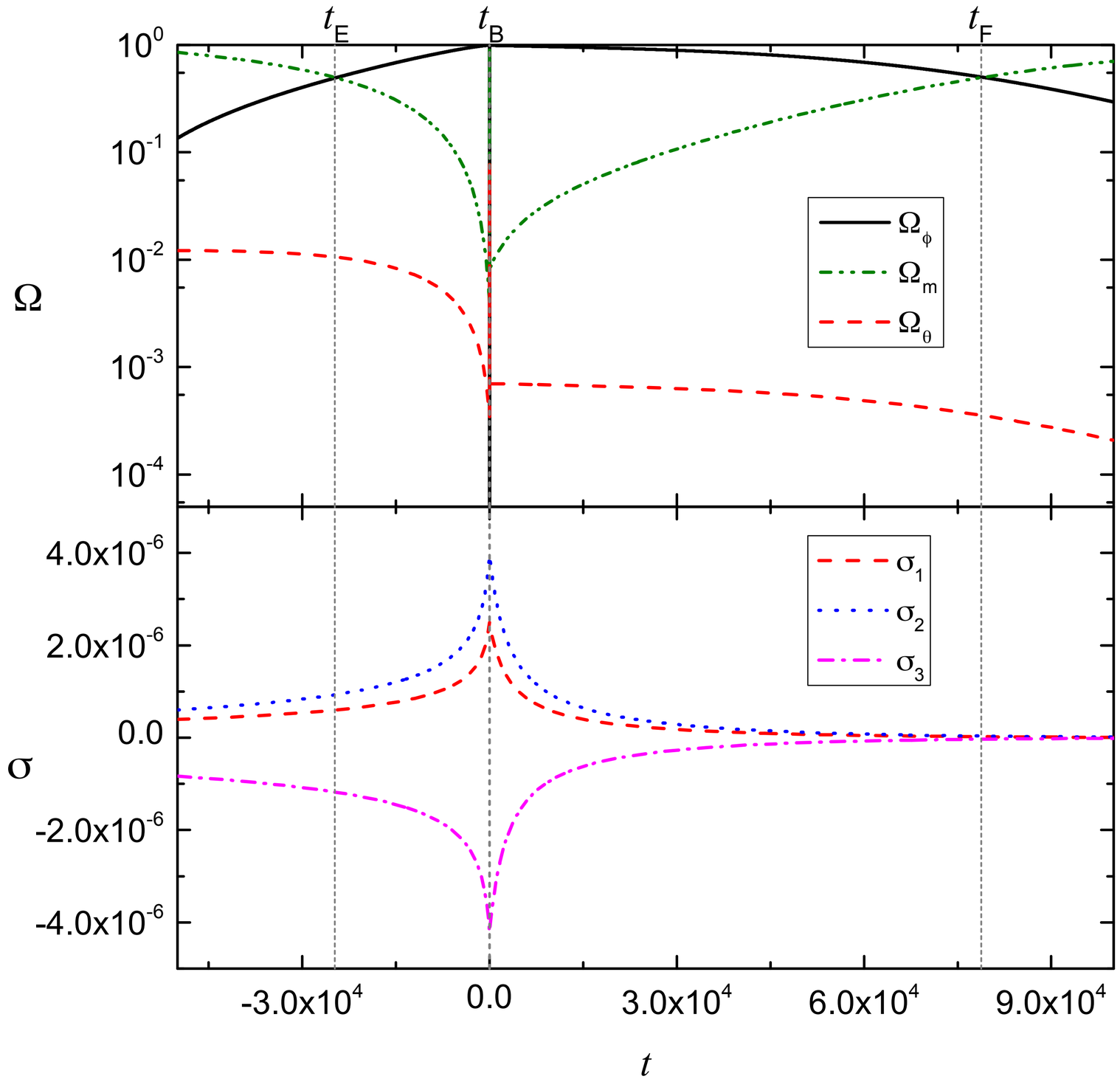
Anisotropies



Density parameters and shears

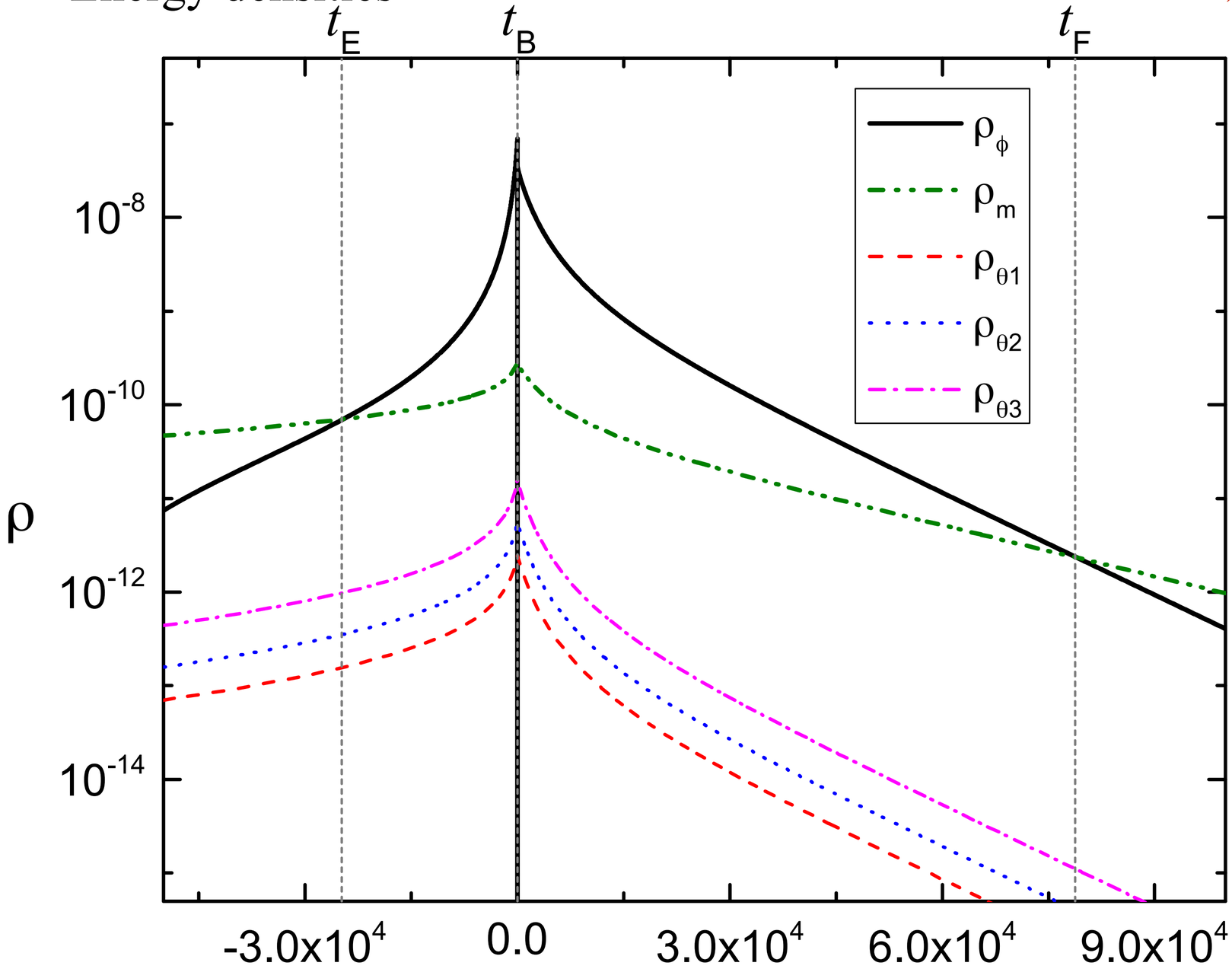
$$\Omega_{\rm I} \equiv \frac{\rho_{\rm I}}{\sum_{\rm I} \rho_{\rm I}}$$

$$\sigma_i \equiv \dot{\theta}_i e^{2\theta_i}$$



Energy densities

Y. Cai, R. Brandenberger & PP, CQG **30,** 075019 (2013)



$\phi_{\rm ini} = -2, \dot{\phi}_{\rm ini} = 7.8 \times 10^{-6}$

Anisotropies can remain small all throughout

Origin of perturbations: quantum fluctuations / time-varying background

Classical temperature fluctuations promoted to quantum operators

$$\widehat{\frac{\Delta T}{T}} \propto \hat{v} \sim \Phi \sim \delta g_{00}$$

second order perturbed Einstein action $${}^{(2)}\delta S = \frac{1}{2}\int \mathrm{d}^4x \left[\left(v'\right)^2 - \delta^{ij}\partial_i v \partial_j v + \frac{\left(a\sqrt{\epsilon_1}\right)''}{a\sqrt{\epsilon_1}} v^2 \right]$$

variable-mass scalar field in Minkowski spacetime

$$\epsilon_1 = 1 - \mathcal{H}'/\mathcal{H}^2$$

slow-roll parameter

+ Fourier transform $$v\left(\eta, \boldsymbol{x}\right) = \frac{1}{\left(2\pi\right)^{3/2}} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\, v_{\boldsymbol{k}}\left(\eta\right) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\Rightarrow \quad {}^{(2)}\delta S = \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{k} \left\{ v'_{\boldsymbol{k}} v^{*\prime}_{\boldsymbol{k}} + v_{\boldsymbol{k}} v^*_{\boldsymbol{k}} \left[\frac{\left(a\sqrt{\epsilon_1}\right)''}{a\sqrt{\epsilon_1}} - k^2 \right] \right\}$$

Lagrangian formulation...

Canonical quantisation

$$v(\boldsymbol{x},\eta) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \left[v_{\boldsymbol{k}}(\eta)\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}} + v^{\star}_{\boldsymbol{k}}(\eta)\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} a^{\dagger}_{\boldsymbol{k}} \right]$$

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^{\dagger}_{\boldsymbol{q}}\right] = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})$$

$$\left[\hat{v}(\boldsymbol{x},\eta), \hat{v}(\boldsymbol{y},\eta)\right] = 0 = \left[\hat{\pi}(\boldsymbol{x},\eta), \hat{\pi}(\boldsymbol{y},\eta)\right] \quad \text{and} \quad \left[\hat{v}(\boldsymbol{x},\eta), \hat{\pi}(\boldsymbol{y},\eta)\right] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$$

Vacuum state $\quad \hat{a}_{\boldsymbol{k}}|0\rangle = 0 \quad$ for all $\boldsymbol{k}$ $\quad \longrightarrow \quad v_{\boldsymbol{k}} \underset{|k\eta|\to\infty}{\longrightarrow} \dfrac{\mathrm{e}^{-ik\eta}}{\sqrt{2k}}$

$\longrightarrow$ Scale invariant spectrum of primordial perturbations...

Hamiltonian

$$H = \int \mathrm{d}^3\boldsymbol{k} \left\{ p_{\boldsymbol{k}} p^*_{\boldsymbol{k}} + v_{\boldsymbol{k}} v^*_{\boldsymbol{k}} \left[k^2 \overbrace{- \frac{\left(a\sqrt{\epsilon_1}\right)''}{a\sqrt{\epsilon_1}}}^{\omega^2(\eta,\boldsymbol{k})} \right] \right\}$$

collection of parametric oscillators with time dependent frequency

factorization of the full wave function

real and imaginary parts

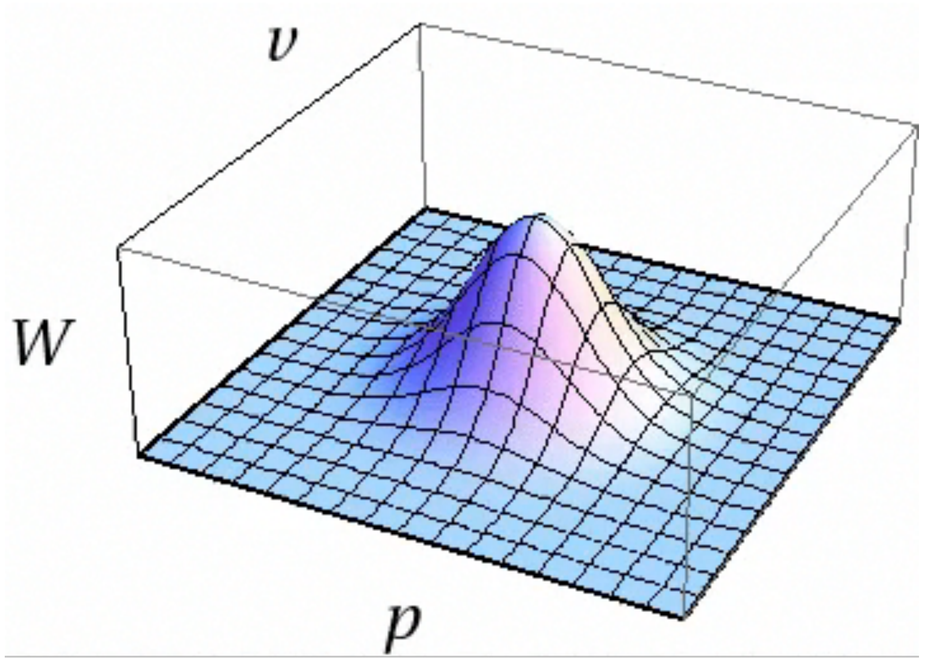
$$\Psi\left[v(\eta,\boldsymbol{x})\right] = \prod_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}\left(v^{\mathrm{R}}_{\boldsymbol{k}}, v^{\mathrm{I}}_{\boldsymbol{k}}\right) = \prod_{\boldsymbol{k}} \Psi^{\mathrm{R}}_{\boldsymbol{k}}\left(v^{\mathrm{R}}_{\boldsymbol{k}}\right) \Psi^{\mathrm{I}}_{\boldsymbol{k}}\left(v^{\mathrm{I}}_{\boldsymbol{k}}\right)$$

$$\left.\begin{aligned} i\frac{\Psi^{\mathrm{R,I}}_{\boldsymbol{k}}}{\partial\eta} &= \hat{\mathcal{H}}^{\mathrm{R,I}}_{\boldsymbol{k}} \Psi^{\mathrm{R,I}}_{\boldsymbol{k}} \\ \hat{\mathcal{H}}^{\mathrm{R,I}}_{\boldsymbol{k}} &= -\frac{1}{2}\frac{\partial^2}{\partial\left(v^{\mathrm{R,I}}_{\boldsymbol{k}}\right)^2} + \frac{1}{2}\omega^2(\eta,\boldsymbol{k})\left(\hat{v}^{\mathrm{R,I}}_{\boldsymbol{k}}\right)^2 \end{aligned}\right\}$$

Gaussian state solution $\Psi\left(\eta, v_{\boldsymbol{k}}\right)=\left[\frac{2\Re\mathrm{e}\,\Omega_{\boldsymbol{k}}\left(\eta\right)}{\pi}\right]^{1/4}\mathrm{e}^{-\Omega_{\boldsymbol{k}}\left(\eta\right)v_{\boldsymbol{k}}^{2}}$

Wigner function $W\left(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}\right)=\int\frac{\mathrm{d}x}{2\pi^{2}}\Psi^{*}\left(v_{\boldsymbol{k}}-\frac{x}{2}\right)\mathrm{e}^{-ip_{\boldsymbol{k}}x}\Psi\left(v_{\boldsymbol{k}}+\frac{x}{2}\right)$

large squeezing limit ⇒ $W\propto\delta\left(p_{\boldsymbol{k}}+k\tan\phi_{\boldsymbol{k}}v_{\boldsymbol{k}}\right)$



