Chapter 7

Cosmological applications of QFT in curved spacetimes
Vladimir M. Mostepanenko

Cosmological Applications of QFT in Curved Spacetimes V.M. Mostepanenko [6]*Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, CEP 58059–970, João Pessoa, Pb—Brazil* (on leave from *A.Friedmann Laboratory for Theoretical Physics, St.Petersburg, Russia)*

Abstract. We consider the Quantum Field Theory in Curved Spacetime and application of its results to the early cosmology. After consideration of classical wave equations in Riemannian spacetime the quantization procedure and construction of the Fock space are presented. The special attention is paid for the definition of the concept of particles, structure of infinities of the vacuum stress-energy tensor and of different regularization and renormalization procedures. The effects of particle creation from vacuum and vacuum polarization are investigated detailly in homogeneous isotropic cosmological models. The problem of back influence of vacuum quantum effects onto the background spacetime is examined. It is shown that the vacuum stress-energy tensor of quantized fields gives rise to inflationary cosmological solutions. The probable mechanisms are analysed of smoth transition between inflationary and Friedmann stages of the Universe evolution including reheating. For this purpose the vacuum stress-energy tensor of nonconformal scalar field is calculated and the problem of the proper choice of initial quantum state of the Universe is discussed.

7.1 Introduction

The classical Newtonian gravity, as it is well known, leads to contradictions in application to all the Universe. Scientific cosmology, which is the science, describing all the Universe, appeared only after Einstein created the General Relativity Theory [18].

The solutions of Einstein equations

$$R_{ik} - \frac{1}{2} R\, g_{ik} + \Lambda\, g_{ik} = -8\pi G\, T_{ik}, \tag{5.1}$$

where T_{ik} is the stress-energy tensor (SET) of the matter, distributed in the Universe, give us the spacetime structure of the Universe described by the metrical tensor g_{ik}. It is the so called "cosmological models". Here and below R_{ik} is the Ricci tensor, R is a scalar curvature, Λ and G are the cosmological and gravitational constants, $\hbar = c = 1$.

There are different cosmological models depending on suppositions made about the spacetime properties and matter distribution.

The first cosmological models, proposed by Einstein himself, were static. In 1922 St.Petersburg scientist Alexander Friedmann discovered firstly the non-stationary cosmological models whose properties are time-dependent. According to these models the space distances between the galaxies should increase with time. This was confirmed experimentally in thirties when Hubble discovered the red shift in the spectra of galaxies: larger shift for more remote galaxy. This means that the Universe is expanding. On the base of Friedmann cosmological models Gamov elaborated the theory of hot Universe [25] according to which expanding started from some initial moment when all the matter was compressed into a singularity with infinite matter density and temperature. Near the singularity matter density, pressure and temperature are also very high. The theory of hot Universe was confirmed experimentally by Penzias and Wilson who discovered in 1965 the relict, or primordial, microwave radiation, i.e. photons, remaining from the very early stage of the Universe evolution.

According to the Friedmann models, the first stage of the Universe evolution is radiation dominated one with matter equation state $P = \varepsilon/3$ and is described by the scale factor of the metric

$$a(t) \sim \sqrt{t}, \tag{5.2}$$

where t is the proper synchronous time, ε is the energy density and P is the pressure of background matter.

When the temperature decreases the Universe evolution becomes dust-like with matter equation state $P = 0$ and the scale factor changes for

$$a(t) \sim t^{2/3}. \tag{5.3}$$

With $t \to 0$ we have $a(t) \to 0$ also, and background energy density $\varepsilon \to \infty$. So the moment $t = 0$ is a real singularity which is anomalous, non-physical state. The presence of singularities in the Friedmann cosmological solutions forces us to conclude that these solutions do not describe correctly the very early state of the Universe.

The other physical contradiction connected with the application of the Friedmann cosmology to the very early Universe is the problem of causality, or of a horizon.

The temperature of primordial radiation observed in different directions is the same and equal approximately to 2.7 K with relative error $\Delta T/T < 10^{-5}$. This fact testifies that there was a thermal equilibrium between different causally non-connected regions in the past which were tranferred into observed in present time part of the Universe of the dimension $\sim 10^{28}$ cm. Actually, the age of the Universe is $\sim 10^{10}$ years. If to suppose that the Universe was expanding according the Friedmann law (2), (3) $a(t) \sim \sqrt{t}, t^{2/3}$, then it occurs by simple calculations that at Planck moment $t_{Pl} \sim 10^{-43}$ s the size of this part of the Universe was $\sim 10^{-3}$ cm. At the same time the size of causally connected region at Planck moment is 10^{-33} cm. Therefore 10^{90} causally non-connected regions at t_{Pl} ought to be somehow correlated to settle a thermal equilibrium between them.

It is impossible to imagine how such a preforeseen harmony could be established physically. That is why Friedmann expansion law $a(t) \sim \sqrt{t}$ could not be valid from the very beginning.

Aside of these phenomenological difficulties there are serious theoretical expectations that the Friedmann cosmology is not applicable for the very early times. Friedmann cosmological models were obtained by solving the classical Einstein equations. But the General Relativity Theory, as the other physical theories, has some application range. It may be applied when the gravitational field is not so strong that the spacetime curvature is characterized by the Planck length, $l_{Pl} = \sqrt{G} \sim 10^{-33}$ cm. Also the processes under consideration should occur at the distances much larger than l_{Pl}. This means that one can not use Friedmann solution to Einstein equations for too early moments $t < t_{Pl} \sim 10^{-43}$ s. For such moments not Einstein gravity but Quantum Theory of Gravitation should be used for the adequate description of the Universe evolution.

From the other side, quantum effects of matter fields manifest themselves at the scale of Compton length $l_C = m^{-1} \sim 10^{-13}$ cm (for the usual elementary particles). This means that actually one can not use the usual General Relativity Theory even for $t < t_C \sim 10^{-23}$ s.

In the wide range of twenty orders of magnitude

$$10^{-43}\,\mathrm{s} \sim t_{Pl} < t < t_C \sim 10^{-23}\,\mathrm{s} \tag{5.4}$$

gravitational field itself may be considered as classical but the matter fields should be considered as quantized. Then in this range some kind of semiclassical Einstein equations should be considered instead of (1):

$$R_{ik} - \frac{1}{2} R\, g_{ik} + \Lambda\, g_{ik} = -8\pi G\, \langle \Psi | \hat{T}_{ik} | \Psi \rangle_{ren}, \tag{5.5}$$

where $\hat{T}_{ik}$ is the operator of the stress-energy tensor of matter fields, $|\Psi\rangle$ is some quantum state, and index "ren" means that all the infinities are removed from the matrix element.

Consideration of quantized matter fields on the classical gravitational background is the subject of Quantum Field Theory in Curved Spacetimes [2,9,10] to which these lectures are devoted. QFT in Curved Spacetime has its own application range which is expressed by the inequalities (4) in application to cosmological evolution.

Generally speaking, for the complete solution of the problems of the early Friedmann cosmology both Quantum Gravity and its limiting case QFT in Curved Spacetimes are desirable. Unfortunately, we have no consistent, renormalizable QG up to now, after the several decades of attempts. In this situation QFT in Curved Spacetimes is the single reliable foundation for solution of the problems of the Friedmann early cosmology.

In these lectures we discuss the main principles of QFT in Curved Spacetimes, and consider the obtaining of inflation cosmological scenario on the basis of this theory without introducing by hands so named "inflaton" field with extremely small self-interaction. In the end we will discuss the difficulties in choice of the initial vacuum state which were met by the QFT in Curved Spacetimes and the prospectives of this theory in solving cosmological problems.

7.2 Wave equations for classical fields in curved spacetimes

Let us consider briefly scalar, vector and spinor fields in external gravitational field. They describe in quantum theory three types of fundamental particles with spin 0, 1 and 1/2.

The simplest generalization of the scalar field equation for the case of gravitational background is

$$\left(\nabla_i \nabla^i + m^2\right) \varphi(x) = 0,$$

where ∇_i is the covariant derivative. It is the so called equation with minimal coupling. This equation is not invariant, even in the massless limit, under conformal transformations of the metric and of the field

$$\begin{aligned} g_{ik} &\to \tilde{g}_{ik} = \exp[-2\sigma(x)]g_{ik}, \\ \varphi &\to \tilde{\varphi} = F[\sigma(x)]\varphi, \end{aligned} \tag{5.6}$$

where $\sigma(x)$ is an arbitrary smooth function of the coordinates.

The scalar wave equation with arbitrary coupling to gravitation

$$\left(\nabla_i \nabla^i + \xi R + m^2\right) \varphi(x) = 0 \tag{5.7}$$

is more general one (ξ is the arbitrary real constant). For

$$\xi = \xi_c = \frac{N-2}{4(N-1)} \tag{5.8}$$

it is conformally invariant (in N-dimensional spacetime) when $m = 0$ with

$$F[\sigma(x)] = \exp\left[\frac{N-2}{2}\sigma(x)\right]. \tag{5.9}$$

Eq. (7) may be obtained from the Lagrangian

$$\mathcal{L}^{(0)} = \sqrt{-g}\left[g^{ik}\,\partial_i\varphi^*\,\partial_k\varphi - \left(m^2 + \xi\,R\right)\,\varphi^*\varphi\right], \tag{5.10}$$

where $g \equiv \det g_{ik}$.

Varying the action with the Lagrangian (10) with respect to g^{ik} we obtain the metric SET

$$T_{ik}{}^{(0)}(x) = \partial_i\varphi^*\,\partial_k\varphi + \partial_k\varphi^*\,\partial_i\varphi - g_{ik}(-g)^{-1/2}\mathcal{L}^{(0)} - \\ - 2\xi\left[R_{ik} + \nabla_i\nabla_k - g_{ik}\nabla_j\nabla^j\right]\varphi^*\varphi, \tag{5.11}$$

which is covariantly conserved:

$$\nabla^i\,T_{ik}{}^{(0)} = 0. \tag{5.12}$$

Proca vector field $\varphi_k(x)$ in curved spacetime may be described by the Lagrangian

$$\mathcal{L}^{(1)} = \sqrt{-g}\left[-\frac{1}{2}f^*_{ik}\,f^{ik} + m^2\,\varphi^*_i\varphi^i\right], \tag{5.13}$$

where $f_{ik} = \partial_i\varphi_k - \partial_k\varphi_i$.

Lagrangian (13) leads to Proca equation

$$\nabla_i\,f^{ik} + m^2\,\varphi^k = 0 \tag{5.14}$$

and (in $m \neq 0$ case) to the constraint $\nabla_k\varphi^k = 0$.

Both together Eq. (14) and constraint give the possibility to get the second order equation for each component of the vector field:

$$\nabla_i\,\nabla^i\varphi_k + R_{ik}\,\varphi^i + m^2\,\varphi_k = 0. \tag{5.15}$$

The covariantly conserved SET of vector field is

$$T_{ik}{}^{(1)}(x) = -g^{jl}\left(f^*_{kl}\,f_{ij} + f^*_{ij}\,f_{kl}\right) + m^2\,\left(\varphi^*_i\,\varphi_k + \varphi^*_k\,\varphi_i\right) \\ - g_{ik}(-g)^{-1/2}\mathcal{L}^{(1)}. \tag{5.16}$$

The description of a spinor field in curved spacetimes is more complicated because the requirement of Lorenz invariance of the Dirac equation may be transferred to Riemannian geometry only locally. For this at every point of

Riemannian space one introduces a tangent pseudo-Euclidean space with a metric tensor η_{ab}. As a basis vectors of the latter, four so-called vierbein (or tetrad) 4-vectors ${h_{(a)}}^k(x)$ may be chosen numbered by the index $a = 0, 1, 2, 3$ and normalized according to

$$ {h_{(a)}}^k \, h_{(b)k} = \eta_{ab}. \tag{5.17} $$

For this vectors

$$ {h^{(a)}}_k \, h_{(a)i} = g_{ik}. \tag{5.18} $$

The covariant derivative of a spinor $\vec{\nabla}_k \psi(x)$ must be a spinor for fixed x relatively to transformations of the vierbein. It must be a 4-vector relative to general coordinate transformations. These requirements lead to:

$$ \vec{\nabla}_k \psi(x) = \left[\partial_k + \frac{1}{4} C_{abc} \, {h^{(c)}}_k \, \gamma^a \gamma^b \right] \psi(x), \tag{5.19} $$

where Ricci rotational coefficients are:

$$ C_{abc} = \left(\nabla_i \, {h_{(a)}}^k \right) \, h_{(b)k} \, {h_{(c)}}^i. \tag{5.20} $$

Then the Dirac equation in curved spacetimes takes the form

$$ \left[i \, \gamma^k(x) \, \vec{\nabla}_k - m \right] \psi(x) = 0, \qquad \gamma^k(x) \equiv {h_{(a)}}^k(x) \, \gamma^a. \tag{5.21} $$

The corresponding Lagrangian and covariantly conserved SET are:

$$ \begin{aligned} \mathcal{L}^{(1/2)}(x) &= \sqrt{-g} \left\{ \frac{i}{2} \left[\bar{\psi} \gamma^k(x) \vec{\nabla}_k \psi - \left(\vec{\nabla}_k \bar{\psi} \right) \gamma^k(x) \psi \right] - m \, \bar{\psi} \psi \right\}, \\ T^{(1/2)}_{kj} &= \frac{i}{4} \left[\bar{\psi} \gamma_j(x) \vec{\nabla}_k \psi + \bar{\psi} \gamma_k(x) \vec{\nabla}_j \psi \right. \\ &\quad \left. - \left(\vec{\nabla}_k \bar{\psi} \right) \gamma_j(x) \psi - \left(\vec{\nabla}_j \bar{\psi} \right) \gamma_k(x) \psi \right]. \end{aligned} \tag{5.22} $$

7.3 Canonical quantization in curved spacetimes

Let $\psi(x)$ be a charged quantized field of spin 0, 1/2 or 1. Let $\left\{ \psi^{(+)}_\alpha(x);\, \psi^{(-)}_\alpha(x) \right\}$ be a complete orthonormal set of solutions to the wave equations (7), (15) or (21) in terms of corresponding scalar product:

$$ \begin{aligned} &\left(\psi^{(+)}_\alpha, \psi^{(+)}_\beta \right) = \mp \delta_{\alpha\beta}, \qquad \left(\psi^{(-)}_\alpha, \psi^{(-)}_\beta \right) = \delta_{\alpha\beta}, \\ &\left(\psi^{(+)}_\alpha, \psi^{(-)}_\beta \right) = 0. \end{aligned} \tag{5.23} $$

Here the upper indices (±) correspond to the positive- and negative-frequency solutions the meaning of which in curved spacetimes is not trivial and is discussed below. The signs $\mp$ hereafter are related to the boson and fermion cases.

The specific definitions of scalar products in (23) are based on the existence of the global spacelike fypersurfaces Σ in the curved spacetimes under consideration. For example, for $s = 0$ case

$$(\varphi_1, \varphi_2) = i \int\limits_{\Sigma} d\sigma^k \left(\varphi_1^* \overset{\leftrightarrow}{\partial}_k \varphi_2 \right), \tag{5.24}$$

where $d\sigma$ is the square element of the hypersurface Σ.

Then the field ψ can be represented as the expansion

$$\psi(x) = \sum_{\alpha} \left[\psi_{\alpha}^{(-)}(x)\, a_{\alpha}^{(-)} + \psi_{\alpha}^{(+)}(x)\, a_{\alpha}^{(+)} \right], \tag{5.25}$$

where the expressions for antiparticle creation operator and particle annihilation operator follow from orthonormality conditions:

$$a_{\alpha}^{(+)} = \mp \left(\psi_{\alpha}^{(+)}, \psi \right), \qquad a_{\alpha}^{(-)} = \left(\psi_{\alpha}^{(-)}, \psi \right). \tag{5.26}$$

Quantization procedure consists in imposing commutation (anticommutation) relations:

$$\begin{aligned} \left[\overset{*}{a}{}_{\alpha}^{(-)}, a_{\beta}^{(+)} \right]_{\mp} &= \left[a_{\alpha}^{(-)}, \overset{*}{a}{}_{\beta}^{(+)} \right]_{\mp} = \delta_{\alpha\beta}, \\ \left[a_{\alpha}^{(\pm)}, a_{\beta}^{(\pm)} \right]_{\mp} &= \left[\overset{*}{a}{}_{\alpha}^{(\pm)}, \overset{*}{a}{}_{\beta}^{(\pm)} \right]_{\mp} = 0. \end{aligned} \tag{5.27}$$

The vacuum state may be defined by

$$a_{\alpha}^{(-)}|0\rangle = \overset{*}{a}{}_{\alpha}^{(-)}|0\rangle = 0, \qquad \langle 0|0\rangle = 1 \tag{5.28}$$

and the Fock space is constructed in an usual way.

Specifically, the operators of density number of particles and antiparticles take the form:

$$N_{\beta} = \overset{*}{a}{}_{\beta}^{(+)}\, a_{\beta}^{(-)}, \qquad \bar{N}_{\beta} = a_{\beta}^{(+)}\ \overset{*}{a}{}_{\beta}^{(-)}. \tag{5.29}$$

The total numbers of particles and antiparticles turned out to be

$$N = \sum_{\beta} N_{\beta}, \qquad \bar{N} = \sum_{\beta} \bar{N}_{\beta}. \tag{5.30}$$

The quantization procedure in terms of creation-annihilation operators (27) is equivalent to canonical commutation (anticommutation) relations of the form:

$$[\psi(t,\boldsymbol{x}),\psi(t,\boldsymbol{x}')]_{\mp} = [\pi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}')]_{\mp} = 0,$$
$$[\psi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}')]_{\mp} = i\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{5.31}$$

where π is the momentum, canonically conjugated to the field operator ψ.

The relations (31) may be rewritten in an invariant form for $s = 0$ and $1/2$ correspondingly:

$$\int_\Sigma f(x')\ [\psi(x), \partial_i\psi^*(x')]_-\ d\sigma^i(x') = i\, f(x),$$
$$\int_\Sigma f(x')\ \left[\bar{\psi}_A(x),\ \left(h_{(a)i}\,\gamma^a\,\psi(x')\right)^B\right]_+\ d\sigma(x') = \delta^B_A\, f(x). \tag{5.32}$$

Here $f(x)$ is a test function, A, $B = 1$, 2, 3, 4 are the spinor indices and point x belongs to Σ. The other commutators (and anticommutators) are equal to zero.

