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Chapter 1

Classical Solutions in Multidimensional Cosmology

1. Multidimensional Cosmology with Multicomponent Perfect Fluid

1.1. Introduction. The model

Multidimensional gravitation and cosmology (see, for example [1-21] and references therein) is a very interesting object of investigations both from physical and mathematical points of view. Here we continue the study of such models started in [21].

Last decade the interest in multidimensional cosmology was stimulated mainly by the Kaluza-Klein and superstring paradigmas [22,23]. The "realistic" multidimensional cosmological models appeared mainly in a context of some unifications theories. Certainly, it is quite natural to believe that the Entire Universe is multidimensional and we live in a some sort of a (3+1)- dimensional layer, that is Our Universe. Of course, at first stage we should try to understand the structure of our 3-dimensional crude (dense) matter and the formation of Our Universe. But it seems to be very likely that at some stage of our development it will be just impossible to describe our (3+1)-dimensional layer (Our Universe) out of touch with other (multidimensional) layers and domains.

A large variety of multidimensional cosmological models is described by pseudo-Euclidean Toda-like systems [19] (see formula (1.1.10) below). These systems are not well studied yet. We note, that the Euclidean Toda-like systems are more or less well studied [24-28] (at least for certain sets of parameters, associated with finite-dimensional Lie algebras or affine Lie algebras). There is also a criterion of integrability by quadrature (algebraic integrability) for these (Euclidean) systems established by Adler and van Moerbeke [28]. Nevertheless, there are some indications that cosmological models may contain rather rich mathematical structures. For example, a self-dual reduction of the Bianchi-IX cosmology [29] leads us to the Halphen system of ordinary differential equations [30]. This system may be integrated in terms of modular forms [31] and is connected with a certain integrable reduction of the self-dual Yang-Mills equation [32] (with the infinite-dimensional group $SDiffSU(2)$). Another example is connected with the Kaluza-Klein dyon solution from [33]. The field equations for a spherically-symmetric Kaluza-Klein dyon in 5-dimensions were reduced in [33] to an open (Euclidean) Toda lattice with three points. Certainly, this problem may be formulated in terms of an appropriate cosmological model described by a pseudo-Euclidean Toda- like Lagrangian. So, we are led to an interesting nontrivial example of an integrable cosmological model.

In this lectures we consider a cosmological model describing the evolution of n Einstein spaces in the presence of m-component perfect-fluid matter. The metric of the model

$$g = -exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} exp[2x^i(t)]g^{(i)}, \tag{1.1}$$

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n, \tag{1.2}$$

where the manifold M_i with the metric $g^{(i)}$ is an Einstein space of dimension N_i, i.e.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \tag{1.3}$$

$i = 1, \ldots, n$; $n \geq 2$. The energy-momentum tensor is adopted in the following form

$$T^M_N = \sum_{\alpha=1}^{m} T^{M(\alpha)}_N, \tag{1.4}$$

$$(T^{M(\alpha)}_N) = diag(-\rho^{(\alpha)}(t), p^{(\alpha)}_1(t)\delta^{m_1}_{k_1}, \ldots, p^{(\alpha)}_n(t)\delta^{m_n}_{k_n}). \tag{1.5}$$

$\alpha = 1, \ldots, m$, with the conservation law constraints imposed:

$$\nabla_M T^{M(\alpha)}_N = 0 \tag{1.6}$$

$\alpha = 1, \ldots, m-1$. The Einstein equations

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N \tag{1.7}$$

(κ^2 is gravitational constant) imply $\nabla_M T^M_N = 0$ and consequently $\nabla_M T^{M(m)}_N = 0$.

We suppose that for any α-th component of matter the pressures in all spaces are proportional to the density

$$p^{(\alpha)}_i(t) = (1 - h^{(\alpha)}_i(x(t)))\rho^{(\alpha)}(t), \tag{1.8}$$

where

$$h_i^{(\alpha)}(x) = \frac{1}{N_i}\frac{\partial}{\partial x^i}\Phi^{(\alpha)}(x), \tag{1.9}$$

$i = 1,\ldots,n$, where $\Phi^{(\alpha)}(x)$ is a smooth function on R^n, $\alpha = 1,\ldots,m$. So, the total model is anisotropic.

In Sec. 1.2 the Einstein equations (1.1.7) for the model are reduced to the equations of motion for some Lagrange system with the energy constraint $E = 0$ imposed. When $m = 1$ and all spaces are Ricci-flat ($\lambda^i = 0$ in (1.1.3), $i = 1,\ldots,n$) such reduction was performed previously in [9].

In Sec. 1.3 we consider the Einstein equations, when all spaces are Ricci-flat and $h_i^{(\alpha)} = const$, $i = 1,\ldots,n$, $\alpha = 1,\ldots,m$. In this case we deal with pseudo-Euclidean Toda-like system with the Lagrangian

$$L_A = \frac{1}{2}G_{ij}\dot{x}^i\dot{x}^j - \sum_{\alpha=1}^{m}\kappa^2 A^{(\alpha)}\exp(u_i^{(\alpha)}x^i), \tag{1.10}$$

where $sign(G_{ij}) = (-,+,\ldots,+)$ [14,15], $u_i^{(\alpha)} = N_i h_i^{(\alpha)}$ and $A^{(\alpha)} = const$ $i = 1,\ldots,n$, $\alpha = 1,\ldots,m$. The Einstein equations are integrated in the following cases: 1) $m = 1$; 2) $n = 2$, $m \geq 2$, $A^{(\alpha)} \neq 0$, $u^{(\alpha)} - u^{(1)} = b^{(\alpha)}u$, $\alpha = 1,\ldots,m$, where $u^2 = G^{ij}u_iu_j = 0$, $u \neq 0$; 3) $u^{(\alpha)} = b^{(\alpha)}u$, $u^2 < 0$, $A^{(\alpha)} > 0$, $\alpha = 1,\ldots,m$.

1.2. The equations of motion

The non-zero components of the Ricci-tensor for the metric (1.1.1) are following

$$R_{00} = -\sum_{i=1}^{n}N_i[\ddot{x}^i - \dot{\gamma}\dot{x}^i + (\dot{x}^i)^2], \tag{1.2.1}$$

$$R_{m_in_i} = g^{(i)}_{m_in_i}[\lambda^i + \exp(2x^i - 2\gamma)(\ddot{x}^i + \dot{x}^i(\sum_{i=1}^{n}N_i\dot{x}^i - \dot{\gamma}))], \tag{1.2.2}$$

$i = 1,\ldots,n$.

We put

$$\gamma = \gamma_0 \equiv \sum_{i=1}^{n}N_ix^i \tag{1.2.3}$$

in (1.1.1) (the harmonic time is used). Then it follows from (1.2.1) and (1.2.2) that the Einstein equations (1.1.7) for the metric (1.1.1) with γ from (1.2.3) and the energy-momentum tensor from (1.1.4), (1.1.5) are equivalent to the following set of equations

$$\frac{1}{2}G_{ij}\dot{x}^i\dot{x}^j + V_c + \kappa^2\sum_{\alpha=1}^{m}\rho^{(\alpha)}\exp(2\gamma_0) = 0, \tag{1.2.4}$$

$$\lambda^i + \ddot{x}^i\exp(2x^i - 2\gamma_0) = \kappa^2\exp(2x^i)\sum_{\alpha=1}^{m}[p_i^{(\alpha)} + (D-2)^{-1}(\rho^{(\alpha)} - \sum_{j=1}^{n}N_j p_j^{(\alpha)})], \tag{1.2.5}$$

$i = 1,\dots,n$. Here

$$G_{ij} = N_i\delta_{ij} - N_iN_j \tag{1.2.6}$$

are the components of the minisuperspace metric,

$$V_c = -\frac{1}{2}\sum_{i=1}^{n}\lambda^i N_i exp(-2x^i + 2\gamma_0) \tag{1.2.7}$$

is the potential and $D \equiv dim M = 1 + \sum_{i=1}^{n} N_i$.

The conservation law constraint (1.1.6) for $\alpha \in \{1,...,m\}$ reads

$$\dot{\rho}^{(\alpha)} + \sum_{i=1}^{n}N_i\dot{x}^i(\rho^{(\alpha)} + p_i^{(\alpha)}) = 0. \tag{1.2.8}$$

We impose the conditions of state in the form (1.1.8), (1.1.9). Then eq. (1.2.8) gives

$$\rho^{(\alpha)}(t) = A^{(\alpha)}exp[-2N_ix^i(t) + \Phi^{(\alpha)}(x(t))], \tag{1.2.9}$$

where $A^{(\alpha)} = const$ and eqs. (1.2.4), (1.2.5) may be written in the following manner

$$\frac{1}{2}G_{ij}\dot{x}^i\dot{x}^j + V_c + \kappa^2\sum_{\alpha=1}^{m}A^{(\alpha)}\exp\Phi^{(\alpha)} = 0, \tag{1.2.10}$$

$$\lambda^i + \ddot{x}^i\exp(2x^i - 2\gamma_0) = -\kappa^2\sum_{\alpha=1}^{m}u^i_{(\alpha)}A^{(\alpha)}\exp(2x^i - 2\gamma_0 + \Phi^{(\alpha)}), \tag{1.2.11}$$

$i = 1,\dots,n$. In (1.2.11) we denote

$$u_i^{(\alpha)} \equiv N_ih_i^{(\alpha)} = \partial_i\Phi^{(\alpha)}, \qquad u^i_{(\alpha)} = G^{ij}u_j^{(\alpha)}, \tag{1.2.12}$$

where [15]

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{1.2.13}$$

are the components of the matrix inverse to the matrix (G_{ij}) (1.2.6).

It is not difficult to verify that equations (1.2.11) are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - V \tag{1.2.14}$$

where

$$V = V(x) = V_c(x) + \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp[\Phi^{(\alpha)}(x)]. \tag{1.2.15}$$

Eq. (1.2.10) is the zero-energy constraint

$$E = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j + V = 0. \tag{1.2.16}$$

Remark 1. In terms of 1-forms $u^{(\alpha)} = u_i^{(\alpha)} dx^i$, the relations (1.1.9) read: $u^{(\alpha)} = d\Phi^{(\alpha)}$, $\alpha = 1, \ldots, m$. In this case

$$du^{(\alpha)} = 0, \tag{1.2.17}$$

$\alpha = 1, \ldots, m$. The set of eqs. (1.2.17) (on R^n) is equivalent to (1.1.9). An open problem is to generalise the considered here formalism for the following cases: a) $du^{(\alpha)} \neq 0$ for some $\alpha \in \{1, \ldots, m\}$; b) $du^{(\alpha)} = 0$ for all $\alpha = 1, \ldots, m$, but $u^{(\alpha)}$ are defined on an open submanifold $\Omega \in R^n$ with the non-trivial cohomology group $H^1(\Omega, R) \neq 0$.

Using eqs. (1.2.1) and (1.2.2), it is not difficult to verify that the Einstein equations (1.1.7) for the metric (1.1.1) and the energy-momentum tensor from (1.1.4), (1.1.5), (1.1.8), (1.1.9) are equivalent to the Lagrange equations for the following degenerate Lagrangian (see also [15])

$$L = \frac{1}{2} \exp(-\gamma + \gamma_0(x)) G_{ij} \dot{x}^i \dot{x}^j - \exp(\gamma - \gamma_0(x)) V(x) \tag{1.2.18}$$

$(L = L(\gamma, x, \dot{x}))$. Fixing the gauge

$$\gamma = \gamma_0(x) - 2f(x), \tag{1.2.19}$$

where $f = f(x)$ is a smooth function on R^n, we get the Lagrangian

$$L_f = \frac{1}{2}\exp(2f(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(-2f(x))V(x). \tag{1.2.20}$$

For $f = 0$ we have the harmonic-time gauge (1.2.3). The set of Lagrange equations for the Lagrangian (1.2.18) (or equivalently the set of the Einstein equations) with γ from (1.2.19) is equivalent to the set of Lagrange equations for the Lagrangian (1.2.20) with the energy constraint imposed

$$E_f = \frac{1}{2}\exp(2f(x))G_{ij}\dot{x}^i\dot{x}^j + \exp(-2f(x))V(x) = 0. \tag{1.2.21}$$

Remark 2. We remind that the action of the relativistic particle of mass m, moving in the pseudo-Euclidean background space with the metric $\hat{G}_{ij}(x)$ has the following form

$$S = \int d\tau[\hat{G}_{ij}(x(\tau))\frac{\dot{x}^i\dot{x}^j}{2e(\tau)} - \frac{m^2}{2}e(\tau)], \tag{1.2.22}$$

where $e = e(\tau)$ is 1-bein. Comparing (1.2.18) and (1.2.22), we find that for $V(x) > 0$ the cosmological model (1.2.18) is equivalent to the model of relativistic particle with the mass $m = 1$, moving in the conformally-flat (pseudo-Euclidean) space with the metric $\hat{G}_{ij}(x) = 2V(x)G_{ij}$). In this case $e = 2V(x)\exp(\gamma - \gamma_0(x))$. For $V(x) < 0$ we have a tachyon. The problem may be also reformulated in terms of a geodesic-flow problem for conformally-flat metric (this follows from (1.2.22) or from a more general scheme).

1.3. Classical solutions

Now, we consider the following case: $\lambda^i = 0$ (all spaces are Ricci-flat), $u_i^{(\alpha)} = N_i h_i^{(\alpha)} = const$, $i = 1, \ldots, n$. Then $V_c = 0$ and we put $\Phi^{(\alpha)} = u_i^{(\alpha)} x^i$ in (1.2.15). In this case the Lagrangian (1.2.14) has the form (1.1.10).

Remark 3. The curvature induced term V_c (1.2.7) may be generated in the framework of the model with the Ricci-flat spaces M_i by the addition of n new components of the perfect fluid with $u_i^{(k)} = 2N_i - 2\delta_i^k$ and $\kappa^2 A^{(k)} = -\lambda^k N_k/2$, $i, k = 1, \ldots, n$. The introduction of the cosmological constant Λ into the model is equaivalent to the addition of a new component with $u_i^{(n+1)} = 2N_i$ and $\kappa^2 A^{(n+1)} = \Lambda$.

One-component matter

We consider the case $m = 1$, $A^{(1)} = A \neq 0$. We denote $h_i^{(1)} = h_i$, $u_i^{(1)} = u_i = N_i h_i$.

We remind [14, 15] that the minisuperspace metric

$$G = G_{ij} dx^i \otimes dx^j \tag{1.3.1}$$

has pseudo-Euclidean signature $(-,+,\ldots,+)$, i.e. there exist a linear transformation

$$z^a = V_i^a x^i, \tag{1.3.2}$$

diagonalizing the minisuperpace metric (1.3.1)

$$G = \eta_{ab} dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{1.3.3}$$

where

$$(\eta_{ab}) = (\eta^{ab}) \equiv diag(-1,+1,\ldots,+1), \tag{1.3.4}$$

$a, b = 0, \ldots, n-1$.

Proposition 1. For any $u = (u_i) \in R^n$, $u \neq 0$, there exists a (nondegenegate) $n \times n$ matrix (V_i^a) such that

$$\eta_{ab} V_i^a V_j^b = G_{ij} \tag{1.3.5}$$

and a) $V_i^0 = u_i/\sqrt{-u^2}$, for $u^2 < 0$; b) $V_i^1 = u_i/\sqrt{u^2}$, for $u^2 > 0$; c) $V_i^0 + V_i^1 = u_i$, for $u^2 = 0$;

Here and below $(u = (u_i) = (N_i h_i))$

$$u^2 \equiv u_i u^i = G^{ij} u_i u_j = \sum_{i=1}^{n} N_i (h_i)^2 + \frac{1}{2-D} (\sum_{i=1}^{n} N_i h_i)^2. \tag{1.3.6}$$

(We note that in notations of [14] $u^2 = \acute{\Delta}(h)/(2-D)$.)

This proposition follows from the fact that $< u, v > \equiv G^{ij} u_i v_j$ is bilinear symmetric 2-form of signature $(-,+,\ldots,+)$ and the following quite obvious.

Proposition 2. Let $v \in E = R^n$, $n \geq 2$, and $< .,. >: E \times E \longrightarrow R$ is a bilinear symmetric 2-form of signature $(-,+,\ldots,+)$. Then there exists a basis $v^0, \ldots, v^{n-1}$ in E, such that $< v^a, v^b > = \eta^{ab}$ and a) $v = v^0$, b) $v = v^1$, c) $v = v^0 + v^1$, in the cases: a) $v^2 \equiv < v, v > = -1$, b) $v^2 = 1$, c) $v^2 = 0$ respectively.

Let $u \neq 0$. In $z = (z^a)$-coordinates (1.3.2) with the matrix (V_i^a) from the Proposition 1 the Lagrangian (1.2.14) has the following form

$$L_A = \frac{1}{2}\eta_{ab}\dot{z}^a\dot{z}^b - V_A = -\frac{1}{2}(\dot{z}^0)^2 + \sum_{i=1}^{n-1}\frac{1}{2}(\dot{z}^i)^2 - V_A, \tag{1.3.7}$$

where

$$\begin{aligned} V_A &= \kappa^2 A \exp(2qz^0), \qquad u^2 < 0, && (1.3.8)\\ &= \kappa^2 A \exp(2qz^1), \qquad u^2 > 0, && (1.3.9)\\ &= \kappa^2 A \exp(z^0 + z^1), \qquad u^2 = 0, && (1.3.10)\end{aligned}$$

is the potential (1.2.15). Here we denote

$$2q \equiv \sqrt{|u^2|}. \tag{1.3.11}$$

The Lagrange equations for the Lagrangian (1.3.7)

$$\ddot{z}^a = -\eta^{ab}\partial_b V_A \tag{1.3.12}$$

with the energy constraint (1.2.16)

$$E_A = \frac{1}{2}\eta_{ab}\dot{z}^a\dot{z}^b + V_A = 0, \tag{1.3.13}$$

can be easily solved. We present the solutions.

a) For $u^2 < 0$

$$\begin{aligned} &z^i = p^i t + q^i, \qquad i = 1, \ldots, n-1, && (1.3.14)\\ &2qz^0 = y(t), && (1.3.15)\end{aligned}$$

where p^i and q^i are constants and

$$\begin{aligned} y(t) = &\quad \ln[C/D \sinh^2(\frac{1}{2}\sqrt{C}(t - t_0))], C \neq 0, D > 0, && (1.3.16)\\ = &\quad \ln[4/D(t - t_0)^2], \qquad C = 0, D > 0, && (1.3.17)\\ = &\quad \ln[-C/D \cosh^2(\frac{1}{2}\sqrt{C}(t - t_0))], C > 0, D < 0, && (1.3.18)\end{aligned}$$

Here t_0 is an arbitrary constant, $D = -2u^2\kappa^2 A$, $C = -u^2(\vec{p})^2$ and $(\vec{p})^2 = \sum_{i=1}^{n-1}(p^i)^2$.

b) For $u^2 > 0$ we have

$$\begin{aligned} &z^i = p^i t + q^i, \qquad i = 0, 2, \ldots, n-1, && (1.3.19)\\ &2qz^1 = y(t), && (1.3.20)\end{aligned}$$

with $(\vec{p})^2 = (p^0)^2 - \sum_{i=2}^{n-1}(p^i)^2$ in (1.3.15)-(1.3.18).

c) $u^2 = 0$, $u \neq 0$. In this case

$$x^i = p^i t + q^i, \qquad i = 2, \dots, n-1, \tag{1.3.21}$$
$$x^+ = x^0 + x^1 = p^+ t + q^+, \tag{1.3.22}$$
$$x^- = x^0 - x^1 = p^- t + q^- + \kappa^2 A z(t), \tag{1.3.23}$$

where for $p^+ \neq 0$

$$z(t) = 2(p^+)^{-2} \exp(p^+ t + q^+), \qquad p^+ p^- = (\vec{p})^2 \tag{1.3.24}$$

$(p^- = 0$ for $n = 2)$ and for $p^+ = 0$

$$z(t) = t^2 \exp q^+, \qquad (\vec{p})^2 + 2\kappa^2 A \exp q^+ = 0. \tag{1.3.25}$$

Here $(\vec{p})^2 = \sum_{i=2}^{n-1}(p^i)^2$.

For $u = 0$ we have

$$x^a = p^a t + q^a, \qquad a = 0, \dots, n-1, \tag{1.3.26}$$
$$\tfrac{1}{2}\eta_{ab} p^a p^b + \kappa^2 A = 0. \tag{1.3.27}$$

Kasner-like parametrization. Here we consider the case $u^2 < 0$, $A \neq 0$. For $C = -u^2(\vec{p})^2 > 0$ we reparametrize the time variable

$$\tau = \frac{T}{\sqrt{\varepsilon}} \ln \frac{\exp(\sqrt{C}(t - t_0)) + \sqrt{\varepsilon}}{\exp(\sqrt{C}(t - t_0)) - \sqrt{\varepsilon}}, \tag{1.3.28}$$

where

$$\varepsilon \equiv A/|A| = \pm 1, \qquad T \equiv (2/\kappa^2 |A| |u^2|)^{1/2}. \tag{1.3.29}$$

We introduce new (Kasner-like) parameters

$$\alpha^i \equiv -2V^i_s p^s / \sqrt{-u^2(\vec{p})^2}, \tag{1.3.30}$$

where $(V^i_s) = (V^s_i)^{-1}$ and the summation parameter s runs: $s = 1, \dots, n-1$. Then, due to relations (1.3.2), (1.3.5), (1.3.14)-(1.3.16), (1.3.18) and Proposition 1 we get the following expression for the metric (1.1.1) [40]

$$= -(\prod_{i=1}^{n}(a_i(\tau))^{2N_i - w_i}) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{1.3.31}$$

where

$$a_i(\tau) = A_i[\frac{\sinh(\tau\sqrt{\varepsilon}/T)}{\sqrt{\varepsilon}}]^{2u^i/u^2}[\frac{\tanh(\tau\sqrt{\varepsilon}/2T)}{\sqrt{\varepsilon}}]^{\alpha^i}, \tag{1.3.32}$$

$i = 1,\dots,n$; $A_i > 0$ are constants and the parameters α^i satisfy the relations

$$u_i\alpha^i = 0, \tag{1.3.33}$$

$$G_{ij}\alpha^i\alpha^j = -4/u^2 \tag{1.3.34}$$

(see Proposition 1 and (3.30)). For the density (2.15) we have

$$\rho(\tau) = A\prod_{i=1}^{n}(a_i(\tau))^{u_i-2N_i}. \tag{1.3.35}$$

We note, that $(\bar{p})^2 = 2\kappa^2|A|\prod_{i=1}^{n}A_i^{u_i}$

For $A > 0$ we have an exceptional solution (1.3.31), (1.3.33), (1.3.34) with the scale factors

$$a_i(\tau) = \bar{A}_i\exp(\pm 2u^i\tau/u^2T), \tag{1.3.36}$$

$\bar{A}_i > 0$, $i = 1,\dots,n$. This solution correspond to $C = 0$ case (1.3.17).

Remark 4. In [19] the Einstein equations (1.2.10), (1.2.11) were solved for $A^{(\alpha)} = 0$, $\alpha = 1,\dots,m$, $\lambda^1 \neq 0$, $\lambda^i = 0$, $i > 1$. The solutions [19] may be also obtained from the formulas (1.3.31)-(1.3.34). We note that the spherically-symmetric analogue of the solution [19] was considered in [36] (the case $d = 2$ was considered previously in [35]). There exists an interesting special case of the solutions [35, 36]. It is the n-time generalisation of the Schwarzschild solution

$g = -[(1-\frac{L}{R})^A]_{ab}dt^a\otimes dt^b + (1-\frac{L}{R})^{-spA}dR\otimes dR + (1-\frac{L}{R})^{1-spA}R^2d\Omega^2$, where $L \neq 0$ and $A = (A_{ab})$ is symmetric $n\times n$ matrix, satisfying the relation $sp(A^2) + (spA)^2 = 2$.

We consider this solution in a separate publication.

Two spaces with m-component matter

Here we consider the following case: $n = 2$, $m \geq 2$, $A^{(\alpha)} \neq 0$,

$$u^{(\alpha)} - u^{(1)} = b^{(\alpha)}u \tag{1.3.37}$$

$\alpha = 1,\dots,m$, where $u^2 = 0$, $u \neq 0$ and $b^{(\alpha)}$ are constants.

In z-coordinates (1.3.2), where the matrix (V_i^a) satisfies the Proposition 1 (see the case c) $u^2 = 0$) we have

$$z^+ = z^0 + z^1 = (V_i^0 + V_i^1)x^i = u_ix^i, \tag{1.3.38}$$

$$\Phi^{(1)} = u_i^{(1)}x^i = \alpha_+z^+ + \alpha_-z^-, \tag{1.3.39}$$

where $2\alpha_+ = - < u^1, u^* >$, $2\alpha_- = - < u^1, u >$, and $u^* = (u_i^*)$ is defined by the relation : $u_i^* x^i = z^-$ (or equivalently $< u^*, u^* >= 0$, $< u^*, u >= -2$).

Due to (1.3.37)-(1.3.39) the potential in (1.1.10) is factorised

$$V = V_+(z^+)V_-(z^-), \tag{1.3.40}$$

where

$$V_+(z^+) = \exp(\alpha_+ z^+)(\kappa^2 A^{(1)} + \sum_{k=2}^{m} \kappa^2 A^{(\alpha)} \exp(b^{(\alpha)} z^+), \tag{1.3.41}$$

$$V_-(z^-) = \exp(\alpha_- z^-). \tag{1.3.42}$$

Let $A^{(\alpha)} > 0$, $\alpha = 1, \ldots, m$, . We consider the f-gauge (1.2.19) with

$$F = e^{2f} = V. \tag{1.3.43}$$

In this gauge the Lagrangian (1.2.20) reads

$$L_f = -\frac{1}{2} V_+(z^+)\dot{z}^+ V_-(z^-)\dot{z}^- - 1. \tag{1.3.44}$$

In the variables

$$w^\pm = w^\pm(z^\pm) = \int_{z_0}^{z^\pm} dz V_\pm(z) \tag{1.3.45}$$

the Lagrangian (1.3.44) has a rather simple form

$$L_f = -\frac{1}{2}\dot{w}^+ \dot{w}^- - 1. \tag{1.3.46}$$

The equations of motion for (1.3.46) give

$$w^\pm(t) = p^\pm t + q^\pm. \tag{1.3.47}$$

The parameters $p^\pm$ satisfy the energy constraint

$$2E_f = -p^+ p^- + 2 = 0. \tag{1.3.48}$$

Remark 5. It is interesting to note that the so-called D-dimensional Schwarzschild-deSitter solution [44,45] may be obtained from the considered here cosmological solution with $n = m = 2$ and $N_1 = 1$, $N_2 = D - 2$.

n spaces with m component matter

Now we consider the simplest case of the multicomponent matter. We put in (1.1.10) $n \geq 2$, $A^{(\alpha)} > 0$, $u^{(\alpha)} = b^{(\alpha)}u$, $u^2 < 0$, where $b^{(\alpha)}$ are constants, $\alpha = 1, \ldots, m$.