Stochastic distribution of classical processes

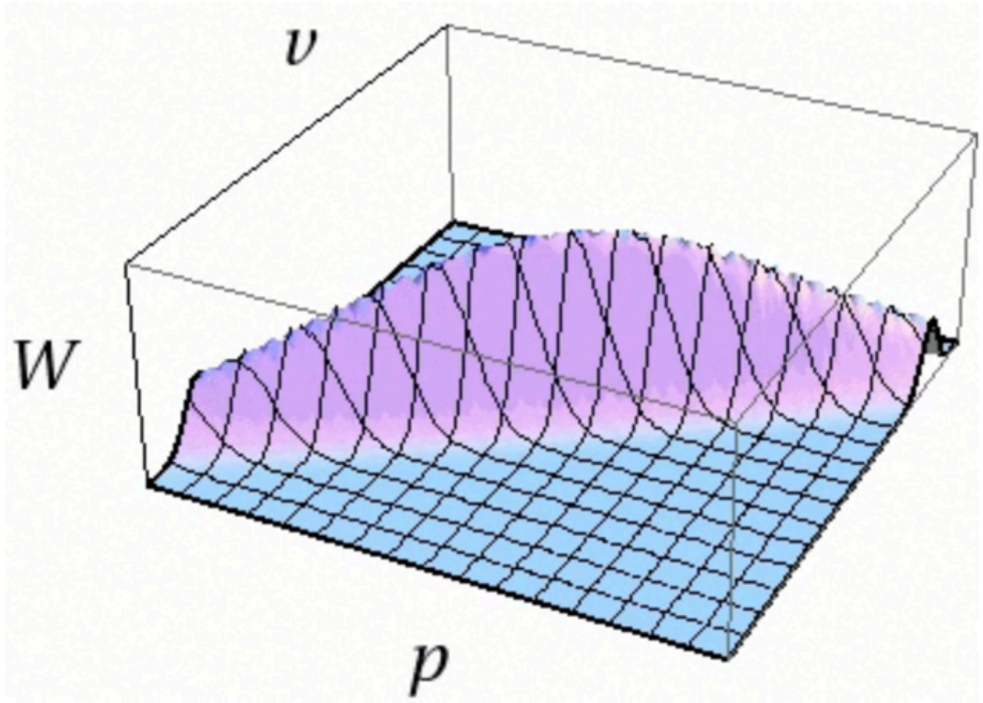
Ergodicity $\left\langle\frac{\Delta T(\xi,\boldsymbol{e})}{T}\right\rangle_{\xi}\simeq\left\langle\frac{\Delta T(\xi,\boldsymbol{e})}{T}\right\rangle_{\boldsymbol{e}}$

realization: ξ ; spatial direction: $\boldsymbol{e}$

Gaussian state solution $\Psi\left(\eta, v_{\boldsymbol{k}}\right)=\left[\frac{2\Re\mathrm{e}\,\Omega_{\boldsymbol{k}}\left(\eta\right)}{\pi}\right]^{1/4}\mathrm{e}^{-\Omega_{\boldsymbol{k}}\left(\eta\right)v_{\boldsymbol{k}}^{2}}$

Wigner function $W\left(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}\right)=\int\frac{\mathrm{d}x}{2\pi^{2}}\Psi^{*}\left(v_{\boldsymbol{k}}-\frac{x}{2}\right)\mathrm{e}^{-ip_{\boldsymbol{k}}x}\Psi\left(v_{\boldsymbol{k}}+\frac{x}{2}\right)$

large squeezing limit ⇒ $W\propto\delta\left(p_{\boldsymbol{k}}+k\tan\phi_{\boldsymbol{k}}v_{\boldsymbol{k}}\right)$



Stochastic distribution of classical processes

Ergodicity

realization, spatial direction

$\left\langle\frac{\Delta T(\xi,\boldsymbol{e})}{T}\right\rangle_{\xi}\simeq\left\langle\frac{\Delta T(\xi,\boldsymbol{e})}{T}\right\rangle_{\boldsymbol{e}}$

Primordial Power Spectrum

Standard case

Quantization in the Schrödinger picture (functional representation)

$$i\frac{\mathrm{d}|\Psi_{\boldsymbol{k}}\rangle}{\mathrm{d}\eta}=\hat{\mathcal{H}}_{\boldsymbol{k}}\,|\Psi_{\boldsymbol{k}}\rangle$$

with $$\hat{\mathcal{H}}_{\boldsymbol{k}}=\frac{\hat{p}_{\boldsymbol{k}}^2}{2}+\omega^2(\boldsymbol{k},\eta)\hat{v}_{\boldsymbol{k}}^2$$

$$\hat{v}_{\boldsymbol{k}}=v_{\boldsymbol{k}}$$

$$\hat{p}_{\boldsymbol{k}}=i\frac{\partial}{\partial v_{\boldsymbol{k}}}$$

and $$\omega^2(\boldsymbol{k},\eta)=k^2-\frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}$$

$$=k^2-\frac{\beta(\beta+1)}{\eta^2}$$

Power-law inflation example

$$a(\eta)=\ell_0(-\eta)^{1+\beta}$$

$$\beta\lesssim-2$$

(de Sitter: $\beta=-2$)

Parametric Oscillator System

Primordial Power Spectrum

Standard case

Quantization in the Schrödinger picture (functional representation)

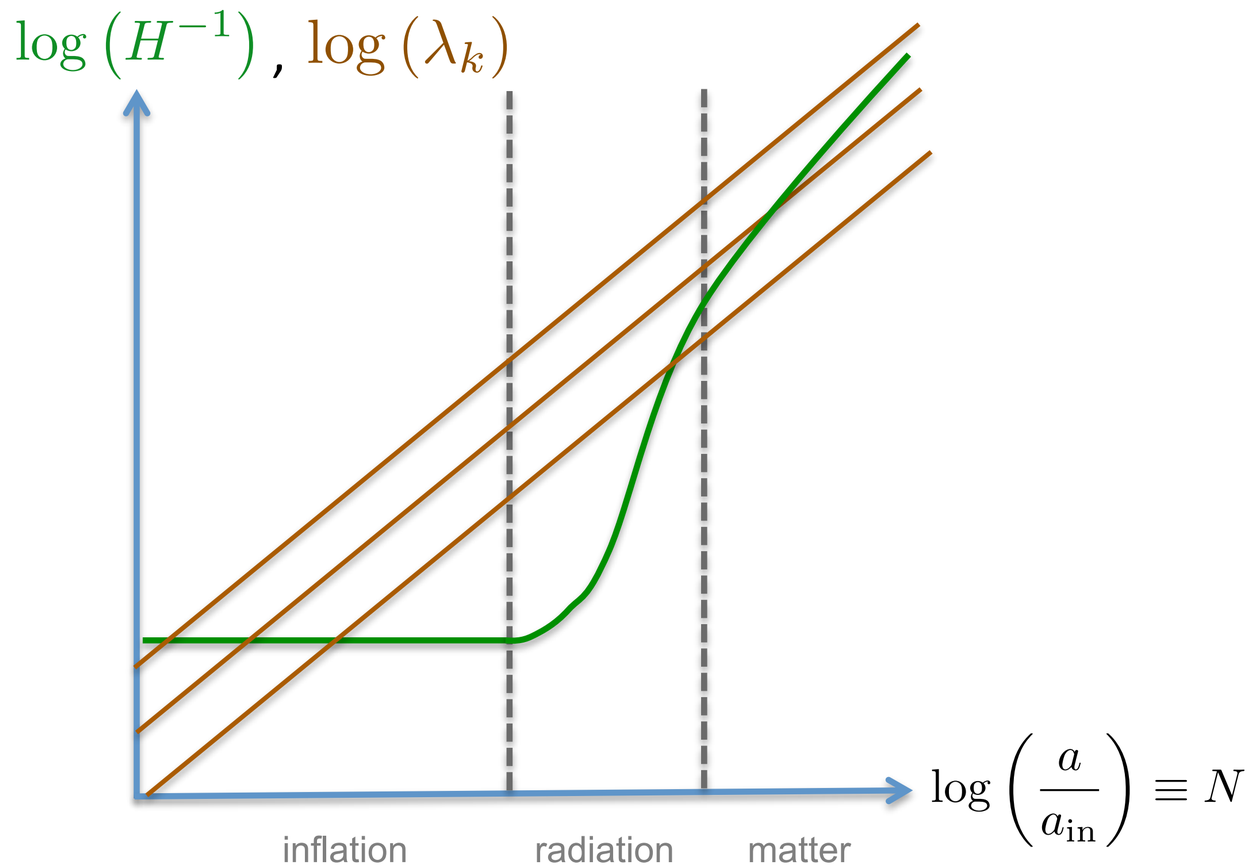
$$\Psi_{\boldsymbol{k}}(\eta, v_{\boldsymbol{k}}) = \left[\frac{2\,\Re\mathrm{e}\Omega_{\boldsymbol{k}}(\eta)}{\pi}\right]^{1/4} \mathrm{e}^{-\Omega_{\boldsymbol{k}}(\eta)\, v_{\boldsymbol{k}}^2}$$

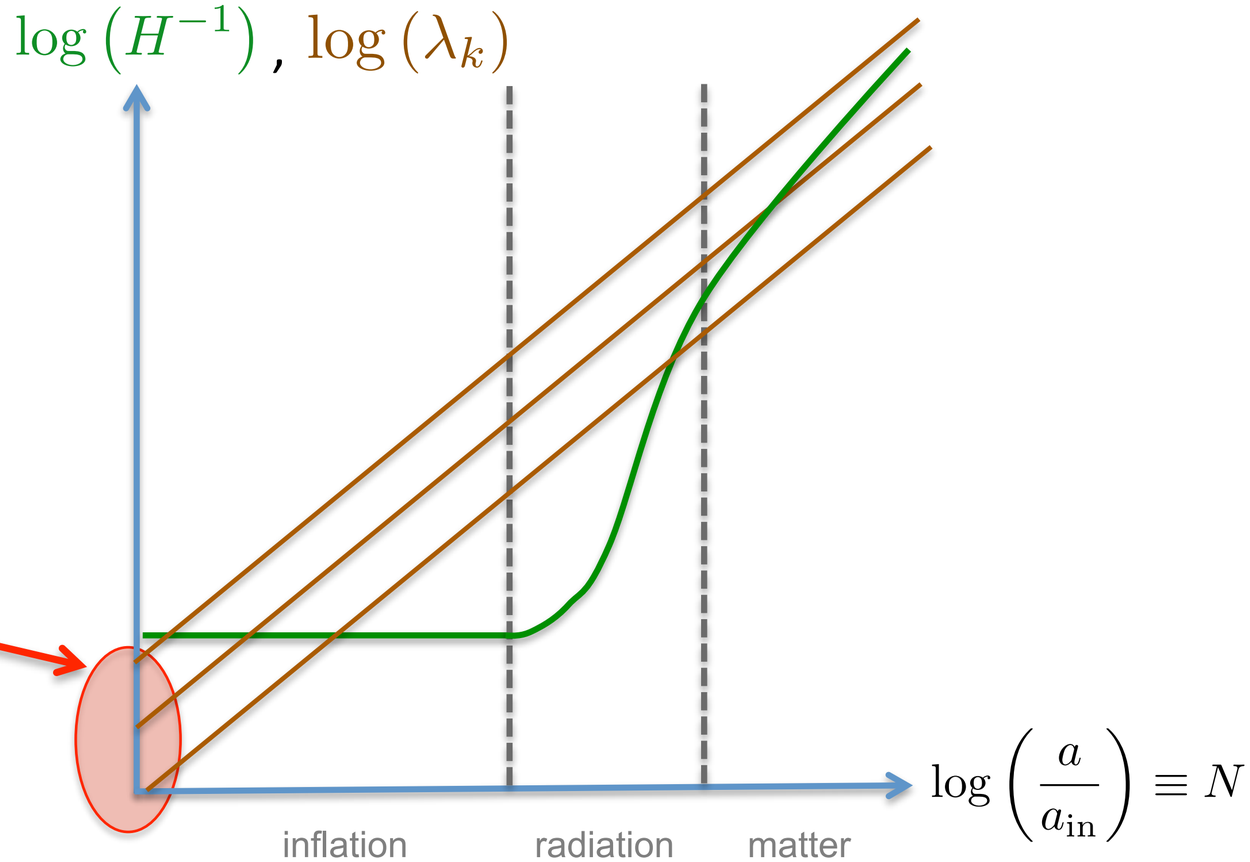
$$i\frac{\mathrm{d}|\Psi_{\boldsymbol{k}}\rangle}{\mathrm{d}\eta} = \hat{\mathcal{H}}_{\boldsymbol{k}}\,|\Psi_{\boldsymbol{k}}\rangle \qquad \text{with} \qquad \hat{\mathcal{H}}_{\boldsymbol{k}} = \frac{\hat{p}_{\boldsymbol{k}}^2}{2} + \omega^2(\boldsymbol{k},\eta)\hat{v}_{\boldsymbol{k}}^2$$

$$\Omega'_{\boldsymbol{k}} = -2i\Omega_{\boldsymbol{k}}^2 + \frac{i}{2}\omega^2(\eta,\boldsymbol{k})$$

$$\Omega_{\boldsymbol{k}} = -\frac{i}{2}\frac{f'_{\boldsymbol{k}}}{f_{\boldsymbol{k}}}$$

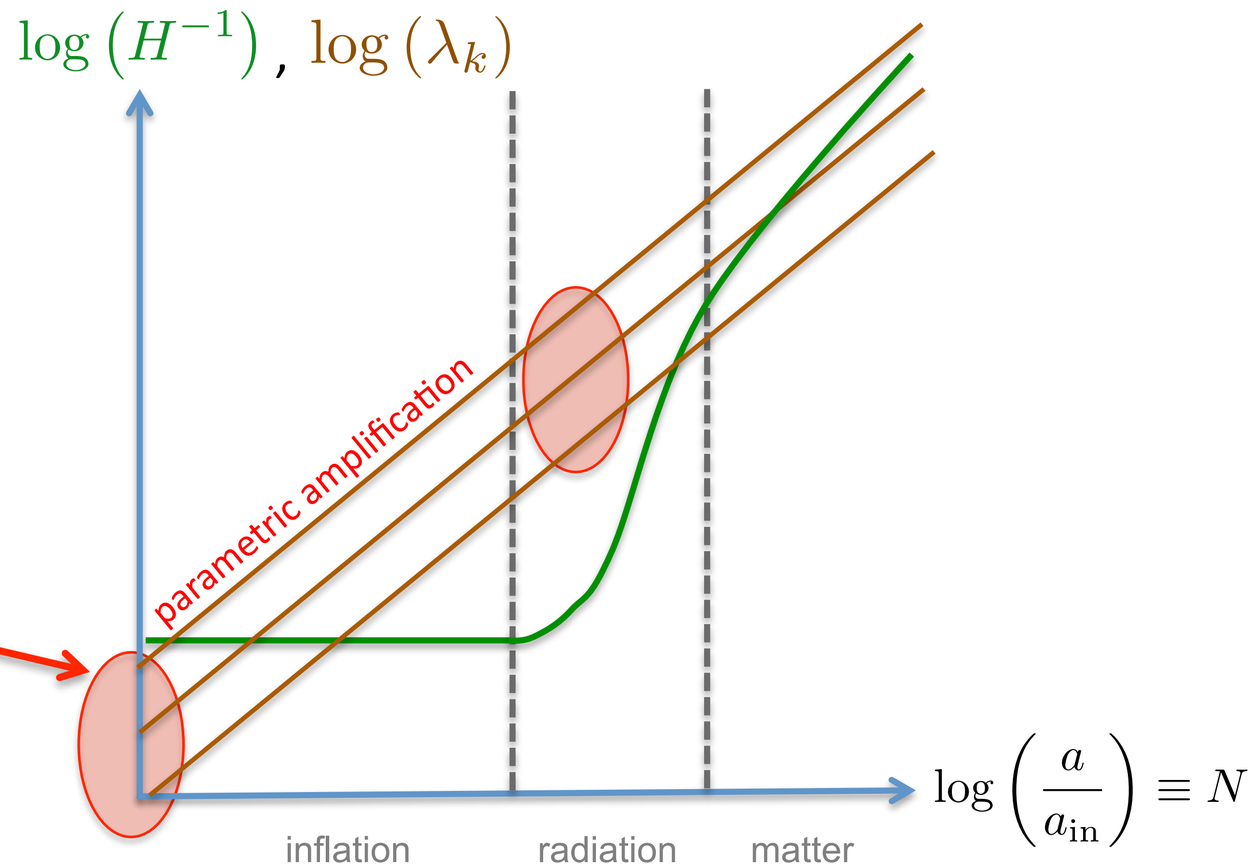
$$f''_{\boldsymbol{k}} + \omega^2(\boldsymbol{k},\eta)\, f_{\boldsymbol{k}} = 0$$


$\log\left(H^{-1}\right)$, $\log\left(\lambda_k\right)$
$\log\left(\frac{a}{a_{\rm in}}\right) \equiv N$
inflation
radiation
matter