Let us now discuss the meaning of the frequency signs ($\pm$) in our set of solutions $\left\{\psi^{(\pm)}_\alpha(x)\right\}$. In the absence of external field in Minkowski spacetime the positive- and negative-frequency solutions are the eigenfunctions of the translation in time operator:

$$\frac{\partial}{\partial t}\psi^{(\pm)}_\alpha(x) \sim \pm i\,\omega_\alpha\,\psi^{(\pm)}_\alpha(x), \tag{5.33}$$

which is the Poincaré group generator. In this case, e.g., the positive-frequency solution at one moment preserves this property for any moment.

The vacuum state $|0\rangle$ defined in (28) is invariant under transformations from Poincaré group. So, the procedure of constructing Fock space for a free quantized field turns out to be highly unambiguous. Specifically, the definition of particles and their number is defined in a unique way.

A different situation takes place in nonstationary curved spaces, which are interesting for cosmological applications. In this case the translation invariance in time is absent, and it is impossible to introduce an operator of time translations the eigenfunctions of which would be $\psi^{(\pm)}_\alpha(x)$. Therefore the interpretation of elements of the complete orthonormal set of solutions to the wave equations as positive- and negative-frequency ones at an arbitrary moment of time loses its sense.

One may define the first complete orthonormal set of solutions by some initial conditions of the form:

$$\frac{\partial}{\partial t}\psi_\alpha^{(\pm)}(\boldsymbol{x},t)\bigg|_{t=t_1} \sim \pm i\,\omega_\alpha(t_1)\,\psi_\alpha^{(\pm)}(\boldsymbol{x},t_1), \tag{5.34}$$

and the second complete orthonormal set of solutions by some other ones:

$$\frac{\partial}{\partial t}\varphi_\alpha^{(\pm)}(\boldsymbol{x},t)\bigg|_{t=t_2} \sim \pm i\,\Omega_\alpha(t_2)\,\varphi_\alpha^{(\pm)}(\boldsymbol{x},t_2). \tag{5.35}$$

In the absence of external field both effective frequencies ω_α and Ω_α turn into the usual one-particle energy.

Strictly speaking, the solutions from the first (second) set are positive- and negative-frequency ones for the moment t_1 (t_2) only. But from the mathematical point of view both these sets of solutions are complete and orthonormal ones and may be used for any t. Then we can expand the functions from the first set in terms of those belonging to the second set. It is some linear transformation:

$$\begin{cases} \psi_\alpha^{(+)} &= \sum\limits_\beta \left(\Phi_{\alpha\beta}\,\varphi_\beta^{(+)} - \Psi_{\alpha\beta}\,\varphi_\beta^{(-)}\right), \\ \psi_\alpha^{(-)} &= \sum\limits_\beta \left(\Phi^*_{\alpha\beta}\,\varphi_\beta^{(-)} \mp \Psi^*_{\alpha\beta}\,\varphi_\beta^{(+)}\right). \end{cases} \tag{5.36}$$

The field operator, instead of (25), can be represented now in the form

$$\psi = \sum_\alpha \left[\varphi_\alpha^{(-)}(x)\,b_\alpha^{(-)} + \varphi_\alpha^{(+)}(x)\,b_\alpha^{(+)}\right], \tag{5.37}$$

defining the new type of particles.

¿From (25), (36), (37) the Bogoliubov transformation follows

$$\begin{cases} a_\alpha^{(-)} &= \sum\limits_\beta \left(\Phi_{\alpha\beta}\,b_\beta^{(-)} + \Psi_{\alpha\beta}\,b_\beta^{(+)}\right), \\ \overset{*}{a}{}_\alpha^{(-)} &= \sum\limits_\beta \left(\Phi_{\alpha\beta}\;\overset{*}{b}{}_\beta^{(-)} \pm \Psi_{\alpha\beta}\;\overset{*}{b}{}_\beta^{(+)}\right), \end{cases} \tag{5.38}$$

which connects old and new definitions of particles.

¿From the orthonormality of two sets of solutions we have

$$\Phi\Phi^+ \mp \Psi\Psi^+ = I, \qquad \Phi\Psi^T = \Psi\Phi^T, \tag{5.39}$$

and also the commutation relations for the operators

$$\left[\overset{*}{b}{}_\alpha^{(-)}, b_\beta^{(+)}\right]_\mp = \left[b_\alpha^{(-)}, \overset{*}{b}{}_\beta^{(+)}\right]_\mp = \delta_{\alpha\beta}, \tag{5.40}$$

which are the same as (27).

¿From the reversiability of transformations (36), (38) the equalities may be obtained

$$\left(\Phi^+\Phi \mp \Psi^T\Psi^*\right)_{\alpha\beta} = \delta_{\alpha\beta}, \qquad \Phi^+\Psi = \Psi^T\Phi^* \tag{5.41}$$

and also

$$\begin{cases} b_\alpha^{(-)} &= \sum\limits_\beta \left(\Phi^+_{\alpha\beta}\, a_\beta^{(-)} - \Psi^T_{\alpha\beta}\, a_\beta^{(+)}\right), \\ \overset{*}{b}{}_\alpha^{(-)} &= \sum\limits_\beta \left(\Phi^+_{\alpha\beta}\ \overset{*}{a}{}_\beta^{(-)} \mp \Psi^T_{\alpha\beta}\ \overset{*}{a}{}_\beta^{(+)}\right). \end{cases} \tag{5.42}$$

To the new definition of particles the new vacuum corresponds:

$$b_\alpha^{(-)}|\tilde{0}\rangle = \overset{*}{b}{}_\alpha^{(-)}|\tilde{0}\rangle = 0, \qquad \langle\tilde{0}|\tilde{0}\rangle = 1. \tag{5.43}$$

It is connected with the one defined in (28) according to:

$$|0\rangle = \frac{1}{N_\mp}\left[\exp\left(\sum_{\alpha,\beta} D_{\alpha\beta}\ \overset{*}{b}{}_\alpha^{(+)}\, b_\beta^{(+)}\right)\right]|\tilde{0}\rangle. \tag{5.44}$$

Here the matrix D is defined by the equations

$$\Phi\, D + \Psi = 0, \qquad D\,\Phi^T + \Psi^T = 0. \tag{5.45}$$

The transition amplitude between two vacua

$$N_\mp^{-1} = \langle\tilde{0}|0\rangle \tag{5.46}$$

may be expressed in terms of eigenvalues of the matrix $D\,D^+$.

Operators of new particles and antiparticles densities are

$$\tilde{N}_\beta = \overset{*}{b}{}_\beta^{(+)}\, b_\beta^{(-)}, \qquad \tilde{\bar{N}}_\beta = b_\beta^{(+)}\ \overset{*}{b}{}_\beta^{(-)}. \tag{5.47}$$

Using (42) it is very easy to calculate the density of new particles in the old vacuum state:

$$\langle 0|\tilde{N}_\beta|0\rangle = \langle 0|\tilde{\bar{N}}_\beta|0\rangle = \sum_\gamma \Psi^+_{\beta\gamma}\,\Psi_{\gamma\beta} \equiv \left[\Psi^+\,\Psi\right]_{\beta\beta}. \tag{5.48}$$

Thus the old vacuum is not the vacuum state for new particles and vice versa. The concepts of particles and vacuum state become umbiguous in nonstationary curved spacetimes. More exactly, different concepts of particles may be used by different observers, and the number of particles loses its invariant sense. Strictly speaking, the quantum particles in strong gravitational field (in cosmology this corresponds to the time interval (4)) are not the well defined fragments of physical reality any more. The better description of reality here is given by the complete renormalized SET of quantized fields in some physical state (see below). All this takes place if the quantity $\Psi_{\alpha\beta}$ in (36), (38) and (42) is not equal to zero (which means the frequency mixing due to nonstationarity of the spacetime). If $\Psi_{\alpha\beta}$ is equal to zero (which is the case, e.g., for the static non-singular spacetimes) then the problems with corpuscular interpretation of the theory do not arise. In such cases the use of two different sets of solutions to the wave equations corresponds simply to the use of two physically equivalent Fock representations (like the states numerated by the definite 3-momentum or by the energy, orbital momentum and its projection in usual quantum mechanics).

7.4 The problem of divergencies

Let us consider now the vacuum expectation values of the SET-operator as quantities characterizing the properties of vacuum in curved spacetimes. Using the expansion of field operator in the creation-annihilation operators (25) we get

$$\langle 0|T_{ij}(x)|0\rangle = \sum_{\alpha} T_{ij}\left\{\psi_{\alpha}^{(-)}(x), \psi_{\alpha}^{(+)}(x)\right\}, \tag{5.49}$$

where $T_{ij}\{f, g\}$ is a bilinear form defined by the classical expressions (11), (16) or (22). Here summation in α should be understood as integration in continuous quantum numbers and summation in discrete ones. The divergence arises even in flat spacetime. For instance, for the SET of a free charged scalar field in Minkowski space the Eq. (49) takes the form

$$\langle 0|T_{ij}(x)|0\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\omega}\, k_i\, k_j, \tag{5.50}$$

where $\omega^2 = k_0^2 = \boldsymbol{k}^2 + m^2$. This expression diverges as k^4 when $k \to \infty$.

The divergence of (50) is connected with the presence of zero vacuum oscillations. In standard quantum field theory the operator of T_{ij} is reduced to the normal form in creation-annihilation operators in order to remove this divergence. This corresponds to the subtraction of contributions of zero oscillations of the form (50). For the interacting fields and also in curved spacetimes there are, generally speaking, other, more weak, divergencies in the vacuum expectation values. To remove all these divergencies, the renormalizations of physical constants should be used. Actually, one subtracts from a given infinite expression some other, also infinite, expression, which is selected according to two demands. The first demand is that the difference should be finite. The second one is that the subtraction can be obtained from the counter terms introduced into the Lagrangian of the theory. To give sense to such formal manipulations with infinities one makes infinite expressions temporarily finite with the help of some regularization. At the end of calculations, when renormalization constants are fixed and finite values for physical observables are obtained, the regularization is removed.

This procedure may be used also for interpretation of the SET normal ordering in Minkowski space. Actually, let us apply to expression (50) a dimensional regularization. For this we calculate (50) in $N = 4 - 2\varepsilon$ dimensional spacetime, where ε is complex. Retaining in the result expanded in powers of ε only terms nonvanishing when $\varepsilon \to 0$, we obtain

$$\langle 0|T_{ij}|0\rangle_{\varepsilon} = -\frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon} + \frac{3}{2} - C - \ln\frac{v^2}{4\pi}\right)\eta_{ij}. \tag{5.51}$$

Here $v \equiv m/M$, M — parameter of the dimension of mass, C is the Euler's constant.

The fact that (51) is proportional to metrical tensor η_{ij} allows one to interpret its subtraction as a renormalization of a cosmological term Λ in the initial action for the gravitational field. Setting $\langle T_{ij}\rangle_{ren}$ equal to zero in Minkowski space, we fix in this way $\Lambda_{ren} = 0$.

Except of dimensional regularization, there exist many other regularization procedures: ζ-function method, covariant point splitting, adiabatic and n-wave regularizations. They will be discussed in more details below.

7.5 Some specific features of QFT in Curved Spacetimes

In standard quantum field theory, the nonlinearities are usually the quantum corrections to the linear classical equations. But the gravitational field is nonlinear by the origin. Additional difficulties which arise in QFT in Curved Spacetimes are the following:

— field equations in gravitational background may have variable coefficients by higher derivatives;

— topology of space may be non-Euclidean;

— space may have event horizons;

— S-matrix picture is possible only in asymptotically flat spaces;

— the construction of Fock space of quantized fields is nonunique;

— the expectation values of local physical observables have more complicated structure of infinities;

— the renormalizable theory of quantized gravitational field is absent.

We will start our discussion with the specific nature of particle creation from vacuum by strong gravitational field and then continue with vacuum polarization.

Let the invariant of the curvature tensor have in the neighbourhood of some spacetime point M the value of order

$$R_{ijkl}\,R^{ijkl} \sim \rho^{-4}, \tag{5.52}$$

where ρ is the characteristic curvature radius. Then one can introduce at the point M a coordinate system that is locally Galilean up to distances of the order ρ from M. One can construct in this system a complete set of solutions $\psi_\alpha^{(\pm)}$, which for the frequencies $\omega_\alpha \gg \rho^{-1}$ will be, approximately, positive- and negative-frequency. However, for the frequencies $\omega_\alpha < \rho^{-1}$ the difference between positive- and negative-frequency functions disappears, that corresponds to the unity order uncertainty for the number of the particles in the mode α.

By the anology with electrodynamics, one can qualitatively describe particle creation as a "breakdown of vacuum loops" by external gravitational field. Since there are only the positive masses, gravitation acts in the similar way on particles and antiparticles. By this reason, the breakdown of virtual pairs is explained by the action of tidal forces.

A characteristic distance between particles of virtual pair is $L_C = m^{-1}$. To define a tidal force breaking this pair, consider the geodesic deviation equation

$$\frac{d^2 n^i}{ds^2} = R^i{}_{jkl}\,u^j\,n^k\,u^l, \tag{5.53}$$

where u^i is 4-velocity of one particle of a pair, n^i is a spacelike vector connecting it with the second particle, $n_i\,n^i \sim -l_C^2$.

To "break" the virtual pair it is necessary that, in the center of mass system, the work of tidal forces at the distance $\sim l_C$ would exceed $2m$. Setting $u_0 = 1$, $u^\alpha = 0$, $n^0 = 0$, $|n^\alpha| \sim l_C$ we obtain:

$$|R^\alpha{}_{0\beta 0}| > l_C^{-2} = m^2. \tag{5.54}$$

Thus to get an essential particle creation, the curvature of spacetime must be at least of the order of the inverse Compton length.

As it was told before the concept of particle in curved spacetimes is not unique. By this reason, a number of different definitions of particles were used in literature. There were formulated several definitions of adiabatic particles; particles defined by some special symmetry group of spacetime; particles, diagonalizing the instantaneous Hamiltonian of quantized field and so on.

Some definitions depend on the choice of time-like vector field ξ^i which is orthogonal to a set of space-like hypersurfaces Σ. Integral curves of the field ξ^i may be considered as the world lines of the system of observers. Thus, the definition of particle also turns out to be dependent on the system of reference. That is quite natural in the absence of Poincaré-invariance.

Now let us discuss some necessary requirements to the renormalized vacuum SET of quantized fields $\langle T_{ik}\rangle$ which describes not only the effect of particle creation but also the vacuum polarization. These requirements are:

— $\langle T_{ik}\rangle$ should be a causal functional of the metric, i.e. it should depend only on the geometry of the spacetime point under consideration and on its causal past;

— the energy conservation condition should be valid:

$$\nabla_k \langle T_i{}^k\rangle = 0; \tag{5.55}$$

— $\langle T_{ik}\rangle$ should not turn to infinity when go back to initial moment $\eta = \eta_0$ at which the vacuum state $|0\rangle$ was defined.

Due to the following theorem the creation of classical matter by gravitational field is impossible.

<u>Theorem:</u> *If, in a closed region of space, at some moment $T_{ik} = 0$ and the flow of matter over the region boundary is absent, then $T_{ik} = 0$ at all the following moments.*

The proof of this theorem uses the energy dominance conditions according to which in any local orthogonal coordinate basis the components of the SET obey the inequality

$$T_{00} \geq |T_{ik}| \cdot \tag{5.56}$$

These conditions are used also in the formulation of the Hawking-Penrose theorems about the inavitability of singularities in the nonstationary cosmological solutions to Einstein equations. For the quantum particles and vacuum polarization SET, the conditions (56) are not valid. This gives the possibility to expect that in QFT in Curved Spacetimes not only particle creation from vacuum takes place but also nonsingular cosmological solutions are possible.

7.6 The character of divergencies

To investigate divergencies of the vacuum SET, it is convenient to suppose that the spacetime is asymptotically static, so that it is possible to introduce vacuum states $|0_{in}\rangle$ and $|0_{out}\rangle$. If the metric is regular everywhere (g_{ik}

are sufficiently smooth function of coordinates) then divergencies have purely local character, and the divergent part of the SET does not depend on the assumption about asymptotic staticity.

The divergencies of the quantity $\langle 0_{in}|T_{ik}|0_{in}\rangle$ are the same as of

$$\langle T_{ik}\rangle_M \equiv \frac{\langle 0_{out}|T_{ik}|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} \tag{5.57}$$

In its own turn

$$\langle T_{ik}\rangle_M = \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{ik}}, \tag{5.58}$$

where the effective action W is connected with the effective Lagrangian by:

$$W = \int d^4x\,\sqrt{-g}\,L_{eff}. \tag{5.59}$$

Thus, the problem is reduced to studying divergencies of W or L_{eff}. But the divergencies of L_{eff} are determined by the behavior of the Green function $G(x,x')$ under rapprochement of arguments. For example, in spin zero case it is valid:

$$\frac{\partial L_{eff}}{\partial m^2} = i\,G(x,x). \tag{5.60}$$

The investigation of the infinities which are contained in the quantity (60) may be performed by the Schwinger-De Witt technique [7] based on the Fock method of proper time. The result of long calculations may be presented in the form (spin zero case):

$$\mathcal{L}_{div} = \sqrt{-g}\,L_{div} = \sqrt{-g}\left[\frac{-2\Lambda_\infty + R}{16\pi G_\infty} + \alpha_\infty\left(R^{ik}R_{ik} - \frac{1}{3}R^2\right) + \beta_\infty R^2\right], \tag{5.61}$$

where

$$\begin{aligned}
&\frac{\Lambda_\infty}{8\pi G_\infty} = -\frac{m^4}{32\pi^2}\,J_3, \qquad G_\infty^{-1} = \frac{m^2}{2\pi}\left(\xi - \frac{1}{6}\right)J_2,\\
&\alpha_\infty = \frac{1}{1920\pi^2}\,J_1, \qquad \beta_\infty = \frac{1}{64\pi^2}\left(\xi - \frac{1}{6}\right)^2 J_1,\\
&J_n \equiv \int_0^\infty \frac{dz}{z^n}\,e^{-z}.
\end{aligned} \tag{5.62}$$

Then, the divergent part of the vacuum SET is:

$$\langle T_{ik}\rangle_{div} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{ik}}\int d^4x\,\mathcal{L}_{div} = \frac{\Lambda_\infty}{16\pi G_\infty}\,g_{ik} + \frac{1}{16\pi G_\infty}\,G_{ik} + \alpha_\infty\,{}^{(2)}H_{ik} + \left(\beta_\infty - \frac{1}{3}\alpha_\infty\right)\,{}^{(1)}H_{ik}, \tag{5.63}$$

where the quadratic in curvature tensors are defined as

$$\begin{aligned} {}^{(1)}H_{ik} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{ik}}\int d^4x\,\sqrt{-g}\,R^2 \\ &= 2\left(\nabla_i\nabla_k R - g_{ik}\,\nabla^l\nabla_l R\right) + 2R\left(R_{ik} - \frac{1}{4}R\,g_{ik}\right), \\ {}^{(2)}H_{ik} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{ik}}\int d^4x\,\sqrt{-g}\,R^{lm}\,R_{lm} \\ &= \nabla_i\nabla_k R - \nabla^l\nabla_l R_{ik} - \frac{1}{2}\left(\nabla^l\nabla_l R + R^{lm}\,R_{lm}\right)\,g_{ik} + 2R^{lm}\,R_{limk}. \end{aligned} \tag{5.64}$$

For the spinor and vector fields one obtains the similar results.