In z-coordinates (1.3.2), corresponding to the case a) from the Proposition 1, the Lagrangian (1.1.10) has the form (1.3.7) with the potential

$$V_A = V_A(z^0) = \sum_{i=1}^{m} \kappa^2 A^{(\alpha)} \exp(2qb^{(\alpha)}z^0), \tag{1.3.49}$$

where q is defined in (1.3.11) ($A = (A^{(\alpha)})$). The solutions of the equations (1.3.12) and (1.3.13) are expressed by the formula (1.3.14) and the following relation

$$\int_{c_0}^{z^0} dx[2\mathcal{E} + 2V_A(x)]^{-1/2} = \pm(t - t_0), \tag{1.3.50}$$

where $2\mathcal{E} = \sum_{i=1}^{n-1}(p^i)^2$, and c_0, t_0 are constants.

1.4. Concluding remarks

In this section we investigated the multidimensional cosmological model with n ($n > 1$) Ricci-flat spaces, filled by m-component perfect fluid. In some sense, this model may be considered as "universal" cosmological model: a lot of cosmological models may be obtained from it under a suitable choice of parameters. This fact may be used for "Toda-like" classification of known exact cosmological (and spherically-symmetric) solutions of the Einstein equations. (We note, that the Bianchi-IX cosmological model is described by the "Toda-like" Lagrangian (1.10) with $n = 3$ and $m = 6$.)

Here we integrated the Einstein equations for some sets of parameters. But an open problem is the problem of integrability of the considered here model (at classical and quantum levels) for arbitrary values of the parameters m, n, N_i and $u_i^{(\alpha)}$. We hope to continue the investigation of this problem in forthcoming publications.

2. Multidimensional Cosmology with Multicomponent Perfect Fluid and Toda Lattices

2.1. Introduction

We consider dynamical systems with $n \geq 2$ degrees of freedom described by the Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij}\dot{x}^i\dot{x}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{i=1}^{n} b_i^s x^i], \quad m \geq 2. \tag{2.1.1}$$

A lot of systems in gravitation [33,44] and as well in multidimensional cosmology [1-21,45-48] reduce to the systems with such a Lagrangian.

Without loss of generality it can be assumed that matrix (η_{ij}) is diagonalized and $\eta_{ii} = \pm 1$ for $i = 1, \ldots, m$. Such system is an algebraic generalization of a well-known Toda lattice [24,39] suggested by Bogoyavlensky [25,40]. We say that it is an Euclidean Toda-like system, if bilinear form of kinetic energy is positively definite, i.e. $\eta_{ij} = \delta_{ij}$. Nearly nothing is known about Euclidean Toda-like systems with arbitrary sets of vectors $b_1, \ldots, b_m$, where $b_s = (b_1^s, \ldots, b_n^s)$ for $s = 1, \ldots, m$. But, if they form the set of admissible roots of a simple Lie algebra, then the system is completely integrable and possesses a Lax representation. Remind, that the set of roots $\alpha_1, \ldots, \alpha_m$ is called admissible [25,40], provided vectors $\alpha_r - \alpha_s$ are not roots for all $r, s = 1, \ldots, m$. Each simple Lie algebra possesses the following set of admissible roots

$$\omega_1, \ldots, \omega_n, -\Omega \tag{2.1.2}$$

where $\omega_1, \ldots, \omega_n$ are simple roots and Ω is the maximal root [41] (usually $\Omega = \omega_1 + \ldots + \omega_n$). Any subset of the set (2.1.2) is also admissible.

If the maximal root holds in the set (2.1.2), then generalised periodic Toda lattices arise. The different $L - A$ pairs for them were found by Bogoyavlensky [25,40]. There were also presented the Hamiltonians of this systems connected with simple Lie algebras.

The further progress in this field was attained by a number of authors (see, for example, [26-28,42,43] and refs. therein). In ref. [28] Adler and van Moerbeke established a criterion of algebraic complete integrability for Euclidean Toda-like systems. (This criterion was formally applied to multidimensional vacuum cosmology with n Einstein spaces in [19].) The explicit integration of the equations of motion for the generalized open Toda lattices (in this case the maximal root is thrown away) was developed by Olshanetsky and Perelomov [27] and Kostant [26]. (See also [42].)

Here we are interested in the problem of integrability of the Toda-like systems with the indefinite bilinear form of the kinetic energy. Let us call such systems pseudo-Euclidean Toda-like systems. To our knowledge, this problem has not been discussed intensively in the mathematical literature before. The reason, as it seems to us, consists in the following. If one try to connect a pseudo-Euclidean Toda-like system by the known manner with simple Lie algebra it reduces to an Euclidean system for the part of coordinates (see Sect. 2.4). Nevertheless, integrable pseudo-Euclidean Toda-like systems and search for their solutions in explicit form evoke a special interest, because such systems arise in cosmology. For instance, 4-dimensional vacuum homogeneous cosmological model of Bianchi IX-type is described by the Lagrangian (2.1.1) with $(\eta_{ij}) = \mathrm{diag}(-1, +1, +1)$ [25,40]. (In [31] it was shown, that this model has a rather rich mathematical structure.)

So, in this section we study integrable pseudo-Euclidean Toda-like systems appearing in multidimensional cosmology. This trend in the modern theoretical physics has appeared within the new paradigm based on unified theories and hypothesis of additional space-time dimensions. According to this hypothesis the physical space-time manifold has the topology $M^4 \times B$, where M^4 is a 4-dimensional manifold, and B is a so-called internal space (or spaces). Nonobservability of additional dimensions is attained in multidimensional cosmology by spontaneous or dynamical compactification of internal spaces to the Planck scale (10^{-33} cm.). Integrable cosmological models are of great interest, because the exact solutions allow to study dynamical properties of the model, in particular compactification of internal spaces, in detail.

In the Sect. 2.2, as in [37], we consider the cosmological model where multidimensional space-time manifold M is a direct product of the time axis R and of the n Einstein spaces $M_1, \dots, M_n$. We remind, that any manifold of constant curvature is the Einstein one. It is shown that Einstein equations for the scale factors with a source in the multicomponent perfect fluid form correspond to the Lagrangian (2.1.1) with (η_{ij}) =diag$(-1, +1, \dots, +1)$. We develop the integration procedure to the case of an orthogonal set of vectors $b_1, \dots, b_m$ in Sect. 2.3. Sect. 2.4 is devoted to the reduction of pseudo-Euclidean Toda-like system to the Euclidean one for a part of coordinates. This reduction allows us to obtain the class of the exact solutions for some nonorthogonal sets of the vectors $b_1, \dots, b_m$. We present the exact solution in the simplest case, when the reducible pseudo-Euclidean system is connected with the Lie algebra A_2. Discussion of results is presented is Sect. 2.5. We single out some interesting solutions, in particular, Euclidean wormholes.

We denote by n the number of Einstein spaces and by m the number of the matter components. Indices i and j run from 1 to n. Index s runs from 1 to m.

2.2. The model

Here we consider a cosmological model describing the evolution of $n \geq 2$ Einstein spaces in the presence of m-component perfect-fluid matter [37] as in Sect. 1.1 with $h_i(x) = const$ (see (1.1.9)). Then

$$\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2\gamma_0 + \sum_{i=1}^{n} u_i^{(\alpha)} x^i]. \tag{2.2.1}$$

where $A^{(\alpha)}$ =const and

$$u_i^{(\alpha)} = N_i h_i^{(\alpha)}. \tag{2.2.2}$$

The Einstein eqs. (1.1.7) may be written in the following manner

$$\frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j + V = 0, \tag{2.2.3}$$

$$\lambda^i + \ddot{x}^i \exp[2x^i - 2\gamma_0] = -\kappa^2 \sum_{\alpha=1}^{m} u^i_{(\alpha)} A^{(\alpha)} \exp[2x^i - 2\gamma_0 + \sum_{j=1}^{n} u^{(\alpha)}_j x^j]. \tag{2.2.4}$$

Here

$$G_{ij} = N_i \delta_{ij} - N_i N j \tag{2.2.5}$$

are the components of the minisuperspace metric,

$$V = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \exp[-2x^i + 2\gamma_0] + \kappa^2 \sum_{\alpha=1}^{m} A^{(\alpha)} \exp[\sum_{i=1}^{n} u^{(\alpha)}_i x^i]. \tag{2.2.6}$$

We denote

$$u^i_{(\alpha)} = \sum_{j=1}^{n} G^{ij} u^{(\alpha)}_j, \tag{2.2.7}$$

where

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{2.2.8}$$

are the components of the matrix inverse to (G_{ij}) [15].

It is not difficult to verify that eqs. (2.2.14) are equivalent to the Lagrange-Euler eqs. for the Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - V. \tag{2.2.9}$$

Eq. (2.2.3) is the zero-energy constraint.

We note, that in the framework of our model the curvature induced terms in the potential (2.2.6) may be considered as additional components of the perfect fluid. The introduction of the cosmological constant Λ into the model is equivalent also to the addition of a new component with $u_i = 2N_i$ and $\kappa^2 A = \Lambda$.

Finally, we present the potential (2.2.6) modified by introduction of Λ-term in the following form

$$\begin{aligned} V = \quad & \sum_{k=1}^{n} (-\frac{1}{2} \lambda^k N_k) \exp[\sum_{i,j=1}^{n} G_{ij} v^i_{(k)} x^j] + \\ & \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp[\sum_{i,j=1}^{n} G_{ij} u^i_{(\alpha)} x^j] + \Lambda \exp[\sum_{i,j=1}^{n} G_{ij} u^i x^j], \end{aligned} \tag{2.2.10}$$

where we denote:

$$v^i_{(k)} = \sum_{j=1}^{n} G^{ij} v^{(k)}_j = -2\frac{\delta^i_k}{N_k}, \quad v^{(k)}_j \equiv 2(N_j - \delta^k_j), \tag{2.2.11}$$

$$u^i = \sum_{j=1}^{n} G^{ij} u_j. \tag{2.2.12}$$

Let $< .,. >$ be a symmetrical bilinear form defined on n-dimensional real vector space R^n with the components $G_{ij} =< e_i, e_j >$ in the canonical basis $e_1, \dots e_n$. ($e_1 = (1, 0, \dots, 0)$ etc.) It was shown [14,15], that the bilinear form $< .,. >$ is pseudo-Euclidean one with the signature $(-, +, ..., +)$. Then the Lagrangian (2.2.9) may be written as:

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >]. \tag{2.2.13}$$

($x = x^1 e_1 + \dots + x^n e_n, \quad x \in R^n$). Here we denoted by m the total number of components, including curvature, perfect fluid and the cosmological term. We note, that for $m = 1$ the Lagrangian system (2.2.13) is always integrable. The exact solutions were obtained in [37]. (Some special cases were considered in [20,48].) In the present paper we consider multicomponent case: $m \geq 2$.

We say that a vector $y \in R^n$ is called time-like, space-like or isotropic, if $< y, y >$ has negative, positive or null values correspondingly. Vectors y and z are called orthogonal if $< y, z >= 0$.

2.3. Exact solutions for orthogonal sets of vectors

Let vectors $b_1, \dots, b_m$ satisfy the conditions: 1. They are linear independent;
2. $< b_\alpha, b_\beta >= 0$ for all $\alpha \neq \beta$, i.e. the set of vectors is orthogonal.

Then $m \leq n$. It is not difficult to prove

Proposition 1. The set of vectors $b_1, \dots, b_n$ may contain at most one isotropic vector.

Proposition 2. The set of vectors $b_1, \dots, b_n$ may contain at most one time-like vector and, if it holds the other vectors must be space-like.

Remark 1. The additional term $a^{(0)} \exp[< b_0, x >]$ with zero-vector $b_0 = 0$ does not change the equations of motion, but changes the energy constraint (2.2.3)

$$\frac{1}{2} < \dot{x}, \dot{x} > + \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >] + a^{(0)} = 0. \tag{2.3.1}$$

It corresponds to the perfect fluid with $h_i^{(0)} = 0$ for all $i = 1, \ldots, n$. Such a perfect fluid is called the stiff or Zeldovich matter [49]. It may be considered also as minimally coupled real scalar field [50]. We take into account this additional component by modification of the energy constraint

$$\frac{1}{2} < \dot{x}, \dot{x} > + \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >] = E_0. \tag{2.3.2}$$

These propositions allow to split the class of exact solutions under consideration into following subclasses:

A. There are one time-like vector and at most $(n-1)$ space-like vectors.

B. There are at most $(n-1)$ space-like vectors.

C. There are one isotropic vector and at most $(n-2)$ space-like vectors (this subclass arises for $n \geq 3$).

To integrate eqs. of motion in all subclasses we consider an orthonormal basis $e'_1, \ldots, e'_n$. These vectors are such that

$$< e'_i, e'_j >= \eta_{ij}, \tag{2.3.3}$$

where we denote by η_{ij} the components of the matrix

$$(\eta_{ij}) = \mathrm{diag}(-1, +1, \ldots, +1). \tag{2.3.4}$$

Let us define coordinates of the vectors in this basis by

$$x = X^1 e'_1 + \ldots + X^n e'_n. \tag{2.3.5}$$

For these new coordinates we have

$$X^i = \eta_{ii} < e'_i, x >, \quad x^i = \sum_{k=1}^{n} t^i_k X^k, \tag{2.3.6}$$

where we denoted by t^i_k the components of a non-degenerate matrix defined by

$$e'_k = \sum_{i=1}^{n} t^i_k e_i. \tag{2.3.7}$$

Components t^i_k satisfy the relations:

$$\sum_{k,l=1}^{n} G_{kl} t^k_i t^l_j = \eta_{ij}. \tag{2.3.8}$$

Let us try to find exact solutions for subclasses A, B and C.

A. Let b_1 be a time-like vector. Then $< b_r, b_r >> 0$ for $r = 2, \ldots, m$ (in this case $m \leq n$). We choose the orthonormal basis $e'_1, \ldots, e'_m$ as

$$e'_s = b_s / | < b_s, b_s > |^{1/2}, \quad s = 1, \ldots, m. \tag{2.3.9}$$

Then we have:

$$< b_s, x >= \eta_{ss} | < b_s, b_s > |^{1/2} X^s. \tag{2.3.10}$$

The Lagrangian (2.2.13) and the energy constraint (2.3.2) for the coordinates $X^1, \ldots, X^n$ have the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\eta_{ss} | < b_s, b_s > |^{1/2} X^s], \tag{2.3.11}$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + \sum_{s=1}^{m} a^{(s)} \exp[\eta_{ss} | < b_s, b_s > |^{1/2} X^s]. \tag{2.3.12}$$

Lagrangian (2.3.11) leads to the set of eqs.

$$\ddot{X}^s = -| < b_s, b_s > |^{1/2} a^{(s)} \exp[\eta_{ss} | < b_s, b_s > |^{1/2} X^s], \tag{2.3.13}$$

$$\ddot{X}^{m+1} = \ldots = \ddot{X}^n = 0, \tag{2.3.14}$$

which is easily integrable. We get

$$X^s = -\eta_{ss} | < b_s, b_s > |^{-1/2} \ln[F_s^2(t - t_{0s})], \tag{2.3.15}$$

$$X^{m+1} = p^{m+1} t + q^{m+1}, \ldots, X^n = p^n t + q^n, \tag{2.3.16}$$

where we denoted

$$\begin{aligned} F_s(t - t_{0s}) \quad &= \sqrt{|a^{(s)}/E_s|} \cosh[\sqrt{|E_s < b_s, b_s > |/2}(t - t_{0s})], \ \text{ if } \eta_{ss} a^s > 0, \ \eta_{ss} E_s > 0, \\ &= \sqrt{|a^{(s)}/E_s|} \sin[\sqrt{|E_s < b_s, b_s > |/2}(t - t_{0s})], \ \text{ if } \eta_{ss} a^s < 0, \ \eta_{ss} E_s < 0, \\ &= \sqrt{|a^{(s)}/E_s|} \sinh[\sqrt{|E_s < b_s, b_s > |/2}(t - t_{0s})], \ \text{ if } \eta_{ss} a^s < 0, \ \eta_{ss} E_s > 0, \\ &= \sqrt{| < b_s, b_s > a^{(s)}|/2}(t - t_{0s}), \ \text{ if } \eta_{ss} a^s < 0, \ E_s = 0. \end{aligned} \tag{2.3.17}$$

By t_{0s}, E_{0s} $(s = 1, \ldots, m)$, $p^{m+1}, \ldots, p^n$, $q^{m+1}, \ldots, q^n$ we denoted the integration constants. Some of them are not arbitrary and connected by the relation

$$E_1 + \ldots + E_m + \frac{1}{2}(p^{m+1})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0. \tag{2.3.18}$$

We have for components t^i_k

$$t^i_s = b^i_s / | < b_s, b_s > |^{1/2}. \tag{2.3.19}$$

It is convenient to present the exact solution in a Kasner-like form. Kasner-like parameters are defined by

$$\alpha^i = t^i_{m+1}p^{m+1} + \ldots + t^i_n p^n, \tag{2.3.20}$$
$$\beta^i = t^i_{m+1}q^{m+1} + \ldots + t^i_n q^n. \tag{2.3.21}$$

Then for the scale factors of the spaces M_i (see (2.3.6)) we get

$$\exp[x^i] = \prod_{s=1}^{m}[F_s^2(t - t_{0s})]^{-b_s^i/<b_s,b_s>} \exp[\alpha^i t + \beta^i]. \tag{2.3.22}$$

Vectors α, $\beta \in R^n$, are defined by

$$\alpha \;\; = \alpha^1 e_1 + \ldots + \alpha^n e_n, \;\; \beta \;\; = \beta^1 e_1 + \ldots + \beta^n e_n \tag{2.3.23}$$

satisfy the relations

$$< \alpha, \alpha >= 2(E_0 - E_1 - \ldots - E_m) \geq 0, \tag{2.3.24}$$
$$< \alpha, b_s >=< \beta, b_s >= 0, \quad s = 1, \ldots, m. \tag{2.3.25}$$

We remind that $< \alpha, \beta >= \sum_{i,j=1}^{n} G_{ij}\alpha^i\beta^j$.

Remark 2. If $m = n$ then $\alpha = \beta = 0$.

Remark 3. The set of constants E_0, E_s, t_{0s}, α^i and β^i is the final set. Only $2n$ constants from them are independent.

Remark 4. The subclass of the solutions may be easily enlarged. It is clear, that the addition of new component inducing a vector collinear to one of $b_1, \ldots, b_m$ leads to the integrable by quadrature model. Let us take into account the following additional terms in the Lagrangian (2.2.13)

$$-\sum_{\alpha=1}^{m(\sigma)} a^{(\sigma\alpha)} \exp[b_{(\sigma\alpha)} < b_\sigma, x >], \tag{2.3.26}$$

where $b_{(\sigma\alpha)}$ =const$\neq 0$ for $\alpha = 1, \ldots, m(\sigma)$, $1 \leq \sigma \leq m$. It is not difficult to show, that the modification of the exact solution (2.3.22) only consists in the replacement of the function $F_\sigma(t - t_{0\sigma})$ by one $F(t - t_{0\sigma})$, satisfying the quadrature

$$\int dF/\sqrt{E_\sigma F^2 - a^{(\sigma)} - \sum_{\alpha=1}^{m(\sigma)} a^{(\sigma\alpha)} F^{2(1-b_{(\sigma\alpha)})}} =< b_\sigma, b_\sigma > (t - t_{0\sigma}). \tag{2.3.27}$$

The additional components with other numbers σ may be taken into account by the same manner.

B. We have the set of m space-like vectors $b_1, \ldots, b_m$ ($m \le n-1$) and the orthonormal basis defined by

$$e'_{s+1} = b_s / \sqrt{< b_s, b_s >}, \quad s = 1, \ldots, m. \tag{2.3.28}$$

The Lagrangian (2.2.13) and the energy constraint (2.3.2) in terms of X-coordinates have the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sqrt{< b_s, b_s >} X^{s+1}], \tag{2.3.29}$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + \sum_{s=1}^{m} a^{(s)} \exp[\sqrt{< b_s, b_s >} X^{s+1}]. \tag{2.3.30}$$

The corresponding eqs. of motion

$$\ddot{X}^1 = \ddot{X}^{m+2} = \ldots = \ddot{X}^n = 0, \tag{2.3.31}$$

$$\ddot{X}^{s+1} = -\sqrt{< b_s, b_s >} a^{(s)} \exp[\sqrt{< b_s, b_s >} X^{s+1}] \tag{2.3.32}$$

lead to the solution

$$X^1 = p^1 t + q^1, \tag{2.3.33}$$

$$X^{s+1} = \frac{-1}{\sqrt{< b_s, b_s >}} \ln[F_s^2(t - t_{0s})], \tag{2.3.34}$$

$$X^{m+2} = p^{m+2} t + q^{m+2}, \ldots, X^n = p^n t + q^n, \tag{2.3.35}$$

where functions $F_s(t - t_{0s})$ are defined by (2.3.17) (in this case all $\eta_{ss} = 1$). Some of integration constants in (2.3.33)-(2.3.35) satisfy the relation

$$E_1 + \ldots + E_m - \frac{1}{2}(p^1)^2 + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0. \tag{2.3.36}$$

To present the scale factors in a Kasner-like form we define the parameters:

$$\alpha^i = t_1^i p^1 + t_{m+2}^i p^{m+2} + \ldots + t_n^i p^n, \tag{2.3.37}$$

$$\beta^i = t_1^i q^1 + t_{m+1}^i q^{m+2} + \ldots + t_n^i q^n. \tag{2.3.38}$$

Then from (2.3.6) we obtain the same formula:

$$\exp[x^i] = \prod_{s=1}^{m} [F_s^2(t - t_{0s})]^{-b_s^i / < b_s, b_s >} \exp[\alpha^i t + \beta^i]. \tag{2.3.39}$$

The relations (2.3.8) lead to the following constraints for the Kasner-like parameters α^i and β^i :

$$< \alpha, \alpha > = 2(E_0 - E_1 - \ldots - E_m), \tag{2.3.40}$$

$$< \alpha, b_s > = < \beta, b_s > = 0, \quad s = 1, \ldots, m. \tag{2.3.41}$$

Remark 5. If $m = n - 1$, then either $< \alpha, \alpha > < 0$ or $\alpha = 0$; and β has the same properties.

Remark 6. We may also consider the enlargement of this subclass by the manner described in Remark 4. If we add to the Lagrangian (2.2.13) the terms (2.3.26) for some $\sigma \le m$, we should replace the function $F_\sigma(t - t_{0\sigma})$ in eq. (2.3.39) by the function $F(t - t_{0\sigma})$, satisfying (2.3.27).

C. Let b_1 be an isotropic vector. Then $< b_r, b_r >> 0$ for $r = 2, \dots, m$ (in this case $m \le n-1$). We choose the orthonormal basis $e'_1, \dots, e'_n$ by

$$e'_r = b_r/\sqrt{< b_r, b_r >}, \quad b_1 = e'_1 + e'_{m+1}. \tag{2.3.42}$$

Then we get

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - a^{(1)}\exp[-X^1 + X^{m+1}] -$$
$$\sum_{r=2}^{m} a^{(r)} exp[\sqrt{< b_r, b_r >}X^r], \tag{2.3.43}$$
$$E_0 = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j + a^{(1)}\exp[-X^1 + X^{m+1}] +$$
$$\sum_{r=2}^{m} a^{(r)}\exp[\sqrt{< b_r, b_r >}X^r], \tag{2.3.44}$$

The corresponding eqs. of motion have the form

$$\ddot{X}^1 = -a^{(1)}\exp[-X^1 + X^{m+1}], \tag{2.3.45}$$
$$\ddot{X}^{m+1} = -a^{(1)}\exp[-X^1 + X^{m+1}], \tag{2.3.46}$$
$$\ddot{X}^r = -\sqrt{< b_r, b_r >}a^{(r)}\exp[\sqrt{< b_r, b_r >}X^r], \tag{2.3.47}$$
$$\ddot{X}^{m+2} = \dots = \ddot{X}^n = 0. \tag{2.3.48}$$

To integrate (2.3.45), (2.3.46) it is useful to consider the eqs. of motion for $X^+ = X^1 + X^{m+1}$ and $X^- = -X^1 + X^{m+1}$. Then we get the solution

$$X^1 = \frac{1}{2}(p^+ - p^-)t + \frac{1}{2}(q^+ - q^-) - 2\ln[f(t)], \tag{2.3.49}$$
$$X^{m+1} = \frac{1}{2}(p^+ + p^-)t + \frac{1}{2}(q^+ + q^-) - 2\ln[f(t)], \tag{2.3.50}$$
$$X^r = \frac{-1}{\sqrt{< b_r, b_r >}}\ln[F_r^2(t - t_{0r})], \tag{2.3.51}$$
$$X^{m+2} = p^{m+2}t + q^{m+2}, \dots, X^n = p^n t + q^n, \tag{2.3.52}$$

Here by $f(t)$ we denoted the function

$$f(t) \quad = \exp[\frac{a^{(1)}}{2(p^-)^2}\exp[p^- t + q^-]], \quad p^- \ne 0, \tag{2.3.53}$$
$$= \exp[\frac{a^{(1)}}{4}\exp[q^-]t^2], \quad p^- = 0. \tag{2.3.54}$$

The integration constants satisfy the relations

$$\frac{1}{2}p^+p^- + E_2 + \ldots + E_m + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0, \;\; p^- \neq 0, \tag{2.3.55}$$

$$a^{(1)}\exp[q^-] + E_2 + \ldots + E_m + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0, \;\; p^- = 0. \tag{2.3.56}$$

The Kasner-like parameters are defined by

$$\alpha^i = \frac{1}{2}t_1^i(p^+ - p^-) + \frac{1}{2}t_{m+1}^i(p^+ + p^-) + t_{m+2}^i p^{m+2} + \ldots + t_n^i p^n, \tag{2.3.57}$$

$$\beta^i = \frac{1}{2}t_1^i(q^+ - q^-) + \frac{1}{2}t_{m+1}^i(q^+ + q^-) + t_{m+2}^i q^{m+2} + \ldots + t_n^i q^n. \tag{2.3.58}$$

Then from (2.3.6) we obtain the scale factors in a Kasner-like form:

$$\exp[x^i] = [f(t)]^{-b_1^i} \prod_{r=2}^{m} [F_r^2(t - t_{0r})]^{-b_r^i/<b_r,b_r>} \exp[\alpha^i t + \beta i]. \tag{2.3.59}$$

The Kasner-like parameters satisfy

$$< \alpha, \alpha > \;\; = 2(E_0 - E_2 - \ldots - E_m), \;\; < \alpha, b_1 > \neq 0 \tag{2.3.60}$$

$$= (E_0 - a^{(1)}\exp[< \beta, b_1 >] - E_2 - \ldots - E_m), \;\; < \alpha, b_1 >= 0, \tag{2.3.61}$$

$$< \alpha, b_r >=< \beta, b_r >= 0, \;\; r = 2, \ldots, m. \tag{2.3.62}$$

Remark 7. For the parameters p^- and q^- we get:

$$p^- =< \alpha, b_1 >, \quad q^- =< \beta, b_1 > . \tag{2.3.63}$$

Remark 8. If $m = n - 1$ and $< \alpha, b_1 >= 0$, then $< \alpha, \alpha >= 0$, i.e. $\alpha = p^+ b_1$. If $m < n - 1$ and $< \alpha, b_1 >= 0$, then $< \alpha, \alpha >\geq 0$.