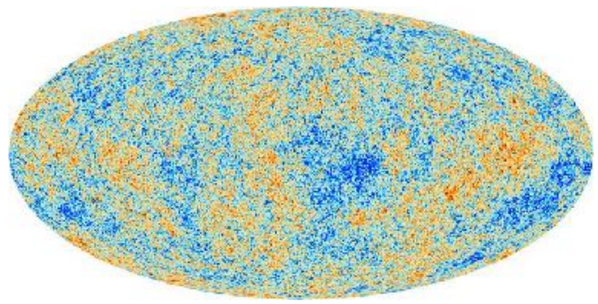
Harmonic oscillator fundamental state

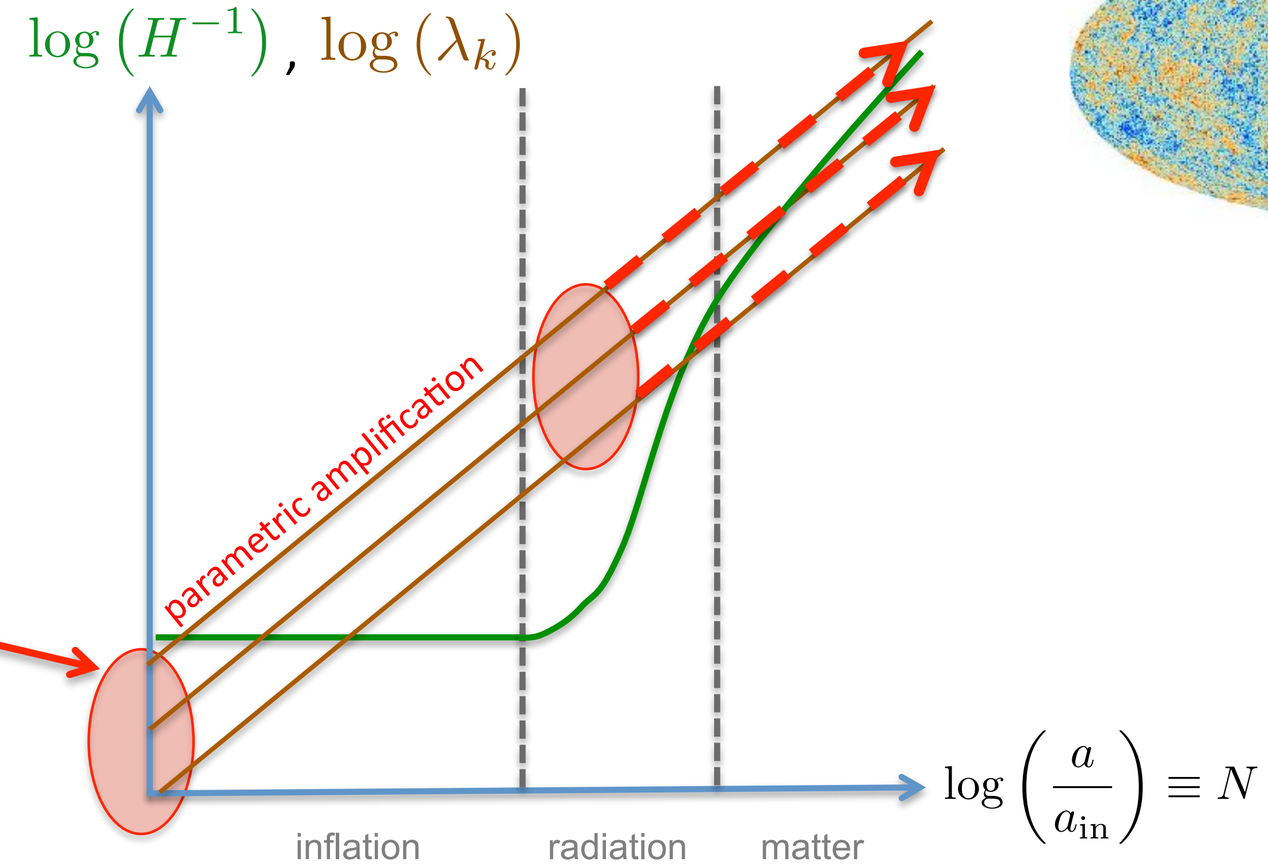
$$\Psi_{\boldsymbol{k}} = \left(\frac{k}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{k}{2}v_{\boldsymbol{k}}^2}$$



Harmonic oscillator fundamental state

$$\Psi_{\boldsymbol{k}} = \left(\frac{k}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{k}{2}v_{\boldsymbol{k}}^2}$$





Harmonic oscillator fundamental state

$$\Psi_{\boldsymbol{k}} = \left(\frac{k}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{k}{2}v_{\boldsymbol{k}}^2}$$

Primordial Power Spectrum

Standard case

Two physical scales

Hubble radius $H^{-1} = \dfrac{a^2}{a'} \underset{\beta\sim -2}{\simeq} \ell_0$

wavelength $\lambda = \dfrac{a}{k} \underset{\beta\sim -2}{\simeq} \dfrac{\ell_0}{-k\eta}$

Sub-Hubble regime

$\lambda \ll H^{-1}$

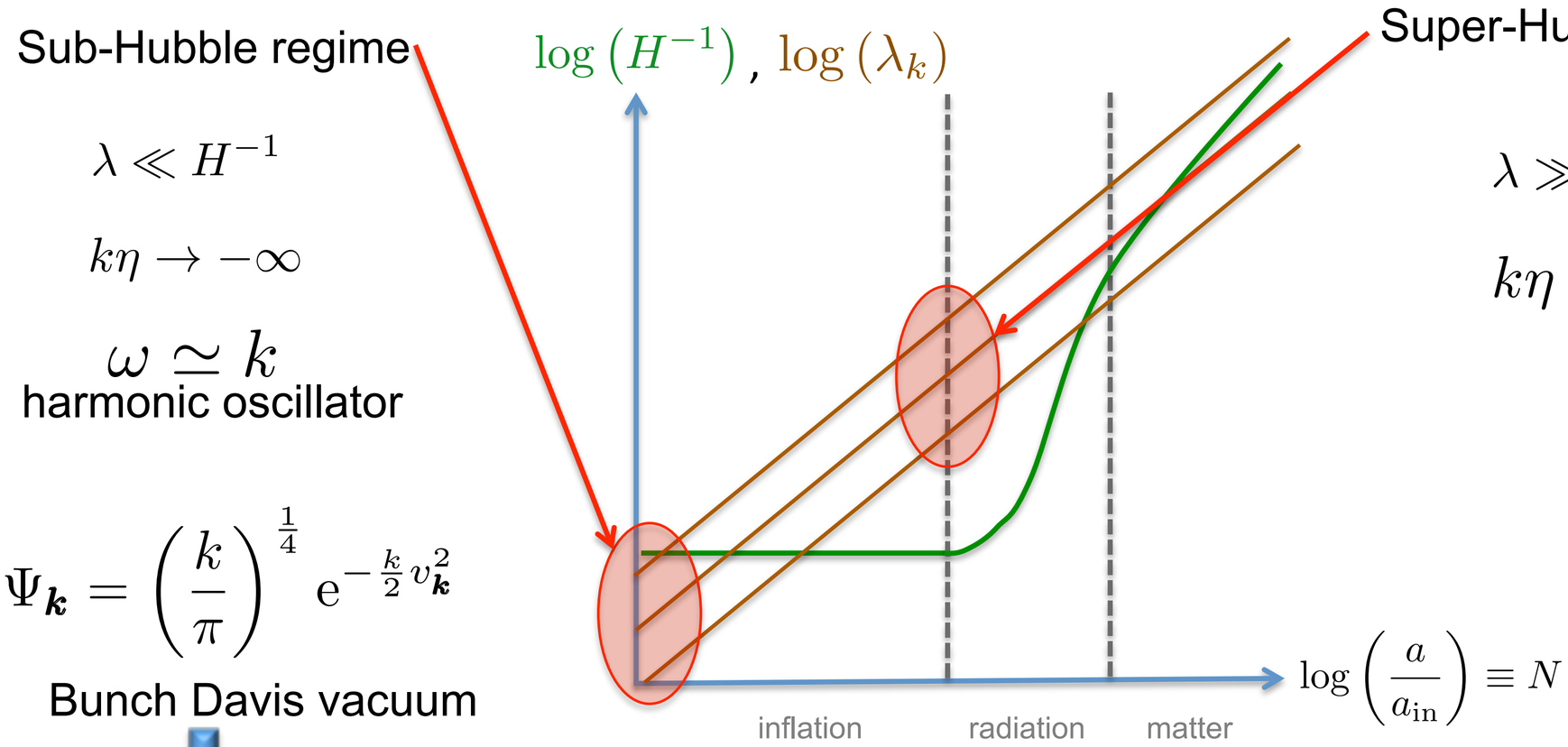
$k\eta \to -\infty$

$\omega \simeq k$

harmonic oscillator

$$\Psi_{\boldsymbol{k}} = \left(\frac{k}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{k}{2}v_{\boldsymbol{k}}^2}$$

Bunch Davis vacuum



Super-Hubble regime

$\lambda \gg H^{-1}$

$k\eta \to 0^-$

sets initial conditions $f_{\boldsymbol{k}}(k\eta \to -\infty) = \mathrm{e}^{ik\eta}/\sqrt{2k}$

$$v_{\boldsymbol{k}}'' + \left[\boldsymbol{k}^2 - U(\eta)\right] v_{\boldsymbol{k}} = 0$$

Vacuum state

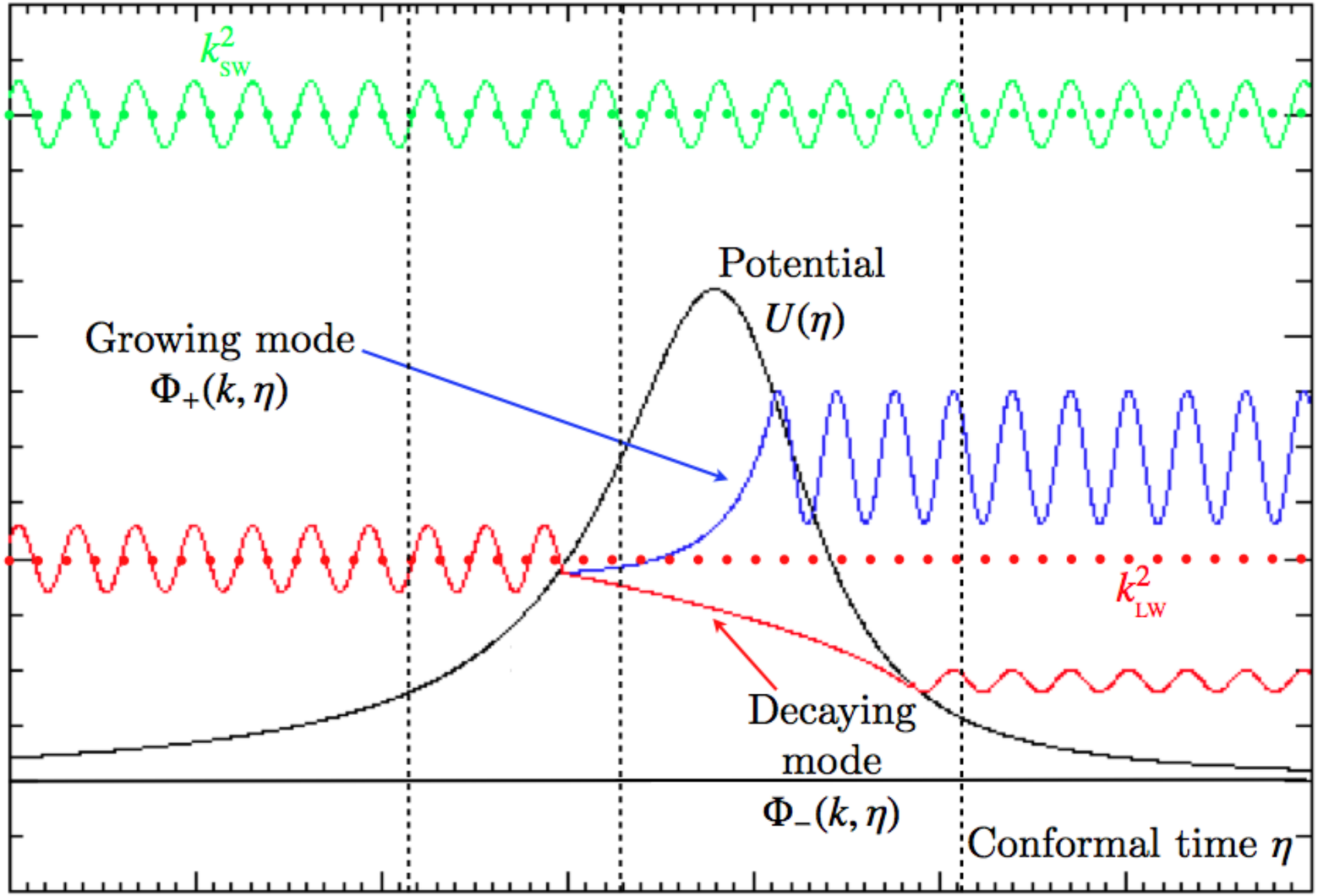
$$v_{\boldsymbol{k}} \underset{|k\eta|\to\infty}{\longrightarrow} \frac{\mathrm{e}^{-ik\eta}}{\sqrt{2k}}$$

Initial conditions fixed!

compare

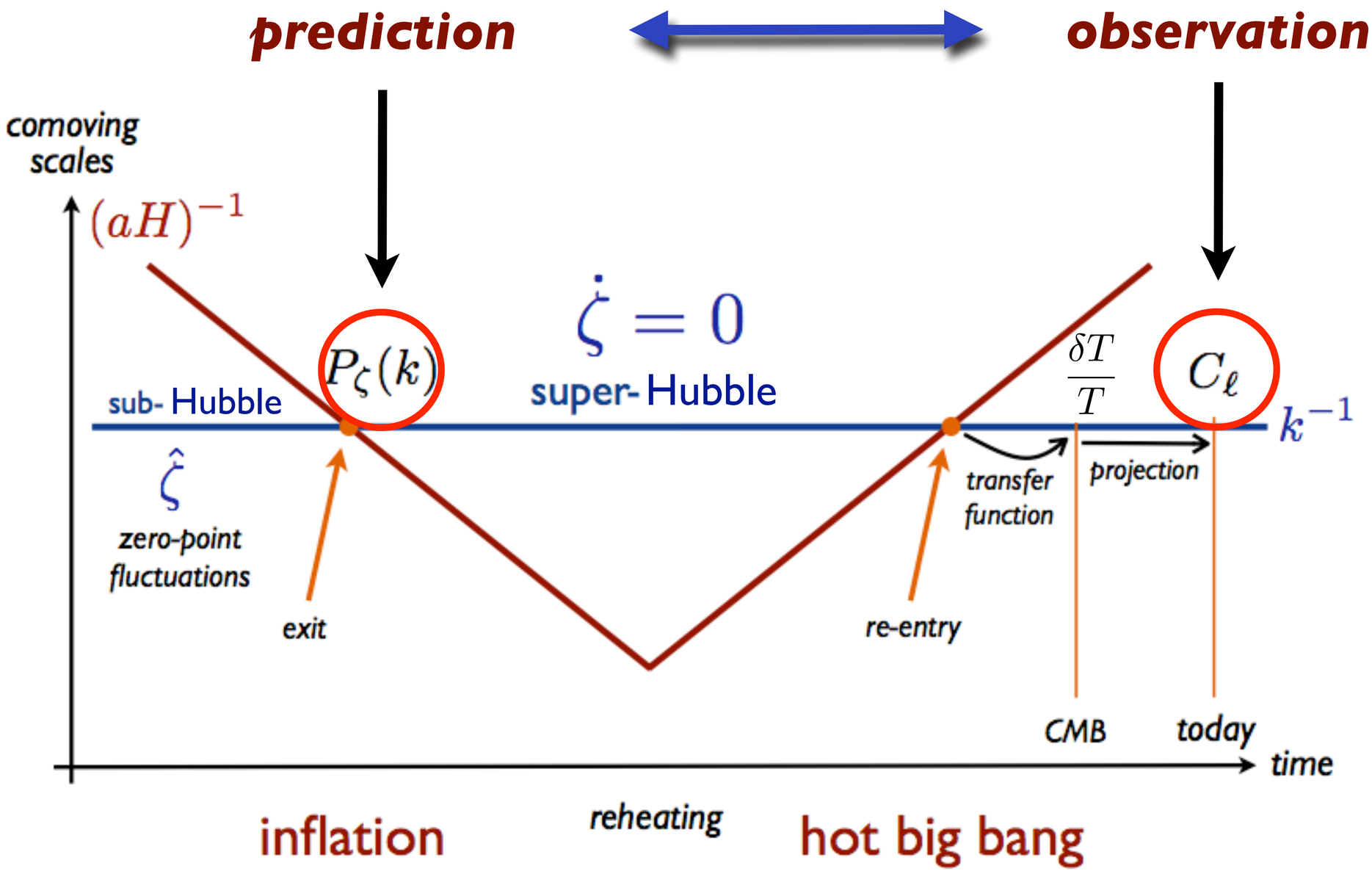
$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}\left[E - U(x)\right]\Psi = 0$$

(time independent
Schrödinger equation)