Therefore, to remove the devergencies from the vacuum SET in curved spacetimes by the renormalization procedure we should assume that the initial gravitational Lagrangian has the form:

$$\mathcal{L}_{gr,0} = \sqrt{-g}\left[\frac{-2\Lambda_0 + R}{16\pi G_0} + \alpha_0\left(R^{ik}R_{ik} - \frac{1}{3}R^2\right) + \beta_0\,R^2\right], \tag{5.65}$$

where Λ_0, G_0, α_0 and β_0 are the initial (bare) values of, respectively, cosmological constant, gravitational constant, and coefficients before quadratic terms. The values of renormalized constants Λ_{ren}, G_{ren}, α_{ren} and β_{ren} ought to be determined from the experiment.

To get the finite results after the subtraction of infinities, as it was discussed above, one should use some regularization method. Let us discuss briefly the main such methods known from the literature. One of the most effective methods is the n-wave regularization [26], which is equivalent to the adiabatic regularization [23]. The main idea of n-wave regularization is the following. If to denote the continuous momentum quantum number by λ, the Eq. (49) for the vacuum SET may be rewritten as some divergent integral of the form:

$$\langle 0|T_{ik}|0\rangle = \int d\lambda\,\tau_{ik}(\lambda, m). \tag{5.66}$$

One can set a "n-mode" $n^{-1/2}\,\psi^{(\pm)}_{n\lambda}$ to each mode $\psi^{(\pm)}_{\lambda}$, so that

$$\tau^{(n)}_{ik}(\lambda, m) = \frac{1}{n}\,\tau_{ik}(n\lambda, nm). \tag{5.67}$$

The regularization of the expressions (49), (66) is carried out by the regularization of the contribution of each mode according to:

$$\tau_{ik}^{(reg)}(\lambda, m) = \lim_{n\to\infty}\left\{\tau_{ik}(\lambda, m) - \sum_{l=0}^{2}\frac{1}{l!}\frac{\partial^l}{\partial(n^{-2})^l}\tau_{ik}^{(n)}(\lambda, m)\right\},$$

$$\langle T_{ik}\rangle \equiv \langle 0|T_{ik}|0\rangle_{ren} \equiv \int d\lambda\, \tau_{ik}^{(reg)}(\lambda, m). \tag{5.68}$$

One can show that subtraction of the terms in (68) with $l = 0, 1$ and 2 is equivalent to the removal of divergencies proportional, respectively, to g_{ik}, G_{ik} and ${}^{(1)}H_{ik}$, ${}^{(2)}H_{ik}$ from (64).

If the spectrum is discrete (e.g., when *3-space is closed*) the Eq. (68) should be changed for:

$$\langle T_{ik}\rangle = \lim_{n\to\infty}\left\{\sum_{\lambda}\tau_{ik}(\lambda, m) - \int d\lambda\sum_{l=0}^{2}\frac{1}{l!}\frac{\partial^l}{\partial(n^{-2})^l}\tau_{ik}^{(n)}(\lambda, m)\right\}. \tag{5.69}$$

The reason is that divergencies are determined by the local properties of spacetime and do not depend on its global, in particular, topological characteristics.

In adiabatic regularization the subtraction from (66) is used of the first terms of its asymptotic expansion in inverse powers of parameter of metric variation adiabacity.

Another regularization method is based on the splitting of arguments of field operators in the bilinear form (49) of the vacuum SET [5]. In four-dimensional spacetime this method is rather complicated. The results obtained by it may depend on splitting direction.

Dimensional regularization [4] is the example of covariant renormalization method. The essence of this method is explained in Sec. 5. We will use it below, together with the n-wave procedure in the case of isotropic metrics.

The other covariant regularization procedure is the method of generalized ζ-function [8]. In this method the Green's function is reproduced as the matrix element of some operator $\hat{G}$:

$$G(x, x') = \langle x|\hat{G}|x'\rangle. \tag{5.70}$$

The operator $\hat{F}$ reciprocal to $\hat{G}$ is defined as:

$$\hat{F}\,\hat{G} = I. \tag{5.71}$$

Let λ_n, $|\varphi_n\rangle$ be eigenvalues and eigenfunctions of $\hat{F}$. Then the spectral expansion of the operator $\hat{G}^\nu$ can be formally written as:

$$\hat{G}^\nu = \hat{F}^{-\nu} = \sum_n \lambda_n^{-\nu}\,|\varphi_n\rangle\langle\varphi_n|. \tag{5.72}$$

Then the trace of $\hat{G}^{\nu}$ is equal to:

$$\mathrm{tr}\left(\hat{G}^{\nu}\right) = \lim_{x'\to x} \int d^4x \sqrt{-g} \langle x|\hat{G}^{\nu}|x'\rangle = \sum_n \lambda_n^{-\nu} \equiv \zeta(\nu). \tag{5.73}$$

This quantuty is called a generalized ζ-function of the operator $\hat{F}$. In complex ν-plane $\zeta(\nu)$ may be analytically continued in such a way that it is regular at $\nu = 0$ (whilst the series (73) with $\nu = 0$ is divergent). As a result, the regularized effective Lagrangian is expressed in terms of the values $\zeta(0)$ and $\zeta'(0)$.

With the use of all regularization methods, mentioned above, it is possible to calculate the trace of renormalized vacuum SET $\langle T_i^i \rangle$ for the massless quantized fields in arbitrary curved spacetime. Unexpectedely, it turned out to be nonzero even for conformally invariant fields for which $T_i^i = 0$ in classical theory. By this reason it was called anomalous. The expression for anomalous trace obtained by different methods and different authors may be presented in the following form (see, e.g., [6])

$$\langle T_i^i \rangle = -\frac{1}{2880\pi^2} \left\{ \begin{pmatrix} 1 \\ 7/2 \\ 33 \end{pmatrix} C^{iklm} C_{iklm} + \begin{pmatrix} 1 \\ 11 \\ -27 \end{pmatrix} \left(R^{ik} R_{ik} - \frac{1}{3} R^2 \right) \right.$$
$$\left. + \begin{pmatrix} 30\xi - 6 \\ -6 \\ -12 \end{pmatrix} \nabla^k \nabla_k R + \begin{pmatrix} 90\left(\xi - \frac{1}{6}\right)^2 \\ 0 \\ 5/2 \end{pmatrix} R^2 \right\}. \tag{5.74}$$

Here the results, correspondingly, for the arbitrary coupled scalar, spinor and vector fields are presented. Scalar and vector fields assumed to be real. C_{iklm} is the Weyl conformal tensor (see, e.g., [12]) and

$$C^{iklm} C_{iklm} = R^{iklm} R_{iklm} - 2 R^{ik} R_{ik} + \frac{1}{3} R^2. \tag{5.75}$$

The expression (74) is known also as the conformal anomaly.

It is impossible to restore the conformal invariance in the quantum theory by introducing some local counter term into the gravitational Lagrangian.

Since for conformal fields the Lagrangian of the considered theory is conformally invariant, the nonzero trace of the vacuum SET in this case means a spontaneous breaking of conformal symmetry in curved spacetime (analogous anomaly arises in flat spacetime in the gauge theories for the axial current, see, e.g., [13]).

7.7 Vacuum quantum effects in isotropic space

Let us now consider the particle creation from vacuum and vacuum polarization in homogeneous isotropic cosmological models. A high symmetry of space allows one, in this case, to carry out total quantitative investigation of quantum effects and to obtain results of much significance for cosmology.

The metric of homogeneous isotropic spacetime in co-moving coordinates has the form

$$ds^2 = g_{ik}\, dx^i\, dx^k = dt^2 - a^2(t)\, dl^2, \tag{5.76}$$

where t is the synchronous proper time of co-moving observers; dl^2 is the metric of 3-space of constant curvature $\kappa = -1,\ 0,\ +1$:

$$dl^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta = d\chi^2 + f^2(\chi)\, \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right); \tag{5.77}$$

$f(\chi) = \sinh\chi$, χ and $\sin\chi$ for $\kappa = -1$, 0 and $+1$ respectively. Coordinates χ, θ and φ are dimensionless and function $a(t)$ called the scale factor has the dimensionality of a length.

It is convenient to use, instead of synchronous time t, the dimensionless "conformal" time η:

$$\eta = \int a^{-1}(t)\, dt, \tag{5.78}$$

In terms of η the metric (76) has conformally static form

$$ds^2 = a^2(\eta)\, d\tilde{s}^2 = a^2(\eta)\, \left(d\eta^2 - dl^2\right), \tag{5.79}$$

i.e., it differs from static metric $d\tilde{s}^2$ by a conformal multiplier $a(\eta)$ only.

The complete set of solutions to Eq. (7) in metric (79) may be found in the form

$$\varphi_J(x) = a^{-1}(\eta)\, g_J(\eta)\, \Phi_J(x). \tag{5.80}$$

Here $\Phi_J(x)$ are the eigenfunctions of the Laplace-Beltrami operator on 3-space, $J = \{\lambda, l, m\}$ and the time function is the solution to oscillator equation

$$g_J''(\eta) + \Omega^2(\eta)\, g_J(\eta) = 0, \tag{5.81}$$

where

$$\Omega^2(\eta) = \omega^2(\eta) - q(\eta); \tag{5.82}$$

$$\omega^2 = \lambda^2 + m^2 a^2; \quad q = -\left(\xi - \frac{1}{6}\right) a^2 R = -6\left(\xi - \frac{1}{6}\right)\left(\frac{a''}{a} + \kappa\right).$$

Note that for conformal coupling in (7) ($\xi = \xi_c = 1/6$ in four-dimensional spacetime) one has $q(\eta) = 0$.

In the conformal case ($\xi = 1/6$) it is natural to understand the quantity ω in (82) as a dimensionless one-particle energy; the physical one-particle energy is $k_0 = \omega/a = (k^2 + m^2)^{1/2}$. In the nonconformal case the role of one-particle energy is played by Ω. In this Section we perform all calculations (except of investigation of the structure of infinities in vacuum SET) for conformal case only. In the separate Sections the isotropic self-consistent models will be considered determined by the vacuum quantum effects of conformal fields and the nonconformal case.

Since the frequency Ω in (81) is determined by the momentum quantum number λ only, we will write hereafter g_λ instead of g_J. Let us normalize g_λ by the condition on the Wronskian

$$g_\lambda \, g_\lambda^{*\prime} - g_\lambda{}' \, g_\lambda{}^* = -2\mathrm{i}. \tag{5.83}$$

Initial conditions on the functions $g_\lambda(\eta)$ set at some initial moment $\eta = \eta_0$

$$g_\lambda(\eta_0) = \omega^{-1/2}(\eta_0); \quad g_\lambda{}'(\eta_0) = \mathrm{i}\,\omega(\eta_0)\, g_\lambda(\eta_0) \tag{5.84}$$

determine a complete set of classical solutions to Eq. (7):

$$\varphi_J^{(+)} = \frac{1}{\sqrt{2}\,a}\, g_\lambda(\eta)\, \Phi_J^*(x); \qquad \varphi_J^{(-)} = \left(\varphi_J^{(+)}\right)^*, \tag{5.85}$$

orthonormalized according to (23) in the sense of scalar product (24). One may consider $\varphi_J^{(+)}$ and $\varphi_J^{(-)}$ as positive- and negative-frequency at the moment $\eta = \eta_0$ functions.

Let us turn now to the case of spinor field. The general form of the Dirac equation and of dynamic quantities for spinor field in an external gravitational field have been represented in Sec. 2. In a homogeneous spacetime with the metric (79) it is natural to choose vierbein vectors $h_{(0)i}$ defined by the equalities (17), (18) to be orthogonal to coordinate lines

$$\begin{aligned} h_{(0)0} &= -h_{(1)1} = a(\eta); \qquad h_{(2)2} = -a(\eta)\, f(\chi); \\ h_{(3)3} &= -a(\eta)\, f(\chi)\, \sin\theta; \qquad h_{(a)l} = 0, \quad a \neq l. \end{aligned} \tag{5.86}$$

Separation of variables in Eq. (21) is carried out in the following way

$$\psi_J(x) = a^{-3/2}\left[f_{\lambda+}(\eta) I \oplus f_{\lambda-}(\eta) I\right] N_J(\chi, \theta, \varphi), \tag{5.87}$$

where bispinors N_J may be expressed in terms of the spherical spinors, associated Legendre functions and gamma-functions, $J = \{\lambda, j, l, M\}$ [10].

The bispinors (87) would satisfy the Dirac equation (21) if the time functions $f_{\lambda\pm}$ obey the system of equations

$$f'_{\lambda\pm} + \mathrm{i}\,\lambda\, f_{\lambda\mp} \pm \mathrm{i}\, ma\, f_{\lambda\pm} = 0. \tag{5.88}$$

Solutions to the system (88) obey the second order equations

$$\begin{aligned} &f''_{\lambda\pm}(\eta) + \Omega^2_{\pm}(\eta)\, f_{\lambda\pm}(\eta) = 0, \\ &\Omega^2_{\pm}(\eta) = \omega^2_{\pm}(\eta) \pm \mathrm{i}\, ma'(\eta), \end{aligned} \tag{5.89}$$

which describe an oscillator with a variable complex frequency.

Let us fix the positive- and negative-frequency at the moment $\eta = \eta_0$ solutions to the system (88) by the initial conditions corresponding to (84):

$$f^{(+)}_{\lambda\pm}(\eta_0) = \pm \left(\frac{\omega \mp ma}{\omega}\right)^{1/2}\Bigg|_{\eta=\eta_0}, \qquad f^{(-)}_{\lambda\pm}(\eta_0) = \left(\frac{\omega \pm ma}{\omega}\right)^{1/2}\Bigg|_{\eta=\eta_0}. \tag{5.90}$$

One can easy check that in this case

$$f^{(-)}_{\lambda+}(\eta) = -\left(f^{(+)}_{\lambda-}(\eta)\right)^*, \qquad f^{(-)}_{\lambda-}(\eta) = \left(f^{(+)}_{\lambda+}(\eta)\right)^*, \tag{5.91}$$

which allow one to exclude from the furthcoming formulae the negative-frequency solutions.

Now let us turn back to the scalar case. The solutions to (81) defined by (84), (85) are positive- and negative-frequency ones at the initial moment η_0 only. To account explicitly frequency mixing, it is worth while to search solutions to the Eq. (81) in the form

$$\begin{aligned} g_\lambda(\eta) &= \frac{1}{\sqrt{\omega(\eta)}}\left[\alpha^*_\lambda(\eta)\,\exp\left(\mathrm{i}\Theta(\eta)\right) + \beta_\lambda(\eta)\,\exp\left(-\mathrm{i}\Theta(\eta)\right)\right], \\ g_\lambda{}'(\eta) &= \mathrm{i}\sqrt{\omega(\eta)}\left[\alpha^*_\lambda(\eta)\,\exp\left(\mathrm{i}\Theta(\eta)\right) - \beta_\lambda(\eta)\,\exp\left(-\mathrm{i}\Theta(\eta)\right)\right], \end{aligned} \tag{5.92}$$

where

$$\Theta(\eta) = \int_{\eta_0}^{\eta} \omega(\eta')\, d\eta', \tag{5.93}$$

$\alpha^*_\lambda(\eta)$ and $\beta_\lambda(\eta)$ are the complex functions to be determined and obeying the initial conditions

$$a(\eta_0) = 1, \qquad \beta(\eta_0) = 0. \tag{5.94}$$

The conditions on $g_\lambda{}'$ in (92) are prompted by the Lagrange method; they remove arbitrariness in the definition of two functions α_λ and β_λ by one function g_λ. The condition on the Wronskian (83) ensures the fulfillment of the equality

$$|\alpha_\lambda(\eta)|^2 - |\beta_\lambda(\eta)|^2 = 1 \tag{5.95}$$

for all η.

The quantities α_λ and β_λ obey the system of first order equations

$$\begin{aligned} \alpha_\lambda^{*\prime} &= -\mathrm{i}\frac{q}{2\omega}\,\alpha_\lambda^* + \frac{1}{2}\left(w^{(0)} - \mathrm{i}\frac{q}{\omega}\right)\beta_\lambda\,\mathrm{e}^{-2i\Theta}, \\ \beta_\lambda' &= \frac{1}{2}\left(w^{(0)} + \mathrm{i}\frac{q}{\omega}\right)\alpha_\lambda^*\,\mathrm{e}^{2i\Theta} + \mathrm{i}\frac{q}{2\omega}\,\beta_\lambda, \end{aligned} \tag{5.96}$$

where q was defined in (82) and

$$w^{(0)} = \frac{\omega'}{\omega}. \tag{5.97}$$

Analogous representation of solutions to the system (88) for spinor field has the form

$$f_{\lambda\pm}(\eta) = \pm N_\mp\,\alpha_\lambda^*\,\exp(\mathrm{i}\Theta) - N_\pm\,\beta_\lambda\,\exp(-\mathrm{i}\Theta), \tag{5.98}$$

where

$$N_\pm = \left(\frac{\omega \pm ma}{\omega}\right)^{1/2}. \tag{5.99}$$

The quantities α_λ and β_λ obey the condition

$$|\alpha_\lambda(\eta)|^2 + |\beta_\lambda(\eta)|^2 = 1 \tag{5.100}$$

and satisfy the system of equations

$$\alpha_\lambda^{*\prime} = \frac{1}{2}\,w^{(1/2)}\,\beta_\lambda\,\exp(-2\mathrm{i}\Theta), \qquad \beta_\lambda' = -\frac{1}{2}\,w^{(1/2)}\,\alpha_\lambda^*\,\exp(2\mathrm{i}\Theta), \tag{5.101}$$

where

$$w^{(1/2)} = \frac{\lambda\,m\,a'}{\omega^2}, \tag{5.102}$$

and obey initial conditions (94).

The analogical formalism may be presented for the case of vector field also [10].

It is convenient to express local observables bilinear in a field through real bilinear combinations of α_λ and β_λ defined by the equalities

$$s_\lambda = |\beta_\lambda|^2, \qquad u_\lambda - \mathrm{i}\, v_\lambda = \pm 2\, \alpha_\lambda\, \beta_\lambda \exp(-2\mathrm{i}\Theta) \tag{5.103}$$

(upper and lower signs correspond to bosons and fermions).