Remark 9. Let us consider the enlargement of this subclass by the addition of the terms (2.3.26) to the Lagrangian. The modification of the exact solution (2.3.59) for each $\sigma = 2, \ldots, m$ is described in the Remark 6. Let us take into account the additional components, induced by isotropic vectors collinear to b_1 . It is not difficult to show that in this case (for $\sigma = 1$) the additional terms (2.3.26) leads to the following modification of the function $f(t)$

$$f(t) \;\; = \exp\{\frac{a^{(1)}}{2(p^-)^2}\exp[p^- t + q^-] + \sum_{\alpha=1}^{m(1)} \frac{a^{(1\alpha)}}{2b_{(1\alpha)}(p^-)^2}\exp[b_{(1\alpha)}(p^- t + q^-)]\}, \; p^- \neq 0,$$

$$= \exp\{(a^{(1)}\exp[q^-] + \sum_{\alpha=1}^{m(1)} a^{(1\alpha)}\exp[b_{(1\alpha)}q^-])\frac{t^2}{4}\}, \; p^- = 0. \tag{2.3.64}$$

In (2.3.56) and (2.3.61) the additional terms appear

$$\sum_{\alpha=1}^{m(1)} a^{(1\alpha)}\exp[b_{(1\alpha)}q^-]. \tag{2.3.65}$$

These are all modifications in this case.

2.4. Reduction of pseudo-Euclidean Toda-like system to Euclidean one

Now we consider the case, when the set of vectors $b_1, \ldots, b_m$ is not orthogonal. It is easily shown that eqs. of motion of our system with the Lagrangian

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >]. \tag{2.4.1}$$

for the new variables

$$p = \dot{x} \in R^n, \tag{2.4.2}$$
$$l_\alpha = a^{(\alpha)} \exp[< b_\alpha, x >] \tag{2.4.3}$$

have the following form

$$\dot{p} = - \sum_{\alpha=1}^{m} l_\alpha b_\alpha, \tag{2.4.4}$$
$$\dot{l}_\alpha = l_\alpha < b_\alpha, p > . \tag{2.4.5}$$

Note that this representation is valid for non-degenerate bilinear form $< .,. >$ with arbitrary signature.

Let us consider a simple complex Lie algebra G. Let H be a Cartan subalgebra, and h_i, e_{ω_γ} be a Weyl-Cartan basis in G [41]. We denote by $h_1, \ldots, h_n$ some basis in H and by $\omega_1, \ldots, \omega_N$ the set of roots ($\omega_\gamma \in H$, $\gamma = 1, \ldots, N$). If the roots $\omega_1, \ldots, \omega_m$ are admissible, then we have [25,40]

$$[h, e_{\omega_\alpha}] = (\omega_\alpha, h) e_{\omega_\alpha}, \quad h \in H \tag{2.4.6}$$
$$[e_{\omega_\alpha}, e_{-\omega_\beta}] = \delta_{\alpha\beta} \omega_\alpha, \quad \alpha, \beta = 1, \ldots, m, \tag{2.4.7}$$

where we denote by $(.,.)$ the Killing-Cartan form. Let us define in the algebra G the vectors ($L - A$ pair) [25,40]

$$L(t) = \sum_{\alpha=1}^{m} f_\alpha(t) e_{-\omega_\alpha} + C \sum_{i=1}^{n} h^i(t) h_i + C^2 \sum_{\alpha=1}^{m} e_{\omega_\alpha}, \tag{2.4.8}$$
$$A(t) = -\frac{1}{C} \sum_{\alpha=1}^{m} f_\alpha(t) e_{-\omega_\alpha}, \tag{2.4.9}$$

where C is arbitrary constant. Using (2.4.6-2.4.7), it can be easily checked that eq.

$$\dot{L}(t) = [L(t), A(t)] \tag{2.4.10}$$

is equivalent to the following set of eqs. for the variables $f_\alpha(t), h^i(t)$

$$\dot{h} = - \sum_{\alpha=1}^{m} f_\alpha \omega_\alpha, \tag{2.4.11}$$
$$\dot{f}_\alpha = f_\alpha (\omega_\alpha, h), \tag{2.4.12}$$

where we denoted $h = h^1(t)h_1 + \ldots + h^n(t)h_n,\ h \in H$.

Consider the real linear subspace of dimension n $H' \in H$ such that the Killing-Cartan form $(.,.)$ on H' is a real non-degenerate bilinear form with the signature $(-,+,\ldots,+)$, i.e. $<.,.>$. It is evident, that the sets of eqs. (2.4.4-2.4.5) and (2.4.11-2.4.12) are identical, if $h, \omega_1, \ldots, \omega_m \in H'$. Thus, if the set of vectors $b_1, \ldots, b_m \in R^n$ equipped with the bilinear form $<.,.>$ may be identified with a set of admissible roots $\omega_1 \ldots, \omega_m \in H'$, then pseudo-Euclidean Toda-like system with the Lagrangian (2.4.1) possesses the Lax representation. If such identification is possible, then the system is called to be connected with the simple complex Lie algebra.

Proposition 3. Let a pseudo-Euclidean Toda-like system is connected with a simple complex Lie algebra. Then it is reducible to an Euclidean Toda-like system for a part of coordinates.

Proof. We get in an arbitrary orthonormal basis $e'_1, \ldots, e'_n$

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{i=1}^{n} B^s_i X^i], \tag{2.4.13}$$

where we denoted

$$B^s_i = \sum_{j=1}^{n} \eta_{ij} B^j_s. \tag{2.4.14}$$

We remind, that $b_s = B^1_s e'_1 + \ldots + B^n_s e'_n$.

It is known [41] that the Killing-Cartan form defined on the real linear span of roots of a simple (or semi-simple) complex Lie algebra is positively definite. But we have the indefinite bilinear form $<.,.>$. Then the first components of the vectors $b_1, \ldots, b_m$ must be zero in a suitably chosen orthonormal basis, i.e. $B^s_1 = 0$ for $s = 1, \ldots, m$. Then, in this basis Lagrangian (2.4.1) has the form

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{k=2}^{m} B^s_k X^k]. \tag{2.4.15}$$

Coordinate X^1 satisfies the eq.: $\ddot{X}^1 = 0$. Eqs. of motion for $X^2, \ldots, X^n$ are followed from the Euclidean Toda-like Lagrangian

$$L_E = \frac{1}{2}\sum_{k,l=2}^{n} \delta_{kl}\dot{X}^k\dot{X}^l - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{k=2}^{m} B^s_k X^k]. \tag{2.4.16}$$

Thus, we obtained the reduction of a pseudo-Euclidean Toda-like system to the Euclidean one.

Integrating the eqs. of an Euclidean Toda-like system by known methods [26,27,42], we obtain the class of exact solutions for some nonorthogonal set of vectors $b_1, \ldots, b_m$. Here we consider this procedure for the simplest 2-component case ($n \geq 3$), when Toda lattice is connected with Lie algebra A_2.

Suppose, that the vectors b_1 and b_2, induced by two components in the Lagrangian

$$L = \frac{1}{2} < \dot{x}, \dot{x} > -a^{(1)} \exp[< b_1, x >] - a^{(2)} \exp[< b_2, x >], \tag{2.4.17}$$

satisfy the following conditions

$$< b_1, b_2 >= -\frac{1}{2} < b_1, b_1 >= -\frac{1}{2} < b_2, b_2 >< 0. \tag{2.4.18}$$

Then, we have two space-like vectors with the same lengths. The angle between them is equal to 120°. We denote

$$\sqrt{< b_1, b_1 >} = \sqrt{< b_2, b_2 >} = b. \tag{2.4.19}$$

Let us define the orthonormal basis $e'_1, \ldots, e'_n$ in R^n by

$$b_1 = be'_2, \tag{2.4.20}$$

$$b_2 = b(-\frac{1}{2}e'_2 + \frac{\sqrt{3}}{2}e'_3). \tag{2.4.21}$$

In this basis the Lagrangian (2.4.17) and corresponding energy constraint have the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - a^{(1)} \exp[bX^2] - a^{(2)} \exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)], \tag{2.4.22}$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + a^{(1)} \exp[bX^2] + a^{(2)} \exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)] \tag{2.4.23}$$

For the coordinates X^1, $X^4, \ldots, X^n$ we get the following eqs. of motion:

$$\ddot{X}^1 = \ddot{X}^4 = \ldots = \ddot{X}^n. \tag{2.4.24}$$

Therefore

$$X^1 = p^1 t + q^1,\ X^4 = p^4 t + q^4, \ldots, X^n = p^n t + q^n, \tag{2.4.25}$$

where $p^1, p^4, \ldots, p^n, q^1, q^4, \ldots, q^n$ are arbitrary integration constants. The eqs. of motion for the coordinates X^2 and X^3 follow from the Lagrangian

$$L_B = \frac{1}{2}((\dot{X}^2)^2 + (\dot{X}^3)^2) - a^{(1)} \exp[bX^2] - a^{(2)} \exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)]. \tag{2.4.26}$$

Let us introduce new coordinates y^1 and y^2 as

$$y_1 = \frac{b}{2\sqrt{2}}X^2, \quad y_2 = \frac{b}{2\sqrt{2}}X^3. \tag{2.4.27}$$

We obtain the Lagrangian of the open Toda lattice connected with the Lie algebra $A_2 = SL(3, C)$

$$L_T = \frac{1}{2}((\dot{y}^1)^2 + (\dot{y}^2)^2) - \epsilon g_1^2 \exp[2\sqrt{2}y_1] - \epsilon g_2^2 \exp[-\sqrt{2}y_1 + \sqrt{6}y_2], \tag{2.4.28}$$

where we denoted

$$b^2 a^{(1)}/8 = \epsilon g_1^2, \quad b^2 a^{(2)}/8 = \epsilon g_2^2, \tag{2.4.29}$$

$$\epsilon = \mathrm{sgn}[a^{(1)}] = \mathrm{sgn}[a^{(2)}] = \pm 1. \tag{2.4.30}$$

To study the open Toda lattice it is useful to add the additional coordinate y_3:

$$L_T = \frac{1}{2}((\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2) - \epsilon g_1^2 \exp[2\sqrt{2}y_1] - \epsilon g_2^2 \exp[-\sqrt{2}y_1 + \sqrt{6}y_2], \tag{2.4.31}$$

After the orthogonal linear transformation

$$q_1 = \frac{1}{\sqrt{6}}(\sqrt{3}y_1 + y_2 + \sqrt{2}y_3),$$

$$q_2 = \frac{1}{\sqrt{6}}(-\sqrt{3}y_1 + y_2 + \sqrt{2}y_3), \tag{2.4.32}$$

$$q_3 = -2y_2 + \sqrt{2}y_3 \tag{2.4.33}$$

the Lagrangian (2.4.31) takes the well-known form [24,26-28,42,43]

$$L_T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \epsilon g_1^2 \exp[2(q_1 - q_2)] - \epsilon g_2^2 \exp[2(q_2 - q_3)]. \tag{2.4.34}$$

In this representation the additional degree of freedom corresponds to the free motion of the center of mass ($\bar{q}_1 + \bar{q}_2 + \bar{q}_3 = 0$). The integrating of the eqs. of motion for this system leads to the result [26,27,42]

$$g_1^2 \exp[2(q_1 - q_2)] = \frac{F_+}{F_-^2}, \quad g_2^2 \exp[2(q_2 - q_3)] = \frac{F_-}{F_+^2}, \tag{2.4.35}$$

where

$$F_\pm = \frac{4}{9A_1A_2(A_1 + A_2)}\{A_1 \exp[\pm(A_1 + 2A_2)t \pm B_1] + \epsilon(A_1 + A_2)\exp[\pm(A_1 - A_2)t \mp (B_1 - B_2)] + A_2 \exp[\mp(2A_1 + A_2)t \mp B_2]\}. \tag{2.4.36}$$

The integration constants B_1, B_2 are arbitrary and A_1, A_2 satisfy the condition: $A_1A_2 > 0$. For the energy of the system with Lagrangian (2.4.24) we have

$$\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)+\epsilon g_1^2\exp[2(q_1-q_2)]+\epsilon g_2^2\exp[2(q_2-q_3)]=\frac{3}{4}(A_1^2+A_1A_2+A_2^2). \tag{2.4.37}$$

Doing the inverse linear transformation

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{2}}[q_1-q_2],\\ y_2 &= \frac{1}{\sqrt{6}}([q_1-q_2]+2[q_2-q_3]),\\ y_3 &= \frac{1}{\sqrt{3}}(q_1+q_2+q_3), \end{aligned} \tag{2.4.38}$$

for the system with Lagrangian (2.4.22) we get the solution

$$X^2=\frac{1}{b}\ln[\frac{8}{b^2|a^{(1)}|}\frac{F_+}{F_-^2}], \tag{2.4.39}$$

$$X^3=\frac{\sqrt{3}}{b}\ln[\frac{8}{b^2(|a^{(1)}|(a^{(2)})^2)^{1/3}}\frac{1}{F_+}], \tag{2.4.40}$$

and the following energy constraint

$$E_0=-\frac{1}{2}(p^1)^2+\frac{1}{2}(p^4)^2+\ldots+\frac{1}{2}(p^n)^2+\frac{6}{b^2}(A_1^2+A_1A_2+A_2^2). \tag{2.4.41}$$

To present the scale factors in the Kasner-like form let us introduce the Kasner-like parameters

$$\alpha^i=t_1^ip^1+t_4^ip^4+\ldots+t_n^ip^n, \tag{2.4.42}$$

$$\beta^i=t_1^iq^1+t_4^iq^4+\ldots+t_n^iq^n, \tag{2.4.43}$$

where components t_k^i are determined by (2.3.7). In this case they satisfy the relations

$$t_2^i=\frac{1}{b}b_1^i,\quad t_3^i=\frac{2}{\sqrt{3}}(\frac{1}{b}b_2^i+\frac{1}{2b}b_1^i). \tag{2.4.44}$$

From (2.3.6) we obtain the coordinates x^i and finally present the exact solution in the form

$$\exp[x^i]=[\bar{F}_-^2]^{-b_1^i/<b_1,b_1>}[\bar{F}_+^2]^{-b_2^i/<b_2,b_2>}\exp[\alpha^it+\beta^i], \tag{2.4.45}$$

where

$$\bar{F}_-=\frac{1}{8}b^2\{(a^{(1)})^2|a^{(2)}|\}^{\frac{1}{3}}F_-, \tag{2.4.46}$$

$$\bar{F}_+=\frac{1}{8}b^2\{(a^{(2)})^2|a^{(1)}|\}^{\frac{1}{3}}F_+. \tag{2.4.47}$$

The vectors α and β defined by (2.3.23) satisfy the relations

$$<\alpha,\alpha>=2(E_0-\frac{6}{b^2}(A_1^2+A_1A_2+A_2^2)), \tag{2.4.48}$$

$$<\alpha,b_r>=<\beta,b_r>=0,\quad r=1,2. \tag{2.4.49}$$

Remark 10. If $n=3$, then $<\alpha,\alpha>\leq 0$ and $<\beta,\beta>\leq 0$.

2.5. Discussion

Let us consider some cosmological models corresponding to the introduced in the Sect. 2.3 integrable subclasses of pseudo-Euclidean Toda-like systems. For this purpose in Table I we present values of the bilinear form $< .,. >$ (see Sect. 2.2) for the vectors

$$v_i \equiv v^1_{(i)} e_1 + \ldots + v^n_{(i)} e_n, \quad v^j_{(i)} = -2\frac{\delta^j_i}{N_i}, \tag{2.5.1}$$

$$u_\alpha \equiv u^1_{(\alpha)} e_1 + \ldots + u^n_{(\alpha)} e_n, \quad u^j_\alpha = h^{(\alpha)}_j + \frac{1}{2-D}\sum_{i=1}^{n} N_i h^{(\alpha)}_i, \tag{2.5.2}$$

$$u \equiv u^1 e_1 + \ldots + u^n e_n, \quad u^j = \frac{2}{2-D}, \tag{2.5.3}$$

induced by curvature, perfect fluid and Λ-term correspondingly.

Within the subclass A we are able to construct the model with one Einstein space of non-zero curvature. Let $(n-1)$ Einstein spaces are Ricci-flat and one, for instance M_1 , have a non-zero Ricci tensor. Then we put $b_1 \equiv v_1$. To get the orthogonality with b_1 for at most $(n-1)$ available components of the perfect fluid ($b_{(\alpha+1)} \equiv u_{(\alpha)}$ for $\alpha \leq n-1$) we put: $h^{(\alpha)}_1 = 0$ (see Table I). Then, these components appeared to be in the manifold M_1 in the Zeldovich matter form (see Remark 1). The model of such a type was integrated in [47]. In the same way the model with all Ricci-flat spaces and Λ-term arises . In this case we put $b_1 = u$. The condition of the orthogonality reads: $\sum_{i=1}^{n} h^{(\alpha)}_i N_i = 0$ for all $\alpha \leq n-1$. Then we get the negative values for the some $h^{(\alpha)}_i$. It means that for such perfect fluids $p > \rho$ in some spaces (see (1.1.8)).

The vectors v_i and u induced by curvature and Λ-term correspondingly are time-like, therefore subclasses B and C correspond to the Ricci-flat models without Λ-term for some multicomponent perfect fluid source. These vectors can not be roots of any simple complex Lie algebra. Therefore, the models with more than one non-zero curvature space and the models with curvature and Λ-term are not trivially reducible to the Euclidean Toda lattices. Some possibilities of integration of these models were studied in [18,46].

In conclusion we discuss the existence of the Euclidean wormholes [51-54] within the class of the obtained exact solutions. We consider the simple model within subclass A with the manifold $R \times M_1 \times M_2$, when M_1 has a nonzero Ricci tensor with $\lambda_1 > 0$ (see 1.1.3) and M_2 is Ricci-flat. The integrable model arises in the presence of the perfect fluid in the Zeldovich matter form for the space M_1. It means $h_1 = 0$ and the other parameter in the equation of state for M_2 (see 1.1.8)) may be arbitrary positive constant h. If we demand the positiveness of the mass-energy density for the perfect fluid ($A > 0$), then from (2.3.22) we get for the scales factors of the M_1 and M_2

$$\exp[x^1] = \{F_1^2(t-t_{01})\}^{-\frac{1}{2(N_1-1)}}\{F_2^2(t-t_{02})\}^{\frac{1}{2(N_1-1)}}, \tag{2.5.4}$$

$$\exp[x^2] = \{F_2^2(t-t_{02})\}^{-\frac{1}{hN_2}}, \tag{2.5.5}$$

where

$$F_1(t-t_{01}) = \sqrt{\frac{1}{2}\lambda_1 N_1/|E_1|}\cosh[\sqrt{2|E_1|(N_1-1)/N_1}(t-t_{01})], \tag{2.5.6}$$

$$F_2(t-t_{01}) = \sqrt{\kappa^2 A/E_2}\cosh[h\sqrt{\frac{1}{2}(N_1-1)N_2|E_2/(N_1+N_2-1)}(t-t_{02})]. \tag{2.5.7}$$

In this case $E_1 < 0$ and $E_2 > 0$. The energy constraint (2.3.24) leads to the condition: $-E_1 = E_2 \equiv E$.

We may suppose that M_1 is 3-dimensional sphere S^3 and M_2 is d-dimensional torus T^d. Then formulas (2.5.4-2.5.7) present the multidimensional generalization of closed Friedmann model. This model may be relevant in the theory of the Early Universe, because the Zeldovich matter equation of state: $p = \rho$ is valid on the earlier stage of its evolution [49].

To prove the existence of the Euclidean wormholes we use the transformation $t \to it$. Then for the case $t_{01} = t_{02} = 0$ we obtain

$$\exp[x^1] = \{\frac{\kappa^2 A}{E}\cos^2[\sqrt{\frac{Ed}{d+2}}ht]\}^{1/(2h)}\{\frac{3\lambda_1}{2E}\cos^2[\sqrt{\frac{4E}{3}}t]\}^{-1/4}, \tag{2.5.8}$$

$$\exp[x^2] = \{\frac{\kappa^2 A}{E}\cos^2[\sqrt{\frac{Ed}{d+2}}ht]\}^{-1/(hd)}. \tag{2.5.9}$$

It is easy to see that when $\frac{d}{d+2}h^2 > \frac{4}{3}$ one has wormhole with respect to the internal space T^d. The case $\frac{d}{d+2}h^2 < \frac{4}{3}$ corresponds to the wormhole for the external space S^3. Note, that for $h = 2$ and $d = 1$ the wormhole for the internal space is accompanied by the static external space. It is not difficult to show that wormhole with respect to the whole space for this model arises in the presence of the additional component in the form of minimally coupled scalar field.

	v_j	u_β	u
v_i	$4(\frac{\delta_{ij}}{N_i} - 1)$	$-2h_i^{(\beta)}$	-4
u_α	$-2h_j^{(\alpha)}$	$\sum_{i=1}^{n} h_i^{(\alpha)} h_i^{(\beta)} N_i +$ $\frac{1}{2-D}[\sum_{i=1}^{n} h_i^{(\alpha)} N_i][\sum_{j=1}^{n} h_j^{(\beta)} N_j]$	$\frac{2}{2-D}\sum_{i=1}^{n} h_i^{(\alpha)} N_i$
u	-4	$\frac{2}{2-D}\sum_{i=1}^{n} h_i^{(\beta)} N_i$	$-4\frac{D-1}{D-2}$

TABLE I

3. Billiard Representation for Multidimensional Cosmology with Multicomponent Perfect Fluid near the Singularity

3.1. Introduction

A lot of interesting topics in multidimensional cosmology were considered: exact solutions and the problem of integrbility, superstring cosmology and the problem of compactification, variation of constants, classical and quantum wormholes, chaotic behaviour near the singularity, etc.

In the present section we deal with a stochastic behavior in multidimensional cosmological models [53-55,18]. This direction in higher-dimensional gravity was stimulated by well-known results for "mixmaster" model [56-59]. We note, that there is also an elegant explanation for stochastic behavior of scale factors of Bianchi-IX model suggested by Chitre [58-59] and recently considered in [60-62]. (For "history" of the problem see also [63].) In the Chitre's approach the Bianchi-IX cosmology near the singularity is reduced to a billiard on the Lobachevsky space H^2 (see Fig. 4 below). The volume of this billiard is finite. This fact together with the well-known behavior (exponential divergences) of geodesics on the spaces of negative curvature leads to a stochastic behavior of the dynamical system in the considered regime [64,65].

Chitre's approach [58] may also be used in the multidimensional case [55]. It allows us to obtain a more evident picture for the origin of the oscillatory behaviour near the singularity using the formation of billiard walls. The present section is devoted to a construction of the "billiard representation" for the multidimensional cosmological model describing the evolution of n Einstein spaces in the presence if $(m + 1)$-component perfect fluid [37] (see section 3.2). One of these components corresponds to the cosmological constant term [66]. In some sense the model [37] may be considered as "universal" cosmological model: a lot of cosmological models (not obviously multidimensional) may be embedded in this model.

We impose certain restrictions on the parameters of the model [37] and reduce its dynamics near the singularity to a billiard on the $(n-1)$-dimensional Lobachevsky space H^{n-1} (Sec. 3.3). The geometrical criterion for the finiteness of the billiard volume and its compactness is suggested. This criterion reduces the considered problem to the geometrical (or topological) problem of illumination of $(n-2)$-dimensional unit sphere S^{n-2} by $m_+ \leq n$ point-like sources located outside the sphere [68-69]. These sources correspond to the components with $(u^{(\alpha)})^2 > 0$ (Sec. 3.3). When these sources illuminate the sphere then, and only then, the billiard has a finite volume and the cosmological model possesses a stochastic behavior near the singularity. (We note, that, for cosmological and curvature terms $(u^{(\alpha)})^2 < 0$ and these terms may be neglected near the singularity). For the case of an infinite billiard volume the cosmological model has a Kasner-like behavior near the singularity. When the minimally coupled massless scalar field is added into consideration, the evolution in time is bounded: $t > t_0$ and the limit $t \to t_0$ corresponds to the approach to the singularity. In this case the stochastic behavior near the singularity is absent.

In Sec. 3.4 we illustrate the suggested approach on an example of the Bianchi-IX cosmology.

3.2. The model

Here we start also from the cosmological model describing the evolution of n Einstein spaces in the presence of $(m+1)$-component perfect-fluid matter (see section 1.2). The metric of the model

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]g^{(i)}, \tag{3.2.1}$$

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n, \tag{3.2.2}$$

where the manifold M_i with the metric $g^{(i)}$ is an Einstein space of dimension N_i, i.e.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \tag{3.2.3}$$

$i = 1, \ldots, n$; $n \geq 2$. The energy-momentum tensor is adopted in the following form

$$T^M_N = \sum_{\alpha=0}^{m} T^{M(\alpha)}_N, \tag{3.2.4}$$

$$(T^{M(\alpha)}_N) = diag(-\rho^{(\alpha)}(t), p^{(\alpha)}_1(t)\delta^{m_1}_{k_1}, \ldots, p^{(\alpha)}_n(t)\delta^{m_n}_{k_n}). \tag{3.2.5}$$

$\alpha = 0, \ldots, m$, with the conservation law constraints imposed:

$$\nabla_M T^{M(\alpha)}_N = 0, \tag{3.2.6}$$

$\alpha = 0, \ldots, m-1$. The Einstein equations

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N \tag{3.2.7}$$

(κ^2 is gravitational constant) imply $\nabla_M T^M_N = 0$ and consequently $\nabla_M T^{M(m)}_N = 0$.

We suppose that for any α-th component of matter the pressures in all spaces are proportional to the density

$$p_i^{(\alpha)}(t) = (1 - \frac{u_i^{(\alpha)}}{N_i})\rho^{(\alpha)}(t), \tag{3.2.8}$$

where $u_i^{(\alpha)} = const$, $i = 1, \ldots, n$; $\alpha = 0, \ldots, m$.

Non-sero components of the Ricci-tensor for the metric (3.2.1) are the following

$$R_{00} = -\sum_{i=1}^{n} N_i[\ddot{x}^i - \dot{\gamma}\dot{x}^i + (\dot{x}^i)^2], \tag{3.2.9}$$

$$R_{m_i n_i} = g^{(i)}_{m_i n_i}[\lambda^i + \exp(2x^i - 2\gamma)(\ddot{x}^i + \dot{x}^i(\sum_{i=1}^{n} N_i\dot{x}^i - \dot{\gamma}))], \tag{3.2.10}$$

$i = 1, \ldots, n$.

The conservation law constraint (3.2.6) for $\alpha \in \{0, ..., m\}$ reads

$$\dot{\rho}^{(\alpha)} + \sum_{i=0}^{n} N_i\dot{x}^i(\rho^{(\alpha)} + p_i^{(\alpha)}) = 0. \tag{3.2.11}$$

From eqs. (3.2.8), (3.2.11) we get

$$\rho^{(\alpha)}(t) = A^{(\alpha)}\exp[-2N_i x^i(t) + u_i^{(\alpha)}x^i(t)], \tag{3.2.12}$$

where $A^{(\alpha)} = const$. Here and below the summation over repeated indices is understood.

We define

$$\gamma_0 \equiv \sum_{i=1}^{n} N_i x^i \tag{3.2.13}$$

in (3.2.1).