Transmission & Reflexion coefficients!

$$\langle \zeta_{\boldsymbol{k}} \zeta_{\boldsymbol{k'}} \rangle = (2\pi)^3 P_\zeta(k) \delta(\boldsymbol{k} + \boldsymbol{k'})$$



Primordial Power Spectrum

Standard case

$\boxed{f_{\boldsymbol{k}}'' + \omega^2(\boldsymbol{k},\eta) f_{\boldsymbol{k}} = 0}$ with $\omega^2(\boldsymbol{k},\eta) = k^2 - \dfrac{\beta(\beta+1)}{\eta^2}$ and $f_{\boldsymbol{k}}(k\eta \to -\infty) = \mathrm{e}^{ik\eta}/\sqrt{2k}$

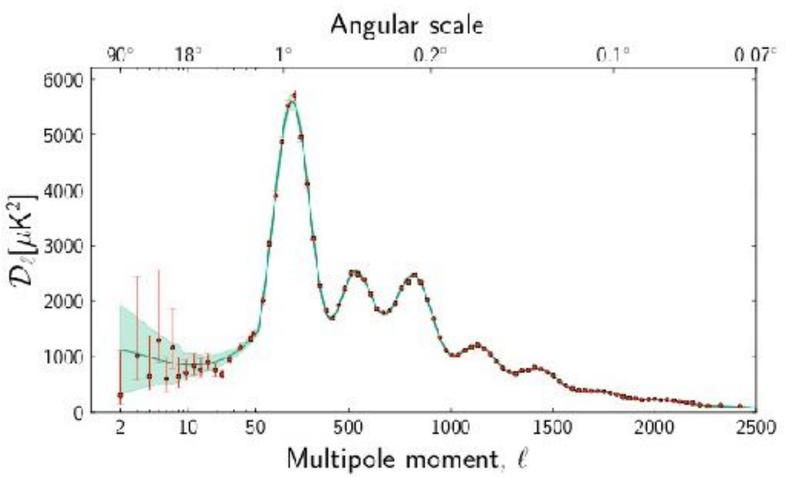
Uniquely determines $f_{\boldsymbol{k}}$ $\xrightarrow{\Omega_{\boldsymbol{k}} = -\frac{i}{2}\frac{f_{\boldsymbol{k}}'}{f_{\boldsymbol{k}}}}$ $\Re\mathrm{e}\,\Omega_{\boldsymbol{k}} = \langle \hat{v}_{\boldsymbol{k}}^2 \rangle - \langle \hat{v}_{\boldsymbol{k}} \rangle^2$

Evaluated at the end of inflation($k\eta \to 0^-$), this gives $P_v(k) = \dfrac{k^3}{2\pi^3}\left(\langle \hat{v}_{\boldsymbol{k}}^2 \rangle - \langle \hat{v}_{\boldsymbol{k}} \rangle^2\right)$

and eventually $P_\zeta(k) = \dfrac{1}{2a^2 M_{\mathrm{Pl}}^2 \epsilon_1} P_v(k) = A_S k^{n_\mathrm{S}-1}$

with $n_\mathrm{S} = 2\beta + 5 \underset{\beta\sim -2}{\simeq} 1$

Planck: $1 - n_\mathrm{s} = 0.0389 \pm 0.0054$



homogeneous

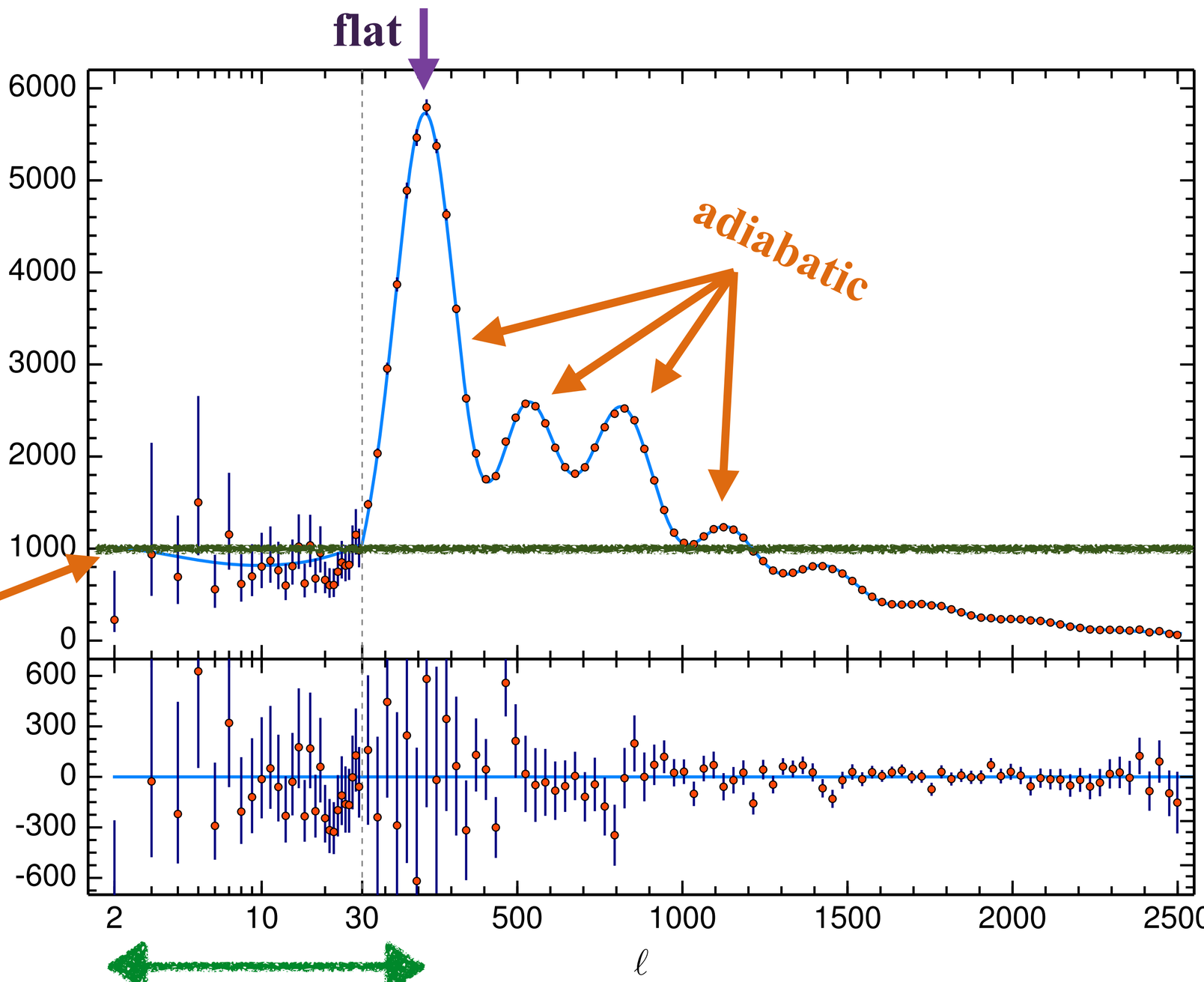
gaussian

scale invariant

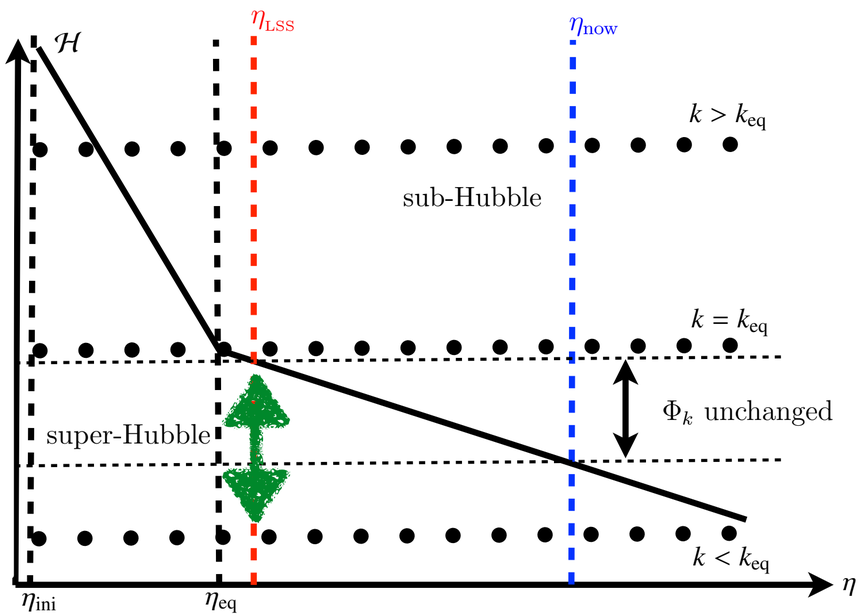
flat

adiabatic

$\ell(\ell+1)C_\ell/2\pi[\mu K^2]$



super Hubble @LSS



A last (potential) issue

- ***Cosmological perturbation theory***

  $\longrightarrow$ **Classical / semiclassical background**

  $\longrightarrow$ **Quantum vacuum initial state**

  $\longrightarrow$ **Unique / unambiguous predictions** $\quad \mathcal{P}_k, \ \dfrac{\Delta T}{T}, \ c_\ell, \ \cdots$

  but see J. Grain & V. Vennin, *JCAP* **02**, 022 (2020)

- ***Inflation***

  $\longrightarrow$ **Monotonic evolution of the scale factor** $\quad a(t); \dot{a} > 0 \ \forall t$

  $\longrightarrow$ $\exists$ **singularity**

  $\longrightarrow$ **resolve by quantum effects?**

- ***Quantum background***

  $\longrightarrow$ **Phase space / coherent state quantization** $\quad a > 0 \ (a \neq 0) \ \forall t$

  $\longrightarrow$ **Semiclassical approximation**

  $\oplus$ **Canonical quantization of perturbations**

  $\Longrightarrow$ *Ambiguities!*

Quantum cosmology

Hamiltonian GR (3+1)

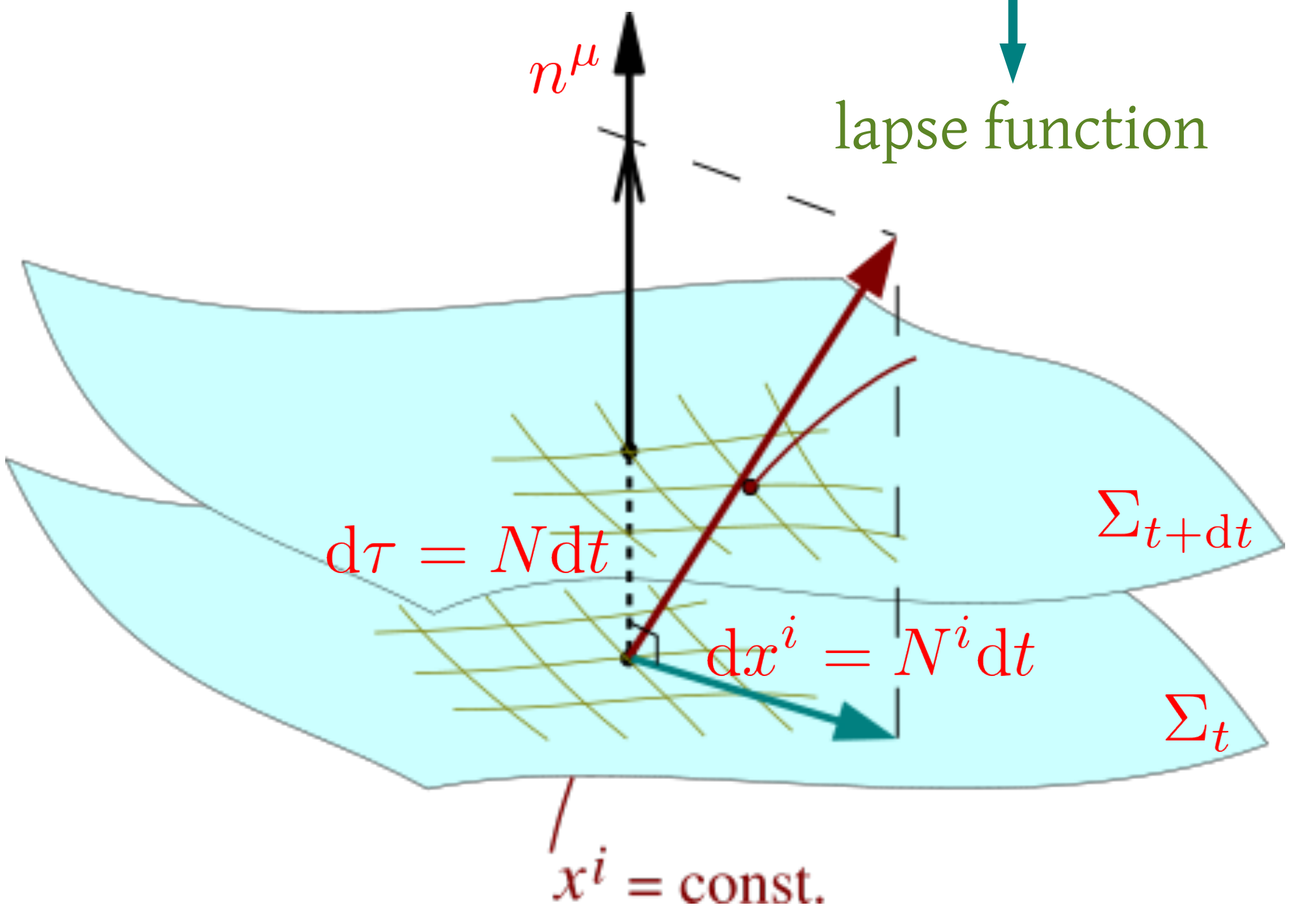
$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

lapse function

intrinsic metric = first fundamental form

shift vector

intrinsic curvature tensor ${}^3R^i{}_{jkl}(h)$



extrinsic curvature = second fundamental form:

$$K_{ij} = -\nabla_j^{(h)} n_i = \frac{1}{2N}\left(\nabla_j^{(h)} N_i + \nabla_i^{(h)} N_j - \frac{\partial h_{ij}}{\partial t}\right)$$

Action (Einstein-Hilbert, compact space):

$$\mathcal{S} = \frac{1}{16\pi G_{\mathrm{N}}}\left[\int_{\mathcal{M}} \sqrt{-g}\ (R - 2\Lambda)\, \mathrm{d}^4x + 2\int_{\partial\mathcal{M}} \sqrt{h} K^i{}_i \mathrm{d}^3x\right] + \mathcal{S}_{\mathrm{matter}}\left[\Phi\,(x)\right]$$

$$\longrightarrow \mathcal{S} = \int L \mathrm{d}t = \frac{1}{16\pi G_{\mathrm{N}}}\int \mathrm{d}t \left[\int \mathrm{d}^3x\, N\sqrt{h}\left(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right) + L_{\mathrm{matter}}\right]$$

Canonical momenta

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{16\pi G_{\mathrm{N}}}\left(K^{ij} - h^{ij}K\right)$$

$$\pi^{\Phi} \equiv \frac{\delta L}{\delta \dot{\Phi}} = -\frac{\sqrt{h}}{N}\left(\dot{\Phi} - N\frac{\partial \Phi}{\partial x^i}\right)$$

$$\left.\begin{aligned} \pi^0 &\equiv \frac{\delta L}{\delta \dot{N}} = 0 \\ \pi^i &\equiv \frac{\delta L}{\delta \dot{N}^i} = 0 \end{aligned}\right\} \text{ primary constraints}$$

Hamiltonian $$H \equiv \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + \pi^{ij} \dot{h}_{ij} + \pi^\Phi \dot{\Phi}\right) - L$$

$$= \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + N\mathcal{H} + N_i \mathcal{H}^i\right)$$

variation wrt lapse: $\mathcal{H} = 0 \rightarrow$ Hamiltonian constraint

variation wrt shift: $\mathcal{H}^i = 0 \rightarrow$ momentum constraint

$\Longrightarrow$ classical description complete

variation wrt 3D metric h_{ij} : dynamical equations

Superspace & canonical quantization

relevant configuration space $\quad \mathrm{Riem}\left(\Sigma\right) \equiv \left\{ h_{ij}\left(\boldsymbol{x}\right), \Phi\left(\boldsymbol{x}\right) \middle| \boldsymbol{x} \in \Sigma \right\}$

parameters

$\mathrm{GR} \Longrightarrow$ invariance/diffeomorphisms $\Longrightarrow \mathrm{Conf} = \dfrac{\mathrm{Riem}\left(\Sigma\right)}{\mathrm{Diff}\left(\Sigma\right)}$: superspace

Wave functional $\Psi\left[h_{ij}\left(\boldsymbol{x}\right), \Phi\left(\boldsymbol{x}\right)\right] \quad (\equiv \text{“}\left\langle h_{ij}, \Phi \middle| \Psi \right\rangle\text{”})$

+ Dirac canonical quantization procedure

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad \pi^{\Phi} \to -i\frac{\delta}{\delta \Phi} \qquad \pi^{0} \to -i\frac{\delta}{\delta N} \qquad \pi^{i} \to -i\frac{\delta}{\delta N_i}$$

primary constraints $\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = \left[-16\pi G_{\rm N}\mathcal{G}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \frac{\sqrt{h}}{16\pi G_{\rm N}}\left(-{}^3R + 2\Lambda + 16\pi G_{\rm N}\hat{T}^{00}\right)\right]\Psi = 0$$

Wheeler - DeWitt equation

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$$

De Witt metric

primary constraints $\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

Wheeler - DeWitt equation

primary constraints $\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

time-independent Schrödinger equation with vanishing energy

primary constraints $\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

TIMELESS Schrödinger equation

mini-superspace

restrict attention from an infinite dimensional configuration space to a 2 dimensional space = mini-superspace

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \mapsto \quad a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]$$

WDW equation becomes Schrödinger like for $\Psi\left[a(t),\phi(t)\right]$

Conceptual & technical issues

infinite # d.o.f. to a few: mathematical consistency?

freeze momenta... Heisenberg uncertainties?