The functions $s_\lambda\, u_\lambda$, v_λ for the scalar field are connected with g_λ by the relations

$$\begin{aligned} s_\lambda &= \frac{1}{4\omega}\left(|g_\lambda{}'|^2 + \omega^2\, |g_\lambda|^2\right) - \frac{1}{2}, \\ u_\lambda &= -\frac{1}{2\omega}\left(|g_\lambda{}'|^2 - \omega^2\, |g_\lambda|^2\right), \qquad v_\lambda = \frac{1}{2}\frac{d}{d\eta}|g_\lambda|^2 \end{aligned} \tag{5.104}$$

and satisfy the system of equations

$$\begin{aligned} s_\lambda{}' &= \frac{1}{2}\, w^{(0)}\, u_\lambda - \frac{q}{2\omega}\, v_\lambda, \\ u_\lambda{}' &= w^{(0)}\,(1 + 2s_\lambda) - \left(2\omega - \frac{q}{\omega}\right) v_\lambda, \\ v_\lambda{}' &= -\frac{q}{\omega}(1 + 2s_\lambda) + \left(2\omega - \frac{q}{\omega}\right) u_\lambda, \end{aligned} \tag{5.105}$$

which is obtained directly from (96). Further we will consider mainly scalar field with conformal coupling ($\xi = 1/6$, $q(\eta) = 0$) and spinor field. For these cases the system of equations for s_λ, u_λ and v_λ have the similar form

$$\begin{aligned} s_\lambda{}' &= \frac{1}{2}\, w^{(s)}\, u_\lambda, \qquad v_\lambda{}' = 2\,\omega\, u_\lambda, \\ u_\lambda{}' &= w^{(s)}\,(1 \pm 2s_\lambda) - 2\,\omega\, v_\lambda. \end{aligned} \tag{5.106}$$

Here the upper (lower) sign in the second line corresponds to bosons (fermions), and the quantities $w^{(s)}$ are defined in (97), (102) for $s = 0,\ 1/2$ respectively.

The initial conditions on s_λ, u_λ, v_λ follow from (103) and (94):

$$s_\lambda(\eta_0) = u_\lambda(\eta_0) = v_\lambda(\eta_0) = 0. \tag{5.107}$$

Now let us start with consideration of field quantization in the homogeneous isotropic spaces. This is carried out in the correspondence with general principles given in Sec. 3. One can represent the quantized field operator in the form of decomposition in basis functions of corresponding classical wave equation:

$$\varphi(x) = \int d\mu(J)\ \left[\varphi_J^{(-)}(x)\, a_J^{(-)} + \varphi_J^{(+)}(x)\, a_J^{(+)}\right] \tag{5.108}$$

(here the unique symbols are used for field operators and for basis function of different spins). If basis functions satisfy initial conditions of the form (84) or (90) given at the moment $\eta = \eta_0$ then the expansion (108) determines the corpuscular interpretation at that moment. The corresponding vacuum state $|0\rangle$ is defined by the equalities $a_J^{(-)}|0\rangle = 0$ for all J. It is easy to check that the Hamiltonian of the field being expressed through the operators $a_J^{(\pm)}$ is diagonal at the moment $\eta = \eta_0$. In [2] a vacuum state defined in this way was called the "adiabatic vacuum" because it coincides with a stable vacuum of static space-time in the limit of infinitely slow variation of scale factor. For simplicity let us discuss the scalar case. Substituting the first equation of (92) into (108) and combining the terms with one sign of frequency one may rewrite (108) in an equivalent form

$$\varphi(x) = \frac{1}{a}\int \frac{d\mu(J)}{\sqrt{2\omega}}\left[\Phi_J(x)\,e^{-i\Theta}\,b_J^{(-)}(\eta) + \Phi_J^*(x)\,e^{i\Theta}\,b_J^{(+)}(\eta)\right], \qquad (5.109)$$

where new creation-annihilation operators depend on time and are connected with the old ones by the canonical Bogoliubov transformation:

$$\begin{aligned} a_J^{(-)} &= \alpha_J^*(\eta)\,b_J^{(-)}(\eta) - (-1)^m\,\beta_J(\eta)\,b_{\bar J}^{(+)}(\eta), \\ \overset{*}{a}{}_J^{(-)} &= \alpha_J^*(\eta)\,\overset{*}{b}{}_J^{(-)}(\eta) - (-1)^m\,\beta_J(\eta)\,\overset{*}{b}{}_{\bar J}^{(+)}(\eta), \\ |\alpha_J|^2 - |\beta_J|^2 &= 1, \qquad \bar J = \{\lambda, l, -m\}. \end{aligned} \qquad (5.110)$$

By chance, for the conformal scalar field the Hamiltonian of quantized field, being expressed in terms of b-operators, turns out to be diagonal at any moment. By this reason the instantaneous vacuum state $|0_\eta\rangle$ annihilated by $b_J^{(-)}(\eta)$ is the vacuum of the Hamiltonian. The same is true for the spinor field.

One can carry out the transition to the Heisenberg representation by introducing Heisenberg quasiparticle operators

$$c_J^{(\pm)} = b_J^{(\pm)}(\eta)\,\exp\left(\pm i\,\Theta(\eta)\right), \qquad (5.111)$$

where $\Theta(\eta)$ was defined in (93). The field operator comes out by the use of (109). For example, for a scalar field it is

$$\varphi(x) = \frac{1}{a(\eta)}\int \frac{d\mu(J)}{\sqrt{2\omega(\eta)}}\left[\Phi_J(x)\,c_J^{(-)}(\eta) + \Phi_J^*(x)\,c_J^{(+)}(\eta)\right]. \qquad (5.112)$$

It is easy to see with the help of the equations for the coefficients $\alpha_\lambda(\eta)$ and $\beta_\lambda(\eta)$ that operators $c_J^{(\pm)}$ satisfy Heisenberg equations of motion:

$$\frac{dc_J^{(\pm)}(\eta)}{d\eta} = \pm\frac{1}{2}w^{(s)}\,c_J^{(\mp)}(\eta) + i\left[H^{(s)}\eta), c_J^{(\pm)}(\eta)\right], \qquad (5.113)$$

where $H^{(s)}(\eta)$ is the Hamiltonian diagonal in $c_J^{(\pm)}$; the quantities $w^{(s)}$ are defined in (97), (102) for $s = 0, 1/2$ respectively. The upper sing by the first term rightwards corresponds to boson fields (the lower one — to fermion fields).

These equations show that evolution of the operators $c_J^{(\pm)}$ is determined by two factors. First, it is the ordinary dependence on time of Heisenberg operators due to the exponential factor in (111) (in Eq. (113) it is described by the last term). Secondly, the dependence on time is connected with the fact that, at every moment, redefinition of the particle notion occurs; it corresponds to the first term in the right-hand side of Eq. (113).

In the Heisenberg picture the state $|0\rangle$, that is a vacuum state at the moment $\eta = \eta_0$, is not a vacuum state for $\eta > \eta_0$. In every mode J it contains

$$n_J^{(s)}(\eta) = \langle 0|\overset{*}{c}{}_J^{(+)}(\eta)\, c_J^{(-)}(\eta)|0\rangle = \langle 0|c_{\bar J}^{(+)}(\eta)\overset{*}{c}{}_{\bar J}^{(-)}(\eta)\,|0\rangle$$
$$= |\beta_\lambda(\eta)|^2 = s_\lambda(\eta) \equiv n_\lambda^{(s)}(\eta) \tag{5.114}$$

pairs of quasiparticles with quantum numbers J and $\bar J$. Due to isotropy, the spectrum of created pairs depends only on the momentum quantum number λ.

The amount of quasiparticle pairs of the spin s per unit space volume is

$$n^{(s)}(\eta) = \frac{2s+1}{2\pi^2\, a^3(\eta)} \int d\mu^{(s)}(\lambda)\, n_\lambda^{(s)}(\eta), \tag{5.115}$$

where $n_\lambda^{(s)}(\eta)$ is defined in (114) and

$$\int d\mu^{(s)}(\lambda) = \begin{cases} \int\limits_0^\infty d\lambda\,(\lambda^2 - \kappa\, s^2), & \kappa = -1, 0; \\ \sum\limits_{\lambda=\lambda_1}^{\infty} (\lambda^2 - s^2), & \kappa = +1, \end{cases} \tag{5.116}$$

here $\lambda_1 = 1 + s$ and summation is carried out with the step $\Delta\lambda = 1$.

In the case of asymptotically static metric (79) one has

$$\lim_{\eta\to\pm\infty} a(\eta) = a_\pm. \tag{5.117}$$

Then the quantity

$$n^{(s)}(\lambda) = \lim_{\eta\to\pm\infty} |\beta_\lambda(\eta)|^2 \equiv |\beta_\lambda|^2 \tag{5.118}$$

describes the real particles created from vacuum by gravitational field during all time of its existence.

Now let us consider renormalization of the vacuum SET $\langle 0|T_{ik}|0\rangle$ in isotropic case. The main problem here is to remove the divergencies and to interpret this procedure in terms of renormalization of constants in the action for the gravitational field. As a vacuum $|0\rangle$ either the in-vacuum at $\eta \to -\infty$ may be chosen or a vacuum defined at some moment η_0. In the letter case we suppose that $a'(\eta)$ and a sufficient number of higher derivatives at $\eta = \eta_0$ vanish.

Here we consider a method of removing divergencies which gives the justification of the subtractive procedure in terms of renormalization and, at the same time, is effective when doing specific calculations. It is based on the combination of n-wave subtractive procedure and of dimensional regularization discussed in Sec. 6.

The main idea of the dimensional regularization method in coordinate representation is the following. All the formulae of the theory are written for a spacetime of the dimensionality N, after what a formal analytic continuation onto a complex plane of N is carried out in such a way that $N = 1 + (3 - 2\varepsilon)$, i.e., to consider that the continuation is carried out in the dimensionality of the space part of the metric. For complex ε the integrals divergent for $N = 4$ become convergent in the sense of distributions boundary values. The divergencies arise in the form of poles of corresponding expressions for $\varepsilon = 0$. As it will be shown below, their geometric structure is unambiguously identified in this case, and they can be consequently removed with the help of renormalization.

Consider at first a scalar field with arbitrary coupling coefficient ξ, and besides, for simplicity let us confine ourselves to the case of quasi-Euclidean metric (77), (79) ($\kappa = 0$) and let us use Cartesian coordinates.

To carry out the dimensional regularization, we need the following formulae for geometric quantities in N-dimensional spacetime with the metric (79) of quasi-Euclidean type:

$$\begin{aligned}
&g = \det g_{ik} = (-1)^{N-1}\, a^{2N};\\
&R_{00} = (N-1)\, c'; \qquad R_{\alpha\alpha} = -\left[c' + (N-2)\, c^2\right];\\
&R = a^{-2}\,(N-1)\,\left[2\, c' + (N-2)\, c^2\right];\\
&G_{00} = -(N-1)(N-2)\frac{c^2}{2}; \qquad G_{\alpha\alpha} = (N-2)\left[c' + (N-3)\frac{c^2}{2}\right],
\end{aligned} \tag{5.119}$$

where $\alpha = 1, 2, \ldots, N-1$ and the following notation was introduced

$$c(\eta) = \frac{a'(\eta)}{a(\eta)}. \tag{5.120}$$

It is convenient to take a complete system of solutions to the Klein-Fock equation (7) in the form

$$\varphi_J^{(+)}(x) = \frac{1}{\sqrt{2\,a^{N-2}}}\, g_\lambda(\eta)\, \Phi_J^*(\boldsymbol{x}),$$
$$\varphi_J^{(-)}(x) = \left(\varphi_J^{(+)}\right)^*, \tag{5.121}$$

where

$$J = \{\lambda_1, \ldots, \lambda_{N-1}\}, \qquad -\infty < \lambda_\alpha < \infty, \qquad \lambda = |\boldsymbol{\lambda}|,$$
$$\Phi_J(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^{N-1}}}\exp(-\mathrm{i}\,\lambda_\alpha\, x^\alpha), \tag{5.122}$$

$g_\lambda(\eta)$ obey the equation (81) and the initial conditions (84). Note that in the N-dimensional case the quantity $q(\eta)$ defined in (82) is equal to

$$q(\eta) = (\xi_c - \xi)\, a^2\, R, \qquad \xi_c = \frac{N-2}{4(N-1)}. \tag{5.123}$$

Expressions for the vacuum expectation values of the SET operator come out according to Eq. (49). Using the notations s_λ, u_λ, v_λ for bilinear combinations of g_λ and g'_λ introduced in Eqs. (104), one can represent the results in the following form

$$\langle 0|T_{ik}|0\rangle = \frac{B_N}{a^{N-2}}\int_0^\infty d\lambda\, \lambda^{N-2}\, \mathcal{T}_{ik}, \tag{5.124}$$

where

$$B_N = \left[2^{N-3}\sqrt{\pi^{N-1}}\,\Gamma\left(\frac{N-1}{2}\right)\right]^{-1}, \tag{5.125}$$

and the components of $\mathcal{T}_{ik}$ are equal to

$$\mathcal{T}_{00} = \frac{\omega}{2} + \omega s_\lambda + (N-1)(N-2)(\xi_c - \xi)\frac{c^2}{2\omega}\left(s_\lambda + \frac{1}{2}u_\lambda + \frac{1}{2}\right)$$
$$+ (N-1)(\xi_c - \xi)c v_\lambda, \tag{5.126}$$

$$\mathcal{T}_{\alpha\beta} = \left\{\frac{1}{N-1}\left[\frac{\lambda^2}{\omega}\left(s_\lambda + \frac{1}{2}\right) - \frac{m^2\,a^2}{2\omega}u_\lambda\right] - 2(\xi_c - \xi)\omega u_\lambda\right.$$
$$+ (\xi_c - \xi)\left[-2\xi\, a^2 R + (N-1)(N-2)\frac{c^2}{2}\right]\frac{1}{\omega}\left(s_\lambda + \frac{1}{2}u_\lambda + \frac{1}{2}\right)$$
$$\left. + (N-1)(\xi_c - \xi)c v_\lambda\right\}\gamma_{\alpha\beta}, \tag{5.127}$$

and $\mathcal{T}_{0\alpha} = \mathcal{T}_{\alpha 0} = 0$. Zero values of nondiagonal components of the SET expectation values are obvious from symmetry of the problem.

It is easy to see that the expressions (124) diverge as λ^N at the upper limit. As the analysis of the behavior of the solutions s_λ, u_λ, v_λ to the system (105) shows for $\lambda \to \infty$ in general case (124) contains, besides this higher divergence, more weak divergencies proportional to $\lambda^{N-2}, \ldots, \lambda^2$ and $\ln\lambda$ (for even N).

Let us continue analytically the expressions (124)–(127) in spacetime dimensionality onto the complex plane by setting $N = 4 - 2\varepsilon$, where ε is a complex parameter. It is worth while to introduce into the integrals a constant M of mass dimensionality in a certain power in order that, for any ε, our expressions would have the same dimensionality which they have in four-dimensional space. Therefore we obtain the regularized expectation values

$$\langle 0|T_{ik}|0\rangle_\varepsilon = \frac{B_{4-2\varepsilon}\,(M\,a)^{2\varepsilon}}{a^2} \int\limits_0^\infty d\lambda\, \lambda^{2-2\varepsilon}\, \mathcal{T}_{ik,\varepsilon}, \tag{5.128}$$

where $\mathcal{T}_{ik,\varepsilon}$ comes from (126), (127) for $N = 4 - 2\varepsilon$. (Note that one should make this replacement also in the expressions for ξ_c and R entering $\mathcal{T}_{ik}$.) For $\mathrm{Im}\varepsilon \neq 0$ the integrals in (128) are convergent in the sense of distributions limiting values.

Now let us apply the subtractive n-wave procedure described in Sec. 6 to the regularized vacuum expectation values. The rule (68) reduces to subtracting from the integrands of (128) three first nonvanishing terms of their asymptotic expansions in powers of ω^{-1} for $\omega \to \infty$ correspondingly to three types of divergencies (λ^4, λ^2 and $\ln\lambda$), which are present in four-dimensional space:

$$\mathrm{ren}\langle 0|T_{ik}|0\rangle_\varepsilon = \langle 0|T_{ik}|0\rangle_\varepsilon - \sum_{l=0}^{2} T_{ik,\varepsilon}(l), \tag{5.129}$$

where the index l has the same sense as in Eq. (68).

To calculate $T_{ik,\varepsilon}(l)$ in explicit form, let us act in the following way. Let us make a replacement in the expressions (126), (127) determining $\mathcal{T}_{ik,\varepsilon}$ for $N = 4 - 2\varepsilon$:

$$\lambda \to n\lambda, \qquad m \to nm, \qquad \omega \to n\omega \tag{5.130}$$

and let us search their expansion in inverse powers of n, after what let us set $n = 1$. To carry out this procedure, one needs to have corresponding expansions for s_λ, u_λ, v_λ entering $\mathcal{T}_{ik}$. The required expansions are, in fact, generalized WKB-solutions to the system (105) for s_λ, u_λ, v_λ with the initial conditions (107).

One can construct generalized WKB-solutions to (105) in the following way [26]. Let us make the replacement (130) in (105) and let us search s_λ, u_λ and v_λ in the form of asymptotic series in powers on n^{-1} (the index λ is omited in the coefficients of these series):

$$s_\lambda = \sum_{k=1}^{\infty} n^{-k} s_k, \qquad u_\lambda = \sum_{k=1}^{\infty} n^{-k} u_k, \qquad v_\lambda = \sum_{k=1}^{\infty} n^{-k} v_k, \tag{5.131}$$

Substituting (131) into (105) and equating terms with the same powers of n, we obtain a getting caught system of equations for s_k, u_k, v_k.