Using relations (3.2.8), (3.2.9), (3.2.10), (3.2.12) it is not difficult to verify that the Einstein equations (3.2.7) for the metric (3.2.1) and the energy-momentum tensor from (3.2.4), (3.2.5) are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2}\exp(-\gamma + \gamma_0(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(\gamma - \gamma_0(x))V(x). \tag{3.2.14}$$

Here

$$G_{ij} = N_i\delta_{ij} - N_iN_j \tag{3.2.15}$$

are the components of the minisuperspace metric,

$$V = V(x) = -\frac{1}{2}\sum_{i=1}^{n}\lambda^i N_i \exp(-2x^i + 2\gamma_0(x)) + \sum_{\alpha=0}^{m}\kappa^2 A^{(\alpha)}\exp(u_i^{(\alpha)}x^i). \tag{3.2.16}$$

is the potential. This relation may be also presented in the form

$$V = \sum_{\alpha=0}^{\bar{m}} A_\alpha \exp(u_i^{(\alpha)}x^i), \tag{3.2.17}$$

where $\bar{m} = m + n$; $A_\alpha = \kappa^2 A^{(\alpha)}$, $\alpha = 0, \ldots, m$; $A_{m+i} = -\frac{1}{2}\lambda^i N_i$ and

$$u_j^{(m+i)} = 2(-\delta_j^i + N_j), \tag{3.2.18}$$

$i, j = 1, \ldots, n$. We also put $A_0 = \Lambda$ and

$$u_j^{(0)} = 2N_j, \tag{3.2.19}$$

$j = 1, \ldots, n$. Thus the zero component of the matter describe a cosmological constant term (Λ-term).

Diagonalization. We remind [14,15] that the minisuperspace metric

$$G = G_{ij}dx^i \otimes dx^j \tag{3.2.20}$$

has a pseudo-Euclidean signature $(-, +, \ldots, +)$, i.e. there exist a linear transformation

$$z^a = e_i^a x^i, \tag{3.2.21}$$

diagonalizing the minisuperspace metric (3.2.20)

$$G = \eta_{ab}dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{3.2.22}$$

where

$$(\eta_{ab}) = (\eta^{ab}) \equiv diag(-1, +1, \ldots, +1), \tag{3.2.23}$$

$a, b = 0, \ldots, n-1$. The matrix of the linear transformation (e^a_i) satisfies the relation

$$\eta_{ab} e^a_i e^b_j = G_{ij} \tag{3.2.24}$$

or equivalently

$$\eta^{ab} = e^a_i G^{ij} e^b_j = < e^a, e^b > . \tag{3.2.25}$$

Here

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{3.2.26}$$

are components of the matrix inverse to the matrix (3.2.15) [15], $D = 1 + \sum_{i=1}^{n} N_i$ is the dimension of the manifold M (3.2.2) and

$$< u, v > \equiv G^{ij} u_i v_j \tag{3.2.27}$$

defines a bilinear form on R^n $(u = (u_i)$, $v = (v_i))$. Inverting the map (3.2.21) we get

$$x^i = e^i_a z^a, \tag{3.2.28}$$

where for the components of the inverse matrix $(e^i_a) = (e^a_i)^{-1}$ we obtain from (3.2.25)

$$e^i_a = G^{ij} e^b_j \eta_{ba}. \tag{3.2.29}$$

Like in [15,21] we put

$$z^0 = e^0_i x^i = q^{-1} N_i x^i, \qquad q = [(D-1)/(D-2)]^{1/2}. \tag{3.2.30}$$

In this case the 00-component of eq. (3.2.25) is satisfied and the set $(e^a, a = 1, \ldots, n-1)$ is defined up to $O(n-1)$-transformation. A special example of the diagonalization with the relations (3.2.30) and

$$z^a = e^a_i x^i = [N_a / (\sum_{j=a}^{n} N_j)(\sum_{j=a+1}^{n} N_j)]^{1/2} \sum_{j=a+1}^{n} N_j (x^j - x^a), \tag{3.2.31}$$

$a = 1, \ldots, n-1$, was considered in [14,15].

In z-coordinates (3.2.21) with z^0 from (3.2.30) the Lagrangian (3.2.14) reads

$$L = L(z^a, \dot{z}^a, \mathcal{N}) = \frac{1}{2}\mathcal{N}^{-1}\eta_{ab}\dot{z}^a\dot{z}^b - \mathcal{N}V(z), \tag{3.2.32}$$

where

$$\mathcal{N} = \exp(\gamma - \gamma_0(x)) > 0 \tag{3.2.33}$$

is the Lagrange multiplier (modified lapse function) and

$$V(z) = \sum_{\alpha=0}^{m} A_\alpha \exp(u_a^\alpha z^a) \tag{3.2.34}$$

is the potential. Here we denote

$$u_a^\alpha = e_a^i u_i^{(\alpha)} = < u^{(\alpha)}, e^b > \eta_{ba}, \tag{3.2.35}$$

$a = 0, \ldots, n-1$, (see (3.2.27) and (3.2.29)). From (3.2.35) we get (see (3.2.26), (3.2.27) and (3.2.30))

$$u_0^\alpha = - < u^{(\alpha)}, e^0 > = (\sum_{i=1}^{n} u_i^{(\alpha)})/q(D-2). \tag{3.2.36}$$

For Λ-term and curvature components (see (3.2.19) and (3.2.18)) we have

$$u_0^0 = 2q > 0, \qquad u_0^{m+j} = 2/q > 0, \tag{3.2.37}$$

$= 1, \ldots, n$. The calculation of

$$(u^\alpha)^2 = \eta^{ab} u_a^\alpha u_b^\alpha = < u^{(\alpha)}, u^{(\alpha)} > = (u^{(\alpha)})^2, \tag{3.2.38}$$

for these components gives

$$(u^0)^2 = 4(D-1)/(2-D) < 0, \qquad (u^{m+j})^2 = 4(\frac{1}{N_j} - 1) < 0, \tag{3.2.39}$$

for $N_j > 1$, $j = 1, \ldots, n$. For $N_j = 1$ we have $\lambda^j = A_{m+j} = 0$.

3.3. Billiard representation

Here we consider the behavior of the dynamical system, described by the Lagrangian (3.2.32) for $n \geq 3$ in the limit

$$z^0 \to -\infty, \qquad z = (z^0, \vec{z}) \in \mathcal{V}_-, \tag{3.3.1}$$

where $\mathcal{V}_- \equiv \{(z^0, \vec{z}) \in R^n | z^0 < -|\vec{z}|\}$ is the lower light cone. For the volume scale factor

$$v = \exp(\sum_{i=1}^{n} N_i x^i) = \exp(q z^0) \tag{3.3.2}$$

(see (3.2.30)) we have in this limit $v \to 0$. Under certain additional assumptions the limit (3.3.1) describes the approaching to the singularity. We impose the following restrictions on the parameters u^α in the potential (3.2.34) for components with $A_\alpha \neq 0$:

$$1) A_\alpha > 0 \; if \; (u^\alpha)^2 = -(u_0^\alpha)^2 + (\vec{u}^\alpha)^2 > 0; \tag{3.3.3}$$

$$2) u_0^\alpha > 0 \; for \; all \; \alpha. \tag{3.3.4}$$

We note that due to (3.2.37) the second condition is always satisfied for Λ-term and curvature components (i.e. for $\alpha = 0, m+1, \ldots, m+n = \bar{m}$).

We restrict the Lagrange system (3.2.32) on $\mathcal{V}_-$, i.e. we consider the Lagrangian

$$L_- \equiv L|_{TM_-}, \qquad M_- = \mathcal{V}_- \times R_+, \tag{3.3.5}$$

where TM_- is tangent vector bundle over M_- and $R_+ \equiv \{\mathcal{N} > 0\}$. (Here $F|_A$ means the restriction of function F on A.) Introducing an analogue of the Misner-Chitre coordinates in $\mathcal{V}_-$ [58-59]

$$z^0 = -\exp(-y^0)\frac{1+\vec{y}^2}{1-\vec{y}^2}, \tag{3.3.6}$$

$$\vec{z} = -2\exp(-y^0)\frac{\vec{y}}{1-\vec{y}^2}, \tag{3.3.7}$$

$|\vec{y}| < 1$, we get for the Lagrangian (3.2.32)

$$L_- = \frac{1}{2}\mathcal{N}^{-1} e^{-2y^0}[-(\dot{y}^0)^2 + h_{ij}(\vec{y})\dot{y}^i \dot{y}^j] - \mathcal{N}V. \tag{3.3.8}$$

Here

$$h_{ij}(\vec{y}) = 4\delta_{ij}(1-\vec{y}^2)^{-2}, \tag{3.3.9}$$

$i, j = 1, \ldots, n-1$, and

$$V = V(y) = \sum_{\alpha=0}^{m} A_\alpha \exp \bar{\Phi}(y, u^\alpha), \tag{3.3.10}$$

where

$$\bar{\Phi}(y, u) \equiv -e^{-y^0}(1 - \vec{y}^2)^{-1}[u_0(1 + \vec{y}^2) + 2\vec{u}\vec{y}], \tag{3.3.11}$$

We note that the $(n-1)$-dimensional open unit disk (ball)

$$D^{n-1} \equiv \{\vec{y} = (y^1, \ldots, y^n) | |\vec{y}| < 1\} \subset R^{n-1} \tag{3.3.12}$$

with the metric $h = h_{ij}(\vec{y})dy^i \otimes dy^j$ is one of the realisation of the $(n-1)$-dimensional Lobachevsky space H^{n-1}.

We fix the gauge

$$\mathcal{N} = \exp(-2y^0) = -z^2. \tag{3.3.13}$$

Then, it is not difficult to verify that the Lagrange equations for the Lagrangian (3.3.8) with the gauge fixing (3.3.13) are equivalent to the Lagrange equations for the Lagrangian

$$L_* = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V_* \tag{3.3.14}$$

with the energy constraint imposed

$$E_* = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V_* = 0. \tag{3.3.15}$$

Here

$$V_* = e^{-2y^0} V = \sum_{\alpha=0}^{m} A_\alpha \exp(\Phi(y, u^\alpha)), \tag{3.3.16}$$

where

$$\Phi(y, u) = -2y^0 + \bar{\Phi}(y, u). \tag{3.3.17}$$

Now we are interested in the behavior of the dynamical system in the limit $y^0 \to -\infty$ (or, equivalently, in the limit $z^2 = -(z^0)^2 + (\vec{z})^2 \to -\infty$, $z^0 < 0$) implying (3.3.1). Using the relations ($u_0 \neq 0$)

$$\Phi(y, u) = -u_0 \exp(-y^0)\frac{A(\vec{y}, -\vec{u}/u_0)}{1 - \vec{y}^2} - 2y^0, \tag{3.3.18}$$

$$A(\vec{y}, \vec{v}) \equiv (\vec{y} - \vec{v})^2 - \vec{v}^2 + 1, \tag{3.3.19}$$

we get

$$\lim_{y^0 \to -\infty} \exp \Phi(y,u) = 0 \tag{3.3.20}$$

for $u^2 = -u_0^2 + (\vec{u})^2 \leq 0$, $u_0 > 0$ and

$$\lim_{y^0 \to -\infty} \exp \Phi(y,u) = \theta_\infty(-A(\vec{y}, -\vec{u}/u_0)) \tag{3.3.21}$$

for $u^2 > 0$, $u_0 > 0$. In (3.3.21) we denote

$$\theta_\infty(x) \equiv \begin{matrix} +\infty, & x \geq 0, \\ 0, & x < 0. \end{matrix} \tag{3.3.22}$$

Using restrictions (3.3.3), (3.3.4) and relations (3.3.16), (3.3.20), (3.3.21) we obtain

$$V_\infty(\vec{y}) \equiv \lim_{y^0 \to -\infty} V_*(y^0, \vec{y}) = \sum_{\alpha \in \Delta_+} \theta_\infty(-A(\vec{y}, -\vec{u}^\alpha/u_0^\alpha)). \tag{3.3.23}$$

Here we denote

$$\Delta_+ \equiv \{\alpha | (u^\alpha)^2 > 0\}. \tag{3.3.24}$$

We note that due to (3.2.39) Λ-term and curvature components do not contribute to V_∞ (i.e. they may be neglected in the vicinity of the singularity).

The potential V_∞ may be also written as following

$$V_\infty(\vec{y}) = V(\vec{y}, B) \equiv \begin{matrix} 0, & \vec{y} \in B, \\ +\infty, & \vec{y} \in D^{n-1} \setminus B, \end{matrix} \tag{3.3.25}$$

where

$$B = \bigcap_{\alpha \in \Delta_+} B(u^\alpha) \subset D^{n-1}, \tag{3.3.26}$$

$$B(u^\alpha) = \{\vec{y} \in D^{n-1} | |\vec{y} + \frac{\vec{u}^\alpha}{u_0^\alpha}| > \sqrt{(\frac{\vec{u}^\alpha}{u_0^\alpha})^2 - 1}\}, \tag{3.3.27}$$

$\alpha \in \Delta_+$. B is an open domain. Its boundary $\partial B = \bar{B} \setminus B$ is formed by certain parts of $m_+ = |\Delta_+|$ (m_+ is the number of elements in Δ_+) of $(n-2)$-dimensional spheres with the centers in the points

$$\vec{v}^\alpha = -\vec{u}^\alpha / u_0^\alpha, \qquad \alpha \in \Delta_+, \tag{3.3.28}$$

($|\vec{v}^{a}| > 1$) and radii

$$r_a = \sqrt{(\vec{v}^a)^2 - 1} \tag{3.3.29}$$

respectively (for $n = 3$, $m_+ = 1$, see Fig. 1).

Fig. 1

So, in the limit $y^0 \to -\infty$ we are led to the dynamical system

$$L_\infty = -\tfrac{1}{2}(\dot{y}^0)^2 + \tfrac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V_\infty(\vec{y}), \tag{3.3.30}$$

$$E_\infty = -\tfrac{1}{2}(\dot{y}^0)^2 + \tfrac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V_\infty(\vec{y}) = 0, \tag{3.3.31}$$

which after the separating of y^0 variable

$$y^0 = \omega(t - t_0), \tag{3.3.32}$$

($\omega \neq 0$, t_0 are constants) is reduced to the Lagrange system with the Lagrangian

$$L_B = \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V(\vec{y}, B). \tag{3.3.33}$$

Due to (3.3.32)

$$E_B = \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V(\vec{y}, B) = \frac{\omega^2}{2}. \tag{3.3.34}$$

We put $\omega > 0$, then the limit $t \to -\infty$ describes the approach to the singularity. When the set (3.3.24) is empty ($\Delta_+ = \emptyset$) we have $B = D^{n-1}$ and the Lagrangian (3.3.33) describes the geodesic flow on the Lobachevsky space $H^{n-1} = (D^{n-1}, h_{ij}dy^i \otimes dy^j)$. In this case there are two families of non-trivial geodesic solutions (i.e. $y(t) \neq const$):

1. $$\vec{y}(t) = \vec{n}_1[\sqrt{v^2 - 1}\cos\varphi(\bar{t}) - v] + \vec{n}_2\sqrt{v^2 - 1}\sin\varphi(\bar{t}), \tag{3.3.35}$$

$$\varphi(\bar{t}) = 2\arctan[(v - \sqrt{v^2 - 1})\tanh(\omega\bar{t})], \tag{3.3.36}$$

2. $$\vec{y}(t) = \vec{n}\tanh(\omega\bar{t}). \tag{3.3.37}$$

Here $\vec{n}^2 = \vec{n}_1^2 = \vec{n}_2^2 = 1$, $\vec{n}_1\vec{n}_2 = 0$, $v > 1$, $\omega > 0$, $\bar{t} = t - t_0$, $t_0 = const$.

Graphically the first solution corresponds to the arc of the circle with the center at point ($-v\vec{n}_1$) and the radius $\sqrt{v^2 - 1}$. This circle belongs to the plane spanned by vectors $\vec{n}_1$ and $\vec{n}_2$ (the centers of the circle and the ball D^{n-1} also belong to this plane). We note, that the solution (3.3.35)-(3.3.36) in the limit $v \to \infty$ coincides with the solution (3.3.37).

We note, that the boundary of the billiard ∂B is formed by geodesics. For some billiards this fact may be used for "gluing" certain parts of boundaries.

When $\Delta_+ \neq \emptyset$ the Lagrangian (3.3.33) describes the motion of the particle of unit mass, moving in the $(n-1)$-dimensional billiard $B \subset D^{n-1}$ (see (3.3.26)). The geodesic motion in B (3.3.35)-(3.3.37) corresponds to a "Kasner epoch" and the reflection from the boundary corresponds to the change of Kasner epochs. For $n = 3$ some examples of (2-dimensional) billiards are depicted in Figs. 2-4.

Figs. 2-4

The billiard B in Fig. 2. has an infinite volume: $volB = +\infty$. In this case there are three open zones at the infinite circle $|\vec{y}| = 1$. After a finite number of reflections from the boundary the particle moves toward one of these open zones. For corresponding cosmological model we get the "Kasner-like" behavior in the limit $t \to -\infty$ [19].

For billiards depicted in Figs. 3 and 4 we have $volB < +\infty$. In the first case (Fig. 3) the closure of the billiard $\bar{B}$ is compact (in the topology of D^{n-1}) and in the second case (Fig. 4) $\bar{B}$ is non-compact. In these two cases the motion of the particle is stochastic.

Analogous arguments may be applied to the case $n > 3$. So, we are interested in the configurations with finite volume of B. We propose a simple geometric criterion for the finiteness of the volume of B and compactness of $\bar{B}$ in terms of the positions of the points (3.3.28) with respect to the $(n-2)$-dimensional unit sphere S^{n-2} $(n \geq 3)$. We say that the point $\vec{y} \in S^{n-2}$ is (geometrically) illuminated by the point-like source located at the point $\vec{v}$, $|\vec{v}| > 1$, if and only if $|\vec{y} - \vec{v}| \leq \sqrt{|\vec{v}|^2 - 1}$. In Fig. 1 the source P illuminates the closed arc $[P_1, P_2]$. We also say that the point $\vec{y} \in S^{n-2}$ is strongly illuminated by the point-like source located at the point $\vec{v}$, $|\vec{v}| > 1$, if and only if $|\vec{y} - \vec{v}| < \sqrt{|\vec{v}|^2 - 1}$. In Fig. 1 the source P strongly illuminates the open arc (P_1, P_2). The subset $N \subset S^{n-2}$ is called (strongly) illuminated by point-like sources at $\{\vec{v}^{\alpha}, \alpha \in \Delta_+\}$ if and only if any point from N is (strongly) illuminated by some source at $\vec{v}^{\alpha}$ $(\alpha \in \Delta_+)$.

Proposition 1. The billiard B (3.3.26) has a finite volume if and only if the point-like sources of light located at the points $\vec{v}^{\alpha}$ (3.3.28) illuminate the unit sphere S^{n-2}. The closure of the billiard $\bar{B}$ is compact (in the topology of $D^{n-1} \simeq H^{n-1}$) if and only if the sources at points (3.3.28) strongly illuminate S^{n-2}.

Proof. We consider the set $\partial^c B \equiv B^c \setminus \bar{B}$, where B^c is the completion of B (or, equivalently, the closure of B in the topology of R^{n-1}). We remind that $\bar{B}$ is the closure of B in the topology of D^{n-1}. Clearly, that $\partial^c B$ is a closed subset of S^{n-2}, consisting of all those points that are not strongly illuminated by sources (3.3.28). There are three possibilities: i) $\partial^c B$ is empty; ii) $\partial^c B$ contains some interior point (i.e. the point belonging to $\partial^c B$ with some open neighborhood); iii) $\partial^c B$ is non-empty finite set, i.e. $\partial^c B = \{\vec{y}_1, \dots \vec{y}_l\}$. The first case i) takes place if and only if $\bar{B}$ is compact in the topology of D^{n-1}. Only in this case the sphere S^{n-2} is strongly illuminated by the sources (3.3.28). Thus the second part of proposition is proved. In the case i) $volB$ is finite. For the volume we have

$$volB = \int_B d^{n-1}\vec{y}\sqrt{h} = \int_0^1 dr(1-r^2)^{1-n} S_r. \tag{3.3.38}$$

The "area" $S_r \to C > 0$ as $r \to 1$ in the case ii) and, hence, the integral (3.38) is divergent. In the case iii)

$$S_r \sim C_1(1-r)^{2(n-2)} \text{ as } r \to 1 \tag{3.3.39}$$

($C_1 > 0$) and, so, the integral (3.3.38) is convergent. Indeed, in the case iii), when $r \to 1$, the "area" S_r is the sum of l terms. Each of these terms is the $(n-2)$-dimensional "area" of a transverse side of a deformed pyramid with a top at some point $\vec{y}_k$, $k = 1, \ldots, l$. This multidimensional pyramid is formed by certain parts of spheres orthogonal to S^{n-2} in the point of their intersection $\vec{y}_k$. Hence, all lengths of the transverse section $r = const$ of the "pyramid" behaves like $(1-r)^2$, when $r \to 1$, that justifies (3.3.39). But the unit sphere S^{n-2} is illuminated by the sources (3.3.28) only in the cases i) and iii). This completes the proof.

The problem of illumination of convex body in multidimensional vector space by point-like sources for the first time was considered in [68,69]. For the case of S^{n-2} this problem is equivalent to the problem of covering the spheres with spheres [70,71]. There exist a topological bound on the number of point-like sources m_+ illuminating the sphere S^{n-2} [69]:

$$m_+ \geq n. \tag{3.3.40}$$

Thus, we are led to the following.

Proposition 2: When $m_+ < n$, i.e. the number of the components with $(u^\alpha)^2 > 0$ is less than the minisuperspace dimension, the billiard B (3.3.26) has infinite volume: vol $B = +\infty$.

In this case there exist an open zone on the sphere S^{n-2} and the stochastic behaviour near the singularity is absent (we get a Kasner-like behaviour for $t \to -\infty$).

Remark 1. Let the points (3.3.28) form an open convex polyhedron $P \subset R^{n-1}$. Then the sources at (3.3.28) illuminate S^{n-2}, if $D^{n-1} \subset P$, and strongly illuminate S^{n-2}, if $\overline{D^{n-1}} \subset P$.

Scalar field generalization. Let us assume that an additional $(m+1)$-th component with the equation of state $p_i^{(m+1)} = \rho^{(m+1)}$ is considered, $i = 1, \ldots, n$. This component describes Zeldovich matter [49] in all spaces and is equivalent to homogeneous massless free minimally coupled scalar field [50]. In this case $u_i^{(m+1)} = 0$, $i = 1, \ldots, n$ and the potential (3.2.17) is modified by the addition of constant $A_{m+1} > 0$. Then the potential V_* (3.3.16) is modified by the addition of the following term

$$\Delta V = A_{m+1} \exp(-2y^0). \tag{3.3.41}$$

This do not prevent from the formation of the billiard walls but change the time dependence of y^0-variable:

$$\exp(2y^0) = 2A_{m+1} \sinh^2[\omega(t-t_0)]/\omega^2, \tag{3.3.42}$$

($\omega > 0$) instead of (3.3.32). In the limit $t \to t_0 + 0$ we have $y^0 \to -\infty$ and $\vec{y}(t) \to \vec{y}_0 \in B$. So, the stochastic behavior near the singularity is absent in this case.

3.4. Bianchi-IX cosmology

Here we consider the well-known mixmaster model [56,57] with the metric

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{3} \exp[2x^i(t)] e^i \otimes e^i, \tag{3.4.1}$$

where 1-forms $e^i = e^i_\nu(\zeta) d\zeta^\nu$ satisfy the relations

$$de^i = \frac{1}{2} \varepsilon_{ijk} e^j \wedge e^k, \tag{3.4.2}$$

$i, j, k = 1, 2, 3$. The Einstein equations for the metric (3.4.1) lead to the Lagrange system (3.2.14)-(3.2.17) with (see, for example, [57]) $n = 3$, $N_1 = N_2 = N_3 = 1$, $m = 6$, $A_1 = A_2 = A_3 = 1/4$, $A_4 = A_5 = A_6 = -1/2$, $A_0 = A_7 = A_8 = A_9 = 0$, and

$$u_i^{(\alpha)} = 4\delta_i^\alpha, \qquad u_i^{(3+\alpha)} = 2(1 - \delta_i^\alpha), \tag{3.4.3}$$

$\alpha = 1, 2, 3$. In this case $\gamma_0 = \sum_{i=1}^{n} x^i$, the minisuperspace metric (3.2.14) is $G_{ij} = \delta_{ij} - 1$ and the potential (3.2.17) reads

$$V = V_{mix} \equiv \frac{1}{4}(e^{4x^1} + e^{4x^2} + e^{4x^3} - 2e^{2x^1+2x^2} - 2e^{2x^2+2x^3} - 2e^{2x^1+2x^3}). \tag{3.4.4}$$

In the z-coordinates (3.2.30), (3.2.31) we have for 3-vectors (3.2.35)

$$u^1 = \frac{4}{\sqrt{6}}(1, 1, -\sqrt{3}),\ u^2 = \frac{4}{\sqrt{6}}(1, 1, +\sqrt{3}),\ u^3 = \frac{4}{\sqrt{6}}(1, -2, 0), \tag{3.4.5}$$

$$u^4 = \frac{1}{2}(u^1 + u^2),\ u^5 = \frac{1}{2}(u^1 + u^3),\ u^6 = \frac{1}{2}(u^2 + u^3), \tag{3.4.6}$$

and, consequently,

$$(u^\alpha)^2 = 8, \qquad (u^{3+\alpha})^2 = 0, \tag{3.4.7}$$

$\alpha = 1, 2, 3$. Thus the conditions (3.3.3), (3.3.4) are satisfied. The components with $\alpha = 4, 5, 6$ do not survive in the approaching to the singularity . For the vectors (3.3.28) we have

$$\vec{v}^1 = (1, -\sqrt{3}),\ \vec{v}^2 = (1, +\sqrt{3}),\ \vec{v}^3 = (-2, 0), \tag{3.4.8}$$

i.e. a triangle from Fig. 4 (see also [60]). In this case the circle S^1 is illuminated by sources at points $\vec{v}^i$, $i = 1, 2, 3$, but not strongly illuminated. In agreement with Proposition the billiard B has finite volume, but $\bar{B}$ is not compact.

3.5. Discussions

We have obtained the "billiard representation" for the asymptotic cosmological model [37] and proved the geometrical criterion for the finiteness of the billiard volume and the compactness of the billiard (Proposition 1, Sec. 3.3). This criterion may be used as a rather effective (and universal) tool for the selection of the cosmological models with a stochastic behavior near the singularity.

For an "isotropic" component: $p_i^{(\alpha)} = (1-h)\rho^{(\alpha)}$, $i = 1,\ldots,n$, with $h \neq 0$ we have $(u^{(\alpha)})^2 = h^2(D-1)/(2-D) < 0$ and, hence, this component may be neglected near the singularity. Only "anisotropic" components with $(u^{(\alpha)})^2 > 0$ take part in the formation of billiard walls near the singularity. According to the topological bound (3.3.40) [69] the stochastic behavior near the singularity in the considered model may occur only if the number of components with $(u^{(\alpha)})^2 > 0$ is not less than the minisuperspace dimension.

We also note that here, like in the Bianchi-IX case [58,59], the considered reduction scheme uses a special time gauge (or parametrization of time). As it was pointed in [60] one should be careful in the interpretations of the results of computer experiments for other choices of time.