A simple example: perfect fluid matter

$$\mathrm{d}s^2 = -N^2(\tau)\mathrm{d}\tau + a^2(\tau)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

perfect fluid formalism: B. Schutz (1970)

→ velocity potentials + canonical transformations

$$H = N\left(-\frac{p_a^2}{4a} - \mathcal{K}a + \frac{p_\tau}{a^{3w}}\right)$$

→ defines time...

→ choose wisely... a^{3w}

WDW $H\Psi = 0$ & $\mathcal{K} \to 0$

$$i\frac{\partial\Psi}{\partial\tau} = \frac{1}{4}\frac{\partial^2\Psi}{\partial x^2}$$

free particle Schrödinger!

$$\left[x = \frac{2a^{3(1-w)/2}}{3(1-w)}\right]$$

$$i\frac{\partial\Psi}{\partial\tau}=\frac{1}{4}\frac{\partial^2\Psi}{\partial x^2}$$

$$\Psi=\sqrt[4]{\frac{8\tau_0}{\pi\left(\tau_0^2+\tau^2\right)^2}}\exp\left(-\frac{\tau_0 x^2}{\tau_0^2+\tau^2}\right)\mathrm{e}^{-iS(x,\tau)}$$

phase $\quad S(x,\tau)=\dfrac{\tau x^2}{\tau_0^2+\tau^2}+\dfrac{1}{2}\arctan\left(\dfrac{\tau_0}{\tau}\right)-\dfrac{\pi}{4}$

Gaussian wave packet

Use trajectory approach (aka dBB)

$x \Longleftrightarrow a$

Scale factor

$$a=a_0\left[1+\left(\frac{\tau}{\tau_0}\right)\right]^{\frac{1}{3(1-w)}}$$

Natural bounce avoiding singularity

Basic model: GR + perfect fluid

$$\mathcal{S}_{\rm EHS} = \underbrace{\frac{1}{16\pi G_{\rm N}}\int \mathrm{d}^4x\sqrt{-g}R}_{\mathcal{S}_{\rm EH}} + \underbrace{\int \mathrm{d}^4x\sqrt{-g}P(w,\phi)}_{\mathcal{S}_{\rm M}}$$

FLRW metric

$$\mathrm{d}s^2 = -N^2(\tau)\mathrm{d}\tau^2 + a^2(\tau)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$-\frac{1}{2\kappa}\int \mathrm{d}\tau N a^3 \underbrace{\int \sqrt{\gamma}\mathrm{d}^3x}_{\mathcal{V}_0}\underbrace{\frac{6\dot{a}^2}{a^2N^2}}_{R}$$

$$P = w\rho$$

$$N = (1+w)a^{3w} \implies H^{(0)}_{\rm M} = p_\tau$$

Use as clock

$$q = \frac{4\sqrt{6}}{3(1-w)\sqrt{1+w}}a^{\frac{3}{2}(1-w)} \equiv \gamma a^{\frac{3}{2}(1-w)}$$

$$p = \frac{\sqrt{6(1+w)}}{2\kappa_0}a^{\frac{3}{2}(1+w)}\frac{\dot{a}}{Na}$$

$\frac{\dot{a}}{Na} \to H$ Hubble rate

New canonical variables

$$H^{(0)} = 2\kappa_0 p^2 \to \kappa/\mathcal{V}_0$$

Hamilton equations $\dfrac{\mathrm{d}q}{\mathrm{d}\tau} = 4\kappa_0 p$ and $\dfrac{\mathrm{d}p}{\mathrm{d}\tau} = 0$

singular solutions

$$q(\tau) = \sqrt{8\kappa_0 H^{(0)}}(\tau - \tau_{\mathrm{s}}) \to q_{\mathrm{B}}\omega\tau \qquad p(\tau) \to \sqrt{\frac{H^{(0)}}{2\kappa_0}} = \frac{q_{\mathrm{B}}\omega}{4\kappa_0}$$

Perturbations: $H_{\mathrm{full}} = H^{(0)} + \sum_{\boldsymbol{k}} H^{(2)}_{\boldsymbol{k}}$

Fluid perturbation $\phi_{\boldsymbol{k}} = \dfrac{p^{\frac{1-w}{1+w}}\delta\phi_{\boldsymbol{k}}}{\sqrt{2w(1+w)\kappa_0}} + \sqrt{\dfrac{3}{w\kappa_0}}\dfrac{a^{-\frac{3w-7}{2}}}{4k^2}\delta R_{\boldsymbol{k}}$
(intrinsic + curvature)

P. Małkiewicz, *CQG* **36**, 215003 (2019)

$$\Longrightarrow H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}|\pi_{\phi,\boldsymbol{k}}|^2 + \frac{1}{2}w(1+w)^2\left(\frac{q}{\gamma}\right)^{\frac{4(3w-1)}{3(1-w)}} k^2|\phi_{\boldsymbol{k}}|^2$$

Hamilton equations

$$\phi''_{\boldsymbol{k}} + \left(\frac{q}{\gamma}\right)^{\frac{4(3w-1)}{3(1-w)}} w(1+w)^2 k^2 \phi_{\boldsymbol{k}} = 0$$

Fluid (F-)parameterization: $(\phi_{\boldsymbol{k}}, \pi_{\phi,\boldsymbol{k}})$

Let's begin the fun… changing the variables! $Z(\tau) = \sqrt{1+w}\left(\frac{q}{\gamma}\right)^{\frac{3w-1}{3(1-w)}}$

$v_{\boldsymbol{k}} = Z\phi_{\boldsymbol{k}}$ and $\pi_{v,\boldsymbol{k}} = Z^{-1}\pi_{\phi,\boldsymbol{k}} + \frac{\dot{Z}}{Z^2}\phi_{\boldsymbol{k}}$ Conformal (C-)parameterization: $(v_{\boldsymbol{k}}, \pi_{v,\boldsymbol{k}})$

$$H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}Z^2\left\{|\pi_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \mathcal{V}_{\rm cl}(\tau)\right]|v_{\boldsymbol{k}}|^2\right\}$$

$$\frac{1}{Z^4}\left[\frac{\ddot{Z}}{Z} - 2\left(\frac{\dot{Z}}{Z}\right)^2\right]$$

Explicitly…

$$H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}(1+w)\left(\frac{q}{\gamma}\right)^{\frac{2(3w-1)}{3(1-w)}}\left\{|\pi_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \frac{8}{9q^2}\left(\frac{q}{\gamma}\right)^{\frac{4(1-3w)}{3(1-w)}}\frac{(2\kappa_0)^2(1-3w)p^2}{(1+w)^2(1-w)^2}\right]|v_{\boldsymbol{k}}|^2\right\}$$

upon using the background e.o.m.

Switch to conformal time $\mathrm{d}\eta = Z^2 \mathrm{d}\tau = (1+w)\left(\dfrac{q}{\gamma}\right)^{\frac{2(3w-1)}{3(1-w)}} \mathrm{d}\tau$ yields $\mathcal{V}_{\rm cl} = \dfrac{Z''}{Z}$

$$v_{\boldsymbol{k}}'' + \left(wk^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}} = 0$$

$$z = q^r$$

$$r_1 = \frac{3w-1}{3(1-w)} \qquad r_2 = \frac{2}{3(1-w)}$$

$$\eta = \frac{1+w}{2r_1+1}\left(\frac{q_{\rm B}\omega}{\gamma}\right)^{2r_1} \tau^{2r_1+1} \longrightarrow q(\eta) = q_{\rm B}\omega\left[\frac{2r_1+1}{1+w}\left(\frac{q_{\rm B}\omega}{\gamma}\right)^{-2r_1}\eta\right]^{1/(2r_1+1)} \propto \eta^{\frac{3(1-w)}{1+3w}}$$

$$\mathcal{V}_{\rm cl} = \frac{(q^{r_1})''}{q^{r_1}} = \frac{(q^{r_2})''}{q^{r_2}} = \frac{2(1-3w)}{(1+3w)^2\eta^2}$$

Solutions = Hankel functions, divergent perturbations at the singularity

Coherent state quantization

Background phase space $(q,p) \in \mathbb{R}^{*+} \times \mathbb{R} = \{(q,p) | q > 0, p \in \mathbb{R}\}$

Natural choice = 2 parameter affine group of the real line $\quad t \in \mathbb{R} \;\mapsto (q,p) \cdot t = \dfrac{t}{q} + p$

$$\{(q_0,p_0)\,,(q,p)\} \mapsto (q',p') = (q_0,p_0) \circ (q,p) = \left(q'q, \frac{p}{q'} + p'\right)$$

left-invariant measure $\quad \mathrm{d}q' \wedge \mathrm{d}p' = \mathrm{d}q \wedge \mathrm{d}p$

Coherent state quantization

Background phase space $(q,p) \in \mathbb{R}^{*+} \times \mathbb{R}$

Natural choice = 2 parameter affine group of the real line $\quad t \in \mathbb{R} \;\mapsto (q,p)\cdot t = \dfrac{t}{q} + p$

unitary, irreducible and square-integrable representation in the Hilbert space $\mathcal{H} = L^2\left(\mathbb{R}^{*+}, \dfrac{\mathrm{d}x}{x}\right)$

$$\langle x|U(q,p)|\psi\rangle = \langle x|q,p\rangle = \frac{\mathrm{e}^{ipx/\hbar}}{\sqrt{q}}\psi\left(\frac{x}{q}\right)$$

$$\psi(x) = \langle x|\psi\rangle$$

covariant integral quantization on (affine) coherent states

$$\mathbb{R}^+ \times \mathbb{R} \ni (q,p) \mapsto |q,p\rangle := U(q,p)|\xi\rangle \in \mathcal{H}$$

normalized fiducial state

$$U(q,p) = \mathrm{e}^{ip\widehat{Q}/\hbar}\mathrm{e}^{-i\ln q\widehat{D}/\hbar}$$

algebra

$$\left[\widehat{Q},\widehat{D}\right] = i\hbar\widehat{Q}$$

dilation $\widehat{D} := \dfrac{1}{2}(\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q})$

define $\quad \rho(\alpha) = \int \frac{\langle\xi|x\rangle\langle x|\xi\rangle}{x^{\alpha+1}}\,\mathrm{d}x = \int \frac{|\xi(x)|^2}{x^{\alpha+1}}\,\mathrm{d}x \;<\infty$

and $\quad \sigma(\alpha) = \int \left|\frac{\mathrm{d}\xi(x)}{\mathrm{d}x}\right|^2 \frac{\mathrm{d}x}{x^{\alpha+1}}$

$\int \frac{\mathrm{d}q\mathrm{d}p}{2\pi\hbar\rho_0}|q,p\rangle\langle q,p| = \mathbb{1}$ (resolution of unity)

affine coherent state quantization: $\boxed{f(q,p) \mapsto A_f := \int_{\mathbb{R}^+\times\mathbb{R}} \frac{\mathrm{d}q\mathrm{d}p}{2\pi\hbar\rho_0}|q,p\rangle\, f(q,p)\,\langle q,p|}$

$\longrightarrow \quad A_1 = \mathbb{1}$

$\longrightarrow \quad A_{q^\alpha} = \mathfrak{l}(\alpha)\widehat{Q}^\alpha$ impose $\mathfrak{l}(1) = 1$ to ensure $[A_q, A_p] = i\hbar$

$\longrightarrow \quad A_p = \widehat{P}$

$\longrightarrow A_{q^\alpha p^2} = \mathfrak{a}(\alpha)\widehat{Q}^\alpha\widehat{P}^2 - i\alpha\hbar\mathfrak{a}(\alpha)\widehat{Q}^{\alpha-1}\widehat{P} + \mathfrak{c}(\alpha)\hbar^2\widehat{Q}^{\alpha-2} = \mathfrak{a}(\alpha)\widehat{P}\widehat{Q}^\alpha\widehat{P} + \hbar^2\mathfrak{c}(\alpha)\widehat{Q}^{\alpha-2}$

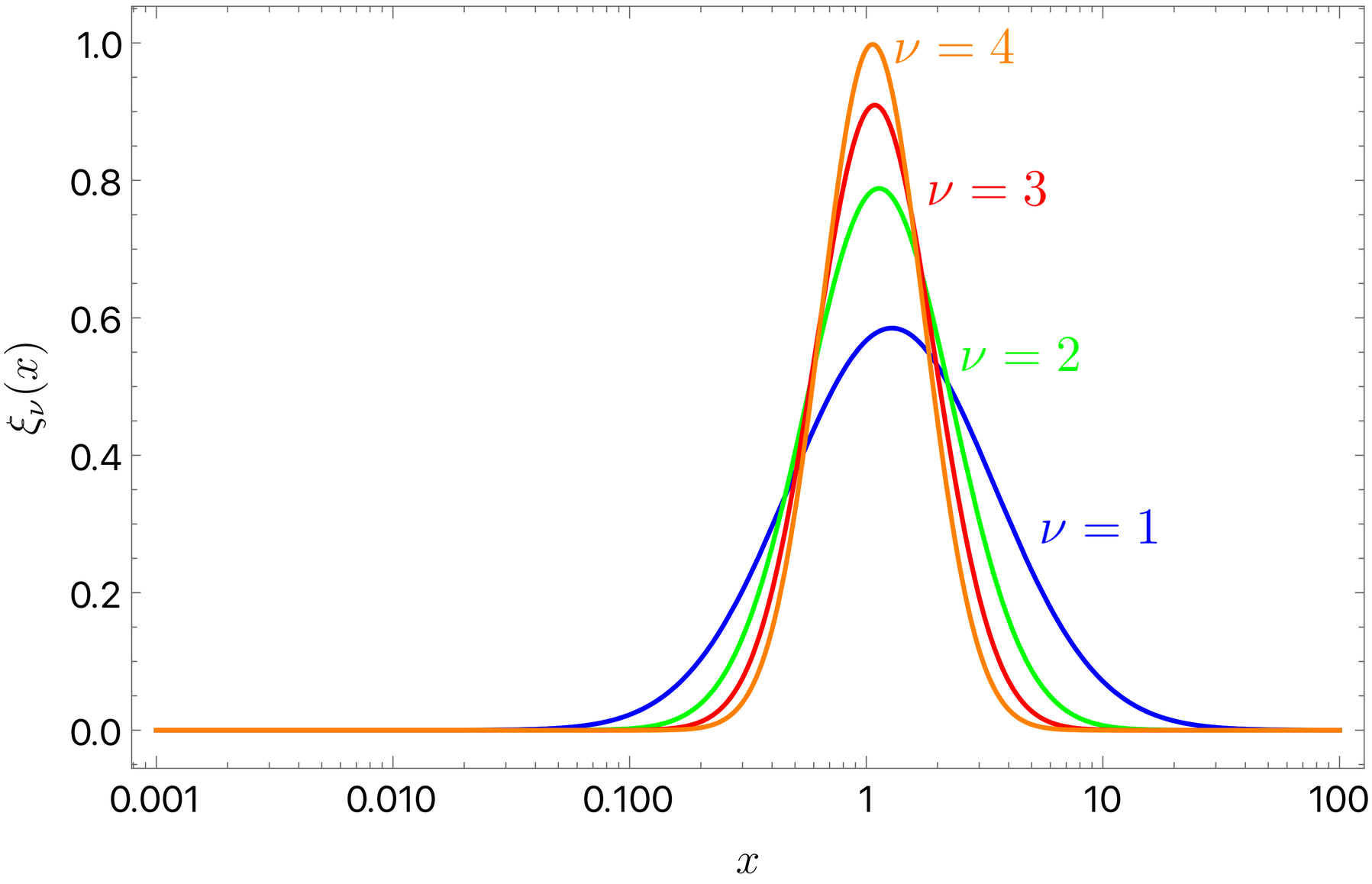
$\mathfrak{l}(\alpha) = \mathfrak{a}(\alpha) = \frac{\rho(\alpha)}{\rho_0}$