Since further we will consider fields with other spins, let us represent at first generalized WKB-solutions to the system (106) having the uniform form for conformal scalar, spinor and vector fields. First nonzero terms in (131) are:

$$\begin{aligned}
v_1 &= \frac{1}{2} W^{(s)}, \qquad u_2 = \frac{1}{4} DW^{(s)}, \qquad s_2 = \frac{1}{16} W^{(s)^2}, \\
v_3 &= -\frac{1}{8} D^2 W^{(s)} \pm \frac{1}{16} W^{(s)^3}, \\
u_4 &= -\frac{1}{16} D^3 W^{(s)} \pm \frac{3}{32} W^{(s)^2} DW^{(s)}, \\
s_4 &= -\frac{1}{32} W^{(s)} D^2 W^{(s)} + \frac{1}{64} \left[DW^{(s)} \right]^2 \pm \frac{3}{256} W^{(s)^4}.
\end{aligned} \tag{5.132}$$

Here the following notations are introduced

$$D = \frac{1}{\omega} \frac{d}{d\eta}, \qquad W^{(s)} = \frac{w^{(s)}}{\omega}.$$

For scalar field with arbitrary coupling (see Eqs. (105)) the quantities v_1 and s_2 have the same form as in (132), and one needs to add the following terms dependent on $(\xi_c - \xi)$ to the remaining terms of the expansions:

$$\begin{aligned}
\Delta u_2 &= \frac{q}{2\omega^2}, \qquad \Delta v_3 = -\frac{\omega}{4} D\left(\frac{q}{\omega^3}\right), \\
\Delta u_4 &= -\frac{1}{8} D\omega D\left(\frac{q}{\omega^3}\right) + \frac{q}{16\omega^2} W^{(0)^2} + \frac{q}{8\omega^2} DW^{(0)^2} + \frac{q^2}{4\omega^4}, \\
\Delta s_4 &= \frac{q^2}{16\omega^4} - \frac{1}{16} W^{(0)} \omega D\left(\frac{q}{\omega^3}\right) + \frac{q}{16\omega^2} DW^{(0)}.
\end{aligned} \tag{5.133}$$

Now one should substitute expansions (131) into the integrands of (128), in which the replacement (130) has been done, and to group terms having the same order in n^{-1}. By setting $n = 1$ in the result, we obtain three terms

$T_{ik,\varepsilon}(l)$, which are to be subtracted according to (129). Let us write them explicitly:

$$T_{ik,\varepsilon}(l) = \frac{B_{4-2\varepsilon}\,(M\,a)^{2\varepsilon}}{a^2}\int\limits_0^\infty d\lambda\,\lambda^{2-2\varepsilon}\,\mathcal{T}_{ik,\varepsilon}(l), \tag{5.134}$$

where for $l=0$ (in the first subtrahend)

$$\mathcal{T}_{00,\varepsilon}(0) = \frac{\omega}{2}, \qquad \mathcal{T}_{\alpha\beta,\varepsilon}(0) = \frac{\lambda^2}{2(3-2\varepsilon)\,\omega}\,\gamma_{\alpha\beta}, \tag{5.135}$$

for $l=1$ (in the second subtrahend)

$$\begin{aligned}
\mathcal{T}_{00,\varepsilon}(1) &= \omega\,s_2 + (\xi_c-\xi)\left[(3-2\varepsilon)(1-\varepsilon)\frac{c^2}{2\omega} + (3-2\varepsilon)\,c\,v_1\right], \\
\mathcal{T}_{\alpha\beta,\varepsilon}(1) &= \left\{\frac{1}{3-2\varepsilon}\left(\frac{\lambda^2}{\omega}\,s_2 - \frac{m^2a^2}{2\omega}\,u_2\right)\right. \\
&\quad + (\xi_c-\xi)\left[-2\omega\,u_2 + \frac{1}{\omega}\left(-\xi a^2 R + \frac{c^2}{2}(3-2\varepsilon)(1-\varepsilon)\right)\right. \\
&\quad\left.\left. +\ (3-2\varepsilon)\,c\,v_1\right]\right\}\gamma_{\alpha\beta},
\end{aligned} \tag{5.136}$$

and, at last, for $l=2$ (in the third subtrahend)

$$\begin{aligned}
\mathcal{T}_{00,\varepsilon}(2) &= \omega\,s_4 + (\xi_c-\xi)(3-2\varepsilon)\left[(1-\varepsilon)\frac{c^2}{\omega}\left(s_2+\frac{1}{2}u_2\right) + c\,v_3\right], \\
\mathcal{T}_{\alpha\beta,\varepsilon}(2) &= \left\{\frac{1}{3-2\varepsilon}\left(\frac{\lambda^2}{\omega}\,s_4 - \frac{m^2a^2}{2\omega}\,u_4\right)\right. \\
&\quad +(\xi_c-\xi)\left[-2\omega\,u_4 + \left(-2\xi a^2 R + (3-2\varepsilon)(1-\varepsilon)\,c^2\right)\frac{1}{\omega}\left(s_2+\frac{1}{2}u_2\right)\right. \\
&\quad\left.\left. +\ (3-2\varepsilon)\,c\,v_3\right]\right\}\gamma_{\alpha\beta},
\end{aligned} \tag{5.137}$$

One can split the integrals in (134) on account of (135)–(137) in elementary terms of the form

$$\int\limits_0^\infty \frac{dx\,x^{2(k-\varepsilon)}}{\sqrt{(x^2+1)^{2m+1}}} = \frac{1}{2}\,B\left(k-\varepsilon+\frac{1}{2}, m-k+\varepsilon\right),$$

where $B(p,q)$ is the beta-function. Analytic continuation of the right-hand side determines the values of the integral even for such parameters k and m for which it diverges in the classical sense as soon as $\mathrm{Im}\varepsilon \neq 0$. The divergencies when the regularization is taken off manifest themselves as poles of beta-function at the point $\varepsilon = 0$.

Let us expand the results of calculation of the values of $T_{ik,\varepsilon}(l)$ in series in powers of ε, and besides, let us write down only those terms, which are nonzero for $\varepsilon \to 0$. In this way we obtain

$$T_{ik,\varepsilon}(0) = -\frac{m^4}{32\pi^2}\left(\frac{1}{\varepsilon}+b-\frac{1}{2}\right) g_{ik}, \tag{5.138}$$

$$T_{ik,\varepsilon}(1) = -\frac{m^2}{144\pi^2} G_{ik} - \left(\frac{1}{6}-\xi\right)\frac{m^2}{8\pi^2}\left(\frac{1}{\varepsilon}+b-1\right) G_{ik},$$

$$T_{ik,\varepsilon}(2) = \frac{1}{1440\pi^2}\left[\frac{1}{6}\,{}^{(1)}H_{ik} - {}^{(3)}H_{ik} + 10\left(\frac{1}{6}-\xi\right){}^{(1)}H_{ik} + 90\left(\frac{1}{6}-\xi\right)^2\left(\frac{1}{\varepsilon}+b-2\right){}^{(1)}H_{ik}\right],$$

where

$$b = -\ln\frac{m^2}{4\pi M^2} + 2 - C,$$

C is the Euler constant, ${}^{(1)}H_{ik}$ is defined in Eq. (64), and ${}^{(3)}H_{ik}$ is quadratic in curvature tensor

$${}^{(3)}H_{ik} = R^l{}_i R_{lk} - \frac{2}{3} R R_{ik} - \frac{1}{2}\left(R^{lm} R_{lm} - \frac{1}{2}R^2\right) g_{ik}. \tag{5.139}$$

Note that, despite that ${}^{(3)}H_{ik}$ has a covariant geometric character, the conservativity condition for it $\nabla_k {}^{(3)}H_i{}^k = 0$ is valid only in conformally flat space and hence, unlike ${}^{(1)}H_{ik}$, it could not be obtained by variation of the action determined by a geometric quantity.

The obtained results allow one unambiguously to solve the problem about the interpretation of the n-wave procedure subtractions (129) in terms of renormalizations of the initial constants in the effective Lagrangian of gravitational field (65). It is obvious from (138) that the first subtraction ($l = 0$) is equivalent to infinite (for $\varepsilon \to 0$) renormalization of cosmological constant and the second subtraction ($l = 1$) — to renormalization of gravitational constant, which is infinite for nonconformal scalar field but becomes finite in the conformal case. Much more complicated situation is for the third subtraction ($l = 2$). Subtraction of those terms in $T_{ik,\varepsilon}(2)$ that are proportional to ${}^{(1)}H_{ik}$

is equivalent to renormalization of the constant β by the term R^2 in Eq. (65). The term with the constant α in this case is absent in the effective Lagrangian because our metric (79) is conformally flat and

$$\frac{\delta}{\delta g^{ik}} \int d^4x \sqrt{-g} \left(R^{ik} R_{ik} - \frac{1}{3} R^2 \right) = 0$$

in it. At the same time finite terms $\sim {}^{(3)}H_{ik}$ are presented in $T_{ik,\varepsilon}(2)$ which could not be obtained by variation of local polynomial action with respect to g^{ik}, so that their subtraction does not correspond to any renormalization.

For spinor and vector fields by analogous way the geometric structure of divergencies is established and the renormalization is carried out. For example, for the spinor case

$$\begin{aligned} \mathcal{T}_{00}^{(1/2)} &= \omega(-1 + 2s_\lambda), \\ \mathcal{T}_{\alpha\beta}^{(1/2)} &= \frac{\lambda^2}{(N-1)\,\omega} \left[-1 + 2s_\lambda - \frac{ma}{\lambda} u_\lambda \right] \gamma_{\alpha\beta}. \end{aligned} \tag{5.140}$$

The subtrahends $T_{\alpha\beta,\varepsilon}^{(1/2)}(l)$ under renormalization (129) are found as in the scalar case. We use the asymptotic series (131) with terms (132), in which for $s = 1/2$ the lower sign is taken. The results have the form

$$\begin{aligned} T_{ik,\varepsilon}^{(1/2)}(0) &= \frac{m^4}{16\pi^2} \left(\frac{1}{\varepsilon} + b - \frac{1}{2} \right) g_{ik}, \\ T_{ik,\varepsilon}^{(1/2)}(1) &= -\frac{m^2}{48\pi^2} \left(\frac{1}{\varepsilon} + b - \frac{1}{2} \right) G_{ik}, \\ T_{ik,\varepsilon}^{(1/2)}(2) &= \frac{1}{1440\pi^2} \left(\frac{1}{2}\, {}^{(1)}H_{ik} - \frac{11}{2}\, {}^{(3)}H_{ik} \right). \end{aligned} \tag{5.141}$$

Note that in this case one needs only finite renormalization of the constant β.

Using the obtained results, we may calculate now the total vacuum SET of quantized fields in isotropic space. Here the results for conformal scalar and spinor fields are presented.

In order to find finite renormalized expectation values, which will be denoted by $\langle T_{ik} \rangle$, one needs to take off the regularization (129) when the pole terms have been cancelled:

$$\langle T_{ik} \rangle = \lim_{\varepsilon \to 0} \left[\langle 0 | T_{ik} | 0 \rangle_\varepsilon - \sum_{l=0}^{2} T_{ik,\varepsilon}(l) \right]. \tag{5.142}$$

Thus one obtains expressions that obey by construction the conservativity condition and contain both local terms describing vacuum polarization by gravitational field and nonlocal (in time) terms connected with particle creation.

When doing explicit calculations, it is convenient to use no dimensional regularization but to make subtractions ordered by the n-wave procedure mode-by-mode (in the integral).

For hyperbolic and quasi-Euclidean space ($\kappa = -1, 0$) one can represent total renormalized vacuum expectation values of the SET in the form

$$\langle T_{ik}^{(s)} \rangle = \frac{1}{\pi^2 a^2} \int_0^\infty d\lambda \left\{ \left(\lambda^2 - \kappa s^2\right) \left[\mathcal{T}_{ik}^{(s)}(s_\lambda, u_\lambda) - \mathcal{T}_{ik}^{(s)}(s_2, u_2) \right] \right. \\ \left. - \lambda^2 \mathcal{T}_{ik}^{(s)}(s_4, u_4) \right\}, \tag{5.143}$$

where s_λ, u_λ are exact solutions to the system (106) with the initial conditions (107), s_k, u_k are first nonvanishing terms in WKB-asymptotics of these solutions written down in (132). The functions $\mathcal{T}_{ik}^{(s)}(s, u)$ come out from the expressions (126), (127) for $\xi = \xi_c$, (140) for $N = 4$, if to delete in them first terms independent of s and u (those having been removed by subtracting $T_{ik}(0)$). In particular, for a scalar conformal field:

$$\mathcal{T}_{00}^{(0)}(s_\lambda, u_\lambda) = \omega\, s_\lambda, \tag{5.144}$$
$$\mathcal{T}_{\alpha\beta}^{(0)}(s_\lambda, u_\lambda) = \frac{1}{3} \left(\frac{\lambda^2}{\omega} s_\lambda - \frac{m^2 a^2}{2\omega} u_\lambda \right) \gamma_{\alpha\beta},$$

and for a spinor field:

$$\mathcal{T}_{00}^{(1/2)}(s_\lambda, u_\lambda) = 2\omega\, s_\lambda, \tag{5.145}$$
$$\mathcal{T}_{\alpha\beta}^{(1/2)}(s_\lambda, u_\lambda) = \frac{2}{3} \left(\frac{\lambda^2}{\omega} s_\lambda - \frac{\lambda m a}{2\omega} u_\lambda \right) \gamma_{\alpha\beta}.$$

The expressions (143) in spherical space is modified in the following way. First, the integral of the first term containing s_λ, u_λ is replaced by the sum over discrete values of λ (whilst in the terms with $s_{2,4}$ and $u_{2,4}$ the integrals are remained). Secondly, "topological" additions, which arise due to difference between the sum and the integral when doing first subtraction in (129) (with $l = 0$) appear here. They have the following form:

$$\Delta T_{ik}^{(s)} = \pm \frac{2s+1}{\pi^2 a^2} \left(\sum_{\lambda=\lambda_0}^{\infty} - \int_0^\infty d\lambda \right) \left(\lambda^2 - s^2\right) \Lambda_{ik} \tag{5.146}$$

with

$$\Lambda_{ik} = \frac{1}{2}\left[\omega \oplus \frac{1}{3}\frac{\lambda^2}{\omega}\,\gamma_{\alpha\beta}\right],$$

where $\lambda_0 = 1+s$ and the upper (lower) sign corresponds to bosons (fermions).

The quantities (146) represent the Casimir contribution to the total vacuum SET of quantized fields in curved spacetime (the detailed calculations of the Casimir SET in spacetimes with different non-Euclidean topology may be found in [20]).

Therefore total vacuum expectation value of the SET in the case of a spherical space are given by the expressions

$$\langle T_{ik}^{(s)}\rangle = \frac{1}{\pi^2 a^2}\left\{\sum_{\lambda=\lambda_0}^{\infty}\left(\lambda^2 - s^2\right)\mathcal{T}_{ik}^{(s)}(s_\lambda, u_\lambda) - \int_0^\infty d\lambda\left[\left(\lambda^2 - s^2\right)\mathcal{T}_{ik}^{(s)}(s_2, u_2) + \lambda^2\,\mathcal{T}_{ik}^{(s)}(s_4, u_4)\right]\right\} + \Delta T_{ik}^{(s)}. \tag{5.147}$$

In general case total renormalized vacuum expectation values $\langle T_{ik}\rangle$ are complicated nonlocal functionals of the metric. The important exception is massless conformally invariant fields in conformally flat spaces, in particular homogeneous isotropic spaces with the metric (79). In this case, as we just show, $\langle T_{ik}\rangle_{m=0} \equiv \langle T_{ik}\rangle_0$ always have a purely local character and can be found by analytical way for any dependencies of $a(\eta)$.

For massless scalar and spinor fields the expressions for $\langle T_{ik}\rangle_0$ come out from general formulae (143), (147) by going to the limit $m \to 0$ that ought to be completed after integrating in λ.

As it seen from (106), the contributions from the terms dependent on s_λ, u_λ in this case vanish (there is one important exception: the metric with $a(\eta) \sim \exp(\eta)$; it will be considered below). For hyperbolic and quasi-Euclidean space ($\kappa = -1, 0$) using the expansions (132) for s_k, u_k after long but elementary calculations we find for a scalar field

$$\langle T_{00}^{(0)}\rangle_0 = \frac{1}{480\pi^2 a^2}\left[2c''c - c'^2 - 2c^4\right], \tag{5.148}$$
$$\langle T_{\alpha\beta}^{(0)}\rangle_0 = \frac{1}{1440\pi^2 a^2}\left[-2c''' + 2c''c - c'^2 + 8c'c^2 - 2c^4\right]\gamma_{\alpha\beta},$$

and for a spinor field

$$\langle T_{00}^{(1/2)}\rangle_0 = \frac{1}{480\pi^2 a^2}\left[6c''c - 3c'^2 - \frac{7}{2}c^4 + 5\kappa c^2\right],$$

$$\langle T^{(1/2)}_{\alpha\beta}\rangle_0 = \frac{1}{1440\pi^2 a^2}\Big[-6\,c''' + 6\,c''\,c - 3c'^2 + 14\,c'\,c^2 - \frac{7}{2}\,c^4 - 10\,\kappa\,c' + 5\,\kappa\,c^2\Big]\,\gamma_{\alpha\beta}. \tag{5.149}$$

In the spherical case one should add to (148), (149) the topological terms (146), which for a massless field are equal to

$$\Delta T^{(s)}_{ik} = \frac{d_s}{240\pi^2}\,J_{ik}, \tag{5.150}$$

where $d_0 = 1$, $d_{1/2} = 17/4$ and J_{ik} is a tensor, which has, in the coordinates (77), (79), the components

$$J_{ik} = \frac{1}{a^2}\left(1 \oplus \frac{1}{3}\,\gamma_{\alpha\beta}\right). \tag{5.151}$$

One can write the results (148)–(150) in terms of introduced above tensors ${}^{(1)}H_{ik}$ and ${}^{(3)}H_{ik}$

$$\langle T^{(s)}_{ik}\rangle_0 = \frac{1}{1440\pi^2}\left[A_s\,{}^{(3)}H_{ik} + B_s\,{}^{(1)}H_{ik} + \delta_{\kappa,-1}\,C_s\,J_{ik}\right], \tag{5.152}$$

where the coefficients are $A_0 = 1$, $B_0 = -1/6$, $C_0 = -6$, $A_{1/2} = 11/2$, $B_{1/2} = -1/2$, $C_{1/2} = -51/2$, and the multiplier $\delta_{\kappa,-1}$ shows that the term with C_s is present only in the case of hyperbolic space.

Let us calculate the trace of (152). From (64) and (139) we have

$$ {}^{(1)}H^i_i = -6\,\nabla^i\nabla_i R, \qquad {}^{(3)}H^i_i = -\left(R^{lm}\,R_{lm} - \frac{R^2}{3}\right).$$

Because the trace $J^i_i = 0$, in all three cases $\kappa = 0, \pm 1$ there is:

$$\langle T^{(s)i}_i\rangle_0 = \frac{1}{1440\pi^2}\left[A_s\left(R^{lm}\,R_{lm} - \frac{R^2}{3}\right) - 6\,B_s\,\nabla^i\nabla_i R\right]. \tag{5.153}$$

This is the conformal anomaly that we have told about in Sec. 6. The result (153) is consistent with (74) because in conformally flat space the Weyl tensor $C_{iklm} = 0$, and here we consider a complex scalar field.