Restrictions on parameters. Here we discuss the physical sense of the restrictions on parameters of the model (3.3.3) and (3.3.4). The condition (3.3.3) means that the densities of the "anisotropic" components with $(u^{(\alpha)})^2 > 0$ should be positive. Using (3.2.8) and (3.2.36) we rewrite the restriction (3.3.4) in the equivalent form

$$\sum_{i=1}^{n} N_i \frac{\rho^{(\alpha)} - p_i^{(\alpha)}}{\rho^{(\alpha)}} > 0, \tag{3.5.1}$$

$(\rho^{(\alpha)} \neq 0)$ $\alpha = 1,\ldots,m$ (for curvature and Λ-terms (3.3.4) is satisfied). For

$$\rho^{(\alpha)} > 0, \qquad p_i^{(\alpha)} < \rho^{(\alpha)}, \tag{3.5.2}$$

$\alpha = 1,\ldots,m$, $i = 1,\ldots,n$, (3.5.1) is satisfied identically.

Remark 2. It may be shown that the condition (3.3.4) may be weakened by the following one

$$u_0^\alpha > 0, \ if \ (u^\alpha)^2 \leq 0. \tag{3.5.3}$$

In this case there exists a certain generalization of the set $B(u^\alpha)$ from (3.3.27) for arbitrary u_0^α ($(u^\alpha)^2 > 0$). The Proposition 1 (Sec. 3.3) should be modified by including into consideration the sources at infinity (for $u_0^\alpha = 0$) and "anti-sources" (for $u_0^\alpha < 0$). For "anti-source" the shadowed domain coincides with the illuminated domain for the usual source (with $u_0^\alpha > 0$). In this case we deal with the kinematics of tachyons. (We may also consider a covariant and slightly more general condition instead of (3.5.3)

$$\mathrm{sign} u_0^\alpha = \varepsilon,\ for\ all\ (u^\alpha)^2 \le 0,\ \varepsilon = \pm 1.) \tag{3.5.4}$$

We note that for the component $\nu \in \Delta_+$ with $u_0^\nu < 0$ or, equivalently, $\sum_{i=1}^n u_i^{(\nu)} < 0$, the relation (3.4.37) should be substituted by

$$\sum_{i=1}^{n} u_i^{(\nu)} \alpha^1 < 0 \tag{3.5.5}$$

4. Dynamics of Inhomogeneities of Metric in the Vicinity of a Singularity in Multidimensional Cosmology

4.1. Introduction

As is well known a number of unified theories predict that dimension of the Universe exceeds that of we normally experience at a macroscopic level [23]. It is assumed that presently additional dimensions are hidden, for they are compactified to the Planckian size, and they do not display themselves in macroscopic and even in microscopic processes. However, the situation must be changed as we come back with time to the very beginning of the evolution of our Universe. Standard cosmological models predict the existence of a singular point at the very beginning and, therefore, the universe size could approach to the Planckian scale. Thus, in the early universe the additional dimensions, if exist, must not be different from ordinary dimensions and should be taken into account. Moreover, one could expect that the existence of additional dimensions may drastically change properties of the singularity and even remove it. The main aim of this section is to construct a general solution of multidimensional Einstein equations near a singularity and to investigate properties of inhomogeneities.

The way to construct a general solution with singularity was indicated first by Belinsky et al.in Ref. [57] for $D = 4$, where D is the dimension of a spacetime. Dynamics of metric at a particular point of space was shown to resemble the behaviour of the well studied "mixmaster" (or of the type-IX) homogeneous model and the last one has a complex stochastic nature [57,74]. Subsequent utilising of that construction has been done in Ref.[53] where the so-called scalar-vector-tensor theory (or the case $D = 5$) was considered and the main feature of the mixmaster model, i.e. the complex oscillatory regime was shown to be also present in the 5-dimensional case.

An investigation of inhomogeneities of metric based on the general solutions has been considered first in Ref. [75]. The case of the scalar-tensor theory (or $D = 4+$ scalar fields) was considered and it turned out that the oscillatory regime leads to the fractioning of the coordinate scale λ of the inhomogeneities of Kasner exponents ($\lambda \approx \lambda_0 2^{-N}$, where N is the number of elapsed Kasner epochs and λ_0 is the initial scale of inhomogeneities). However, the methods by means of which the properties and statistics of the inhomogeneities were investigated turned out to be unapplicable for general case (i.e. for the absence of scalar fields as well as for the expanding universe). This problem has been solved recently in Ref. [61]. In this paper we generalise the results obtained in Ref. [61] to the case of arbitrary number of dimensions D.

As it was mentioned above the main features of the dynamics of an inhomogeneous gravitational field nearby the singularity in 4-dimensional case may be summarised as follows:

1. Locally dynamics of metric functions resembles the behaviour of the most general homogeneous "mixmaster" model [57], which has stochastic behaviour [74]. Just the stochastic behaviour leads to a monotonic decrease of the coordinate scale of the metric inhomogeneities [61,75].

2. In the vicinity of a singularity a scalar field is the only kind of matter effecting the dynamics of metric [53].

These facts may be simply understood under the following qualitative estimates (that is confirmed by subsequent consideration). As is well known in cosmology the horizon size l_h is a natural scale measuring a distance from the singularity. Therefore, inhomogeneities may be divided into the large-scale ($l_i \gtrsim l_h$) and small-scale ($l_i \ll l_h$) ones. The horison size varies with time as $l_h \sim t$ (where t is the time in synchronous reference system) whereas the characteristic spatial dimension of the inhomogeneity may be estimated as $l_i \sim t^\alpha$ (as $t \to 0$). In a linear theory for an isotropic background the exponent α may be expressed via the state equation of matter as $\alpha = \frac{2\epsilon}{3(p+\epsilon)}$ and what is important $\alpha < 1$. Thus, it is clear that an arbitrary inhomogeneous field becomes large-scale in the sufficient closeness to the singularity. Since the inhomogeneities are large-scale there are no effects connected with propagating of gravitational waves etc, and this would mean that inhomogeneities become passive. Consequently, dynamics of the field may be approximately described by the most general homogeneous model depending parametrically upon the spatial coordinates. Note, however, that the homogeneous model would appear to be in a general non-diagonal form.

The second fact may be understood in the same way. As it was shown in Ref. [76] the gravitational part of the Einstein equations at the singular point varies with time, in the leading order, as $R^\alpha_\beta \sim t^{-2}$ whereas the matter has the order $T^\alpha_\beta \sim t^{-2k}$, where k depends upon the state equation as $k = \frac{\epsilon+p}{2\epsilon}$. Thus, one can see that for the equation of state satisfying the inequality $p < \epsilon$ we have $k < 1$ and only for the limiting case $p = \epsilon$ ($k = 1$) the both sides turn out to be of the same order. We note that in the vicinity of a singularity scalar fields give just this equation of state.

As it is well known (see for example Ref.[21,37,53,77,78]) additional dimensions may be treated in ordinary gravity as a set of nonminimally coupled scalar and vector fields. Therefore, one could expect that the main contribution to dynamics in the vicinity of a singularity would be given by those dynamical functions which are connected with scalar fields, whereas other functions would play a passive role.

Thus, one could expect that in multidimensional cosmology local behaviour of the metric functions (at a particular point of space) will be described by a most general homogeneous model. Here, it is necessary to recall the important property of the mixmaster universe that is the stochastic behaviour. The problem of stochasticity of homogeneous multidimensional cosmological models has been investigated in a number of papers [18,54]. In particular, in Refs. [54] the result was obtained that chaos is absent in the spaces whose dimension $D \geq 11$, since in this case the last stage of a cosmological collapse is described by a minimally-coupled scalar field [53]. Therefore it seems to be sufficient to consider the Einstein equation of $D < 10$ dimensions with the scalar field matter source.

Thus, here we consider the D-dimensional Einstein equations with the matter source given by a minimally-coupled scalar field. Using generalised Kasner variables we divide the dynamical functions connected with physical degrees of freedom into two parts. One part has a simple behaviour while the other is described by a billiard on an appropriate Lobachevsky space. In dimensions $D < 11$ the billiard has a finite volume and shows stochastic properties. This stochasticity causes the degree of inhomogeneity of the part of dynamical functions and leads to the formation of spatial chaos. The presence of a scalar field results in the fact that lengths of trajectories on the billiard take finite values. This destroys the chaotic properties which, however, are restored in the limit when the ADM energy density for the scalar field turns out to be small as compared with that of the gravitational variables.

4.2. Generalised Kasner Solution, Generalized Kasner Variables

We consider the theory in canonical formulation. Basic variables are the Riemann metric components $g_{\alpha\beta}$ with signature $(+,-,...,-)$ and a scalar field ϕ specified on the n-manifold S, and its conjugate momentum $\Pi^{\alpha\beta} = \sqrt{g}(K^{\alpha\beta} - g^{\alpha\beta}K)$ and Π_ϕ, where $\alpha = 1, ..., n$ and $K^{\alpha\beta}$ is the extrinsic curvature of S. For the sake of simplicity we shall consider S to be compact i.e. $\partial S = 0$. The action has in Planck units the following form

$$I = \int_S (\Pi^{ij}\frac{\partial g_{ij}}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{4.2.1}$$

where

$$H^0 = \frac{1}{\sqrt{g}}\left\{ \Pi^\alpha_\beta \Pi^\beta_\alpha - \tfrac{1}{n-1}(\Pi^\alpha_\alpha)^2 + \tfrac{1}{2}\Pi^2_\phi + g(W(\phi) - R) \right\}, \tag{4.2.2}$$

$$H^\alpha = -2\Pi^{\alpha\beta}_{|\beta} + g^{\alpha\beta}\partial_\beta \phi \Pi_\phi, \tag{4.2.3}$$

here

$$W(\phi) = \frac{1}{2}\left\{ g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi) \right\}. \tag{4.2.4}$$

A generalised Kasner solution is realised under the following assumption

$$\sqrt{g}T \sim (\Pi^{\alpha}_{\beta}, \Pi_{\phi}) \bar{g}V = g(W - R), \tag{4.2.5}$$

where $\sqrt{g}T$ denotes the first three terms in (4.2.2). Then, using (4.2.1) one can find the following solution of the multidimensional Einstein equations

$$ds^2 = dt^2 - \sum_{a=0}^{n-1} t^{s_a} l^a_{\alpha}, l^a_{\beta} dx^{\alpha} dx^{\beta} \tag{4.2.6}$$

where l^a_{α}, s_a are functions of space coordinates. Kasner exponents s_a satisfy the identities $\sum s_a = \sum s_a^2 + q^2 = 1$, and run the domain $-\frac{n-2}{n} \leq s_a \leq 1$ (here $q^2 = \frac{(n-1)^2}{2} \frac{\Pi^2_{\phi}}{(\Pi^{\alpha}_{\alpha})^2}$). Since, as it was shown in Ref.[53,57] the generalised Kasner solution takes a substantial portion of the evolution of metric it is convenient to introduce a Kasner-like parametrization of the dynamical variables [61]. We consider the following representation for metric components and their conjugate momenta

$$g_{\alpha\beta} = \sum_a \exp\{q^a\} l^a_{\alpha} l^a_{\beta}, \tag{4.2.7}$$

$$\Pi^{\alpha}_{\beta} = \sum_a p_a L^{\alpha}_a l^a_{\beta}, \tag{4.2.8}$$

here $L^{\alpha}_a l^b_{\alpha} = \delta^b_a$ $(a, b = 0, ..., (n-1))$, and the vectors l^a_{α} contain only $n(n-1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the following form

$$l^a_{\alpha} = U^a_b S^b_{\alpha},\ U^a_b \in SO(n),\ S^a_{\alpha} = \delta^a_{\alpha} + R^a_{\alpha} \tag{4.2.9}$$

where R^a_{α} denotes a triangle matrix ($R^a_{\alpha} = 0$ as $a \leq \alpha$). Substituting (4.2.7) - (4.2.9) into (4.2.1) one gets the following expression for the action functional

$$I = \int_S (p_a \frac{\partial q^a}{\partial t} + T^{\alpha}_a \frac{\partial R^a_{\alpha}}{\partial t} + \Pi_{\phi} \frac{\partial \phi}{\partial t} - N H^0 - N_{\alpha} H^{\alpha}) d^n x dt, \tag{4.2.10}$$

here $T^{\alpha}_a = 2 \sum_b p_b L^{\alpha}_b U^b_a$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}} \{ \sum p_a^2 - \tfrac{1}{n-1} (\sum p_a)^2 + \tfrac{1}{2} \Pi^2_{\phi} + V \}. \tag{4.2.11}$$

In the case of $n = 3$ the functions R^a_{α} are connected purely with transformations of a coordinate system and may be removed by solving momentum constraints $H^{\alpha} = 0$. In the multidimensional case the functions R^a_{α} contain $\frac{n(n-3)}{2}$ dynamical functions as well. Now it is easy to see that the choice of Kasner-like parametrisation simplifies the procedure of the constructing of the generalized Kasner solution. Indeed, if we now neglect the potential term in (4.2.10) and put $N^{\alpha} = 0$ we find that Hamiltonian does not depend on the scale functions and other dynamical variables contained in Kasner vectors introduced by expressions (4.2.7) (4.2.8).

4.3. The asymptotic model in the vicinity of a cosmological singularity

As it is well known, [53], [57], the Kasner regime (4.2.6) turns out to be unstable in a general case. This happens due to the violation of the condition (4.2.5) because the potential V contains increasing terms which lead to replacement of Kasner regimes. To find out the law of replacement it is more convenient to use an asymptotic expression for the potential [61], [55]. For this aim we put the potential in the following form

$$V = \sum_{A=1}^{k} \lambda_A g^{u_A}, \tag{4.3.1}$$

here λ_A is a set of functions of all dynamical variables and of their derivatives and u_A are linear functions of the anisotropy parameters $Q_a = \frac{q^a}{\sum q}$ $(u_A = u_A(Q))$. Assuming the finiteness of the functions λ and considering the limit $g \to 0$ we find that the potential V may be modeled by potential walls

$$g^{u_A} \to \theta_\infty[u_A(Q)] = \begin{cases} +\infty, & u_A < 0, \\ 0, & u_A > 0 \end{cases} \tag{4.3.2}$$

Thus, putting $N^\alpha = 0$ we can remove the passive dynamical function T^α_a, R^a_α from the action (4.2.10) and get the reduced dynamical system

$$I = \int_S \left\{ p_a \frac{\partial q^a}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - \lambda \left\{ \sum p^2 - \frac{1}{n-1}(\sum p)^2 + \frac{1}{2}\Pi^2_\phi + U(Q) \right\} \right\} d^n x dt, \tag{4.3.3}$$

here λ is expressed via the lapse function as $\lambda = \frac{N}{\sqrt{g}}$. In harmonic variables the action (4.3.3) takes the form formally coincided with the action for a relativistic particle

$$I = \int_S \{ P_r \frac{\partial z^r}{\partial t} - \lambda'(P_i^2 + U - P_0^2) \} d^n x dt, \tag{4.3.4}$$

here $r = 0, ..., n$, $i = 1, ..., n$, $q^a = A^a_j z^j + z^0$ $(j = 1, ..., n-1)$, $z^n = \sqrt{\frac{n(n-1)}{2}}\phi$ and the constant matrix A^a_j obeys the following conditions

$$\sum_a A^a_j = 0, \sum_a A^a_j A^a_k = n(n-1)\delta_{jk} \tag{4.3.5}$$

and can be expressed in the following form

$$A^a_j = \sqrt{\frac{n(n-1)}{j(j-1)}}(\theta^a_j - j\delta^a_j),$$

where $\theta^a_j = \begin{cases} 1, j > a \\ 0, j \le a \end{cases}$.

Since the timelike variable z^0 varies during the evolution as $z^0 \sim \ln g$ the positions of potential walls turn out to be moving. It is more convenient to fix the positions of walls. This may be done by using the so-called Misner-Chitre like variables [55] ($\vec{y} = y^j$)

$$z^0 = -e^{-\tau}\frac{1+y^2}{1-y^2}, \vec{z} = -2e^{-\tau}\frac{\vec{y}}{1-y^2}, \qquad y = |\vec{y}| < 1. \tag{4.3.6}$$

Using these variables one can find the following expressions for the anisotropy parameters

$$Q_a(y) = \frac{1}{n}\left\{1 + \frac{2A_j^a y^j}{1+y^2}\right\}, \tag{4.3.7}$$

which are now independent of timelike variable τ. From (4.3.7) one can find the range of the anisotropy functions $-\frac{n-2}{n} \leq Q_a \leq 1$.

Choosing as a time variable the quantity τ (i.e. in the gauge $N = \frac{n(n-1)}{2}\sqrt{g}\exp(-2\tau)/P^0$) we put the action (4.3.4) into the ADM form

$$I = \int_S \{\vec{P}\tfrac{\partial}{\partial\tau}\vec{y} + P^n \tfrac{\partial}{\partial\tau}z^n - P^0(P,y)\} d^n x d\tau, \tag{4.3.8}$$

where the quantity

$$P^0(P,y) = (\epsilon^2(\vec{y},\vec{P}) + V[y] + (P^n)^2 e^{-2\tau})^{1/2}, \tag{4.3.9}$$

plays the role of the ADM Hamiltonian density and

$$\epsilon^2 = \frac{1}{4}(1-y^2)^2\vec{P}^2. \tag{4.3.10}$$

The part of the configuration space connected with the variables $\vec{y}$ is a realisation of the $(n-1)$ -dimensional Lobachevsky space [64] and the potential V cuts a part of it. Thus, locally (at a particular point of S) the action (4.3.9) describes a billiard on the Lobachevsky space. The positions of walls which form the boundary of the billiard are determined, due to (4.3.1) by the inequalities

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, a \neq b \neq c \tag{4.3.11}$$

and the total number of walls is $\frac{n(n-1)(n-2)}{2}$. Using the matrix (4.3.5) one can find that the walls are formed by spheres determined by the equations

$$\sigma_{abc} = \frac{n-1}{n(1+y^2)}\{(\vec{y} + \vec{B}_{abc})^2 + 1 - B_{abc}^2\}, \vec{B}_{abc} = \frac{1}{n-1}(\vec{A}^a - \vec{A}^b - \vec{A}^c), \tag{4.3.12}$$

here for arbitrary a, b, c we have $B^2 = 1 + \frac{2n}{n-1}$. In a general case n points of the billiard having the coordinates $\vec{P}_a = \frac{1}{n-1}\vec{A}^a$ lie on the absolute (at infinity of the Lobachevsky space). The trajectories which end with these points correspond to the set of Kasner exponent $(0, \cdots 0, 1)$. When $n = 9$ there appear additional isolated points S_{ab} lying on the absolute. The coordinates of these points are given by the vectors $\vec{S}_{abc} = \frac{1}{12}(\vec{A}_a + \vec{A}_b + \vec{A}_c)$, $a \neq b \neq c$ (see appendix). In the case of $n \geq 10$ in addition to the points P_a and S_{abc} there appear open accessible domains on the absolute (see appendix of Refs. [54] where has been used another approach) and the volume of the billiard becomes infinite. If on the contrary $n < 10$, the volume of the billiard is finite and the billiard turns out to be a mixing one. We give two simplest examples for illustration of the billiards on fig.5. The case $n = 3$ on fig.5a coincides with the well-known "mixmaster" model and on fig.5b we illustrate the case of $n = 4$ considered in Ref.[53].

4.4. Dynamics of inhomogeneities

The system (4.3.8) has the form of the direct product of "homogeneous" local systems. Each local system in (4.3.8) has two variables ϵ and P^m as integrals of motion. The solution of this local system for remaining functions represents a geodesic flow on a manifold with negative curvature. As it is well known the geodesic flow on a manifold with negative curvature is characterized by exponential instability [64]. This means that during the motion along a geodesic the normal deviations grow no slower than the exponential of the traversed path $\xi \simeq \xi_0 e^s$), where the traversed path is determined by the expression

$$s = \int_{\tau_0}^{\tau} dl = \int_{\tau_0}^{\tau} \frac{2 \mid \frac{\partial y}{\partial \tau} \mid}{(1-y^2)} d\tau = \frac{1}{2} \ln \mid \frac{P^0 - \epsilon}{P^0 + \epsilon} \mid_{\tau_0}^{\tau} . \tag{4.4.1}$$

This instability leads to the stochastic nature of the corresponding geodesic flow. The system possesses the mixing property [65] and an invariant measure induced by the Liouviulle one

$$d\mu(y, P) = const\delta(E - \epsilon) d^{n-1} y d^{n-1} P, \tag{4.4.2}$$

where E is a constant. Integrating this expression over ϵ we find

$$d\mu(y, s) = const \frac{d^{n-1} y d^{n-2} s}{(1-y^2)^n}, \tag{4.4.3}$$

where $\vec{s} = \frac{\vec{P}}{\epsilon}$, $|s| = 1$.

Since the inhomogeneous system (4.3.8) is the direct product of "homogeneous" systems one can simply describe its behaviour as in ref [61]. In particular, the scale of the inhomogeneity decreases as

$$\lambda_i \sim (\frac{\partial y}{\partial x})^{-1} \sim \lambda_i^0 \exp(-s) \tag{4.4.4}$$

and after sufficiently large time ($s(\tau) \to \infty$) the dynamical functions $\vec{y}(x)$, $\vec{P}(x)$ become a random functions of the spatial coordinates. In order to calculate different mean values one can use the following n-point distribution functions [61]

$$\rho_{x_1 \cdots x_n}(y_1, \cdots, y_n, m_1 \cdots, m_n) = < \prod_{i=1}^{n} \delta(y_i - y(x_i))\delta(m_i - m(x_i)) >, \tag{4.4.5}$$

where the angular brackets can denote the averaging out either over an initial distribution or over a certain coordinate volume $\Delta V >> (\lambda_i^0)^3$. The mixing results in the relaxation of initial functions (4.4.5) to the limiting ones which have the form of the direct product of measures (4.4.3): $d\mu = \prod_i d\mu_i$. Thus, asymptotic expressions for averages and correlating functions have the form

$$< y(\vec{x}) > = < P(\vec{x}) > = 0, < y_k(x), y_l(x') > = < y_k, y_l > \delta(x, x'), \tag{4.4.6}$$

for $|x - x'| \overline{\gamma} \lambda_i^0 \exp(-s)$.

Here it is necessary to point out a role of the scalar field in dynamics and statistical properties of inhomogeneities. As may be easily seen from (4.4.1) in the absence of a scalar field (i.e. $P^n = 0$) the transversed path coincides with the duration of motion (we have $s = \Delta\tau = \tau - \tau_0$ instead of (4.4.1)). Thus, the effect of scalar fields is displayed in the replacement of the dependence for transversed path of time variable and, therefore, in the replacement of the rate of increasing of the inhomogeneities. This replacement does not change qualitatively the evolution of the universe in the case of cosmological expansion. But in the case of the contracting universe the situation changes drastically. Indeed, in the limit $\tau \to -\infty$ from (4.4.1) we find that the transversed path s takes a limited value s_0 and therefore the increasing of inhomogeneities turns out to be finite. One of consequences of such behaviour is the fact that at the singularity the functions $\vec{y}$ and $\vec{P}$ take constant values. In other words in the presence of scalar fields a cosmological collapse ends with a stable Kasner-like regime (4.2.6). This fact may be seen in the other way. Indeed, in the limit $\tau \to -\infty$ the scalar field gives the leading contribution in ADM Hamiltonian (4.3.9) and P^0 does not depend on gravitational variables at all.

The finiteness of the transversed path $s(\tau)$ leads, generally speaking, to the destruction of the mixing properties [65], since for establishment of the invariant measure it is necessary to satisfy the condition $s_0 \to \infty$. Evidently, this condition requires the smallness of the energy density for scalar field as compared with the ADM energy of gravitational field (the last term in (4.3.9) in comparison with the first ones). Indeed, in this case s_0 is determined by the expression $s_0 = -\ln \frac{P^n e^{\tau_0}}{2\epsilon}$, which follows from (4.4.1), and as $P^n \to 0$ one get $s_0 \to \infty$ (i.e. s can have arbitrary large values).

Thus, in the case of cosmological contraction one may speak of the mixing and, therefore, of establishment of the invariant statistical distribution just only for those spatial domains which have suficiently small energy density of the scalar field.

4.5. Estimates and concluding remarks

In this manner the large-scale structure of the space in the vicinity of singularity acquires a quasi-isotropic nature. A distribution of inhomogeneities is determined by the set of functions of spatial coordinates $\epsilon(x)$, $\Pi_\phi(x)$ and R^a_α which conserve during the evolution a primordial degree of inhomogeneity of the space. The scale of inhomogeneity of other functions grows as $\lambda \approx \lambda_0 e^{-s(\tau)}$. In this section we give some estimates clarifying the behaviour of the inhomogeneities. For simplicity we consider the case when the scalar field is absent.

To find the estimate for the inhomogeneity growth in a synchronous time t $(dt = Nd\tau)$ we put $y = 0$. Then for variation of the variable τ one may find the following estimate $\sqrt{g} \sim \exp(-\frac{n}{2}e^{-\tau}) \sim P^0 t$, (here the point $t = 0$ corresponds to the singularity). According to (4.4.4) the dependence of the coordinate scale of inhomogeneity upon the time t takes the form

$$\lambda \approx \lambda_0 \ln(1/g_0)/\ln(1/g)$$

in the case of contracting $(g \to 0)$ and

$$\lambda \approx \lambda_0 \ln(1/g)/\ln(1/g_0)$$

in the case of the expanding universe.

A rapid generation of the more and more small scales leads to the formation of spatial chaos in metric functions and so the large-scale structure acquires a quasi-isotropic nature. Speeds of the scale growing (Hubble constants) for different directions turn out to be equal after averaging over a spatial domains having the size $\approx \lambda_0$. Indeed, using (4.3.7) one may find the expressions for averages $< Q_a >= 1/n$.

Besides, it is necessary to mention one more characteristic feature of the oscillatory regime in the inhomogeneous case. This is the formation of a cellular structure in the scale functions Q_a during the evolution which demonstrate explicitely the stochastic process of development of inhomogeneities. Indeed, let us consider some region of coordinate space ΔV. Two functions $y(\vec{x})$ define the map of that region on some square $\Sigma \in K$ (see fig.1c). During the evolution the size of the square Σ grows $\simeq e^{s(\tau)}$ and Σ covers the domain of the billiard K many times. Each covering determines its own preimage in ΔV. In this manner the initial coordinate volume is splitted up in "cells" $\Delta V = \bigcup_i \Delta V_i$. In the every cell the vector $y(\vec{x})$ takes almost all admissible values $\vec{y} \in K$ and that of the functions $Q_a(Q_a \in [Q_{min}, 1]$ where $Q_{min} = -\frac{(n-1)^2-(n+1)}{n(n+1)}$. To illustrate this process let us consider the case $n = 3$. In this case it is convenient to use the Poincaré model of the Lobachevsky plane on the upper complex half-plane $H = \{W = U + iV, V \geq 0\}$ (see fig.5c). The line $V = 0$ is called the absolute and its points lie at infinity. Geodesics in H are given by semi-circles with centers on the absolute, or by rays perpendicular to the absolute. The billiard constitutes the region $K \in H$, bounded by geodesics triangle $\partial K = [|W| = 1, U = \pm 1]$. The area of the billiard is equal to π. The motion can be continued to the whole plane H. For this aim one needs to reflect the domain of the billiard with respect to one of the boundary walls and make iteration of such procedure. In this way the Lobachevsky plane will be covered by a set of domains K^n each of which is connected with the region of the billiard K by a one-to-one mapping. During the evolution an arbitrary initial square Σ^0 begins to grow and covers the more and more number of the domains K^n (see fig.5c). Such cellular structure turns out to be depending on time and the number of cells increases as $N \approx N_0 e^{s(\tau)}$. However, the situation will be changed if we consider a contracting space filled with a scalar field. Then the evolution of this structure in the limit $g \to 0$ ends, because the functions Q_a become independent of time, and on the final stage of the collapse one would have a real cellular structure [75].