$\mathfrak{c}(\alpha) = \frac{1}{2}\alpha(1-\alpha)\mathfrak{a}(\alpha) + \frac{\sigma(\alpha-2)}{\rho_0}$

Example of fiducial vector family $\xi_\nu(x) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp\left[-\frac{\nu}{2}\left(\ln x - \frac{3}{4\nu}\right)^2\right]$

$\sigma_\nu(\alpha) = \left[\frac{\nu}{2} + \left(\frac{\alpha+2}{2}\right)^2\right] \exp\left[\frac{\alpha(\alpha+3)}{4\nu}\right]$

$\rho_\nu(\alpha) = \exp\left[\frac{(\alpha-2)(\alpha+1)}{4\nu}\right]$



$\rho_\nu(1) = \rho_\nu(0) = \mathrm{e}^{-1/(2\nu)}$

$\mathfrak{l}(\alpha) = \frac{\rho(\alpha)}{\rho_0} \Longrightarrow \mathfrak{l}(1) = 1$ ✓

Quantum Background Hamiltonian

$$H^{(0)} \mapsto \widehat{H}^{(0)} = 2\kappa_0 \left(\widehat{P}^2 + \hbar^2 \mathfrak{c}_0 \widehat{Q}^{-2} \right)$$

$$\mathfrak{c}_0 = \mathfrak{c}(0) = \frac{\sigma(-1)}{\rho_0} \;\rightarrow\; \frac{1}{2}\left(\nu + \frac{1}{2}\right)$$

$$\mathfrak{c}_0 \geqslant \frac{3}{4} \;\Leftrightarrow\; \nu \geqslant 1$$

$\widehat{H}^{(0)}$ essentially self-adjoint

$\mathfrak{c}_0 \to 0$ canonical quantization

plug back in perturbation

$$H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}|\pi_{\phi,\boldsymbol{k}}|^2 + \frac{1}{2} w (1+w)^2 \left(\frac{q}{\gamma}\right)^{4r_1} k^2 |\phi_{\boldsymbol{k}}|^2$$

⊕ Canonical quantization of perturbations

F-*parameterization quantization*

$$\phi_{\boldsymbol{k}} \mapsto \widehat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}} \left[a_{\boldsymbol{k}} \phi_k^*(\tau) + a^\dagger_{-\boldsymbol{k}} \phi_k(\tau) \right]$$

C-*parameterization quantization*

$$v_{\boldsymbol{k}} \mapsto \widehat{v}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}} \left[a_{\boldsymbol{k}} v_k^*(\tau) + a^\dagger_{-\boldsymbol{k}} v_k(\tau) \right]$$

F-*parameterization quantization*

$$\phi_{\boldsymbol{k}} \mapsto \widehat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\phi_k^*(\tau) + a^\dagger_{-\boldsymbol{k}}\phi_k(\tau)\right]$$

$$\mathfrak{L}_{\mathrm{Q}} = \mathfrak{l}(4r_1)$$

One free parameter

$$\widehat{H}^{(2)}_{\boldsymbol{k}} = \frac{1}{2}|\widehat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\mathrm{Q}}}{2} w(1+w)^2 \left(\frac{\widehat{Q}}{\gamma}\right)^{4r_1} k^2|\widehat{\phi}_{\boldsymbol{k}}|^2$$

C-*parameterization quantization*

$$v_{\boldsymbol{k}} \mapsto \widehat{v}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}v_k^*(\tau) + a^\dagger_{-\boldsymbol{k}}v_k(\tau)\right]$$

$$\widehat{H}^{(2)}_{\boldsymbol{k}} = \frac{1}{2}(1+w)\left(\frac{\widehat{Q}}{\gamma}\right)^{2r_1} \mathfrak{M}_{\mathrm{Q}} \left\{|\widehat{\pi}_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \frac{8\mathfrak{M}_{\mathrm{Q}}^{-1}}{9\widehat{Q}^2}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2(1+w)^2}\left(\frac{\widehat{Q}}{\gamma}\right)^{-4r_1}\left(\mathfrak{N}_{\mathrm{Q}}\widehat{P}^2 + i\hbar\mathfrak{R}_{\mathrm{Q}}\widehat{Q}^{-1}\widehat{P} + \hbar^2\mathfrak{T}_{\mathrm{Q}}\widehat{Q}^{-2}\right)\right]|\widehat{v}_{\boldsymbol{k}}|^2\right\}$$

$$\mathfrak{M}_{\mathrm{Q}} = \mathfrak{l}(2r_1)$$

$$\mathfrak{N}_{\mathrm{Q}} = \mathfrak{a}(-2r_2) \qquad \mathfrak{T}_{\mathrm{Q}} = \mathfrak{c}(-2r_2)$$

$$\mathfrak{R}_{\mathrm{Q}} = \mathfrak{b}(-2r_2) = 2r_2\mathfrak{N}_{\mathrm{Q}}$$

Three free parameters

Coherent state semiclassical approximation

new fiducial state $|\tilde{\xi}\rangle$ satisfying $\langle\tilde{\xi}|\widehat{Q}|\tilde{\xi}\rangle = 1$ (physical centering condition)

time-dependent functions $q(\tau)$ and $p(\tau)$

define time-dependent coherent states $|q(\tau), p(\tau)\rangle = \mathrm{e}^{ip(\tau)\widehat{Q}/\hbar}\mathrm{e}^{-i\ln q(\tau)\widehat{D}/\hbar}|\tilde{\xi}\rangle$

$$\left.\begin{aligned}\langle\widehat{Q}\rangle &= q(\tau)\\ \langle\widehat{P}\rangle &= p(\tau)\end{aligned}\right\}$$

$$\mathcal{S}_{\mathrm{Sch}} = \int \langle q(\tau), p(\tau)| \left(i\hbar\frac{\partial}{\partial\tau} - \widehat{H}\right) |q(\tau), p(\tau)\rangle \mathrm{d}\tau = \int \{\dot{q}(\tau)p(\tau) - H_{\mathrm{sem}}[q(\tau), p(\tau)]\}\,\mathrm{d}\tau$$

$$\langle q(\tau), p(\tau)|\widehat{H}|q(\tau), p(\tau)\rangle$$

$$\left.\begin{aligned}\dot{q} &= \frac{\partial H_{\mathrm{sem}}}{\partial p}\\ \dot{p} &= -\frac{\partial H_{\mathrm{sem}}}{\partial q}\end{aligned}\right\}$$

define $\quad \tilde{\rho}(\alpha) = \int \frac{\langle\tilde{\xi}|x\rangle\langle x|\tilde{\xi}\rangle}{x^{\alpha+1}}\,\mathrm{d}x = \int \frac{|\tilde{\xi}(x)|^2}{x^{\alpha+1}}\,\mathrm{d}x < \infty$

and $\quad \tilde{\sigma}(\alpha) = \int \left|\frac{\mathrm{d}\tilde{\xi}(x)}{\mathrm{d}x}\right|^2 \frac{\mathrm{d}x}{x^{\alpha+1}}$

expectation values of operators

$\longrightarrow \quad \langle q,p|\widehat{Q}^\alpha|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha$

$\longrightarrow \langle q,p|\widehat{Q}^\alpha\widehat{P}|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha p + i\frac{\alpha}{2}\tilde{\rho}(-\alpha)q^{\alpha-1}$

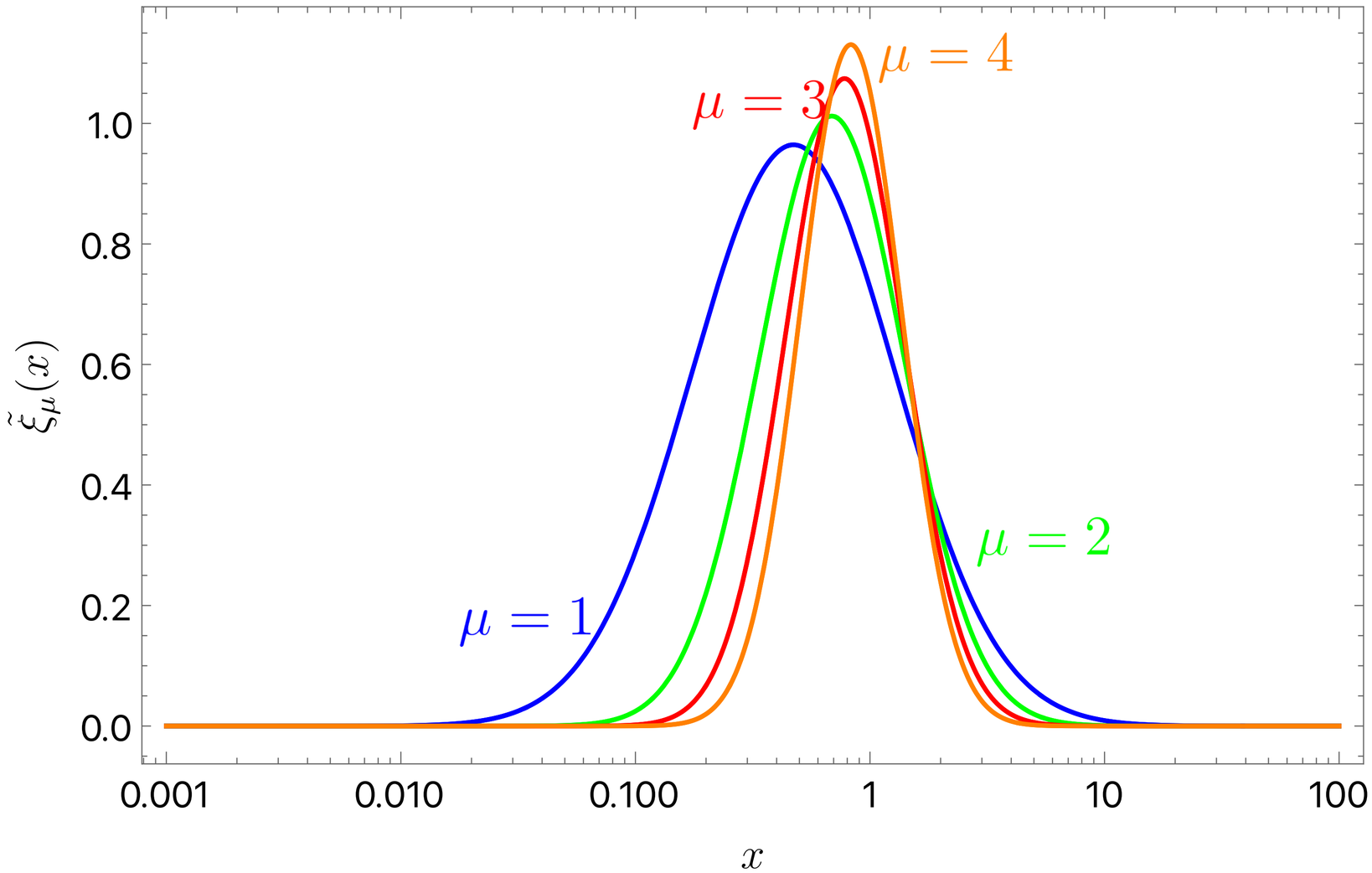
$\longrightarrow \langle q,p|\widehat{Q}^\alpha\widehat{P}^2|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha p^2 + i\alpha\tilde{\rho}(-\alpha)q^{\alpha-1}p + \left[\tilde{\sigma}(-\alpha-1) + \frac{\alpha(1-\alpha)}{2}\tilde{\rho}(-\alpha+1)\right] q^{\alpha-2}$

Necessary choice: $\tilde{\rho}(-2) = 1$ to ensure $\langle\tilde{\xi}|\widehat{Q}|\tilde{\xi}\rangle = \int |\tilde{\xi}(x)|^2 x\mathrm{d}x = 1$

Example of fiducial vector family $\tilde{\xi}_\mu(x) = \left(\frac{\mu}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp\left[-\frac{\mu}{2}\left(\ln x + \frac{1}{4\mu}\right)^2\right]$

$\sigma_\nu(\alpha) = \left[\frac{\nu}{2} + \left(\frac{\alpha+2}{2}\right)^2\right] \exp\left[\frac{\alpha(\alpha+3)}{4\nu}\right]$

$\tilde{\rho}_\mu(\alpha) = \exp\left[\frac{(\alpha+1)(\alpha+2)}{4\mu}\right]$



$\tilde{\rho}(1) = \mathrm{e}^{3/(2\mu)}$

$\neq \mathrm{e}^{1/(2\mu)} = \tilde{\rho}(0)$

not for quantization

$\tilde{\rho}_\mu(-2) = 1$ ✓

back to the quantum Hamiltonian $\widehat{H}^{(0)} = 2\kappa_0 \left(\widehat{P}^2 + \hbar^2 \mathfrak{c}_0 \widehat{Q}^{-2} \right)$

$$\longrightarrow \quad H_{\rm sem} = 2\kappa_0 \left(p^2 + \frac{\hbar^2 \mathfrak{K}}{q^2} \right) \qquad \text{with} \qquad \mathfrak{K} = \mathfrak{c}_0 \tilde{\rho}(1) + \tilde{\sigma}(-1)$$

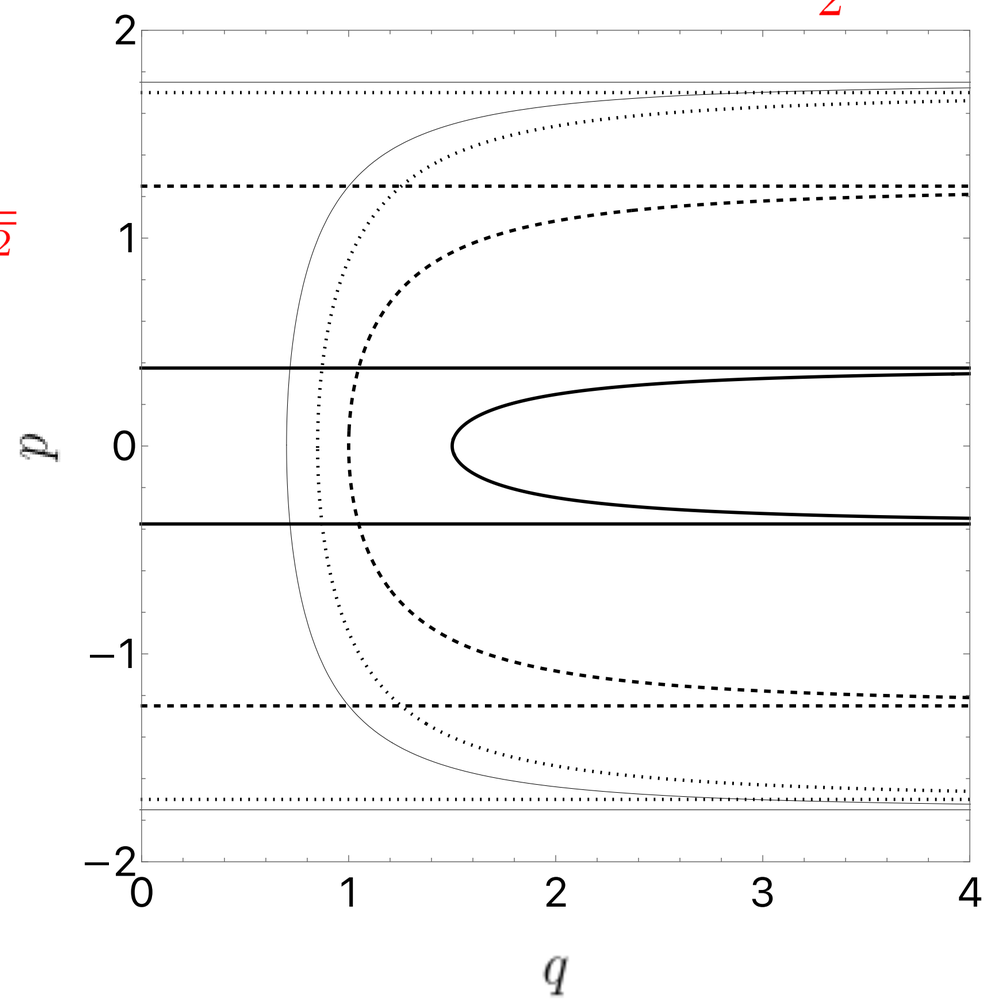
$$\mathfrak{K} = \frac{\nu}{2} + \frac{2\mu+1}{4} \exp\left(\frac{3}{2\mu}\right)$$

semiclassical trajectory in phase space

$$\begin{cases} q = q_{\rm B} \sqrt{1 + (\omega\tau)^2} \\ p = \dfrac{q_{\rm B}\omega^2}{4\kappa_0} \dfrac{\tau}{\sqrt{1 + (\omega\tau)^2}} \end{cases}$$