Note that if we consider from the beginning a field as a massless one, then just after the first (with $l = 0$) subtraction in (129) the expectation values of the SET would vanish because in this case s_λ, $u_\lambda = 0$, and there is no need in further renormalization. This property, however, is not preserved for arbitrary small deviation from the field comformal invariance or from the space isotropy, when for a massless field s_λ, $u_\lambda \neq 0$ and there are all the types of divergencies.

Thus, one may consider both a nonzero value of $\langle T_{ik}\rangle_0$ and the conformal anomaly as a phenomenon of spontaneous conformal symmetry breaking.

Coming back to the general expression (152), let us note that because the tensor ${}^{(1)}H_{ik}$ comes out by varying the integral of R^2, the second term with this tensor can be removed by the additional renormalization of the constant β in the initial Lagrangian (65). The rest terms in (152) represent genuine vacuum polarization of massless fields in gravitational field. When the above renormalization has been done the vacuum SET of massless field takes the form

$$\langle T_{ik}^{(s)}\rangle_0 = \frac{1}{1440\pi^2}\left[A_s\,{}^{(3)}H_{ik} + \delta_{\kappa,-1}\,C_s\,J_{ik}\right]. \tag{5.154}$$

Consider now massive fields. The vacuum expectation values of the SET (143) in this case can be represented in the form of the sum

$$\langle T_{ik}^{(s)}\rangle = \langle T_{ik}^{(s)}\rangle_0 + \langle T_{ik}^{(s)}\rangle_m, \tag{5.155}$$

where the first terms are local expressions (152) independent on the mass, and $\langle T_{ik}^{(s)}\rangle$ are, in general case, nonlocal causal functionals of the scale factor $a(\eta)$.

To calculate $\langle T_{ik}^{(s)}\rangle_m$ let us start from the consideration of "Big-Bang" cosmological models, which form a base of modern cosmology.

As an important particular case, we consider cosmological Friedmann models which are the solutions to Einstein equations in the right-hand side of which the SET of classical matter stands with the state equation

$$P_b = (\gamma - 1)\,\varepsilon_b. \tag{5.156}$$

The spatial curvature κ is determined by the sign of the difference $\varepsilon_b - \varepsilon_{cr}$ where $\varepsilon_{cr} = 3h^2/(8\pi G)$ is the critical density corresponding to quasi-Euclidean model, and $h = a'/a^2$ is the Hubble parameter.

Friedmann models possess a singularity in classical theory: when $t \to 0$ the scale factor vanishes by the power law

$$a(t) = b_0\,t^q, \tag{5.157}$$

where $q = 2/(3\gamma)$. We suppose that $1/3 \le q < 1$; the value $q = 1/3$ corresponds to the most rigid state equation $P_b = \varepsilon_b$, and $q = 1$ corresponds to so-called Miln metric, which will be considered below. The considered interval includes the case of both radiation dominated ($\gamma = 4/3$, $q = 1/2$) background and the dust-type ($\gamma = 1$, $q = 2/3$) background. In terms of "conformal" time η

$$a(\eta) = b_1\,\eta^p, \tag{5.158}$$

where

$$b_1 = [(1-q)^q\, b_0]^{\frac{1}{1-q}}\,, \qquad p = \frac{q}{1-q}, \quad 0 < p < \infty.$$

Expressions (156), (157) are exact in the quasi-Euclidean model; in hyperbolic and spherical cases they are valid only for $\eta \ll 1$. However for realistic models of the Universe (where ε_b does not much differ from ε_{cr}) in the epoch $t \sim m^{-1}$ when nonlocal quantum effects were essential, the curvature of 3-space does not yet influence on the expansion law because the condition $ma_1 \gg 1$ is fulfilled with great supply. Therefore, for the models with $\kappa = \pm 1$, one may consider $\eta \gg 1$ and use the power laws (157) and (158).

Let us turn now to calculation of renormalized expectation values of the SET

$$\langle T_0^{(s)0} \rangle \equiv \varepsilon^{(s)}, \qquad \langle T_\alpha^{(s)\alpha} \rangle \equiv -P^{(s)}.$$

We represent details of calculations only for the energy density $\varepsilon^{(s)}$; the pressure $P^{(s)}$ can be easy obtained from the conservativity condition which reduces to the equality

$$\varepsilon' = -3\frac{a'}{a}\,(\varepsilon + P)\,.$$

Let us represent $\varepsilon^{(s)}$, similarly to (155), in the form of sum

$$\varepsilon^{(s)} = \varepsilon_0^{(s)} + \varepsilon_m^{(s)}, \tag{5.159}$$

where $\varepsilon_m^{(s)}$ is the term dependent on the field mass; $\varepsilon_0^{(s)}$ is the polarization term nonvanishing for $m \to 0$. The latter is determined by expression (154) (we suppose that the term with R^2 is absent in the renormalized effective Lagrangian). Using explicit expressions for ${}^{(3)}H_{00}$ and J_{00} we find

$$\varepsilon_0^{(s)} = \frac{1}{480\,\pi^2\,a^4}\left[A_s\,\left(c^2+\kappa\right)^2 + \delta_{\kappa,-1}\frac{C_s}{3}\right]. \tag{5.160}$$

For the power law (157), accounting that for $\eta \ll 1$ one can neglect the spatial curvature, we have

$$\varepsilon_0^{(s)} \approx \frac{q^4\,A_s}{480\,\pi^2\,t^4}, \qquad P_0^{(s)} \approx \left(\frac{4}{3q} - 1\right)\,\varepsilon_0^{(s)}. \tag{5.161}$$

The term $\varepsilon_m^{(s)}$ in Eq. (159) is determined by two first terms of the integrand of (143). To calculate it one needs to know s_λ, besides the quantity s_2 defined in (132). To find s_λ one should either directly solve the system (106) or search solutions to the systems (96), (101) for α_λ and β_λ (remind that $s_\lambda = |\beta_\lambda(\eta)|^2$). It is possible to carry out the calculations in analytical way for two important limiting cases: $t \ll m^{-1}$ and $t \gg m^{-1}$ (we suppose that $t = 0$ corresponds to $\eta = 0$).

Let us consider at first the early epoch $t \ll m^{-1}$. Here the gravitational field is strong because in this case $|R_i^k| \gg m^2$. One can find asymptotics of s_λ for $\lambda \ll \eta^{-1}$ and for $\lambda \gg ma$, and besides, these regions overlap each other because

$$m\, a(\eta)\, \eta \sim m \int_0^\eta a(\eta')\, d\eta' = m\, t \ll 1.$$

In the region $\lambda \ll \eta^{-1}$ the following estimation is valid for Θ from Eq. (93):

$$\Theta \equiv \int_0^\eta \omega(\eta')\, d\eta' < \lambda\eta + mt \ll 1,$$

and the systems (96), (101) can be solved by the method of sudden perturbations. For a scalar field we obtain

$$s_\lambda = \frac{(\omega - \lambda)^2}{4\,\omega\,\lambda} \tag{5.162}$$

and for a spinor field

$$s_\lambda = \frac{\omega - \lambda}{2\,\omega}. \tag{5.163}$$

In the region $\lambda \gg ma$ it is convenient to use Volterra equations equivalent to (106), (107) and besides, one may consider that $\omega \approx \lambda$, $\Theta \approx \lambda\eta$. In the scalar case the first iteration gives

$$s_\lambda = \frac{m^4}{16\lambda^4} \left| \int_0^\eta d\eta_1\, \frac{da^2(\eta_1)}{d\eta_1}\, \exp(2i\,\lambda\,\eta_1) \right|^2, \tag{5.164}$$

and in the spinor case, after integrating the result of the first iteration by parts, on account of smoothness of $a(\eta)$ at zero:

$$s_\lambda = s_2 - \frac{m^4}{16\lambda^4} \left\{ 2\,a'(\eta) \int_0^\eta d\eta_1\, a''(\eta_1)\, \cos 2\lambda(\eta - \eta_1) - \left| \int_0^\eta d\eta_1\, a''(\eta_1)\, \exp(2i\,\lambda\,\eta_1) \right|^2 \right\}. \tag{5.165}$$

It is easy to check smooth joining of represented asymptotics in the region

$$ma \ll \lambda \ll \eta^{-1}. \tag{5.166}$$

The quantity $\varepsilon_m^{(s)}$ comes out by integrating in λ two terms of (143) dependent on s_λ and s_2. Choosing the transitional momentum λ_0 by proceeding from the condition (166), we can use in the interval $(0, \lambda_0)$ the asymptotics (162), (163), and in the interval (λ_0, ∞) — (164), (165). Moreover, the quantity λ_0 is absent from the result.

Therefore for a scalar field we find [10]

$$\varepsilon_m^{(0)} = -\frac{m^2}{48\pi^2}\left(\frac{a'}{a^2}\right)^2 + \frac{m^4}{16\pi^2}\left\{ \ln\frac{1}{m\,a} - C - \frac{1}{4} - \frac{1}{a^4}\int_0^\eta d\eta_1 \frac{da^2}{d\eta_1}\int_0^\eta d\eta_2 \frac{da^2}{d\eta_2}\ \ln|\eta_1-\eta_2| \right\}, \tag{5.167}$$

and for a spinor field [10]

$$\varepsilon_m^{(1/2)} = -\frac{m^2}{4\,\pi^2\,a^4}\left[\frac{a'^2}{2}\left(\ln\frac{1}{m\,a} - C - \frac{4}{3}\right) - a'\,\Phi_1(\eta) + \Phi_2(\eta)\right]. \tag{5.168}$$

Here the notations are used

$$\Phi_1(\eta) = \int_0^\eta d\eta_1\, a''(\eta_1)\ \ln(\eta-\eta_1), \qquad \Phi_2(\eta) = \int_0^\eta d\eta_1\, a''(\eta_1)\,\Phi_1(\eta_1).$$

For power laws of expansion we obtain from (167), (168):

$$\begin{aligned}\varepsilon_m^{(0)} &= -\frac{q^2\,m^2}{48\,\pi^2\,t^2} + \frac{m^4}{16\pi^2}\left[\ln\frac{1}{mt} + D^{(0)}(q)\right],\\ \varepsilon_m^{(1/2)} &= -\frac{q^2\,m^2}{8\,\pi^2\,t^2}\left[\ln\frac{1}{mt} + D^{(1/2)}(q)\right],\end{aligned} \tag{5.169}$$

where

$$\begin{aligned}D^{(0)}(q) &= \psi\left(\frac{1+q}{1-q}\right) + \ln(1-q) + \frac{1-2q}{4q},\\ D^{(1/2)}(q) &= \psi\left(\frac{q}{1-q}\right) + \ln(1-q) - \frac{4}{3} - \frac{1-q}{2(2q-1)},\\ \psi(z) &= \frac{\Gamma'(z)}{\Gamma(z)}.\end{aligned} \tag{5.170}$$

For the epoch $t \gg m^{-1}$ the approximative calculation uses expansion in small parameter $(mt)^{-1}$. The obtained results

$$\varepsilon_m^{(s)} \approx \frac{\text{const}}{a^3}, \qquad P_m^{(s)} \ll \varepsilon_m^{(s)} \tag{5.171}$$

describe particles of a nonrelativistic gas. The contribution from the polarization terms (160) at that epoch is small.

Let us turn now to the study of particle creation effect itself in Friedmann models (157). The density of quasiparticle pairs $n_\lambda^{(s)}(t)$ as a function of time is defined by the formula (115) where $n_\lambda^{(s)}(\eta) = s_\lambda(\eta)$. When $t \ll m^{-1}$ we have

$$n^{(0)} = \frac{m^3}{24\pi^2}, \qquad n^{(1/2)} = \frac{q^2}{3q-1}\frac{m^2}{t}. \tag{5.172}$$

The ratio $n^{(1/2)}/n^{(0)} \sim (mt)^{-1} \gg 1$, such a difference in creation of scalar and spinor quasiparticles is explained by action of the Pauli principle for spinor quasiparticles.

Let us turn to the epoch $t \gg m^{-1}$. It is natural to split the quantity $n^{(s)}$ in two components

$$n^{(s)}(t) = n_1^{(s)}(t) + n_2^{(s)}(t) \tag{5.173}$$

where

$$n_1^{(s)}(t) = \frac{2s+1}{2\pi^2 a^3(t)} \int d\lambda\, \lambda^2 n^{(s)}(\lambda) \tag{5.174}$$

represents the pair density of real created particles. The component $n_2^{(s)}(t)$ determined by s_2 is equal to

$$n_2^{(s)}(t) = K_s\, m\, h^2(t), \tag{5.175}$$

where $K_0 = 1/512$, $K_{1/2} = 3/256$. This formula describes the virtual pairs with the characteristic correlation length $r \sim m^{-1}$.

It is obvious that the term $n_1^{(s)}$ corresponding to real particles dominates for $1/3 < q < 2/3$, i.e. in the range of realistic equations of the background state (156). The corresponding amount of particles in the Lagrange volume $N^{(s)} \sim n^{(s)}(t)\, a^3(t)$ =const. This means that particle creation have been ceased for $t \gg m^{-1}$.

Let us turn now to the particular case of isotropic metric with $\kappa = -1$ and

$$a(t) = t = e^{\eta}. \tag{5.176}$$

Its main feature is that it describes usual flat space in some noninertial coordinates. Therefore it may be a test for examining the validity of the procedure of finding $\langle T_{ik}\rangle$: if the state $|0\rangle$ is chosen in such a way that it coincides with the usual vacuum of Minkowski space $|0_M\rangle$ then the exact equality $\langle T_{ik}\rangle = 0$ must take place. Moreover, in spite of the nonstationarity of $a(t)$ here no particle creation must take place.

The vacuum state $|0_M\rangle$ is characterized by the requirement of positive frequency of solutions to the Eq. (81) for $t \to \infty$ (for shortness let us consider here a scalar field only). Such solutions have the form

$$g_\lambda(t) = -\mathrm{i}\sqrt{\frac{2}{\pi}}\, K_{\mathrm{i}\lambda}(-\mathrm{i}\, mt). \tag{5.177}$$

One can exactly find the values of s_λ in (143) with the help of (104). The contribution from "polarization" terms (152) into $\langle T_{ik}\rangle$ in this case is equal to

$$\langle T_{ik}\rangle_0 = -\frac{1}{240\pi^2} J_{ik}. \tag{5.178}$$

Therefore the zero value of the total SET (155) is ensured by the condition

$$\int_0^\infty d\lambda\, \lambda^2\, \omega\, [s_\lambda(t) - s_2(t)] = \frac{1}{240}. \tag{5.179}$$

It guarantees $\langle T_{00}\rangle = 0$. The equality $\langle T_{\alpha\beta}\rangle = 0$ comes out automatically from the conservativity condition.

One may check that the equality (179) is actually correct by substitution of explicit expressions for s_λ and s_2. Thus, all three subtractions should be made in (142) to get the correct result, even if after, e.g., first subtraction the obtained expression is already finite.

7.8 Self-consistent cosmological models

One of the main problems in modern gravitational theory is the problem of cosmological singularity. As it is known, singularities are general properties of classical gravitational theory. At the same time, quantum gravitational effects inevitably cause violation of energy dominance conditions for the SET (see Sec. 5) leading to singularities arising in the solutions to the Einstein equations. Thus the hopes were uttered repeatedly that the account of quantum effects might lead to removal of gravitational singularities and therefore might point the way how to solve the problem of initial conditions in cosmology.

To construct the self-consistent cosmological models we should look for the solutions of semiclassical Einstein equations with a vacuum SET of quantized fields as a source:

$$G_{ik} = -8\pi G \sum_s \langle T_{ik}^{(s)} \rangle. \tag{5.180}$$

Here G is a physical value of the gravitational constant and the cosmological constant is supposed to be zero.

Vacuum expectation values $\langle T_{ik}^{(s)} \rangle$ entering into the right-hand side of (180) can be represented according to (155) in the form of independent on the mass local polarization terms $\langle T_{ik}^{(s)} \rangle_0$ and of the terms $\langle T_{ik}^{(s)} \rangle_m$ determined by the masses of accounted fields which are nonlocal causal functionals of $a(\eta)$.

Let us assume that proportional to the tensor ${}^{(1)}H_{ik}$ terms in $\langle T_{ik}^{(s)} \rangle_0$ are removed by the renormalization of the constant β by the term $\sim R^2$ in gravitational Lagrangian so that $\beta_{ren} = 0$ (see Sec. 7). Thus $\langle T_{ik}^{(s)} \rangle_0$ are determined by the equations (154).

In the class of isotropic metrics it is convenient to confine ourselves to the 00-component of the equations (180) which can be written in the form

$$c^2 + \kappa = \frac{8\pi G}{3} (\varepsilon_0 + \varepsilon_m)\, a^2, \tag{5.181}$$

where

$$\varepsilon_0 = \sum \langle T_0^{(s)0} \rangle_0, \qquad \varepsilon_m = \sum \langle T_0^{(s)0} \rangle_m.$$

With the help of explicit expressions for ${}^{(3)}H_{ik}$ and J_{ik} one can represent the quantity ε_0 in the form

$$\varepsilon_0 = \frac{1}{480\,\pi^2\, a^4} \left[A \left(c^2 + \kappa \right)^2 + \frac{1}{3}\, C\, \delta_{\kappa,-1} \right], \tag{5.182}$$

where $A = \Sigma N_s A_s$, $C = \Sigma N_s C_s$, and N_s is the number of fields of the spin s. The quantity ε_m is determined by the sum of expressions of the form (167), (168).

It is convenient now to fix temporarily the unit of the lengh by setting $G/(180\pi) = 1$.

Let us start from the case of massless fields for which, generally speaking, $\varepsilon_m = 0$. Eq. (181) reduces in this case to

$$c^2 + \kappa = \frac{A}{a^2} \left[\left(c^2 + \kappa \right)^2 + \frac{C}{3A}\, \delta_{\kappa,-1} \right]. \tag{5.183}$$

For a flat 3-space (when $\kappa = 0$), besides the trivial solution a =const which corresponds to Minkowski space, Eq. (183) has a nontrivial solution

$$a(\eta) = \frac{\sqrt{A}}{\eta}. \tag{5.184}$$

This solution descibes the de Sitter space in orispherical coordinates. Its curvature (in usual units) is $R = 2160\pi/(GA)$, i.e., the space has dimensions of the Planck order. When $\kappa = +1$ the solution to Eq. (183) is

$$a(\eta) = \frac{\sqrt{A}}{\cos\eta}. \tag{5.185}$$

Such a metric describes the same de Sitter space but here spatial sections are spherical.