In spite of the isotropic nature of the spatial distribution of the field the large local anisotropy displays itself in the anomalous dependence of spatial lengths upon time variable for vectors and curves. Indeed, a moment of scale function $< g^{MQ_a} >$ (where $M > 0$) decreases in the asymptotic $g \to 0$ as the Laplace integral $\int_{Q_{min}}^{1} g^{MQ_a} \rho(Q_a) dQ_a$, where $\rho(Q_a)$ is the distribution which follows from (4.4.3). The main contribution in this integral is given by the point $Q = Q_{min}$ and $Q = Q_{min}$ and in the case of $n > 3$ in the limit $(Q - Q_{min}) \to 0$ one can find $\rho(Q) \approx C(Q - Q_{min})^{n-1}$, where C is a constant and we obtain the estimate

$$< g^{MQ_a} > \approx \frac{g^{MQ_{min}}}{(M \ln 1/g)^{n-1}}. \tag{4.5.1}$$

This expression shows that for $n > 3$ average lengths even increase while approaching the singularity. The case $n = 3$ must be considered separately. In this case we have $Q_{min} = 0$ and the explicit form of the distribution function $\rho(Q_a)$, as it follows from (4.4.3), is

$$\rho(Q) = \frac{2}{\pi}(Q(1-Q))^{-1/2}(1+3Q)^{-1}. \tag{4.5.2}$$

As $Q \ll 1$ one has $\rho(Q_a) \approx \frac{2}{\pi}(Q_a)^{-1/2}$ and, thus, in the limit $g \to 0$ we get the estimate

$$< g^{MQ_a} > \approx (M \ln(1/g))^{-1/2}. \tag{4.5.3}$$

In conclusion we briefly repeat the main results. The general ihomogeneous solution of D-dimensional Einstein equations with any matter sources satisfying the inequality $\epsilon \geq p$ near the cosmological singularity is constructed. It is shown that near the singularity a local behavior of metric functions (at a particular point of the coordinate space) is described by a billiard on the $(D-1)$-dimensional Lobachevsky space. In the case of $D < 11$ the billiard has a finite volume and consequently a mixing one. The rate of growth of inhomogeneities of metric is obtained. Statistical properties of inhomogeneities are described by the invariant measure. It is shown that a minimally-coupled scalar field leads, in general, to the distruction of stochastic properties of the inhomogeneous model.

Appendix

Here we show that the billiards in the dimensions exceeding $n = 9$ become infinite. Let us introduce a new set of variables connected with the old ones as $\vec{x} = \frac{2\vec{y}}{1+y^2}$. Within these variables the absolute of the Lobachevsky space keeps the old position $|x^2| = 1$ and the walls become planes (see (4.3.7), (4.3.12)). Furthermore, it will be more convenient to select a region on the Lobachevsky space on which the anisotropy parameters are in the increasing order $Q_0 \leq Q_1 \leq \cdots \leq Q_{n-2} \leq Q_{n-1}$ and which is restricted by the only wall (see (4.3.11)) $\sigma(\vec{x}) = \sigma_{1,n-2,n-1}$. This region is formed by the vectors of the type $\vec{x} = \Sigma_{i=1}^{n-1} u^i \vec{e}_i$, where the parameters $0 \leq u^i \leq 1$ and the set of basic vectors is given by: $\vec{e}_i = \frac{1}{n+1} \sum_{a=i}^{n-1} \vec{A}^a$ for $i \leq n-2$, $\vec{e}_{n-2} = \frac{1}{2(n-1)}(\vec{A}^{n-2} + \vec{A}^{n-1})$ and $\vec{e}_{n-1} = \frac{1}{n-1}\vec{A}^{n-1}$. They are normalised so that $\sigma(\vec{e}_i) = 0$. It is easy to find that the wall causes the restrictions on the parameters $u^i : \sum u^i \leq 1$. The Euclidian norms of the basic vectors are $e_i^2 = \frac{i(n-i)(n-1)}{(n+i)^2}$ for $i \leq n-2, e_{n-2}^2 = \frac{n-2}{2(n-1)}$ and $|e_{n-1}| = 1$ (here we used the following property of $\vec{A}^a : \sum_{k=1}^{n-1} A_k^a A_k^b = n(n-1)\delta^{ab} - (n-1)$). Now, it is easy to find that for $n < 9$ all basic vectors except $\vec{e}_{n-1}$ have norms less than unity and we have $|\vec{x}| \leq 1$ (equality is achieved only when $\vec{x} = \vec{e}_{n-1}$). In the case $n = 9$ we get $e_3^2 = e_8^2 = 1$, all the other vectors have norms less than unity and we have the similar situation as above (i.e., $|x| = 1$ only when $\vec{x} = \vec{e}_3$ and $\vec{x} = \vec{e}_8$). In the case $n > 9$ a number of basic vectors have norms exceeding unity, e.g., $\vec{e}_i$ for $i = \left[\frac{n}{3}\right] + 1$ or $i = \left[\frac{n}{3}\right] + 1$, where $\left[\frac{n}{3}\right]$ denotes the entire part of the number $\frac{n}{3}$. This means that the wall in these directions lies outside the absolute of the Lobachevsky space and there appears an open accessible domain. In other words, the trajectories do not meet any obstacle in these directions and run to the infinity. This proves the statement made in Sec. 4.3.

5. Multidimensional Cosmology and the Time Variation of G: a Dynamical System Approach [79]

5.1. Introduction

Multidimensional cosmology has since long ago attracted the attention of cosmologists, who were stimulated initially mainly by the Kaluza-Klein theory [80-81] and more recently by superstrings models [23]. The idea that the Universe we live in can be represented as a 4-dimensional hypersurface imbedded in a (4+n)-spacetime manifold has actually different versions. In particular, we could mention the one put forward by Wesson, who has developed an embedding scheme in which the Friedmann-Robertson-Walker-Lemaitre cosmology can be entirely obtained in a rather simple and elegant way from (4+1)-dimensional Ricci-flat spacetimes [82-83]. Further generalisation of this theory to arbitrary dimensionality with applications to multidimensional cosmology and lower dimensional gravity was later carried out by Rippl et al [84]. General multidimensional and multicomponent schemes were studied in [21] (see also refs. therein).

In addition to the role multidimensional theories might play in providing a theoretical framework in which the most fundamental laws of physics appear to be unified, another motivation may come from a conjecture - originally proposed by Dirac [85] - regarding the time variation of the Newtonian gravitational constant G. Indeed, this idea, which was to be taken seriously by superstrings theory and recent inflationary models, is also present in the context of multidimensional cosmological models where G is considered not as a fundamental constant of Nature, but as a cosmological function depending on the geometry of an 'internal space' [12,50,86].

Among the several attempts to construct gravity theories with varying G is Brans-Dicke theory, where the strength of the gravitational force is determined by a scalar field [87,88]. Here we find again the same idea underlying the connection between higher dimensions and time variation of G, as it can be shown that n-dimensional Kaluza-Klein models reduce to Brans-Dicke vacuum models for $w = 0$. Other theories with scalar field (especially conformal) see in [50].

In this section we consider, as in [21], a (4+n)-spacetime manifold defined by the topological product $M^{4+n} = R \times M_k^3 \times K^n$, where M_k^3 is a 3-dimensional space of constant curvature (i.e., $M_k^3 = S^3, R^3, L^3$ according to $k = +1, 0, -1$, respectively), and K^n is a n-dimensional Ricci-flat manifold. We assume also that this spacetime is generated by a (4+n)-dimensional multicomponent perfect fluid.

Now, it turns out that the field equations for the special case $k = 0$ may be reduced to an autonomous homogeneous system of the second order. This system contains some free parameters, one of them being n (the dimensionality of the internal space) and the others come from the equations of state of the multicomponent-fluid. However, by restricting ourselves to 'dust-like' matter, we are left with n as the only parameter of the system. Then, we construct the phase diagram of the system to obtain a general picture of the solutions. As a by-product of the analysis we also obtain analytical solutions of the equations for arbitrary values of n (see also [14]).

5.2. The field equations

The gravitational field equations in a (4+n)-dimensional gravity are postulated to be

$$ {}^{(4+n)}R_{\mu\nu} = \kappa^2 \left({}^{(4+n)}T_{\mu\nu} - g_{\mu\nu}\frac{T}{(n+2)} \right), \tag{5.2.1} $$

where all the geometric quantities are defined in $(4+n)$ dimensions and κ^2 is the generalised Einstein constant [21]. We take the metric tensor to be given by the line element

$$ ds^2 = dt^2 - R^2(t)^{(3)}g_{ij}(x^k)dx^i dx^j - b^2(t)^{(n)}g_{pq}(y^r)dy^p dy^q, \tag{5.2.2} $$

where $i, j, k = 1, 2, 3; p, q, r = 4, ..., n+3$; ${}^{(3)}g_{ij}$, ${}^{(n)}g_{pq}$, $R(t)$ and $b(t)$ are, respectively, the metrics and scale factors for ${}^{(3)}M_k$ and K^n. The $(4+n)-$dimensional energy-momentum tensor for a multicomponent perfect fluid is taken to be

$$ T^{\mu}_{\nu} = diag(\rho(t), -p_3(t)\delta^i_j, -p_n(t)\delta^m_n) \tag{5.2.3} $$

From (5.2.2) and (5.2.3) the Einstein equations become:

$$3\frac{\ddot{R}}{R}+n\frac{\ddot{b}}{b}=\frac{\kappa^2}{n+2}\left(-(n+1)\varrho-3p_3-np_n\right), \tag{5.2.4}$$

$$\frac{2k}{R^2}+\frac{\ddot{R}}{R}+n\frac{\dot{b}}{b}\frac{\dot{R}}{R}+2\frac{\dot{R}^2}{R^2}=\frac{\kappa^2}{n+2}\left(\varrho+(n-1)p_3-np_n\right) \tag{5.2.5}$$

$$\frac{\ddot{b}}{b}+(n-1)\frac{\dot{b}^2}{b^2}+3\frac{\dot{R}}{R}\frac{\dot{b}}{b}=\frac{\kappa^2}{n+2}\left(\varrho-3p_3+2p_n\right) \tag{5.2.6}$$

At this point it is worthwhile mentioning the way by which higher dimensional gravity theories of this type can be naturally related to their 4-dimensional counterparts with varying G [21]. This is simply done by integrating the $(4+n)$-dimensional energy density over the K^n compact space and equating the result to ${}^{(4)}\varrho(t)$, thereby defining the energy density in 4-dimensional spacetime:

$${}^{(4)}\varrho(t)=\int_{K^n}dy^n\sqrt{{}^{(n)}g}\,b^n(t)\varrho(t)=\varrho(t)b^n(t), \tag{5.2.7}$$

where $\sqrt{{}^{(n)}g}$ is the determinant of ${}^{(n)}g_{pq}$. It is convenient to 'normalise' the scale factor $b(t)$ by imposing the condition $\int_{K^n}\sqrt{{}^{(n)}g}dy^n=1$. Thus, in order to get the equations of the 4-dimensional gravity we put

$$8\pi G(t)\left[{}^{(4)}\varrho(t)\right]=\kappa^2\varrho(t). \tag{5.2.8}$$

This procedure leads us to the definition of an effective gravitational 'constant' $G(t)$ given by $8\pi G(t)=\kappa^2 b^{-n}(t)$. In this way the time variation of G is directly related to the time variation of the internal space scale factor $b(t)$ by

$$\frac{\dot{G}}{G}=-n\frac{\dot{b}}{b} \tag{5.2.9}$$

Clearly for $n=0$ the Friedmann cosmology in ordinary 4-dimensional spacetime is recovered.

5.3. The dynamical system and the phase portraits

In this section we let $M_k^3=R^3$ and assume that the multicomponent fluid satisfies the equations of state $p_3=p_n=0$, i.e., we assume that matter behaves as a $(n+4)$-dimensional 'dust'. Then, letting $x=\frac{3\dot{R}}{R}$ and $y=\frac{\dot{b}}{b}$ the equations (5.2.4-6) become

$$\dot{x}+\frac{x^2}{3}+n\dot{y}+y^2=-\frac{n+1}{n+2}\kappa^2\varrho \tag{5.3.1}$$

$$\dot{x} + x^2 + Nxy = \frac{3\kappa^2\varrho}{n+2} \tag{5.3.2}$$

and

$$\dot{y} + ny^2 + xy = \frac{\kappa^2\varrho}{n+2}. \tag{5.3.3}$$

Eliminating ϱ from these equations results in

$$\dot{x} = \frac{1}{2(n+2)}\left[-2(n+1)x^2 + 2n(1-n)xy + 3n(n-1)y^2\right] \tag{5.3.4}$$

and

$$\dot{y} = \frac{1}{2(n+2)}\left[\frac{2x^2}{3} - 4xy - n(n+5)y^2\right] \tag{5.3.5}$$

Defined[1] in this way x can be interpreted as a measure of the usual cosmological expansion of the 4-dimensional observable Universe, while y is a measure of the time variation of the gravitational constant G or, equivalently, the expansion of the compact space K^n (see eq.(5.2.9)). The above system of equations represents a homogeneous autonomous dynamical system of the second-order. To carry out an analysis of this system we first note that, as the system is homogeneous, the origin of the phase space $x = y = 0$ corresponds to an equilibrium point (in fact , an isolated equilibrium point) [89]. Physically, this point represents nothing else but the flat Minkowski spacetime of General Relativity, with $\varrho = 0$.

In order to construct the phase diagram of a homogeneous dynamical system we first determine the *invariant rays* of the system [89] by introducing the polar coordinates in the phase plane: $x = r\cos\theta, y = r\sin\theta$. In these coordinates a general homogeneous dynamical system of order m of the form

$$\dot{x} = X_m(x,y), \dot{y} = Y_m(x,y)$$

is transformed into

$$\dot{r} = r^m Z(\theta), \dot{\theta} = r^{m-1}N(\theta),$$

where the functions $Z(\theta)$ and $N(\theta)$ are given by

$$Z(\theta) = Y_m(\cos\theta, \sin\theta)\sin\theta + X_m(\cos\theta, \sin\theta)\cos\theta \tag{5.3.6}$$

$$N(\theta) = Y_m(\cos\theta, \sin\theta)\cos\theta - X_m(\cos\theta, \sin\theta)\sin\theta. \tag{5.3.7}$$

[1] It is possible, of course, to absorb the factor $\frac{1}{2(n+2)}$ defining a new time $d\tau = 2(n+2)dt$. However, nothing gained by this in terms of simplicity.

Then, the invariant rays of the system are obtainded by solving the equation $N(\theta) = 0$. Clearly, in the phase plane they will be depicted as straight semi-lines starting from the origin and it is not difficult to see that if they do exist then they are automatically solutions of the dynamical system [89]. In our case $m = 2$ and a straightforward calculation leads to

$$\begin{aligned} Z(\theta) &= \frac{1}{2(n+2)}\Big[-n(n+5)\sin^3\theta + (3n^2 - 3n - 4)\sin^2\theta\cos\theta \\ &+ (2n - 2n^3 + \frac{2}{3})\sin\theta\cos^2\theta - 2(n+1)\cos^3\theta\Big] \end{aligned} \tag{5.3.8}$$

$$\begin{aligned} N(\theta) &= \frac{1}{2(n+2)}\Big[-3n(n-1)\sin^3\theta + n(n-7)\sin^2\theta\cos\theta \\ &+ 2(n-1)\cos^2\theta\sin\theta + \frac{2}{3}\cos^3\theta\Big]. \end{aligned} \tag{5.3.9}$$

Here let us make some comments. First, we should point out that the dynamical system (5.3.4-5) is not defined for $n = 0$, since in this case we would not have equation (5.2.6). If $n = 1$, then the solutions of the equation $N(\theta) = 0$ yield six invariant rays which correspond to the angles $\theta_i = \pm\frac{\pi}{2}$ and $\arctan(\pm\frac{1}{3})$, with $i = 1, ..., 6$. For an arbitrary $n > 1$ we can put the equation (5.3.9) in the following factorized form:

$$N(\theta) = \frac{\cos^3\theta}{2n+4}\left\{(\frac{1}{3} - a)[3n(n-1)a^2 + 6na + 2]\right\} \tag{5.3.10}$$

where we have defined $a = \tan\theta$. Then, for $n > 1$ we have again six invariant rays, now corresponding to the angles $\theta_i = \arctan a_i$, with

$$a_0 = \frac{1}{3}, a_\pm = \frac{1}{n-1}\left(-1 \pm \sqrt{\frac{1}{3}(1 + \frac{2}{n})}\right)$$

See figs. 6 and 7. The knowledge of the invariant rays as well as the analytic expressions for the functions $N(\theta)$ and $Z(\theta)$ allow us to draw separately the following phase diagrams for the two cases $n = 1$ and $n > 1$ (for details see appendix). These diagrams show the behaviour of all solutions of the equations (5.3.4-5) which make up our dynamical system. Each curve corresponds to a specific cosmological model satisfying the field equations (5.3.4-5), the origin representing the Minkowski spacetime M. In order to know the behaviour of the solutions at the infinity we employed a method due to Poincare', consisting of projecting the phase plane onto a plane circle [93]. In this compactified phase plane the points at infinity correspond to points located in the border of the circle. The directions of the invariant rays are not affected by the transformation (see appendix 5.8).

5.4. The physical picture

Let us begin our analysis considering $n > 1$, and leave the comments on the case $n = 1$ to the end of this section. In figure 6 we have a typical diagram for arbitrary $n > 1$. First we note that the invariant rays divide up the phase plane in six topologically distinct regions (or sectors) A,B,...,F. Each of these regions contains an infinite number of solutions which represent cosmological models with different physical properties. The arrows in the curves are to be interpreted as the time evolution of the corresponding models.
Since there is no closed curve in the phase plane we can conclude that all models are singular (the expansion parameter x tends to infinity either in the past or in the future), some of them starting from a big-bang ($x \to +\infty$)while others collapsing to a big-crunch ($x \to -\infty$). In this sense the solutions represented by the invariant rays exhibit the same behaviour. It would be rather tedious to describe exhaustively the time evolution of the models corresponding to all the curves of the phase diagram. So, we will pick up some illustrative cases, although the complete informations about all solutions are provided by the phase portrait.

To begin with let us consider the solution represented by the invariant ray depicted in figure 61 as the semi-line I^+. This curve clearly describes a universe starting from a big-bang ($x = +\infty$) and evolving towards the Minkowski spacetime (depicted in the diagram as the fixed point M located at the origin) . Since $y > 0$ along this trajectory we see that as time goes by the gravitational constant G decreases. This is in agreement with the known hypothesis formulated by Dirac who, postulated, inspired on a different reasoning (*the large numbers conjecture*), that Newtonian gravitational constant should decrease as the Universe expands [85].

Analogously, the same analysis shows us that the invariant ray II^+ corresponds to an expanding universe starting from a big-bang and tending to Minkowski spacetime. Since y is negative in this *anti-Dirac* universe the gravitational constant G increases with the cosmic time.

The invariant rays I^+ and II^+ encloses an infinite class of solutions all lying within the region A. A typical solution of this class describes an expanding and singular universe undergoing a transition from an increasing G (*anti-Dirac*) to an decreasing G era (*Dirac phase*).

A quite different situation arises when one examines the solution corresponding to the invariant ray III^+. Here we observe an initially static universe ($x = 0$) entering an expansion regime during which the gravitational constant increases with time.
At this point it is interesting to note that one might look alternatively at the dynamics of the models corresponding to II^+ and III^+ as describing the usual cosmic expansion taking place in ordinary 4-dimensionality (here expressed by the variable x) followed by a contraction of the internal n-dimensional space (represented here by y). The sector B, which is delimited by II^+ and III^+, contains only solutions which do not approach Minkowski spacetime, neither in the future nor in the past. On the other hand, the solutions lying in sector F all tend to M and start their trajectories as contracting universes, slowing down before enter an expanding era. In this class of models the gravitational constant is an ever decreasing function of the cosmic time.

We shall not carry out a detailed analysis of the solutions lying in sectors D and E as these describe only contracting universes, *ipso facto* not being physically relevant. (As we shall see later, in section 5.6, sector E as well as sector B both represent classes of solutions with negative energy density.) In sector C a typical universe comes from Minkowski spacetime in the past and has a contracting era followed by further expansion.

In the case $n = 1$ (see figure 7) the physical picture is very similar. However, now as two of the invariant rays, namely III^+ and III^- lie exactly on the y-axis they represent vacuum flat solutions with a time-varying G.(In fact, an identical configuration has been already found in the context of Brans-Dicke theory by Romero-Barros [90]). An alternative way to look at these solutions is to consider them as a topological product of a static Minkowski spacetime by a time-dependent (expanding or contracting) compact internal space.

5.5. Exact solutions of the field equations

Often the knowledge of the invariant rays present in a homogeneous dynamical system is helpful in obtaining exact analytical solutions of the system. In that case the problem of finding the solutions corresponding to the invariant rays reduces to solving an algebraic equation of one order higher as the system itself. In our particular case we will have to solve a cubic polynomial equation, the roots of which are nothing more than the already known tangents a_i of the arcs defined by the invariant rays. Let us express the equations of the invariant rays simply by $y = ax$, where clearly a generically denotes a_i. Now, putting this into the equations (5.3.4-5) we get

$$\dot{x} = \frac{x^2}{2(n+2)}\left[-2(n+1) + 2n(1-n)a + 3n(n-1)a^2\right] \tag{5.5.1}$$

$$\dot{y} = a\dot{x} = \frac{x^2}{2(n+2)}\left[\frac{2}{3} - 4a - n(n+5)a^2\right] \tag{5.5.2}$$

The condition for (5.5.1) and (5.5.2) to be consistent is the algebraic equation

$$3n(n-1)a^3 + n(7-n)a^2 + 2(1-n)a - \frac{2}{3} = 0 \tag{5.5.3}$$

which is, in fact, equivalent to eq.(5.3.9). Again, we have to consider the two cases a) $n > 1$ and b) $n = 1$:

a) If $n > 1$ then the roots of (5.5.3) are given by

$$a_0 = 1/3, a_\pm = \frac{1}{n-1}\left[-1 \pm \sqrt{\frac{1}{3}(1+\frac{2}{n})}\right]$$

Now, going back to equation (5.3.4) and putting $y = ax$, with $a = a_0, a_\pm$, we get respectively:

$$\dot{x} = \gamma x^2 \tag{5.5.4}$$

where $\gamma = \gamma_0, \gamma_\pm$ and

$$\gamma_0 = -\frac{(n+3)}{6} \tag{5.5.5}$$

$$\gamma_\pm = -(1 + na_\pm) \tag{5.5.6}$$

These last equations can be immediately integrated to give $R(t)$ and $b(t)$.Then, corresponding to the three values of $a = a_0, a_\pm$ we have respectively (after suitable coordinate transformations):

$$R(t) \sim t^{-\frac{1}{3\gamma_0}} = R_0 t^{\frac{2}{n+3}} \tag{5.5.7}$$

$$b(t) \sim [R(t)]^{3a_0} = b_0 t^{\frac{2}{n+3}} \tag{5.5.8}$$

$$R(t) \sim t^{\frac{-1}{3\gamma_\pm}} = R_0 t^{\frac{-1}{3(1+na_\pm)}} \tag{5.5.9}$$

$$b(t) \sim [R(t)]^{3a_\pm} = b_0 t^{\frac{a_\pm}{1+na_\pm}} \tag{5.5.10}$$

where R_0 and b_0 are constants (see also 12).
b) If $n = 1$ then the equation (5.5.3) has two solutions, namely, $a = \pm\frac{1}{3}$. Naturally, these solutions correspond to the invariant rays defined by $\theta_i = \arctan \pm\frac{1}{3}$ in section 5.3. The third solution, corresponding to the other invariant rays, $\theta_i = \pm\frac{\pi}{2}$ can be obtained directly from the dynamical system (eqs.(5.3.4-5)) just putting $n = 1$ and $x = 0$. This procedure leads us back to the static solution referred earlier in section 5.4:

$$R(t) = constant, \tag{5.5.11}$$

$$b(t) = b_0 t \tag{5.5.12}$$

The other solutions are:

$$R(t) = R_0 t^{\frac{1}{4}} \tag{5.5.13}$$

$$b(t) = b_0 t^{\frac{1}{4}} \tag{5.5.14}$$

$$R(t) = R_0 t^{\frac{1}{2}} \tag{5.5.15}$$

$$b(t) = b_0 t^{-\frac{1}{2}}. \tag{5.5.16}$$

We conclude this section by noting that equations (5.5.7-16) actually represent six distinct pair of solutions $R(t)$, $b(t)$, each being singular at $t = 0$. Indeed, after integrating (5.5.4) we obtain (apart from a constant of integration which can be further eliminated by a coordinate transformation)

$$x = -\frac{1}{\gamma t}, \tag{5.5.17}$$

which,in fact, has to be understood as representing different solutions (for the same γ) according to $t \in (-\infty, 0)$ or $t \in (0, +\infty)$. In the phase diagrams these *twofold degeneracy* is reflected by the presence of distinct solutions (including the equilibrium point M) all lying on the same line $y = ax$. Finally, we should mention that if $n = 0$ in (5.5.7) we recover Friedmann's solution for a dust filled universe.

5.6. The energy density

So far we have not been concerned with the energy density predicted by the models. A brief look into the field equations shows us that ϱ must be given by

$$\varrho = \frac{1}{6\kappa^2}\left[2x^2 + 3n(n-1)y^2 + 6nxy\right]. \tag{5.6.1}$$

If $n > 1$ the above equation however can be put into the factorised form :

$$\varrho = \frac{1}{6\kappa^2}(y - a_+ x)(y - a_- x), \tag{5.6.2}$$

with $a_\pm$ as defined in section 5.5. This last equation allows us to draw the following conclusions:

i) For $n > 1$ we verify that the solutions lying on the invariant rays corresponding to $a_\pm$ are vacuum solutions.

ii) All solutions lying on the sector B and F are non-physical (in the sense that they have negative energy, which classically is forbidden). Incidentally, these are the only solutions which never tend to Minkowski spacetime neither in the past nor in the future.

iii) Solutions lying on the invariant ray corresponding to a_0 have positive energy density for arbitrary value of $n > 1$. This can be easily verified by computing ϱ for this case as we have $\varrho = \frac{s^2}{36 a^3} \left[2n^2 + n + 12\right]$.

All the properties mentioned above are ilustrated in figure 8. [2]

For $n = 1$ the same procedure leads to the picture displayed by fig. 9.