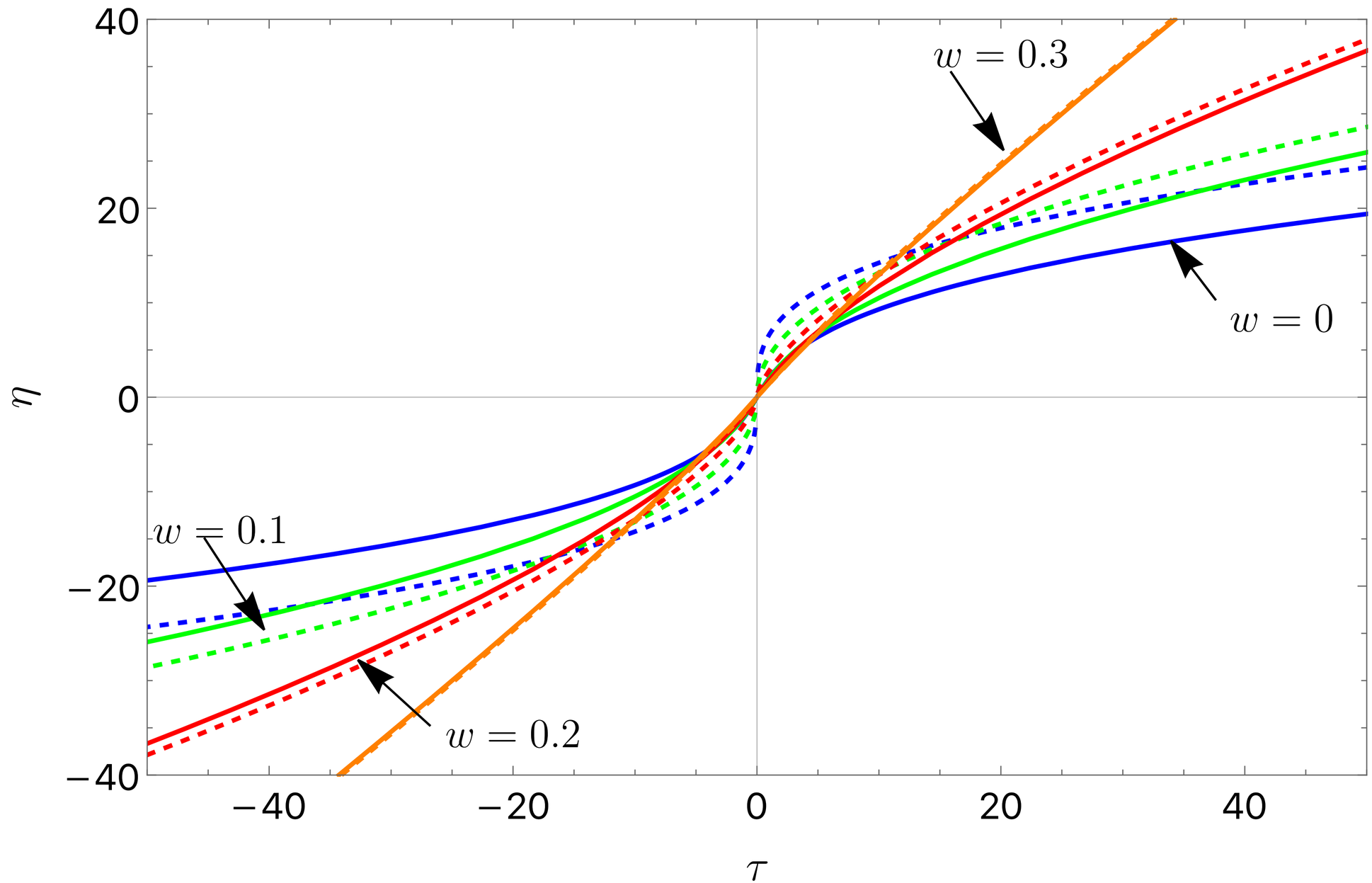
$$q_{\rm B}^2 = \frac{2\kappa_0 \hbar^2 \mathfrak{K}}{H_{\rm sem}}$$

$$\omega = \frac{2H_{\rm sem}}{\hbar\sqrt{\mathfrak{K}}}$$



conformal time $\eta(\tau)$

$$\eta = (1+w)\tau \left(\frac{q_{\rm B}}{\gamma}\right)^{2r_1} F\left[\frac{1}{2}, -r_1; \frac{3}{2}; -(\omega\tau)^2\right]$$



Full quantum dynamics: $|\psi(\tau)\rangle = |\psi_{\rm B}(\tau)\rangle \otimes |\psi_{\rm P}(\tau)\rangle$

$|q(\tau), p(\tau)\rangle$ $\qquad$ $|\psi_{\rm P}\rangle = \prod_{\boldsymbol{k}} |\psi_{\boldsymbol{k}}\rangle$

2nd order quantum action

$$\mathcal{S}^{(0)+(2)} = \mathcal{S}_{\rm B} + \mathcal{S}_{\rm P} = \int \langle\psi(\tau)| \left(i\hbar\frac{\partial}{\partial\tau} - \widehat{H}^{(0)} - \sum_{\boldsymbol{k}} \widehat{H}^{(2)}_{\boldsymbol{k}} \right) |\psi(\tau)\rangle \mathrm{d}\tau$$

$$\int \langle\psi_{\rm P}| \left(i\hbar\frac{\partial}{\partial\tau} - \sum_{\boldsymbol{k}} \underbrace{\langle q(\tau), p(\tau)|\widehat{H}^{(2)}_{\boldsymbol{k}}|q(\tau), p(\tau)\rangle}_{\widehat{H}_{\boldsymbol{k}}} \right) |\psi_{\rm P}\rangle \mathrm{d}\tau$$

Schrödinger equations for the perturbations $\quad i\hbar\frac{\partial}{\partial\tau}|\psi_{\boldsymbol{k}}\rangle = \widehat{H}_{\boldsymbol{k}}|\psi_{\boldsymbol{k}}\rangle$

From that point on, business as usual...

F-*parameterization quantization*

$$\mathfrak{L}_{\rm S} = \mathfrak{L}_{\rm Q}\tilde{\rho}\,(-4r_1 - 1)$$

$$\widehat{H}_{\boldsymbol{k}} = \frac{1}{2}|\widehat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\rm S}}{2} w(1+w)^2 \left(\frac{q}{\gamma}\right)^{4r_1} k^2 |\widehat{\phi}_{\boldsymbol{k}}|^2$$

Heisenberg e.o.m.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\phi}_{\boldsymbol{k}} = -\widehat{\pi}_{\phi,\boldsymbol{k}}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\pi}_{\phi,\boldsymbol{k}} = \mathfrak{L}_{\rm S} w(1+w)^2 \left(\frac{q}{\gamma}\right)^{4r_1} k^2 \widehat{\phi}_{\boldsymbol{k}}$$

mode expansion

$$\widehat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\phi_k^*(\tau) + a^\dagger_{-\boldsymbol{k}}\phi_k(\tau)\right]$$

$$\widehat{\pi}_{\phi,\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\dot{\phi}_k^*(\tau) + a^\dagger_{-\boldsymbol{k}}\dot{\phi}_k(\tau)\right]$$

} canonical commutation relations → $\dot{\phi}_k\phi_k^* - \phi_k\dot{\phi}_k^* = 2i$

} → 2nd order o.d.e $\ddot{\phi}_k \cdots$

Mukhanov-Sasaki variable in the F-parameterization $v_k^{\rm F} = Z\phi_k$

$$\frac{\mathrm{d}^2 v_k^{\rm F}}{\mathrm{d}\eta^2} + \left[k_{\rm F}^2 - \mathcal{V}_{\rm F}(\eta)\right] v_k^{\rm F} = 0$$

F-*parameterization quantization*

$$\widehat{H}_{\boldsymbol{k}} = \frac{1}{2}|\widehat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\rm S}}{2} w(1+w)^2 \left(\frac{q}{\gamma}\right)^{4r_1} k^2 |\widehat{\phi}_{\boldsymbol{k}}|^2$$

$$\frac{{\rm d}^2 v_k^{\rm F}}{{\rm d}\eta^2} + \left[k_{\rm F}^2 - \mathcal{V}_{\rm F}(\eta)\right] v_k^{\rm F} = 0$$

$$\mathfrak{L}_{\rm S} w k^2$$

$$\frac{8}{9q^2 Z^4} \frac{(2\kappa_0)^2(1-3w)}{(1-w)^2} \left[p^2 - \frac{3(1-w)\mathfrak{K}}{2q^2}\right]$$

$$= \frac{(q^{r_1})''}{q^{r_1}}$$ upon using the background e.o.m.

$$\lim_{\eta\to\infty} \mathcal{V}_{\rm F}(\eta) = \mathcal{V}_{\rm cl}(\eta)$$

C-*parameterization quantization*

$$\widehat{H}_{\boldsymbol{k}} = \frac{1}{2} Z^2 \mathfrak{M}_{\rm S} \left\{ |\widehat{\pi}_{v,k}|^2 + \left[w k^2 - \frac{8 \mathfrak{M}_{\rm S}^{-1}}{9 q^2 Z^4} \frac{(2\kappa_0)^2 (1-3w)}{(1-w)^2} \left(\mathfrak{N}_{\rm S} p^2 + \frac{\hbar^2 \mathfrak{T}_{\rm S}}{q^2} \right) \right] |\widehat{v}_k|^2 \right\}$$

$$\mathfrak{M}_{\rm Q} \tilde{\rho} \, (-2 r_1 - 1)$$

$$\mathfrak{N}_{\rm Q} \tilde{\rho} \, (2 r_2 - 1)$$

$$\mathfrak{N}_{\rm Q} \tilde{\sigma} (2 r_2 - 1) + \mathfrak{T}_{\rm Q} \tilde{\rho} \, (2 r_2 + 1)$$

Wronskian $\dot{v}_k v_k^* - v_k \dot{v}_k^* = 2 i Z^2 \mathfrak{M}_{\rm S}$

$$v_k' v_k^* - v_k v_k^{*\prime} = 2 i \mathfrak{M}_{\rm S}$$

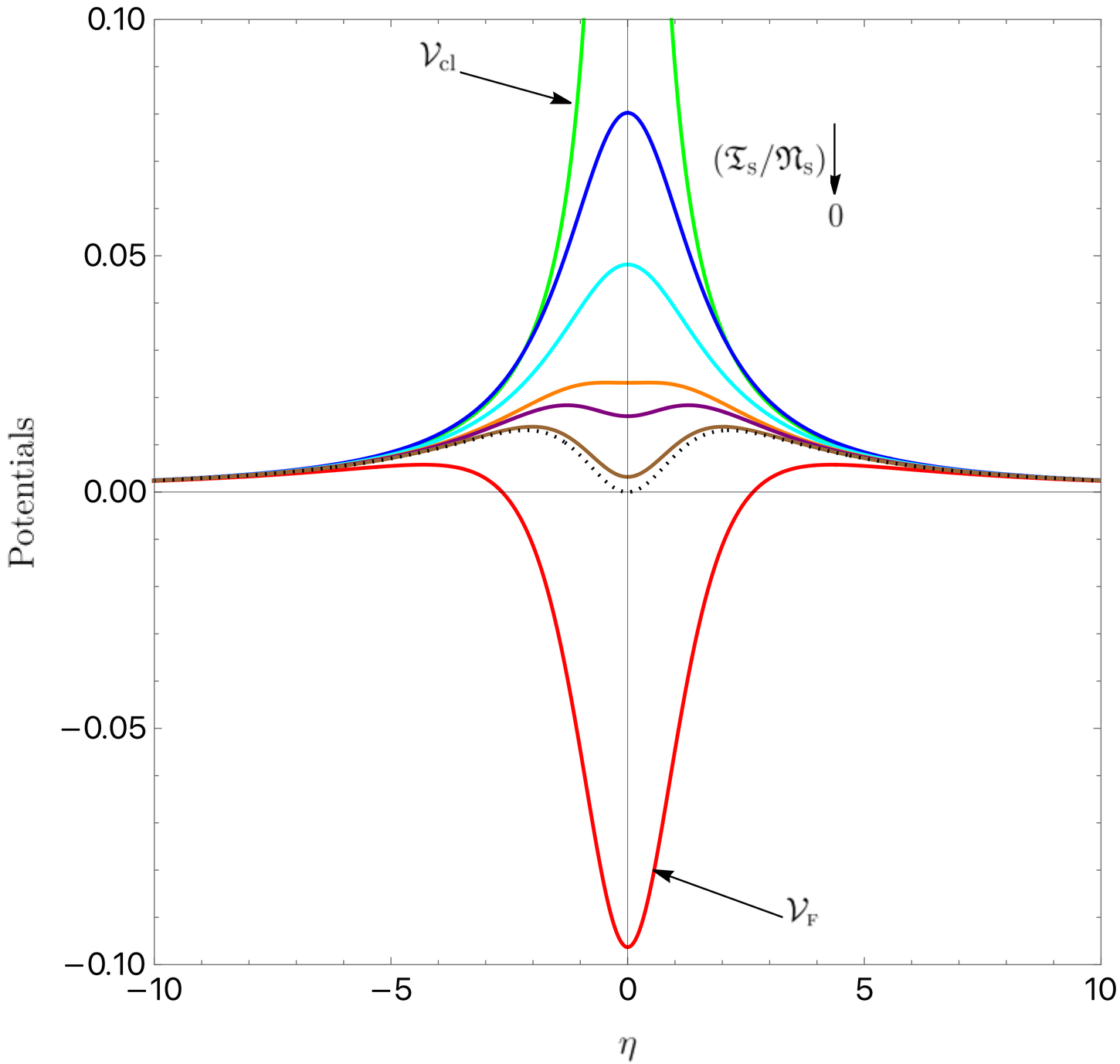
$$\frac{{\rm d}^2 v_k^{\rm C}}{{\rm d}\varsigma^2} + \left[k_{\rm C}^2 - \mathcal{V}_{\rm C}(\varsigma) \right] v_k^{\rm C} = 0$$

$$\varsigma = \sqrt{\mathfrak{M}_{\rm S} \mathfrak{N}_{\rm S}} \, \eta$$

rescaled

$$w \frac{\mathfrak{M}_{\rm S}}{\mathfrak{N}_{\rm S}} k^2$$

$$\frac{8}{9 q^2 Z^4} \frac{(2\kappa_0)^2 (1-3w)}{(1-w)^2} \left(p^2 + \frac{\hbar^2 \mathfrak{T}_{\rm S} / \mathfrak{N}_{\rm S}}{q^2} \right)$$


0.10
0.05
0.00
−0.05
−0.10
−10
−5
0
5
10
Potentials
η
$\mathcal{V}_{\rm cl}$
$(\mathfrak{T}_{\rm s}/\mathfrak{N}_{\rm s})$
0
$\mathcal{V}_{\rm F}$

special case

$$\frac{\mathfrak{T}_{\mathrm{S}}}{\mathfrak{N}_{\mathrm{S}}} = \frac{3\mathfrak{K}(1-w)}{1-3w}$$