In terms of the proper synchronous time t the solutions (184), (185) have the form

$$a(t) = \sqrt{A}\exp\left(\frac{t}{\sqrt{A}}\right), \qquad a(t) = \sqrt{A}\cosh\left(\frac{t}{\sqrt{A}}\right), \tag{5.186}$$

i.e. depend exponentially on time. (The self-consistent solutions (184), (185) to Einstein equations (180) were firstly obtained in 1980 in the paper [17] and a bit later independently in the paper [24].) Such a strong time dependence of a on time during the very early stage of the Universe evolution helps to solve the problem of causality, or of a horizon, discussed in the Introduction. A year later expansion of the Universe according to (186) was called "inflation" and the corresponding cosmological models – the "inflationary" ones [11]. But in [11] inflation was caused by the especially invented classical "inflaton" field, not the vacuum quantum effects of known fields and elementary particles. By this reason obtaining inflation as the self-consistent solutions to semiclassical Einstein equations (180) looks much more fundamental.

Note that found solutions satisfy Eq. (180) even if the term with the tensor ${}^{(1)}H_{ik}$ is included into $\langle T_{ik}\rangle_0$ because the latter is equal to zero identically in the de Sitter space.

It is known that the presence of the tensor ${}^{(1)}H_{ik}$ (dependent on the third and fourth derivatives of g_{ik} and originated from the terms $\sim R^2$ in the Lagrangian) in the equations leads to arising of scalar and tensor instabilities. In particular, the de Sitter solution is unstable in this case relative to spatially homogeneous massive scalar mode (scalarons).

On the basis of this instability, a cosmological model of inflationary type has been constructed [24] in which initial nonsingular de Sitter Universe came at radiative dominated Friedmann expansion regime due to generation of scalarons with very large masses and their decay into the usual particles. The existence of so heavy particles is, however, under question.

Let us come back to Eq. (183) and turn to the case of hyperbolic space ($\kappa = -1$). There are two solutions here, just as in the case $\kappa = 0$. One of them, which disappears when $A = 0$, is a spacetime evolving between two singularities and having a minimal curvature of the Planck order. The second solution does not disappear when $A = 0$. It turns out to be nonsingular on fulfillment of the condition $|C| > 3A$. Introducing notations

$$\delta = \sqrt{\frac{|C|}{3A}}, \qquad x = \frac{a^2}{2A\delta}, \qquad u = \sqrt{x^2+1} - x,$$

one can represent this second solution in an unexplicit form as

$$\pm\eta = \text{Arctanh}\sqrt{1-\delta u} - \frac{1}{\sqrt{1+\delta}}\,\text{Arctanh}\sqrt{\frac{1-\delta u}{1+\delta}} + \frac{1}{\sqrt{\delta-1}}\left(\frac{\pi}{2} - \arctan\sqrt{\frac{\delta^2-1}{1-\delta u}}\right). \tag{5.187}$$

It has one branch symmetric relative to the replacement $\eta \to -\eta$ and when $|\eta| \to \infty$ it behaves like

$$a(\eta) \approx \sqrt{\frac{|C|}{3}}\cosh\left(|\eta| - \eta_0\right), \tag{5.188}$$

$$\eta_0 = \frac{1}{\sqrt{\delta-1}}\left(\frac{\pi}{2} - \arctan\sqrt{\delta^2-1}\right) - \frac{1}{\sqrt{1+\delta}}\,\text{Arctanh}\sqrt{\frac{1}{1+\delta}}.$$

When $\eta = 0$ a "rebound" from a singularity takes place according to the law

$$a(\eta) \sim a_m + \frac{1}{2}\frac{\delta^2-1}{\delta^2+1}\eta^2 \qquad \text{with} \quad a_m = \sqrt{\frac{|C|}{3} - A}.$$

This rebound takes its origin from the second term in Eq. (183), which plays for $C < 0$ the role of background with a negative energy density ensuring violation of energy dominance conditions when $|C| > 3A$.

For $|\eta| \gg 1$ the asymptotic (188) corresponds with exponential accuracy to the Miln metric, i.e., the model during short time interval transfers into empty Minkowski space (the curvature is $R \sim t^{-8}$ in terms of the proper time t).

Let us fix our attention to corrections to the obtained results due to nonzero field mass. These corrections are negligible for de Sitter self-consistent model on condition $GM^2 \ll 1$ where M is the sum of masses of accounted fields.

Consider now corrections to the solution (187). Calculation of ε_m according to (167) on condition $\varepsilon(\eta = 0) = 0$ gives in the region $|t| < m^{-1}$ for each field

$$a^2\,\varepsilon_m \sim \begin{cases} m^2\,\eta^2 & \text{for } |\eta| < 1, \\ m^2 & \text{for } 1 < |\eta| < 50. \end{cases}$$

At the same time local terms in Eq. (181), when $a(\eta)$ is determined by the equality (187), have the form

$$a^2\,\varepsilon_0 \sim \begin{cases} G^{-1} & \text{for } |\eta| < 1, \\ t^{-2} & \text{for } 1 < |\eta|. \end{cases}$$

It is obvious that here one can neglect the contribution from ε_m with respect to ε_0.

When $t > m^{-1}$, nonlocal terms $\varepsilon_m^{(s)}$ describe the contribution of real created particles. There is in a given metric $a(t)$

$$\varepsilon_m^{(s)} = \frac{K_s\,m_s}{a^3},$$

where K_s is a constant dependent on the field spin (see Sec. 7). Let us consider for the sake of simplicity that all the massive particles are created at $t \sim m_s^{-1}$ and from this moment they start to influence on metric evolution. Summing contributions from all massive fields one can write Eq. (181) in the form

$$c^2 - 1 = \frac{1}{a^2}\left[A\,(c^2-1)^2 + \frac{C}{3} + \mu\,a^2\right], \tag{5.189}$$

where $\mu = 480\pi^2 \sum K_s N_s m_s$, and N_s is the number of fields of the mass m_s (let us remind that we consider the case $\kappa = -1$ and suppose $G/(180\pi) = 1$). It follows from the results of Sec. 7 that $480\pi^2 K_s \sim 1$ for all the spins. Solving Eq. (189) asymptotically, we find for $t \gg m_s^{-1}$

$$a(t) \approx t + a_1\,M\,G\;\ln(M\,t), \tag{5.190}$$

where $a_1 \sim 1$, $M = \sum N_s m_s$ is the total mass of all the fields and we came back to the previous units. It is obvious that, when $t \gg (GM^2)/M$, the second term in the right-hand side of Eq. (190) describing the deviation of the metric from the Miln one is negligible.

Thus, taking into account non-zero masses of the usual elementary particles, it is impossible to get the smooth transition between inflationary and Friedmann stages of the Universe evolution due to the vacuum quantum effects. In inflationary scenaria with inflaton field the transition to the Friedmann stage occurs through the intermediate stage of periodical oscillations of inflaton field. During this stage the density number of created scalar particles depends exponentially on the number of periods. The back reaction of these particles onto the background metric leads to the Friedmann expansion law [15]. This scenario, however, supposes very specific character of the inflaton field potential introduced into the theory by hands.

In the end of this section let us suppose that there is some self-consistent solution to Einstein equations (180) with oscillating asymptotic regime after the finishing of inflationary stage (the possible situation in where such solutions may arise is discussed in the next Section). Then, starting from some η, the self-consistent scale factor is periodic function with some period T:

$$a(\eta + T) = a(\eta). \tag{5.191}$$

As a result it follows also that

$$\Omega(\eta + T) = \Omega(\eta) \tag{5.192}$$

and the oscillatory Eq. (81) describing the nonconformal scalar field belongs to the Hill class equations.

As it was shown firstly in two independent papers [19,21], the density number of scalar particles created from vacuum by periodic external field may depend exponentially on the number of field periods. This takes place for some values of quantum numbers corresponding to the instability zones of the field equation.

Let us introduce even (u) and add (v) solutions to Eq. (81) obeying the initial conditions

$$\begin{cases} u(0) = 1, \\ u'(0) = 0, \end{cases} \qquad \begin{cases} v(0) = 0, \\ v'(0) = 1 \end{cases} \tag{5.193}$$

(here we transfered the zero point of conformal time into the initial moment of the oscillating regime).

As it was shown in the paper [19], the momentum density of scalar particles created during n periods of $a(\eta)$ oscillations is:

$$n_\lambda^{(0)} = \frac{\sinh^2(nD)}{\sinh^2 D} \frac{\left[\Omega_+^2\, v(T) + u'(T)\right]^2}{4\Omega_+^2}. \tag{5.194}$$

Here the quantity D is defined by: $\cosh D \equiv u(T)$, and Ω_+ is the value of Ω after n periods of oscillations.

For the Hill-type equations there exist three possibilities:

$$u(T) = \pm 1, \qquad n_\lambda^{(0)} \sim n^2, \tag{5.195}$$

$$|u(T)| > 1, \qquad n_\lambda^{(0)} \sim \exp\left[\varepsilon(\lambda)\, n\right] \quad \text{with} \quad \varepsilon(\lambda) > 0, \tag{5.196}$$

$$|u(T)| < 1, \qquad n_\lambda^{(0)} \sim \sin^2(nD). \tag{5.197}$$

It is clearly seen that in the cases (195), (196) the density number of created particles increases with the number of periods. This corresponds to the zones of unstable solutions to Eq. (81). In the case (197) the solutions are stable and the density of created particles depends periodically on the number of periods. The resonance situation takes place when the characteristics of the created particles (m, λ) and of external field (amplitude, period) are in a certain relation. It is notable, that for any amplitude and period of a oscillations there exist such λ which correspond to the instability zone.

The exponential creation of particles by the oscillations of the inflaton field in inflationary cosmology gives the mechanism of reheating after inflation [15]. The oscillations of a scale factor $a(\eta)$, as was shown above, may lead to the same effect. By this reason the existence of self-consistent solutions to Einstein equations describing both inflationary and oscillation stage is of much interest for applications in cosmology.

7.9 Quantized scalar field with arbitrary coupling in curved spacetime

In the previous Section we investigated the structure of infinities for the vacuum expectation values of the SET of nonconformal field in isotropic gravitational background. At the same time the total vacuum SET of nonconformal field was not calculated explicitly due to additional mathematical difficulties. Here we show that it may be calculated under rather general suggestions and gives much more possibilities for obtaining new self-consistent cosmological models compairing the conformal case.

After the separation of variables one gets once more the Eq. (81) with oscillation frequency (82). In order to keep the discussion as general as possible, we do not fix the values of the spacetime parameters $a(\eta)$ and $q(\eta)$ at the initial moment η_0:

$$a(\eta_0) \equiv a_0, \qquad q(\eta_0) \equiv q_0. \tag{5.198}$$

In nonconformal case it is reasonable to consider the quantity Ω as a natural dimensionless one-particle energy. This definition corresponds to the naive concept of adiabatic particles (for more sophisticated concept of adiabatic particles see below).

Instead of (84) we specify a complete orthonormal set of solutions to Eq. (81) by the following initial conditions:

$$g_\lambda(\eta_0) = \Omega^{-1/2}(\eta_0), \qquad g'_\lambda(\eta_0) = \mathrm{i}\,\Omega(\eta_0)\, g_\lambda(\eta_0). \tag{5.199}$$

The field operator is given by Eq. (108) once more and the equations

$$a_j^{(-)}|0\rangle = \overset{*}{a}_j^{(-)}|0\rangle = 0 \tag{5.200}$$

define the adiabatic vacuum state.

The solutions of (81) with initial conditions (199) may be equivalently represented in the form (compare with (92))

$$\begin{aligned} g_\lambda(\eta) &= \frac{1}{\sqrt{\Omega(\eta)}}\left[\alpha_\lambda^*(\eta)\,\exp\left(\mathrm{i}\Theta(\eta)\right) + \beta_\lambda(\eta)\,\exp\left(-\mathrm{i}\Theta(\eta)\right)\right], \\ g_\lambda{}'(\eta) &= \mathrm{i}\sqrt{\Omega(\eta)}\left[\alpha_\lambda^*(\eta)\,\exp\left(\mathrm{i}\Theta(\eta)\right) - \beta_\lambda(\eta)\,\exp\left(-\mathrm{i}\Theta(\eta)\right)\right], \end{aligned} \tag{5.201}$$

where α_λ, β_λ are the solutions to the first-order differential equations

$$\alpha_\lambda^{*\prime} = \frac{\Omega'}{2\Omega}\exp\left(-2\,\mathrm{i}\,\Theta\right)\beta_\lambda, \qquad \beta_\lambda' = \frac{\Omega'}{2\Omega}\exp\left(2\,\mathrm{i}\,\Theta\right)\alpha_\lambda^* \tag{5.202}$$

with the initial conditions

$$a(\eta_0) = 1, \qquad \beta(\eta_0) = 0 \tag{5.203}$$

and

$$\Theta(\eta) = \int_{\eta_0}^{\eta}\Omega(\eta_1)\,d\eta_1. \tag{5.204}$$

Equations (201) lead once more to a redefinition of a concept of particles according the Bogoliubov transformation (110). The spectral density of created adiabatic particles is given by (114) and the density per unit space volume — by (115) (with the new understanding of $\beta_\lambda(\eta)$).

For the non-renormalized vacuum SET we get the following expressions by the use of (11), (108) and (200):

$$\begin{aligned} \langle 0|T_{00}|0\rangle &= \frac{1}{\pi^2 a^2}\int d\mu(\lambda)\,\lambda^2\,\Big[\Omega S_\lambda + \frac{1}{2}\Omega + 3\Delta\xi\,c\,V_\lambda \\ &\quad + 3\Delta\xi\,\left(c' + 2c^2\right)\frac{1}{\Omega}\left(S_\lambda + \frac{1}{2}V_\lambda + \frac{1}{2}\right)\Big], \\ \langle 0|T_{\alpha\beta}|0\rangle &= \frac{\gamma_{\alpha\beta}}{\pi^2 a^2}\int d\mu(\lambda)\,\lambda^2\,\Big[\frac{\lambda^2}{3\Omega}\left(S_\lambda + \frac{1}{2}\right) - \frac{\Omega^2 - \lambda^2}{6\Omega}U_\lambda \\ &\quad - \Delta\xi\,\left(3c' + 2\kappa\right)\frac{1}{\Omega}\left(S_\lambda + \frac{1}{2}U_\lambda + \frac{1}{2}\right) - 2\,\Delta\xi\,\Omega\,U_\lambda + 3\,\Delta\xi\,c\,V_\lambda\Big]. \end{aligned} \tag{5.205}$$

Here the quantities S_λ, U_λ, V_λ are defined once more by the Eqs. (104) with the change of ω for Ω,

$$\Delta\xi \equiv \frac{1}{6} - \xi,$$

and Eqs. (106) are valid for them (with the change of $w^{(0)}$ for $w = \Omega'/\Omega$).

Renormalization of the matrix elements (205) is performed once more according to (142), where the terms to be subtracted are the same as in Sec. 7. However, they have a bit different form being expressed in terms of expansions of S_λ, U_λ, V_λ in powers of ω^{-1}. For the brevity we present here 00-components only:

$$\begin{aligned}
T_{00}(0) &= \frac{1}{2\pi^2 a^2}\int\limits_0^\infty d\mu(\lambda)\,\lambda^2\,\omega,\\
T_{00}(1) &= \frac{1}{\pi^2 a^2}\int\limits_0^\infty d\mu(\lambda)\,\lambda^2\left[\omega\,S_2 + 3\,c\,\Delta\xi\,V_1 + \frac{3}{2\omega}\,\Delta\xi\,\left(c^2-\kappa\right)\right],\\
T_{00}(2) &= \frac{1}{\pi^2 a^2}\int\limits_0^\infty d\mu(\lambda)\,\lambda^2\left\{\omega\left(S_4 + \frac{q^2}{16\omega^4} + \frac{q}{4\omega^2}\,U_2\right)\right.\\
&\quad\left. +3\,\Delta\xi\,\left[c\,V_3 + \frac{1}{\omega}\left(c^2-\kappa\right)\left(S_2 + \frac{1}{2}U_2 + \frac{q}{4\omega^2}\right)\right]\right\}.
\end{aligned} \tag{5.206}$$

Here the local expansion coefficients are:

$$\begin{aligned}
V_1 &= \frac{1}{2}W, \qquad U_2 = \frac{1}{2}DW, \qquad S_2 = \frac{1}{16}W^2,\\
V_3 &= \frac{1}{16}W^3 - \frac{1}{8}D^2W - \frac{\omega}{4}D\left(\frac{q}{\omega^3}\right),\\
U_4 &= \frac{1}{32}DW^3 - \frac{1}{16}D^3W - \frac{1}{8}D\left(\omega\,D\frac{q}{\omega^3}\right) + \frac{1}{8}\,\frac{q}{\omega^2}\,DW,\\
S_4 &= \frac{3}{256}W^4 - \frac{1}{32}W\,D^2W + \frac{1}{64}\,(DW)^2 - \frac{\omega}{16}\,W\,D\left(\frac{q}{\omega^3}\right),
\end{aligned} \tag{5.207}$$

where

$$W \equiv \frac{\omega'}{\omega^2}, \qquad D \equiv \frac{1}{\omega}\,\frac{d}{d\eta}.$$

The subtraction of the terms (206) is equivalent to renormalization as it was shown in Sec. 7.

Performing all three subtractions we get the finite result (and analogical for $\alpha\beta$-components)

$$\langle T_{00}\rangle = \frac{1}{\pi^2 a^2}\int d\lambda\,\lambda^2 \left\{\Omega\left(\frac{1}{2}+S_\lambda\right)\right. \tag{5.208}$$
$$-\omega\left(\frac{1}{2}-\frac{q}{4\omega^2}-\frac{q^2}{16\omega^4}-\frac{q}{2\omega^2}S_2+S_2+S_4\right)+3c\,\Delta\xi\,\left(V_\lambda - V_1 - V_3\right)$$
$$\left.+3\left(c'+2c^2\right)\Delta\xi\left[\frac{1}{\Omega}\left(\frac{1}{2}+S_\lambda+\frac{1}{2}U_\lambda\right)-\frac{1}{\omega}\left(\frac{1}{2}+S_2+\frac{1}{2}U_2+\frac{q}{4\omega^2}\right)\right]\right\}.$$

Here we omit topological terms which appear in the case $\kappa = 1$.

Notice that the expression (208) is much more complicated than in conformal case because the integrals containing S_λ, U_λ, V_λ and the integrals of local geometrical quantities S_i, U_i, V_i are infinite separately.

As it is shown in [3] by exploiting the early time approximation which reduces to two inequalities

$$mt \ll 1, \qquad \int\limits_{t_0}^{t} dt_1\sqrt{|\Delta\xi R(t_1)|} \ll 1 \tag{5.209}$$

one may obtain the asymptotic representations for g_λ in two overlaping momentum regions $(0,\lambda_0)$ and (λ_0,∞). This, in its turn, gives the possibility to calculate all the momentum integrals in the vacuum SET and to get the explicit cancellation of all the infinities. The first of the inequalities (209) simply means one of the conditions of applicability of semiclassical theory with Eq.(180). The second inequality of (209) restricts the value of the time t to which the obtained results may be applied for a given $\Delta\xi$.