5.7. Conclusions

The idea that the Newtonian constant of gravitation G could indeed vary with time on a cosmic scale, which seems to have ocurred first to Dirac, in 1938, is far from being supported by current experimental data. Recent results [91] based on solar-system experiments tend to indicate an upper limit given by $\left|\dot{G}/G\right| < 10^{-12}$ to any possible variation of G. Yet even this rather stringent condition has not prevented cosmologists to speculate and investigate what theoretical consequences would such hypothesis lead to (for a list of references on past and recent works see [12,21,50,86,92]). Among other attempts to insert G in gravity theories as a scalar field (e.g. , Brans-Dicke-Jordan theories), is the multidimensional cosmology approach [21] which was described in section 5.2. The fact that in this scheme the field equations plus some symmetry assumptions may be tractable by mathematical techniques of dynamical system theory led us to obtain a whole spectrum of cosmic configurations where the matter of the Universe is regarded as a multicomponent perfect fluid in higher dimensions. It turns out that in this scheme some solutions exhibit a non-physical behaviour (at least from a classical standpoint). However, other solutions seem not to be in contradiction with generally accepted and standard models of the Universe, as they manifest properties such as cosmic expansion and the existence of an initial singularity. Also, in some of these expanding solutions the gravitational constant G decreases with time, a property which may justify calling them *Dirac universes* (we detect the presence of *anti-Dirac* models as well). Evidently, it was not our aim here to provide a quantitative discussion of the solutions, even of the more physically relevant ones, trying to square them in the context of present observational and experimental data. Rather, our interest in this paper was actually to call the attention of theorists for the extremely rich scenario which arises when one allows for higher dimensionality and the varying gravitational constant hypothesis.

[2] One could argue that it is not exactly ϱ , but ${}^{(4)}\varrho$ the physical quantity which would be actually measured. However, from equation (5.2.7) we see that all that has been said in this section of ϱ is also true for ${}^{(4)}\varrho$.

5.8. Appendix

In order to construct the phase diagrams corresponding to the figures 6 and 7 all we need is to calculate the values of the functions $N^l(\theta)$, and $Z(\theta)$ at $\theta = \theta_i$, where θ_i is an invariant ray and the superscript l refers to the first non-vanishing derivative evaluated at θ_i [89]. Since the system is quadratic the phase portraits are symmetric by plane reflections ($x \to -x$, $y \to -y$), although the time orientation of the curves must be reversed in this operation. Such property means we only need carrying out our analysis in the neighbourhood of just three of the six invariant rays. Then, let us summarise the results which come from straightforward calculations.

For both cases $n > 1$ and $n = 1$, we obtain the following:

$l = 1$, $N^1(\theta_1) < 0$, $N^1(\theta_2) < 0$, $N^1(\theta_3) > 0$, $Z(\theta_1) < 0$, $Z(\theta_2) < 0$, and $Z(\theta_3) > 0$; where for the case $n > 1$ the invariant rays are: $\theta_1 = \arctan \frac{1}{3}$, $\theta_2 = \arctan a_+$, $\theta_3 = \arctan a_-$, whereas for the case $n = 1$, $\theta_1 = \arctan +\frac{1}{3}$, $\theta_2 = \arctan -\frac{1}{3}$ and $\theta_3 = -\frac{\pi}{2}$. With these results we can classify for arbitrary values of n the invariant rays θ_1 and θ_2 as being of type (β), while θ_3 is of type (α) [89]. From this classification we are led to the diagrams displayed in figs. 6 and 7.

To carry out the Poincare' compactification of phase plane we perform the transformations of variables $u = \frac{y}{x}$ and $z = \frac{1}{x}$. Then, starting from the equations (5.5.1) and (5.5.2), we end up with the dynamical system:

$$\frac{du}{d\tau} = \frac{1}{2(n+2)} \left[(\frac{1}{3} - u) \left(3n(n-1)u^2 + 6nu + 2 \right) \right] \tag{5.8.1}$$

$$\frac{du}{d\tau} = \frac{z}{2(n+2)} \left[2(n+1) + 2n(n-1)u + 3n(1-n)u^2 \right], \tag{5.8.2}$$

where $z d\tau = dt$. The equilibrium points of the dynamical system in the plane uz are: $(1/3, 0), (u_\pm, 0)$, with $u_\pm = a_\pm$. A simple analysis of the topological character of these points reveals that they correspond to a saddle-point and two nodes (unstable and stable), respectively [93].

6. Bulk Viscosity and Entropy Production in Multidimensional Integrable Cosmology

6.1. Introduction

Up till now we studied different properties of multidimensional cosmology using the matter source of multidimensional Einstein equations in the form of the perfect fluid [37-38]. But, of course, more realistic may be the model which incorporates some viscosity effects. Within 4-dimensional cosmology the viscous Universe was considered by a number of authors from quite different points of view. Without carrying of a detailed review of the subject (extensive review was given by Gron [94]), we mention some main trens in cosmology with viscous fluid as a source.

First, Misner [95] considered neutrino viscosity as a mechanism for reducing the anisotropy in the Early Universe. Stewart [96] and Collins and Stewart [97] proved that it is possible only if initial anisotropies are small enough. Another series of papers was started by Weinberg [98] which concerns the production of entropy in the viscous Universe. Both isotropization and production of entropy during lepton era in models of Bianchi types I,V were considered by Klimek [99]. Caderni and Fabbri [100] calculated coefficients of shear and bulk viscosity in plasma and lepton eras within the model of Bianchi type I. The next trend is connected with obtaining of singularity free viscous solutions. The first nonsingular solution was obtained by Murphy [101] within flat Friedman-Robertson-Walker model with fluid possessing a bulk viscosity. Murphy supposed that the coefficient of a bulk viscosity is proportional to the density of a fluid. However, Belinsky and Khalatnikov [102,103] showed that this solution corresponds to the very peculiar choice of parameters and is unstable with respect to the anisotropy perturbations. Other nonsingular solutions with bulk viscosity were obtained by Novello and Araújo [104], Romero [105], Oliveira and Salim [106].

In this section we study the multidimensional cosmological model with a chain of Ricci-flat spaces for the source in the form of a fluid possessing bulk viscosity. In section 6.2 we describe the model and get basic equations. For their integration we develop some vector formalism proposed in our previous papers. In section 6.3 we summarize thermodynamics in multidimensional cosmology and obtain the formula for the rate of change of entropy. In section 6.4 we integrate equations of motion for special set of parameters in the first and second equations of state. Exact solutions are presented in the Kasner-like form and their properties are studied.

6.2. The model

As in previous sections we consider here a multidimensional cosmological model with the metric (1.1.1) defined on the D-dimensional manifold (1.1.2). We consider only Ricci-flat spaces $M_1, \dots, M_n$, i.e.

$$R_{m_i l_i}[g^{(i)}] = 0, \quad m_i, l_i = 1, \dots, N_i. \tag{6.2.1}$$

It is easy to obtain in the usual way the following non-zero components of the Ricci-tensor for the metric (1.1.1)

$$R_0^0 = e^{-2\gamma(t)} \left(\sum_{i=1}^{n} N_i (\dot{x}^i)^2 + \ddot{\gamma}_0 - \dot{\gamma}\dot{\gamma}_0 \right), \tag{6.2.2}$$

$$R_{k_i}^{m_i} = e^{-2\gamma(t)} \left(\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i \right) \delta_{k_i}^{m_i}, \tag{6.2.3}$$

where we denoted $\gamma_0 = \sum_{i=1}^{n} N_i x^i$. Indices m_i and k_i run over from $D - \sum_{j=i}^{n} N_j$ to $D - \sum_{j=i}^{n} N_j + N_i$ for $i = 1, \ldots, n$ $(D = 1 + \sum_{i=1}^{n} N_i = \mathrm{dimM})$.

We take the energy-momentum tensor for a viscous fluid in the standard form (without shear)

$$T_B^A = \rho u^A u_B + (p - \zeta\theta) P_B^A, \tag{6.2.4}$$

where ρ and p are the fluid density and the pressure, respectively, ζ is the bulk viscosity coefficient. Vector u^A is the D-dimensional velocity of a fluid and $P_B^A = \delta_B^A + u^A u_B$ is the projector on the $(D-1)$-dimensional space orthogonal to u^A. By θ we denote the scalar expansion $\theta = u^A{}_{;A}$.

We impose the comoving observer condition for the D-dimensional velocity: $u^A = \delta_0^A e^{-\gamma(t)}$. Then

$$(u^A u_B) = \mathrm{diag}(-1, 0, \ldots, 0), \tag{6.2.5}$$

$$(P_B^A) = \mathrm{diag}(0, 1, \ldots, 1), \tag{6.2.6}$$

$$\theta = \dot{\gamma}_0 e^{-\gamma(t)}. \tag{6.2.7}$$

Let us remark that the function $\gamma(t)$ in (1.1.1) determines a time gauge for the comoving observer. We have the harmonic time gauge for $\gamma(t) = \gamma_0$ and the proper time gauge for $\gamma(t) = 0$. Harmonic time t and proper time τ are connected by $d\tau = \exp[\gamma_0] dt$.

We admit that the pressure and the bulk viscosity term in (6.2.4) are anisotropic with respect to the whole space $M_1 \times \ldots \times M_n$. Such an admission leads to the following generalisation of the expression (6.2.4)

$$(T_B^A) = \mathrm{diag}(-\rho, (p_1 - \theta\zeta_1)\delta_{k_1}^{m_1}, \ldots, (p_n - \theta\zeta_n)\delta_{k_n}^{m_n}), \tag{6.2.8}$$

where p_i and ζ_i are the pressure and the bulk viscosity coefficient in the space M_i. Furthermore, we suppose that the barotropic equations of state holds

$$p_i = (1 - h_i)\rho(t), \tag{6.2.9}$$

where h_i=const for $i = 1, \ldots, n$.

It is easy to show that the equation of motion $\nabla_M T_0^M = 0$ for the viscous fluid with the tensor (6.2.8) looks as follows

$$\dot{\rho} + \sum_{i=1}^{n} N_i \dot{x}^i (\rho + p_i - \zeta_i \theta) = 0. \tag{6.2.10}$$

The Einstein equations $R^A_B - \frac{1}{2}\delta^A_B R = \kappa^2 T^A_B$ (κ^2 is gravitational constant) may be written as $R^A_B = \kappa^2(T^A_B - \frac{T}{D-2}\delta^A_B)$. Further, we employ the equation $R^0_0 - \frac{1}{2}\delta^0_0 R = \kappa^2 T^0_0$ and the equations $R^{m_i}_{k_i} = \kappa^2(T^{m_i}_{k_i} - \frac{T}{D-2}\delta^{m_i}_{k_i})$. Using (6.2.2), (6.2.3) and (6.2.8) we get

$$\sum_{i=1}^{n} N_i(\dot{x}^i)^2 - \dot{\gamma}_0^2 = -2\kappa^2 e^{2\gamma}\rho, \tag{6.2.11}$$

$$\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i = \kappa^2 \left[\left(-h_i + \frac{\sum_{k=1}^{n} N_k h_k}{D-2} \right) \rho e^{2\gamma} + \left(-\zeta_i + \frac{\sum_{k=1}^{n} N_k \zeta_k}{D-2} \right) \dot{\gamma}_0 e^{\gamma} \right]. \tag{6.2.12}$$

To develop the integration procedure for the equations of motion (6.2.11),(6.2.12) we introduce the n-dimensional real vector space R^n. By $e_1, \ldots, e_n$ we denote the canonical basis in R^n, i.e. $e_1 = (1, 0, \ldots, 0)$ etc.

Let $< ., . >$ be a symmetric bilinear form defined on R^n, such that

$$< e_i, e_j >= \delta_{ij} N_j - N_i N_j \equiv G_{ij}. \tag{6.2.13}$$

In our previous papers this form was introduced as a minisuperspace metric for the cosmological models. It was shown that it is a nongenerate form with the pseudo-Euclidean signature $(-, +, \ldots, +)$. So, for vectors $a = a^1 e_1 + \ldots + a^n e_n$ and $b = b^1 e_1 + \ldots + b^n e_n$ we have

$$< a, b >= \sum_{i,j=1}^{n} G_{ij} a^i b^j. \tag{6.2.14}$$

The form $< a, b >$ may be also written as

$$< a, b >= \sum_{i=1}^{n} a_i b^i = \sum_{i=1}^{n} a^i b_i = \sum_{i,j=1}^{n} G^{ij} a_i b_j, \tag{6.2.15}$$

if we introduce the covariant components of vectors by

$$a_i = \sum_{j=1}^{n} G_{ij} a^j. \tag{6.2.16}$$

By $G^{ij} = \delta^{ij}/N_i + 1/(2 - D)$ we denote components of a matrix inverse to (G_{ij}).

We call a vector $y \in R^n$ time-like, space-like or isotropic, if $< y, y >$ takes negative, positive or null values, respectively. Vectors y and z are called orthogonal if $< y, z >= 0$.

In our model the following vectors are used

$$x = x^1 e_1 + \ldots + x^n e_n, \tag{6.2.17}$$

$$u = u^1 e_1 + \ldots + u^n e_n, \quad u^i = h_i - \frac{\sum_{k=1}^n N_k h_k}{D-2}, \quad u_i = N_i h_i \tag{6.2.18}$$

$$\xi = \xi^1 e_1 + \ldots + \xi^n e_n, \quad \xi^i = \zeta_i - \frac{\sum_{k=1}^n N_k \zeta_k}{D-2}, \quad \xi_i = N_i \zeta_i. \tag{6.2.19}$$

If $h_i = 1$ for $i = 1, \ldots, n$, we have dust in the whole space ($p_i = 0$, see (6.2.9)). The vector (6.2.18) corresponding to dust in the whole space is denoted by u_d. We note that

$$(u_d)_i = N_i, \quad u_d^i = \frac{-1}{D-2}, \quad < u_d, u_d > = -\frac{D-1}{D-2}, \quad < u_d, x > = \gamma_0. \tag{6.2.20}$$

Thus, using (6.2.14), (6.2.17)-(6.2.19) we obtain the Einstein equations in the form

$$< \dot{x}, \dot{x} > = -2\kappa^2 e^{2\gamma} \rho, \tag{6.2.21}$$

$$\ddot{x} + (< u_d, \dot{x} > - \dot{\gamma})\dot{x} = -\kappa^2 \left(\rho e^{2\gamma} u + < u_d, \dot{x} > e^{\gamma} \xi \right). \tag{6.2.22}$$

The equation of motion (6.2.10) can be written as

$$\dot{\rho} + \rho < 2u_d - u, \dot{x} > - e^{-\gamma} < u_d, \dot{x} >< \xi, \dot{x} > = 0. \tag{6.2.23}$$

Excluding the density ρ from (6.2.22) by (6.2.21) we get the following equation

$$\ddot{x} + (< u_d, \dot{x} > - \dot{\gamma})\dot{x} = \frac{1}{2} < \dot{x}, \dot{x} > u - \kappa^2 < u_d, \dot{x} > e^{\gamma} \xi. \tag{6.2.24}$$

To integrate (6.2.24) we need a second equation of state for the bulk viscosity coefficients ζ_i. To obtain an exact solution in a 4-dimensional flat Friedman-Robertson-Walker model with bulk viscosity Murphy [101] used the second equation of state of the form $\zeta = \mathrm{const}\,\rho$. Belinsky and Khalatnikov [107] studied the qualitative behavior of this model with a more general equation: $\zeta = \alpha \rho^{\nu}$, where α, ν=const. It is easy to show that for this model on manifold $R \times M_1^3$ for $\gamma(t) = 0$ the set of equations (6.2.23),(6.2.24) may be written as

$$3H^2 = \kappa^2 \rho, \tag{6.2.25}$$

$$\dot{H} = \frac{\alpha}{2} 3^{\nu+1} H^{2\nu+1} + \frac{3}{2}(h-2)H^2, \tag{6.2.26}$$

where H is the Hubble parameter of the 3-dimensional Ricci-flat manifold M_1^3, i.e. $H = \dot{x}^1$. The set of equations (6.2.25)-(6.2.26) coincides with the one obtained by Belinsky and Khalatnikov [107]. It is easy to see that equation (6.2.26) for H is always integrable by quadrature. In the simplest case with $\nu = 1$ we get the exact solution obtained by Murphy [101]. Other solutions for special parameters ν and h and a solution for arbitrary ν and h were also obtained (see [94] for details).

For multidimensional cosmological model with manifold $M = R \times M_1 \times \ldots \times M_n$ the set of equations (6.2.21)-(6.2.22) is more complicated. Obviously , we have the set of nonlinear differential equations (6.2.24) for scale factors $\exp[x^i]$ of the spaces $M_1, \ldots, M_n$. If we adopt Belinsky and Khalatnikov's condition: $\zeta \sim \rho^\nu$, then rather complicated equations arise. In particular, for $\nu = 1$ Appel and Ricatti equations appear. Chakraborty and Nandy [108] within a 5-dimensional model with manifold $R \times M_1^3 \times S_2^1$ avoided this difficulty by imposing an additional constraint for the scale factors: $\exp[x^2] = \mu \exp[\omega x^1]$, $\mu, \nu =$const.

Here, with no loss of generality, we consider an integration of the set of equations (6.2.22) for another second equation of state.We suppose that the bulk viscosity coefficient ζ_i corresponding to the space M_i is proportional to $\exp[-\gamma_0]$, i.e.

$$\zeta_i \sim [\text{scale factor of } \mathrm{M}_1]^{-\dim \mathrm{M}_1} \cdot \ldots \cdot [\text{scale factor of } \mathrm{M}_n]^{-\dim \mathrm{M}_n}. \qquad (6.2.27)$$

Physically, the assumption (6.2.27) means that the expansion of the spaces $M_1, \ldots, M_n$ is accompanied by a decreasing of the bulk viscosity effect.

Let us notice that the metric dependence of the bulk viscosity coefficient was also considered by other authors. Lukacs [109] integrated the homogeneous and isotropic 4-dimensional model with a viscous dust for such second equation of state: $\zeta = \text{const}[\text{scale factor}]^{-1}$. Curvature-dependent bulk viscosity was studied in a multidimensional cosmology by Wolf [110]. Recently Motta and Tomimura [111] studied a 4-dimensional inhomogeneous cosmology with some metric dependence of the bulk viscosity coefficient.

6.3. Thermodynamics of viscous fluid in multidimensional Universe

We first summarize thermodynamics in multidimensional cosmology on the manifold $M = R \times M_1 \times \ldots \times M_n$ following papers [112,113]. The first law of thermodynamics can be written as follows

$$TdS = d(\rho V) + V \sum_{i=1}^{n} p_i \frac{dV_i}{V_i}, \qquad (6.3.1)$$

where V_i is any fluid volume in the space M_i, V is a fluid volume in the whole space: $V = V_1 \cdot \ldots \cdot V_n$ and S is an entropy in the volume V. We suppose the conservation law for the baryon particle number N_B in volume V. Then, for entropy per baryon $s = S/N_b$ and baryon number density $n = N_b/V$ we obtain from (6.3.1)

$$nT\dot{s} = \dot{\rho} + \rho \sum_{i=1}^{n} N_i \dot{x}^i + \sum_{i=1}^{n} p_i N_i \dot{x}^i, \qquad (6.3.2)$$

We remind that $\exp[x^i]$ is the scale factor of the space M_i of the dimension N_i.

For the perfect fluid ($\zeta_i = 0$) comparing (6.3.2) and equation of motion (6.2.10) we get the conservation of entropy : $s =$const and by the barotropic equations of state (6.2.9) the integral of motion

$$\rho \exp[\sum_{i=1}^{n}(2 - h_i)N_i x^i] = \text{const}. \tag{6.3.3}$$

Temperature of the perfect fluid can be obtained in such a way [113]. From (6.3.2) we have

$$(\frac{d\rho}{dx^i})_{s,x^j} = -\rho N_i - p_i N_i = (h_i - 2)N_i\rho, \quad j \neq i. \tag{6.3.4}$$

Then

$$\rho = K(s)\exp[\sum_{i=1}^{n}(h_i - 2)N_i x^i], \tag{6.3.5}$$

where $K(s)$ is an unknown function of the entropy s. By inverting (6.3.5) we get

$$s = s(\rho\exp[\sum_{i=1}^{n}(2 - h_i)N_i x^i]). \tag{6.3.6}$$

Substituting (6.3.6) for s in (6.3.2), we obtain

$$nT\frac{ds}{d(\rho\exp[(2 - h_i)N_i x^i])} = \exp[\sum_{i=1}^{n}(h_i - 2)N_i x^i]. \tag{6.3.7}$$

For the perfect fluid we have $ds/d(\rho\exp[2 - h_i)N_i x^i]) = B = \text{const}$ (see (6.3.3)), then

$$nT = \frac{1}{B}\exp[\sum_{i=1}^{n}(h_i - 2)N_i x^i] = \frac{1}{B}\exp[< u - 2u_d, x >]. \tag{6.3.8}$$

Now we consider the fluid with a bulk viscosity. Comparing (6.2.10) and (6.3.2) we obtain

$$nT\dot{s} = \theta\sum_{i=1}^{n} N_i\zeta_i\dot{x}^i. \tag{6.3.9}$$

Using (6.2.7),(6.2.14),(6.2.17)and (6.2.19) we get

$$\dot{s} = \frac{\dot{\gamma}_0 e^{-\gamma}}{nT}\sum_{i=1}^{n} N_i\zeta_i\dot{x}^i = \frac{e^{-\gamma}}{nT} < u_d, \dot{x} >< \xi, \dot{x} > . \tag{6.3.10}$$

This formula gives the rate of change of entropy per baryon in multidimensional cosmology on the manifold $M = R \times M_1 \times \ldots \times M_n$ with anisotropic bulk viscosity. The production of entropy in the model can be calculated if the temperature of a fluid is known. Further, we suppose that the temperature is given by the perfect fluid formula (6.3.8). Then we get

$$\dot{s} = B\exp[< 2u_d - u, x > -\gamma] < u_d, \dot{x} >< \xi, \dot{x} > . \tag{6.3.11}$$

6.4. Exact solutions

In this section we consider only the model with identical pressures and identical bulk viscosity coefficients in each space M_i, i.e.

$$p_i = (1-h)\rho \quad \text{or} \quad u = hu_d, \tag{6.4.1}$$
$$\zeta_i = \frac{\zeta_0}{\kappa^2} e^{-\gamma_0} \ \text{ or } \ \xi = \frac{\zeta_0}{\kappa^2} e^{-\gamma_0} u_d, \ \ i = 1, \ldots, n, \tag{6.4.2}$$

where ζ_0 and h are constants. Here we suppose that

$$h > 0, \qquad \zeta_0 > 0. \tag{6.4.3}$$

Then, the set of equations (6.2.24) in the harmonic time gauge ($\gamma = \gamma_0$) looks as follows

$$\ddot{x} = \frac{h}{2} < \dot{x}, \dot{x} > u_d - \zeta_0 < u_d, \dot{x} > u_d. \tag{6.4.4}$$

(We remind that $\gamma_0 =< u_d, x >$.) To integrate (6.4.4) we use the following decomposition of the vector x

$$x =< u_d, x > \frac{u_d}{< u_d, u_d >} + \sum_{i=2}^{n} < e'_i, x > e'_i. \tag{6.4.5}$$

The vectors $u_d, e'_2, \ldots, e'_n$ form an orthogonal basis in R^n, i.e.

$$< u_d, e'_i >= 0, \ \ < e'_i, e'_j >= \delta_{ij}, \ \ i, j = 2, \ldots, n. \tag{6.4.6}$$

We notice that in this basis any vector e_i can not be time-like or isotropic because the vector u_d is time-like. The set of equations (6.4.4) may be written as

$$< u_d, \ddot{x} >=< u_d, u_d > \left[\frac{h}{2}\left(\frac{< u_d, \dot{x} >^2}{< u_d, u_d >} + \sum_{i=2}^{n} < e'_i, \dot{x} >^2\right) - \zeta_0 < u_d, \dot{x} >\right], \tag{6.4.7}$$
$$< e'_i, \ddot{x} >= 0, \ \ i = 2, \ldots, n. \tag{6.4.8}$$

Integration of (6.4.8) leads to the results

$$< e'_i, x >= p^i t + q^i, \ \ i = 2, \ldots, n, \tag{6.4.9}$$

where p^i and q^i are arbitrary constants. To present the scale factors $\exp[x^i]$ in a Kasner-like form, we introduce the vectors $\alpha, \beta \in R^n$

$$\alpha = p^2 e'_2 + \ldots + p^n e'_n \equiv \alpha^1 e_1 + \ldots + \alpha^n e_n, \tag{6.4.10}$$
$$\beta = q^2 e'_2 + \ldots + q^n e'_n \equiv \beta^1 e_1 + \ldots + \beta^n e_n. \tag{6.4.11}$$

We remind that the vectors $e_1, \ldots, e_n$ form the canonical basis in R^n. The coordinates α^i and β^i are the Kasner-like parameters. Integration of (6.4.7) results in

$$< u_d, x >= -\frac{1}{h}\ln[Cf^2] + \frac{\zeta_0}{h} < u_d, u_d > t, \tag{6.4.12}$$

where $C > 0$ is an integration constant.

Using (6.4.5),(6.4.9)-(6.4.12) we obtain the exact solution in the Kasner-like form

$$e^{x^i} = (Cf^2)^{-\frac{1}{h(D-1)}} \exp[(\alpha^i - \frac{\zeta_0}{h(D-2)})t + \beta_i]. \tag{6.4.13}$$

The Kasner-like parameters obey the relations

$$< \alpha, u_d >= \sum_{i=1}^{n} \alpha^i N_i = 0, \quad < \beta, u_d >= \sum_{i=1}^{n} \beta_i N_i = 0, \tag{6.4.14}$$

$$< \alpha, \alpha >= \sum_{i=1}^{n} (\alpha^i)^2 N_i = \sum_{j=2}^{n} (p^j)^2. \tag{6.4.15}$$

Using (6.2.21) we obtain the density

$$\rho = \frac{a^2 + < \alpha, \alpha >}{2\kappa^2}(Cf^2)^{\frac{1}{2}} \exp[\frac{2a^2 h}{\zeta_0}t] \left(F + \frac{a - \sqrt{< \alpha, \alpha >}}{\sqrt{a^2 + < \alpha, \alpha >}}\right) \left(F + \frac{a + \sqrt{< \alpha, \alpha >}}{\sqrt{a^2 + < \alpha, \alpha >}}\right). \tag{6.4.16}$$

For the functions f and F in (6.4.12),(6.4.13) and (6.4.16) we have the following variants

$$f = \sinh[Ah(t - t_0)/2], \quad F = \coth[Ah(t - t_0)/2], \quad C > 0, \tag{6.4.17}$$

$$f = \cosh[Ah(t - t_0)/2], \quad F = \tanh[Ah(t - t_0)/2], \quad C > 0, \tag{6.4.18}$$

$$f = \exp[Ah(t - t_0)/2], \quad F = 1, \quad C = \exp[-\frac{\zeta_0(D-1)}{D-2}t_0], \tag{6.4.19}$$

$$f = \exp[-Ah(t - t_0)/2], \quad F = -1, \quad C = \exp[-\frac{\zeta_0(D-1)}{D-2}t_0]. \tag{6.4.20}$$

Constants A and a are such that

$$a = \frac{\zeta_0}{h}\sqrt{\frac{D-1}{D-2}}, \quad A = \frac{D-1}{D-2}\sqrt{\frac{\zeta_0^2}{h^2} + \frac{D-2}{D-1} < \alpha, \alpha >} \tag{6.4.21}$$

Using (6.3.11) we obtain the rate of change of entropy per baryon in this model

$$\dot{s} = \frac{B}{\kappa^2}\zeta_0 C f^2 \exp[\zeta_0 \frac{D-1}{D-2}t] \left(\frac{\zeta_0}{h}\frac{D-1}{D-2} + AF\right). \tag{6.4.22}$$

Let us consider the properties of this model. Further we consider only solutions with

$$< \alpha, \alpha >> 0. \tag{6.4.23}$$

Condition $<\alpha,\alpha>=0$ means that all Kasner-like parameters are zero, then the identical dynamics follows for all spaces $M_1,\ldots,M_n$. Such solutions in the framework of multidimensional cosmology are out of interest. Indeed, the observable distinction between external and internal dimensions demands the stage of various dynamics for the external and internal spaces. In this connection the solutions with expansion of the 3-dimensional external space and simultaneous contraction of the internal space (or spaces) are mostly attractive.