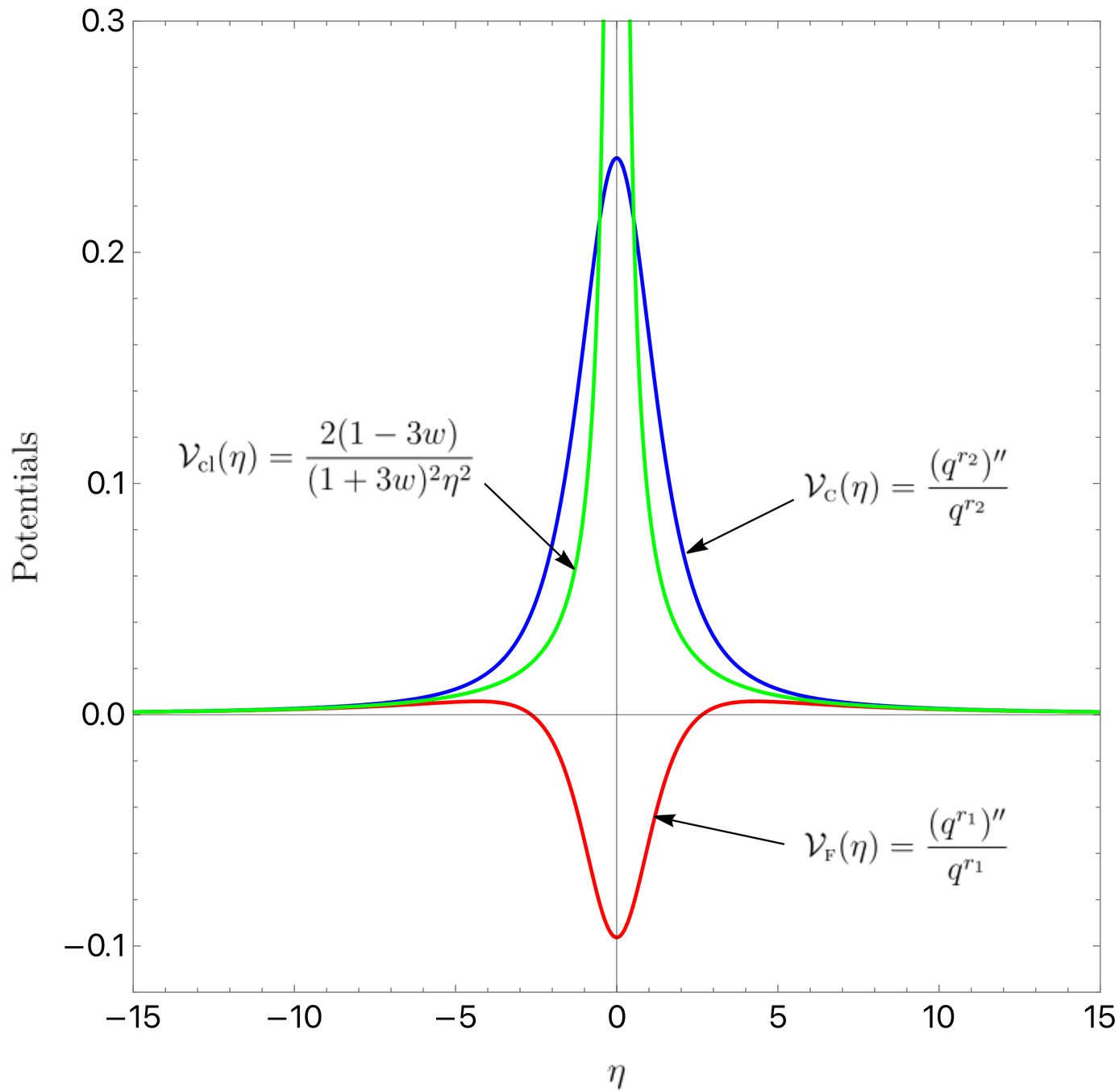


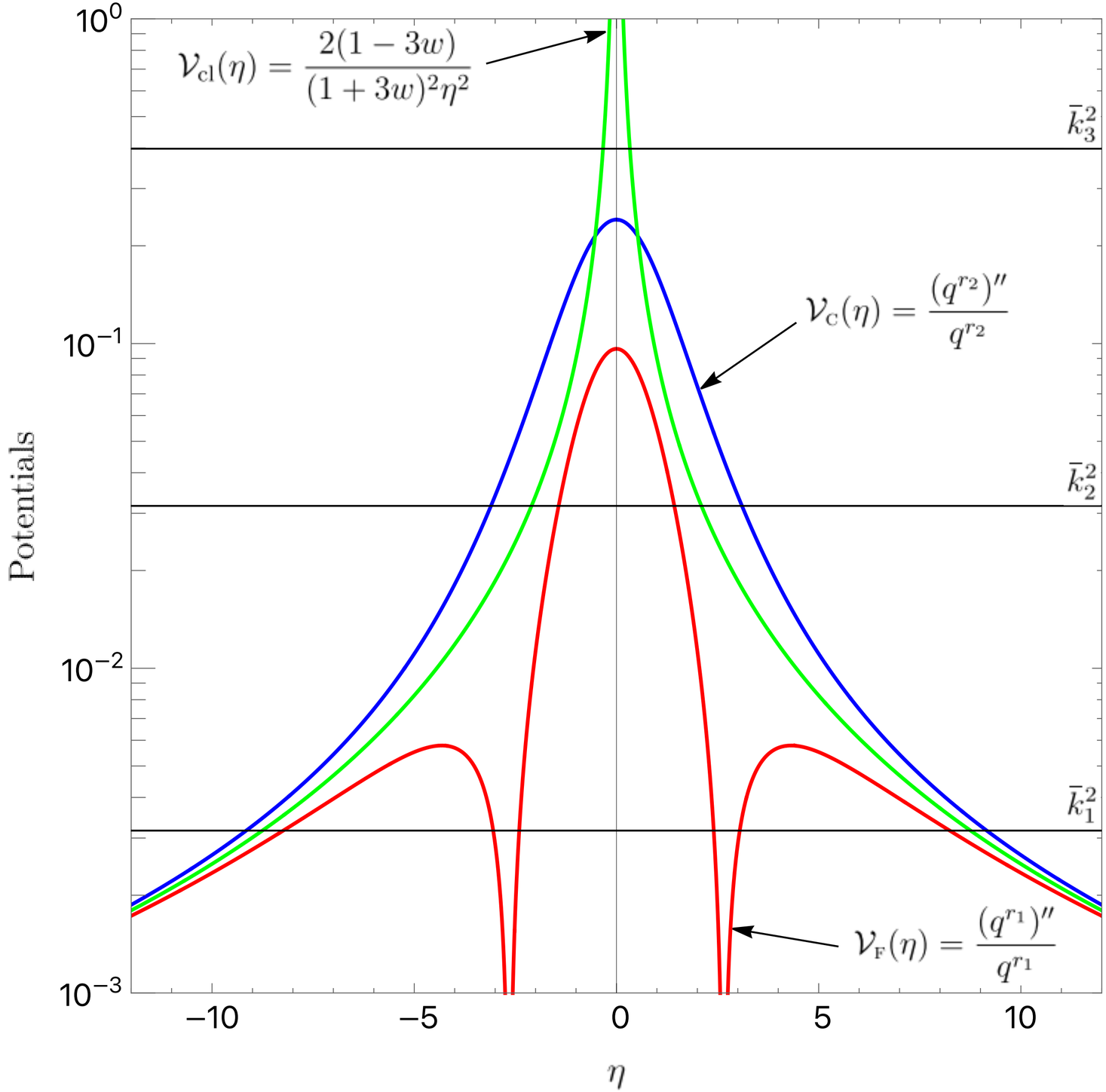
$$\mathcal{V}_{\mathrm{C}} = \frac{(q^{r_2})''}{q^{r_2}}$$

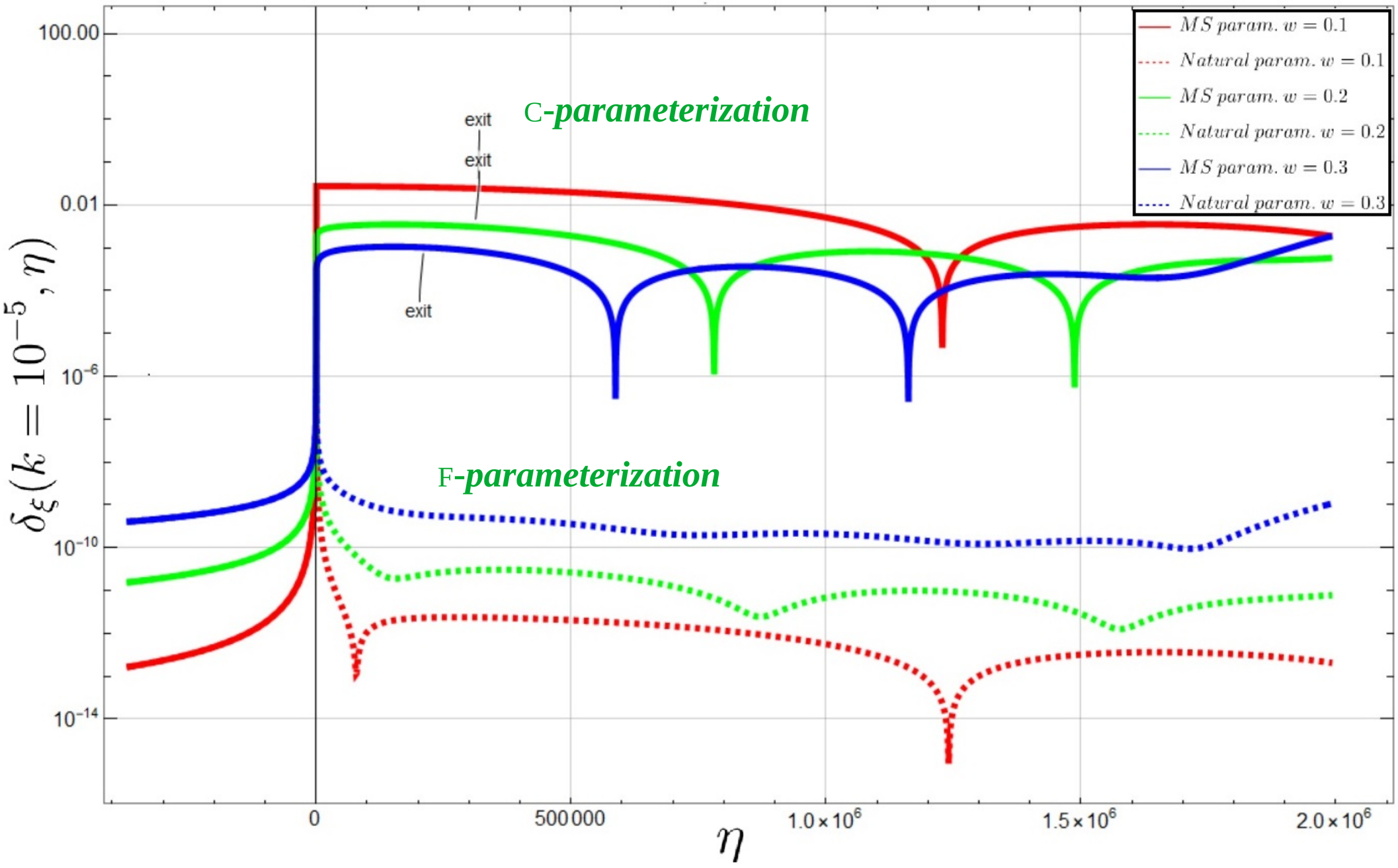
special case

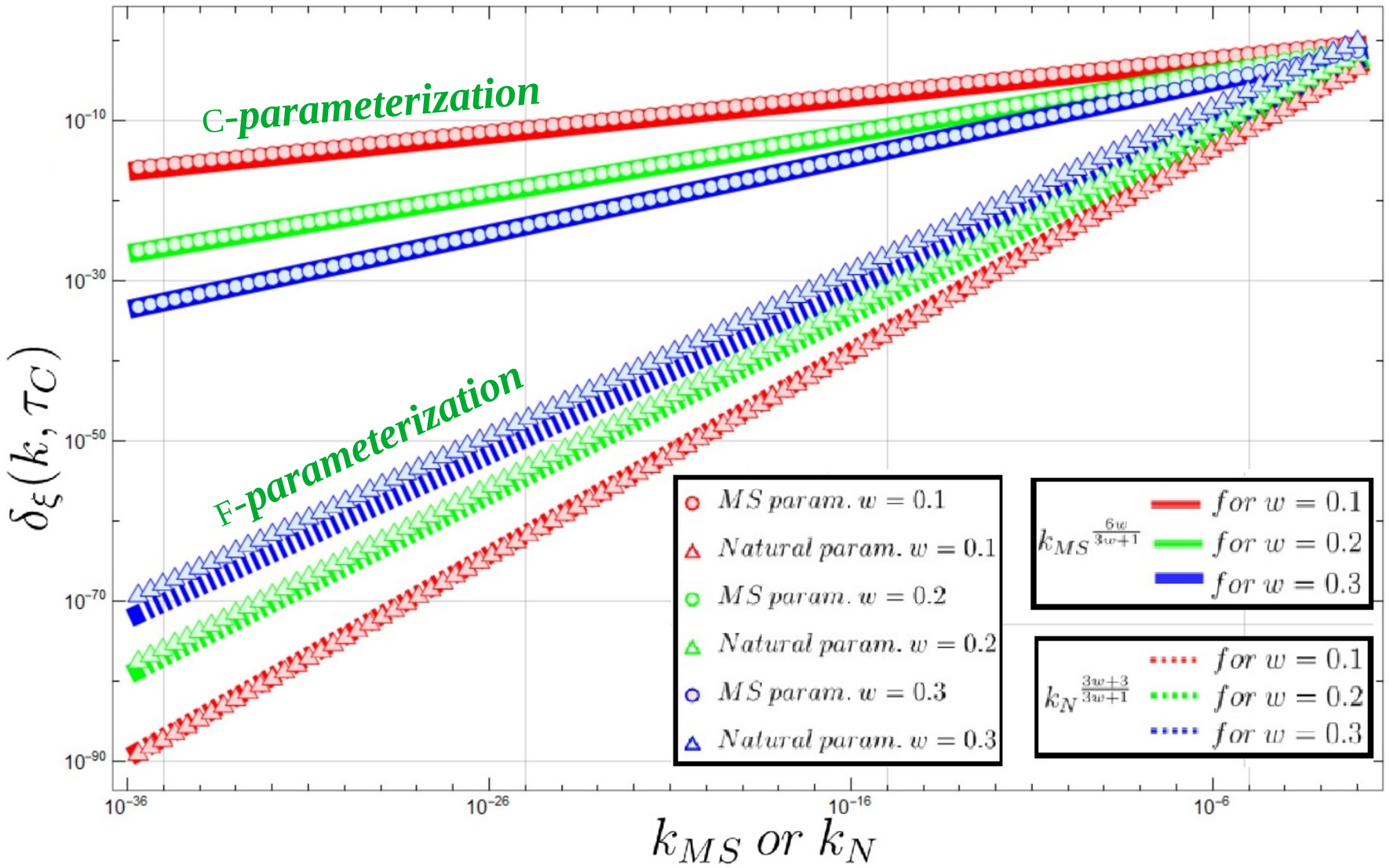
$$\frac{\mathfrak{T}_{\rm S}}{\mathfrak{N}_{\rm S}} = \frac{3\mathfrak{K}(1-w)}{1-3w}$$

$$\mathcal{V}_{\rm C} = \frac{(q^{r_2})''}{q^{r_2}}$$




10^0
10^{-1}
10^{-2}
10^{-3}
Potentials
$\mathcal{V}_{\rm cl}(\eta) = \dfrac{2(1-3w)}{(1+3w)^2\eta^2}$
$\mathcal{V}_{\rm C}(\eta) = \dfrac{(q^{r_2})''}{q^{r_2}}$
$\mathcal{V}_{\rm F}(\eta) = \dfrac{(q^{r_1})''}{q^{r_1}}$
$\bar{k}_3^2$
$\bar{k}_2^2$
$\bar{k}_1^2$
−10
−5
0
5
10
η


C-parameterization
F-parameterization
exit
exit
exit
$MS\ param.\ w = 0.1$
$Natural\ param.\ w = 0.1$
$MS\ param.\ w = 0.2$
$Natural\ param.\ w = 0.2$
$MS\ param.\ w = 0.3$
$Natural\ param.\ w = 0.3$
100.00
0.01
10^{-6}
10^{-10}
10^{-14}
$\delta_\xi(k = 10^{-5}, \eta)$
0
500000
1.0×10^6
1.5×10^6
2.0×10^6
η


C-parameterization
F-parameterization
$\delta_\xi(k, \tau_C)$
10^{-10}
10^{-30}
10^{-50}
10^{-70}
10^{-90}
10^{-36}
10^{-26}
10^{-16}
10^{-6}
k_{MS} or k_N
$MS\ param.\ w = 0.1$
$Natural\ param.\ w = 0.1$
$MS\ param.\ w = 0.2$
$Natural\ param.\ w = 0.2$
$MS\ param.\ w = 0.3$
$Natural\ param.\ w = 0.3$
$k_{MS}{}^{\frac{6w}{3w+1}}$ for $w = 0.1$ for $w = 0.2$ for $w = 0.3$
$k_N{}^{\frac{3w+3}{3w+1}}$ for $w = 0.1$ for $w = 0.2$ for $w = 0.3$

Conclusions?