With inequalities (209) the total renormalized vacuum SET was calculated for the arbitrary $a(\eta)$. The result can be displayed as [3]

$$<T_{ik}> = \sum_{a=1}^{5} <T_{ik}^{(a)}>, \tag{5.210}$$

where every contribution is covariantly conserved:

$$\left(\frac{d}{d\eta}+c\right)<T_{00}^{(a)}> + c\gamma^{\alpha\beta}<T_{\alpha\beta}^{(a)}> = 0. \tag{5.211}$$

The explicit form of different contributions to (210) is as follows. The first of them is:

$$<T_{ik}^{(1)}> = -\frac{m^4}{16\pi^2}\left(C+\frac{1}{4}\right)g_{ik}+\frac{m^2}{144\pi^2}\left[1-36\Delta\xi\left(C+\frac{3}{2}\right)\right]G_{ik} \tag{5.212}$$
$$+\frac{\Delta\xi}{144\pi^2}\left[-1+18\Delta\xi\left(C+1\right)\right]{}^{(1)}H_{ik}+\frac{1}{1440\pi^2}\left(-\frac{1}{6}\,{}^{(1)}H_{ik}+{}^{(3)}H_{ik}\right).$$

This addend to (210) consists of generally covariant tensors only.

The second contribution to the total vacuum SET is given by

$$<T_{00}^{(2)}>=\frac{1}{4\pi^2}\left[-\frac{m^4}{4}g_{00}-\Delta\xi m^2 G_{00}+\frac{1}{2}(\Delta\xi)^2\,{}^{(1)}H_{00}\right]\ \ln(ma),$$

$$<T_{\alpha\beta}^{(2)}>=\frac{1}{4\pi^2}\left[-\frac{m^4}{4}g_{\alpha\beta}-\Delta\xi m^2 G_{\alpha\beta}+\frac{1}{2}(\Delta\xi)^2\,{}^{(1)}H_{\alpha\beta}\right]\ \ln(ma)$$
$$-\frac{\gamma_{\alpha\beta}}{12\pi^2}\left[-\frac{m^4}{4}g_{00}-\Delta\xi m^2 G_{00}+\frac{1}{2}(\Delta\xi)^2\,{}^{(1)}H_{00}\right] \qquad (5.213)$$

and contains logarithmic terms describing the dependence of the vacuum SET on the remormalization point. The logarithm in this term can identically be transformed according to

$$\ln(ma)=\ \ln\frac{m}{\mu}+\ \ln(\mu a)$$

with μ being an arbitrary mass scale. Then, the term proportional to $\ln(m/\mu)$,

$$\frac{1}{4\pi^2}\left[-\frac{m^4}{4}g_{ik}-\Delta\xi\, m^2\, G_{ik}+\frac{1}{2}\,(\Delta\xi)^2\,{}^{(1)}H_{ik}\right]\ \ln\frac{m}{\mu}$$

represents a local, geometrical tensor that is solely made up of g_{ik}, G_{ik} and ${}^{(1)}H_{ik}$. Hence, the removal of this term from the vacuum SET can be interpreted as a finite renormalization of the respective constants. The remaining dependence of $\langle T_{ik}^{(2)}\rangle$ on $\ln(\mu a)$ is the renormalization scale dependence of the vacuum SET in nonconformal case.

The third contribution contains geometrical terms connected with non-zero value of κ:

$$<T_{00}^{(3)}>=\frac{3m^2\kappa}{144\pi^2}-\frac{3\kappa^2}{720\pi^2a^2}+\frac{\kappa}{4\pi^2}\Delta\xi\left[-3m^2+\frac{1}{2a^2}(c^2-\kappa)\right]$$
$$-\frac{9}{2\pi^2a^2}(\Delta\xi)^2c^2(c'+c^2+\kappa),$$

$$<T_{\alpha\beta}^{(3)}>=\gamma_{\alpha\beta}\left\{-\frac{m^2\kappa}{144\pi^2}-\frac{\kappa^2}{720\pi^2a^2}+\frac{\kappa}{4\pi^2}\Delta\xi\left[m^2+\frac{1}{6a^2}(-2c'+c^2-\kappa)\right]\right.$$
$$\left.+\frac{3}{2\pi^2a^2}(\Delta\xi)^2\left[c''c+2c'^2+c'c^2+(c^2+\kappa)(2c'-c^2)\right]\right\}. \qquad (5.214)$$

The fourth contribution to (210) describes dependence on initial conditions:

$$<T_{00}^{(4)}>= -\frac{Q_0^4}{64\pi^2 a^2} + \frac{Q_0^2}{16\pi^2 a^2}\left(C + \frac{1}{2} + \ln Q_0\right) \times \left[2m^2a^2 + 12\Delta\xi(c^2 - \kappa) - Q_0^2\right],$$

$$<T_{\alpha\beta}^{(4)}>= \gamma_{\alpha\beta}\left\{-\frac{Q_0^4}{192\pi^2 a^2} + \frac{Q_0^2}{48\pi^2 a^2}\left(C + \frac{1}{2} + \ln Q_0\right) \times \left[-2m^2a^2 + 12\Delta\xi(-2c' + c^2 - \kappa) - Q_0^2\right]\right\}, \qquad (5.215)$$

where $Q = \sqrt{m^2a^2 - q}$, $Q_0 = Q(\eta_0)$.

The last, fifth contribution consists of nonlocal integral terms and may be associated with the SET of particles created from vacuum by the gravitational field:

$$<T_{00}^{(5)}>= -\frac{1}{16\pi^2 a^2}\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\int_{\eta_0}^{\eta} d\eta_2\, Q^{2'}(\eta_2)\,\ln|\eta_1 - \eta_2|$$

$$+ \frac{3}{4\pi^2 a^2}\Delta\xi\left[cQ_0^{2'}\ln|\eta - \eta_0| + c\int_{\eta_0}^{\eta} d\eta_1\, Q^{2''}(\eta_1)\,\ln|\eta - \eta_1|\right.$$

$$\left. - (c' + 2c^2)\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\,\ln|\eta - \eta_1|\right],$$

$$<T_{\alpha\beta}^{(5)}>= \frac{\gamma_{\alpha\beta}}{48\pi^2 a^2}\left[4m^2a^2\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\,\ln|\eta - \eta_1|\right. \qquad (5.216)$$

$$\left.-\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\int_{\eta_0}^{\eta} d\eta_2\, Q^{2'}(\eta_2)\,\ln|\eta_1 - \eta_2|\right] + \frac{\gamma_{\alpha\beta}}{4\pi^2 a^2}\Delta\xi\left[3cQ_0^{2'}\ln|\eta - \eta_0|\right.$$

$$-\frac{Q_0^{2'}}{|\eta - \eta_0|} - Q_0^{2''}\ln|\eta - \eta_0| - \int_{\eta_0}^{\eta} d\eta_1\, Q^{2'''}(\eta_1)\,\ln|\eta - \eta_1|$$

$$\left. + 3c\int_{\eta_0}^{\eta} d\eta_1\, Q^{2''}(\eta_1)\,\ln|\eta - \eta_1| + (c' - 2c^2)\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\,\ln|\eta - \eta_1|\right].$$

Expressions (210)–(216) contain the conformal anomaly. Direct calculation leads to the result

$$\langle T_i{}^i\rangle^A \equiv \langle T_i{}^i\rangle(m \to 0) - \langle T_i{}^i(m \to 0)\rangle \qquad (5.217)$$

$$= -\frac{1}{1440\pi^2}\left[R^{ik}R_{ik} - \frac{1}{3}R^2 + (30\,\xi - 6)\,\nabla^k\nabla_k R + 90(\Delta\xi)^2 R^2\right],$$

which is in agreement with (74).

The density of created particles is

$$n(\eta) = \frac{Q}{48\pi\, a^3}\left[\left(Q^2 + Q_0^2\right)\, F\left(-\frac{1}{2},\frac{1}{2};1;z\right) - 2\,Q_0^2\, F\left(\frac{1}{2},\frac{1}{2};1;z\right)\right], \quad (5.218)$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function, and a notation is used

$$z \equiv 1 - \frac{Q_0^2}{Q^2}.$$

For example, for $\kappa = a_0 = q_0 = 0$ we have

$$n(\eta) = \frac{1}{24\pi^2}\left[m^2 - \Delta\xi\, R(\eta)\right]^{3/2} \quad (5.219)$$

and for the conformal case $\xi = 1/6$ the known result

$$n = \frac{m^3}{24\pi^2}$$

is reproduced.

Let us calculate the vacuum energy density from (210)–(216) for the power-law scale factors of the Friedmann cosmology (157), (158). The initial moment t_0 is suggested to be larger than the Planck time. The result is:

$$\begin{aligned}\langle {T_0}^0\rangle = \langle {T_0}^0\rangle_C &+ \frac{m^4}{16\pi^2}\frac{a_0^2}{a^2}\left(2 - \frac{a_0^2}{a^2}\right)\,\ln\frac{Q_0}{ma_0} + \frac{q^2}{8\pi^2 t^4}\Delta\xi\left(3 - 6q - \frac{\kappa t^2}{a^2}\right)\\ &- \frac{3\,m^2\,q^2}{4\pi^2\,t^2}\,\Delta\xi\,\left[\,\ln\frac{1}{mt} - C - \frac{1}{2q} - C_1(\kappa) - D_1(T)\right] \qquad (5.220)\\ &+ \frac{27q^2(1-2q)}{4\pi^2\,t^4}\,(\Delta\xi)^2\,\left[\,\ln\frac{1}{mt} - \frac{1-2q}{12} + \frac{2}{3} - C - \frac{1}{2q} - C_2(\kappa) - D_2(T)\right].\end{aligned}$$

Here $T \equiv (t/t_0)^{1-q}$ and $\langle {T_0}^0\rangle_C$ is the corresponding quantity for the case of conformal field. The dominating contributions here come from the terms $D_{1,2}(T)$ because for $t \gg t_0$ we have $T \gg 1$ (the explicit expressions for these terms may be found in [3]).

The quantity (220) changes drastically when ξ relaxes from its conformal value $1/6$.

For the specific case $q = 1/2$, $t_0 \to 0$ we have

$$\langle T_0{}^0 \rangle_{ren} = \frac{1}{7680\pi^2\, t^4} - \frac{m^2}{192\pi^2\, t^2} + \frac{m^4}{16\pi^2}\left(\ln\frac{1}{mt} - \ln 2 - C + \frac{3}{2}\right)$$
$$- \frac{3m^2}{16\pi^2\, t^2}\,\Delta\xi\left(\ln\frac{1}{mt} - \ln 2 - C - 1 + \frac{2\kappa\, t}{b_0^2}\,\ln\frac{(6\xi-1)\kappa}{m^2\, b_0^2\, t}\right)$$
$$- \frac{9}{16\pi^2}\,(\Delta\xi)^2\,\frac{\kappa^2}{b_0^4 t^2}\left(1 + \frac{2\kappa\, t}{b_0^2}\right)\left(1 - \ln\frac{(6\xi-1)\kappa}{m^2\, b_0^2\, t}\right). \tag{5.221}$$

If we additionally put $\kappa = 0$ in (221), we obtain the energy density for the quasi-Euclidean, radiation dominated Universe, in agreement with the known result [1].

The calculation of the total vacuum SET (210)–(216) in the non-singular cosmological model with the scale factor $a = a_0 \cosh\eta$ is presented in the paper [22].

As it was shown above, the vacuum SET of nonconformal scalar field is much more rich in content than of conformal one. It contains new (and large) contributions depending on coupling coefficient ξ. In addition it depends on the value of renormalization scale μ. These free parameters may be fixed in order to obtain the appropriate solutions to Einstein equations with a vacuum SET of nonconformal field as a source. Hopefully, the inflation-type solutions with oscillatory asymptotics may be contained among them.

But there is also one disadvantage in the expressions (210)–(216) for vacuum SET. As it is seen from (216), both the energy density and pressure contain terms which do not turn into zero and even turn into infinity when $\eta \to \eta_0$ and $\xi \neq 1/6$. This means that the vacuum SET of nonconformal field calculated in the naive adiabatic vacuum state developes an initial singularity and should not be considered physically reasonable. Specifically, it can not be substituted into the right-hand side of Einstein equations. This means that we should look for the more appropriate initial state to calculate the expectation values of the SET of nonconformal field.

With this aim let us discuss several latest results concerning the more rigorous understanding of adiabatic vacua.

Let us look for the solution of Eq. (81) in the WKB form:

$$\tilde{g}^*_\lambda(\eta) = \frac{1}{\sqrt{2\, W_\lambda(\eta)}}\, \exp\left[-\mathrm{i}\int_{\eta_0}^{\eta} W_\lambda(\eta')\, d\eta'\right], \tag{5.222}$$

where W_λ is some unknown frequency (not coinciding with the effective oscillator frequency Ω).

Substituting (222) to (81), we get the equation for the determination of W_λ

$$W_\lambda^2 = \Omega^2 - \frac{1}{2}\left(\frac{W_\lambda''}{W_\lambda} - \frac{3}{2}\frac{W_\lambda'^2}{W_\lambda^2}\right). \tag{5.223}$$

This equation may be solved by iteration:

$$W_\lambda^{(N+1)^2} = \Omega^2 - \frac{1}{2}\left[\frac{W_\lambda^{(N)''}}{W_\lambda^{(N)}} - \frac{3}{2}\frac{W_\lambda^{(N)\prime 2}}{W_\lambda^{(N)^2}}\right], \tag{5.224}$$

where (N) means the iteration order. One may put $W_\lambda^{(0)} \equiv \Omega$.

By the definition, an adiabatic vacuum state of iteration order N is determined by a complete set of solutions (g_λ, g_λ^*) satisfying initial conditions:

$$g_\lambda^*(\eta_0) = \tilde{g}_\lambda^{(N)^*}(\eta_0), \qquad g_\lambda^{*\prime}(\eta_0) = \tilde{g}_\lambda^{(N)^{*\prime}}(\eta_0). \tag{5.225}$$

¿From (222) it follows:

$$g_\lambda^*(\eta_0) = \frac{1}{\sqrt{2\,W_\lambda^{(N)}(\eta_0)}}, \tag{5.226}$$

$$g_\lambda^{*\prime}(\eta_0) = -\left[\mathrm{i}\,W_\lambda^{(N)}(\eta_0) + \frac{W_\lambda^{(N)\prime}(\eta_0)}{2\,W_\lambda^{(N)}(\eta_0)}\right] g_\lambda^*(\eta_0).$$

According to the theorem proved in [14] all adiabatic vacuum states are Hadamard states, i.e., the corresponding two-point functions possess the standard Hadamard singularity structure which allow the usual renormalization procedure in quantum field theory. Recently it was proved also that an adiabatic vacuum state must be at least of iteration order one for the renormalized vacuum SET to be finite on the initial Cauchy surface [16]. This important result shows that there is a set of physically acceptable vacua which are adiabatic vacuum states in the rigorous sense $|0_N\rangle$ of iteration order $N = 1, 2, 3, \ldots$ The problem to be solved is the calculation of the total renormalized SET of nonconformal scalar field in vacua $|0_N\rangle$ for homogeneous isotropic models of the Universe with arbitrary scale factor $a(\eta)$. The results should be used for obtaining of new self-consistent solutions to Einstein equations depending on three parameters: ξ, μ and N. It is hoped that in such a way it will be possible to construct the complete cosmological scenario describing inflationary, reheating and Friedmann stages of the Universe evolution and based on the first principles of QFT in curved spacetime.

Bibliography

[1] Bezerra V.B., Mostepanenko V.M., & Romero C., 1996 *Gravit. Cosmol.* **2**, 206

[2] Birrell N.D., Davies P.C.W., 1982 *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge

[3] Bordag M, Lindig J., & Mostepanenko V.M., 1998, *Clas. Quant. Grav.* **15**, 581

[4] Brown L.S., Cassidy J.P., 1977, *Phys. Rev.* **D15**, 2810

[5] Christensen S.M., 1976, *Phys. Rev.* **D14**, 2490

[6] Christensen S.M., 1979, *Nucl. Phys.* **B154**, 301

[7] De Witt B.S., 1975, *Phys. Rept.*, **C19**, 296

[8] Dowker J.S., Critchley R., 1976, *Phys. Rev.*, **D13**, 3224; 1977, **D16**, 3390

[9] Fulling S.A., 1989, *Aspects of Quantum Field Theory in Curved Space-Time*, Cambridge Univ. Press, Cambridge

[10] Grib A.A., Mamayev S.G., & Mostepanenko V.M., 1994, *Vacuum Quantum Effects in Strong Fields*, Friedmann Laboratory Publishing, St.Petersburg

[11] Guth A.H., 1981, *Phys. Rev.* **D23**, 347

[12] Hawking S.W., Ellis G.F.R., 1973, *The Large Scale Structure of Space-Time*, Cambridge Univ. Press, Cambridge

[13] Itzykson C., Zuber J.-B., 1980, *Quantum Field Theory*, Mc Grow-Hill, New York

[14] Junker W., 1995, Preprint DESY–95–144; e-print hep-th/9507097

[15] Kofman L., Linde A.,& Starobinsky A.A., 1997, *Phys. Rev.* **D56**, 3258

[16] Lindig J., e-print hep-th/9808133

[17] Mamayev S.G., Mostepanenko V.M., 1980, *Sov. Phys.—JETP(USA)*, **51**, 9

[18] Misner C.W., Thorne K.S., & Wheeler J.A., 1973, *Gravitation*, W.H. Freeman, New York

[19] Mostepanenko V.M., Frolov V.M., 1974, *Sov. J. Nucl. Phys. (USA)* **19**, 451

[20] Mostepanenko V.M., Trunov N.N., 1997, *The Casimir Effect and Its Applications*, Oxford Univ. Press, Oxford

[21] Narozhnyi N.B., Nikishov A.I., 1974, *Sov. Phys.—JETP(USA)*, **38**, 427

[22] Novello M., Bezerra V.B., & Mostepanenko V.M., 1998 Preprint CBPF-NF-028/98, Rio de Janeiro; *Int. J. Mod. Phys.* **D7**

[23] Parker L., Fulling S.A., 1974, *Phys. Rev.* **D9**, 341

[24] Starobinsky A.A., 1980, *Phys. Lett.*, **91B**, 99

[25] Zeldovich Ya.B., Novikov I.D., 1983, *The Structure and Evolution of the Universe*, Cambridge Univ. Press, Cambridge

[26] Zel'dovich Ya.B., Starobinsky A.A., 1972, *Sov. Phys. — JETP (USA)*, **34**, 1159