Also we suppose the weak energy condition for the solutions obtained, i.e. $\rho(\tau)\geq 0$ for any proper time τ. It is not hard to prove that only solutions with $f=\exp[Ah(t-t_0)/2]$ and $f=\sinh[Ah(t-t_0)/2]$ satisfy the weak energy condition under the condition (6.4.23).

We first consider the properties of the solution with $f=\exp[Ah(t-t_0)/2]$. In the proper time τ it can be written as follows

$$e^{x^i(\tau)}=e^{\bar\beta^i}\left(\frac{\tau_0-\tau}{T_0}\right)^{1/(D-1)-T_0\alpha^i}, \quad \tau<\tau_0, \tag{6.4.24}$$

$$\rho(\tau)=\frac{\zeta_0 T_0}{\kappa^2 h}\frac{1}{(\tau_0-\tau)^2}, \tag{6.4.25}$$

where τ_0 is arbitrary constant and parameters $\bar\beta^i$ obey the relations (6.4.14). For constant T_0 we have

$$\frac{1}{T_0}=\frac{D-1}{D-2}\left(\frac{\zeta_0}{h}+\sqrt{\frac{\zeta_0^2}{h^2}+\frac{D-2}{D-1}<\alpha,\alpha>}\right). \tag{6.4.26}$$

The formula (6.4.22) for the rate of change of entropy per baryon is easily integrable in this case

$$s(\tau)=s(-\infty)+\frac{B\zeta_0}{\kappa^2 T_0 h}\left(\frac{T_0}{\tau_0-\tau}\right)^h. \tag{6.4.27}$$

It is evident from (6.4.25) that this solution is singular at the final point of evolution $\tau=\tau_0$, because $\rho(\tau)\to+\infty$ as $\tau\to\tau_0-0$. We also notice that $\rho(\tau)\to 0$ as $\tau\to-\infty$, so this solution can be interpreted as that describing creation of matter in the Universe.

The entropy per baryon $s(\tau)$ under the conditions (6.4.3) is monotonically increasing to infinity function on the interval $(-\infty,\tau_0)$. Existence of the solutions with similar unbounded production of entropy at the final stage of evolution within 4-dimensional viscous models of Bianchi types I,IX with the second equation of state $\zeta=\alpha\rho^\nu$ was proved by Belinsky and Khalatnikov [107]. Such solutions can be considered in connection with the problem of extremely large entropy per baryon in the present Universe. Indeed, it is evident that such solutions (multidimensional or not) are applicable up to some proper time τ_c. From the time τ_c other equations of state are valid, then the evolution of the Universe is described by another model. However, it is possible that on reaching the time τ_c the entropy per baryon (6.4.27) is large enough (see fig.10).

It is also worth noticing, that this solution describes contraction of at least one space of $M_1, \ldots, M_n$. Indeed, due to the relations (6.4.14) at least one of the Kasner-like parameters is non-positive, so the corresponding scale factor monotonically decreases on the interval $(-\infty, \tau_0)$. This process can be interpreted as contraction of the internal space (or spaces) to the Planck scale ($10^{-33}cm.$). In fact the unbounded production of entropy arises due to the necessary contraction of part of the spaces, which we interpret as internal. Moreover, it can be shown that for some set of Kasner-like parameters the solution describes expansion of one part of spaces and simultaneous contraction of the other part.

Let us consider this property for a simplest model on the manifold $R \times R^3 \times T^d$, where R^3 is a 3-dimensional flat external space and T^d is an internal space having the shape of d-dimensional torus. The exact solution (6.4.24) gives

$$e^{x^1(\tau)} = e^{\tilde{\beta}^1} \left(\frac{\tau_0 - \tau}{T_0}\right)^{1/(d+3) - T_0 \alpha^1}, \tag{6.4.28}$$

$$e^{x^2(\tau)} = \exp[-\frac{3}{d}\tilde{\beta}^1] \left(\frac{\tau_0 - \tau}{T_0}\right)^{1/(d+3) + \frac{3}{d} T_0 \alpha^1}, \tag{6.4.29}$$

where

$$\frac{1}{T_0} = \frac{d+3}{d+2} \left(\frac{\zeta_0}{h} + \sqrt{\frac{\zeta_0^2}{h^2} + 3\frac{d+2}{d}(\alpha^1)^2} \right), \tag{6.4.30}$$

τ_0, $\tilde{\beta}^1$ and α^1 are arbitrary constants. If $\alpha^1 > 0$ then the internal space monotonically contracts. It is not difficult to show that under the condition

$$\frac{(d+3)(d-1)}{d} \alpha^1 > 2\frac{\zeta_0}{h} \tag{6.4.31}$$

we obtain the monotonic expansion of the external space on the interval $(-\infty, \tau_0)$ (see fig. 11). This condition can be satisfied for $d \geq 2$.

Let us suppose that the solution (6.4.28),(6.4.29) describes the evolution of the multidimensional Universe on the time interval $(\tau_0 - T_0, \tau_c)$. Also we put $s(-\infty) = 0$ in (6.4.27). Then under the condition of expansion of the external space (6.4.31) we obtain

$$\left(\frac{\exp[x^2(\tau_0 - T_0)]}{\exp[x^2(\tau_c)]}\right)^{hd} > \frac{s(\tau_c)}{s(\tau_0 - T_0)}, \tag{6.4.32}$$

i.e. if the internal space T^d contracts on the time interval $(\tau_0 - T_0, \tau_c)$ in K times then the entropy per baryon increases on this interval less then in K^{hd} times. Thus, there exists the upper limit for the production of entropy provided the expansion of the external space. This limit depends on the final sizes of the internal space T^d and can be removed to infinity as $d \to +\infty$.

The exact solution (6.4.13),(6.4.16) with $f = \sinh[Ah(t - t_0)]$ under the condition (6.4.23) satisfies the weak energy condition for any $t \in (t_0, +\infty)$ and this interval corresponds to the proper time interval $(-\infty, \tau_0)$. It follows from (6.4.13),(6.4.16) that this solution and that with $f = \exp[Ah(t - t_0)/2]$ have identical behavior near the singularity point $\tau = \tau_0$. So, they have the same main properties. We only note, that for $2/h > 1$ we have $\rho(\tau) \to 0$ as $\tau \to -\infty$, then this solution also can be interpreted as that describing creation of matter.

7. Inflationary Solutions in Multidimensional Cosmology with Perfect Fluid

7.1. The model

It is of interest to study also inflationary solutions in multidimensional cosmology which [119-120]. We consider a cosmological model describing the evolution of n Ricci-flat spaces in the presence of the 1-component perfect-fluid matter [37] and a homogeneous massless minimally coupled scalar field. The metric of the model and the manifold are taken as (1.1.1-2)

We take the field equations in the following form:

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N, \tag{7.1.1}$$

$$\Box\varphi = 0, \tag{7.1.2}$$

where κ^2 is the gravitational constant, $\varphi = \varphi(t)$ is scalar field, $\Box$ is the d'Alembert operator for the metric (1.1.1) and the energy-momentum tensor is adopted in the following form

$$T^M_N = T^{M(pf)}_N + T^{M(\phi)}_N, \tag{7.1.3}$$

$$(T^{M(pf)}_N) = diag(-\rho, p_1\delta^{m_1}_{k_1}, \ldots, p_n\delta^{m_n}_{k_n}), \tag{7.1.4}$$

$$T^{M(\phi)}_N = \partial^M\varphi\partial_N\varphi - \frac{1}{2}\delta^M_N(\partial\varphi)^2. \tag{7.1.5}$$

We put pressures of the perfect fluid in all spaces to be proportional to the density

$$p_i(t) = (1 - \frac{u_i}{N_i})\rho(t), \tag{7.1.6}$$

where $u_i = const$, $i = 1, \ldots, n$.

We impose also the following restriction on the vector $\mathbf{u} = (u_i) \in R^n$

$$< u, u >_* < 0. \tag{7.1.7}$$

Here bilinear form $< .,. >_*: R^n \times R^n \to R$ is defined by the relation

$$< u, v >_* = G^{ij}u_iv_j, \tag{7.1.8}$$

$u, v \in R^n$, where

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{7.1.9}$$

are components of the matrix inverse to the matrix of the minisuperspace metric [8,9]

$$G_{ij} = N_i\delta_{ij} - N_iN_j. \tag{7.1.10}$$

In (7.1.9) $D = 1 + \sum_{i=1}^{n} N_i$ is the dimension of the manifold M (1.1.2).

7.2. Classical solutions

We get the following non-exceptional solutions of the field equations (7.1.1-2) [121]

$$g = -(\prod_{i=1}^{n}(a_i(\tau))^{2N_i - u_i})d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{7.2.1}$$

$$a_i(\tau) = A_i[\sinh(r\tau/T)/r]^{2u^i/<u,u>_*}[\tanh(r\tau/2T)/r]^{\beta^i}, \tag{7.2.2}$$

$$\exp(\kappa\varphi(\tau)) = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \tag{7.2.3}$$

$$\kappa^2\rho(\tau) = A\prod_{i=1}^{n}(a_i(\tau))^{u_i - 2N_i}, \tag{7.2.4}$$

$i = 1, \ldots, n$; where $r = \sqrt{A/|A|}$, $T = (\frac{1}{2}|A < u, u >_* |)^{-1/2}$. $A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\sum_{i=1}^{n} u_i\beta^i = 0, \qquad \sum_{i,j=1}^{n} G_{ij}\beta^i\beta^j + (\beta_\varphi)^2 = -4/ < u, u >_* . \tag{7.2.5}$$

Here $\tau > 0$ for $A > 0$ and $0 < \tau < \pi T$ for $A < 0$.

For positive energy density ($A > 0$), see (7.2.4), we have a family of exceptional solutions with the constant real scalar field [37]

$$g = -(\prod_{i=1}^{n}(a_i(\tau))^{2N_i - u_i})d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{7.2.6}$$

$$a_i(\tau) = \bar{A}_i \exp[\pm 2u^i\tau/(T < u, u >_*)], \tag{7.2.7}$$

$$\varphi(\tau) = const, \tag{7.2.8}$$

and $\rho(\tau)$ is defined by (7.2.4). Here $\bar{A}_i > 0$ ($i = 1, \ldots, n$) are constants, and T is defined as in (7.2.1-4).

We note that for $A > 0$ the solution (7.2.7) with the sign "+" is an attractor for the solutions (7.2.2).

Inflationary solutions. First we consider the case

$$< u^{(\Lambda)} - u, u >_* \neq 0, \tag{7.2.9}$$

where $u_i^{(\Lambda)} = 2N_i$ correspond to the cosmological term. The solution (7.2.6), (7.2.7) in synchronous time parametruization reads as

$$g = -dt_s \otimes dt_s + \sum_{i=1}^{n} a_i(t_s) g^{(i)}, \tag{7.2.10}$$

$$a_i(t_s) = A_i t_s^{\nu^i}, \tag{7.2.11}$$

$$\kappa^2\rho = \frac{-2<u,u>_*}{<u^{(\Lambda)} - u, u>_*^2 t_s^2}. \tag{7.2.12}$$

where

$$\nu^i = 2u^i / < u^{(\Lambda)} - u, u >_* . \tag{7.2.13}$$

$i = 1, \ldots, n$. Thus, formulas (7.2.10)-(7.2.13) and $\varphi = const$ describe exceptional solutions for the case (7.2.9). We call these solutions as the power-law inflationary solutions.

The solution is a self-similar one.

Now we consider the case

$$< u^{(\Lambda)} - u, u >_* = 0. \tag{7.2.14}$$

In this case

$$\kappa^2 \rho = const \tag{7.2.15}$$

and

$$a_i(t_s) = \bar{A}_i \exp[\mp \frac{u^i}{\sqrt{- < u, u >_*}} \frac{t_s}{T_0}], \tag{7.2.16}$$

where

$$T_0 = (2\kappa^2 \rho)^{-1/2}. \tag{7.2.17}$$

The relations (7.2.10), (7.2.15)-(7.2.17) and $\varphi = const$ describe the exponential-type inflation for the case (7.2.14). In the special case $u = u^{(\Lambda)}$ (cosmological constant case) this solution was considered in [48].

The corresponding quantum solutions were considered in [121]. Applying the arguments considered in [67] one may show that the ground state wave function

$$\Psi_0^{(HH)} = \quad I_0\left(\frac{\sqrt{2|A|}}{q}\exp(qz^0)\right), \qquad A < 0, \tag{7.2.18}$$

$$J_0\left(\frac{\sqrt{2A}}{q}\exp(qz^0)\right), \qquad A > 0, \tag{7.2.19}$$

satisfies the Hartle-Hawking boundary condition. Here $2q = \sqrt{- < u, u >_*}$ and $\exp(qz^0) = \prod_{i=1}^n a_i^{u_i/2}$ is quasivolume.

7.3. Some Examples

Let us consider the isotropic case when pressures in all spaces are equal. Then

$$u_i = hN_i = \frac{h}{2}\, u_i^{(\Lambda)} , \tag{7.3..1}$$
$$p_i = (1-h)\rho = p \tag{7.3..2}$$

For this case

$$< u, u >_* = -h\, \frac{D-1}{D-2} < 0 \tag{7.3..3}$$
$$\text{if } h \neq 0 \quad \text{or} \quad p \neq \rho .$$

The cosmological constant corresponds to $h = 2$, and the dust-like matter to $h = 1$.

Then,

$$u^i = G^{ij} u_j = h/(2-D) , \tag{7.3..4}$$
$$\nu^i = 2/h(D-1) = \nu$$

We see that for $h > 0$ (or $p < \rho$) we have according to (7.2.11) the isotropic expansion and for $h < 0$ ($p > \rho$) the isotropic contraction. We may calculate also for this isotropic case

$$< u^{(\Lambda)} - u, u >_* = \frac{1}{4}\,(2-h) < u^{(\Lambda)}, u^{(\Lambda)} >_* , \tag{7.3..5}$$

which for $h = 2$ is equal to zero.

Accordingly, we have the power-law (in general) and the exponential law ($h = 2$) inflations here as well.

8. Integrable Weyl Geometry in Multidimensional Cosmology. Numerical Investigation [122]

8.1. Introduction

The multidimensional gravitation theories are very attractive in the context of the unification of fundamental interactions. Moreover, several modern theories require space-time to have more than four dimensions [23,123-128]. The nonobservability of additional dimensions in such theories needs an explanation. Among different possible ways of such explanation the hypothesis about dynamical contraction of internal manifold during expansion of the universe is very popular. This idea is realised in many exact cosmological solution of multidimensional Einstein's equations [21,129-139]. As a rule such models require additional fields and do not avoid initial big bang singularity. The introduction of additional fields in multidimensional gravitation theories destroy their pure geometrical character and require an additional motivation [126]. Such motivation may be done in the framework of some generalizations of Riemannian geometry. In four dimensional case such generalization in several cases leads to removing of cosmological big bang singularity [140-142]. That is why the unification of generalized geometric structures and multidimensional gravity seems to be very attractive. Unfortunately, only in several papers the multidimensional gravitation theory and cosmology are considered in the scope of some generalization of Riemannian geometry [143-145].

One of the simplest generalisation of the Riemannian geometry is the integrable Weyl geometry with the connection components

$$\Gamma^{\alpha}_{\beta\gamma} = \tilde{\Gamma}^{\alpha}_{\beta\gamma} - \frac{1}{2}\left(\omega_\beta \delta^\alpha_\gamma + \omega_\gamma \delta^\alpha_\beta - g_{\beta\gamma}\omega^\alpha\right), \tag{8.1.1}$$

where $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ are the Christoffel symbols, $\omega_\alpha = \omega_{,\alpha}$, ω is a scalar field, δ^α_β are the Kroneker symbols, $g_{\alpha\beta}$ is a metric tensor; the small Greek indices take values from 0 to $n-1$, n is a dimension of space-time. The Ricci tensor and the curvature scalar of the connection (1) are equal to

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{n-2}{2}\omega_{\mu||\nu} + \frac{1}{2}g_{\mu\nu}\Box\omega + \frac{n-2}{4}\left(\omega_\mu\omega_\nu - g_{\mu\nu}\omega^\lambda\omega_\lambda\right), \tag{8.1.2}$$

$$R = \tilde{R} + (n-1)\Box\omega - \frac{(n-1)(n-2)}{4}\omega^\lambda\omega_\lambda, \tag{8.1.3}$$

where the tildes denote the quantities calculated in the connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$, two parallel vertical bars and $\Box$ denote the covariant derivative and the d'Alembert operator of this connection. It is necessary to note that the integrable Weyl space-time is also conformally-Riemannian, since there is a conformal transformation of metric tensor $g_{\alpha\beta}$ which maps the Riemannian space-time into integrable Weyl space-time. As the integrable Weyl space-time is defined by the pair $(g_{\alpha\beta}, \omega)$ the gravitation theory in this space-time does not coincide with Einsteinian general relativity because the field ω must be contained in the Lagrangian independently from $g_{\alpha\beta}$ and cannot be excluded by the conformal transformation.

Some features of the Einsteinian cosmological models with scalar fields were recently considered by several authors [21,129,131,137,139,145-150] both in 4-dimensional and in (4+d)-dimensional space-times. The cosmological models in four-dimensional Weyl-integrable space-time were recently considered by Novello et al. in [140], where the existence of nonsingular open cosmological models was shown. The appearance of Weyl geometry in multidimensional cosmology was discussed also in [144].

In this paper we consider the influence of Weyl geometry on the evolution of Friedman-Robertson-Walker (FRW) cosmological models in multidimensional gravitation theory. As usually the space-time is assumed to have the structure of direct product $M^4 \times V^d$ of four-dimensional

FRW space-time M^4 and d-dimensional interior space V^d that is supposed to be d-sphere S^d or d-torus T^d. The metric of space-time is supposed to be block-diagonal

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - \bar{g}_{ab} du^a du^b, \tag{8.1.4}$$

where $k = +1,\ 0,\ -1$ for closed, plane and open models, $d\Omega^2$ is a line element on two-sphere, u^a, $a = 1, ..., d$, and $\bar{g}_{\alpha\beta}$ are the coordinates and metric tensor of the interior space V^d. Once we consider only spatially homogeneous FRW cosmologies, it is natural to make the Weyl scalar field ω to be a function of cosmic time t only: $\omega = \omega(t)$. We consider both vacuum case and non vacuum case with the additional scalar field φ with non minimal coupling. The 4-dimensional case will be briefly considered also for completeness. The existence of the conformal map between Riemannian and integrable Weyl space-times may be used for generation of exact solutions from the known solutions of general relativity. Such approach admits obtaining only the particular solutions. Therefore to demonstrate general qualitative behavior of the models we solve the system of cosmological equations numerically with initial values given at $t = 0$ and satisfying the constraint equation. For that purpose we use adaptive numerical methods with automatic choice of integration step and with the stiffness checking. The geometrical units where $G = c = 1$ are used in what follows.

8.2. Integrable Weyl cosmology in vacuum

Following [140] we shall consider the vacuum cosmological models in the gravitation theory with the Lagrangian

$$L = R + \xi\omega_\alpha\omega^\alpha \tag{8.2.1}$$

where R is defined by (8.1.3) and $\xi = const$. After excluding the total derivatives of the scalar field Lagrangian (8.2.1) takes the form

$$L = \tilde{R} - \frac{(n-1)(n-2) - 4\xi}{4}\omega^\alpha\omega_\alpha \tag{8.2.2}$$

So, the theory differs from the Einstein theory with the massless scalar field by the coefficient before the square of the scalar field gradient and has different geodesic lines. Note also that due to the definition of the Weyl connection (8.1.1) the scalar field ω cannot be renormalised and hence the coefficient ξ before $\omega^\alpha\omega_\alpha$ cannot be put to ±1 as it may be done in the pure Einstein theory with massless scalar field. Variation of (8.2.2) with respect to the pair $(g_{\alpha\beta}, \omega)$ of independent variables yields the equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R} - \frac{(n-1)(n-2) - 4\xi}{4}\left(\omega_\mu\omega_\nu - \frac{1}{2}g_{\mu\nu}\omega^\alpha\omega_\alpha\right) = 0, \tag{8.2.3}$$

and

$$\Box\omega = 0 \tag{8.2.4}$$

The equations (8.2.3-4), coincide with the Einstein equations for the massless scalar field, whose solutions for the FRW cosmological models were investigated both in four-dimensional [148] and multidimensional cases [78]. By this reason here we only summarise briefly the main results.

8.2.1 Four-dimensional case. As the scalar field ω is a function on t only, equation (8.2.4) yields the first integral

$$\dot{\omega} = \frac{\gamma}{a^3} \tag{8.2.5}$$

where overdot denotes time differentiation and $\gamma = const$ is the integration constant. Due to (8.2.5) equations (8.2.3) take the form

$$\dot{a}^2 + k - \frac{\lambda\gamma^2}{12a^4} = 0 \tag{8.2.6}$$

and

$$2a\ddot{a} + \dot{a}^2 + k + \frac{\lambda\gamma^2}{4a^4} = 0 \tag{8.2.7}$$

where $\lambda = (3 - 2\xi)$. As it is easy to see from (8.2.6), only singular and static solution of equations (8.2.6-7) exist if $\lambda > 0$. For negative values of λ solution exists only for the open models. In this case $a(t) \geq a_0 = (\xi - 3)\gamma^2/12$ and so the cosmological singularity is absent. The qualitative behavior of scale factor $a(t)$ for negative λ is shown in figure 12 and its features are discussed in detail in [140].

8.2.2. Multidimensional case. In the multidimensional case the behavior of the model depends not only on the parameter ξ, as in the previous case, but on the structure of the interior space also. For simplicity only 5- and 6-dimensional models will be considered in the following. We consider these two cases separately. The main qualitative features of models in general n-dimensional $(n > 6)$ case are the same as in 5- and 6-dimensions.

8.2.2.1. 5-dimensional models. In 5-dimensions space-time interval (8.1.4) reads

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - s^2(t)du^2 \tag{8.2.8}$$

where $u \in S^1$ is the interior space coordinate. Assuming, as above, a scalar field ω to be a function of the cosmological time only, the first integral of equation (8.2.4) takes the form

$$\dot{\omega} = \frac{\gamma_1}{a^3(t)s(t)}, \tag{8.2.9}$$

where $\gamma_1 = const$. Due to (8.2.8-9) equations (8.2.3) become after simplification

$$3\frac{\dot{a}}{a}\frac{\dot{s}}{s} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} - \frac{\gamma_1^2(3-\xi)}{2a^6s^2} = 0, \tag{8.2.10}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{\dot{s}}{s} + \frac{k}{a^2} = 0, \tag{8.2.11}$$

and

$$\frac{\ddot{s}}{s} + 3\frac{\dot{a}}{a}\frac{\dot{s}}{s} = 0 \tag{8.2.12}$$

The last equation has the first integral

$$\dot{s} = \frac{\gamma_2}{a^3} \tag{8.2.13}$$

where $\gamma_2 = const$. It is easy to see that analogous to the four-dimensional case the nonsingular solutions of equations (15), (17) exist only for the open models ($k = -1$). In this case for $t < 0$ the scale factor of 3-space $a(t)$ decreases monotonically from infinity to its minimal value a_0 and then grows to infinity at $t > 0$, while the Weyl field $\omega(t)$ and the scale factor of interior space evolve monotonous from $\omega_- = \lim_{t\to-\infty}\omega(t)$ and $s_- = \lim_{t\to-\infty} s(t)$ to $s_+ = \lim_{t\to\infty} s(t)$, where a_0, $\omega_\pm$ and $s_\pm$ are defined by the integration constants and may have arbitrary values. Note that if $\gamma_2 < 0$ than the constants s_- and s_+ satisfy the condition $s_- > s_+$ and so the standard dimensional reduction scenario is realised. The typical shape of the functions $a(t)$ and $s(t)$ are shown in the figures (13.a,b).

The figure (13a) shows that unlike the 4-dimensional case the evolution of 3-space in 5-dimensional model is time-asymmetric. This asymmetry appears because the equation (8.2.11) depends not only on $\dot{s}(t)$ but also on the time-asymmetric interior space scale factor $s(t)$.

8.2.2.2. 6-dimensional models. In 6-dimensional case we consider two types of topological structures for the interior space: the 2-sphere S^2 and 2-dimensional torus T^2. Therefore, the space-time metric (8.1.4) may have one of two forms

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - s^2(t)\left(\frac{du^2}{1-u^2} + u^2 dv^2\right) \tag{8.2.14}$$

or

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - s_1^2(t)du^2 - s_2^2(t)dv^2, \tag{8.2.15}$$

where $\{u, v\}$ are the coordinates on S^2 or T^2 respectively. First integrals of equation (8.2.4) take the form

$$\dot{\omega} = \frac{q_1}{a^3 s^2}, \tag{8.2.16}$$

for metric (8.2.14) and

$$\dot{\omega} = \frac{q_2}{a^3 s_1 s2}, \tag{8.2.17}$$

for metric (8.2.15).

Equations (8.2.3) for the metric (8.2.14) after simplification take the form

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{s}}{s} + \frac{2k}{a^2} = 0, \tag{8.2.18}$$

$$\frac{\ddot{s}}{s} + \left(\frac{\dot{s}}{s}\right)^2 + 3\frac{\dot{a}}{a}\frac{\dot{s}}{s} + \frac{1}{s^2} = 0, \tag{8.2.19}$$

and the constraint equation

$$3\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + 2\frac{\dot{s}}{s}\right) + \frac{3k}{a^2} + \frac{1}{s^2} - \frac{q_1(5-\xi)}{2a^6 s^4} = 0. \tag{8.2.20}$$

The first two equations are dynamical and the last is the constraint.

It is easy to see that only singular solutions of equations (8.2.18-20) exist: the scale factor $s(t)$ of the interior space evolves from zero at $t = t_0$ to its maximal value s_{max} and return to zero at $t = t_1 > t_0$. The behavior of $a(t)$ depends on the sign of k. Namely, if $k = +1$ then the qualitative evolution of $a(t)$ is the same as the evolution of $s(t)$. If $k = 0$ than $a(t)$ increase from zero at $t = t_0$ to infinity at $t = t_1$ or decrease from infinity to zero; the unstable solutions with $a(t) = const$ are also exist. Finally, if $k = -1$ then $a(t)$ evolves from infinity at $t = t_0$ to its minimum a_{min} and then grows to infinity at $t = t_1$.

For the metric (8.2.15) equations (8.2.3) after simplification read

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{s}_1}{s_1} + \frac{\dot{s}_2}{s_2}\right) + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 0, \tag{8.2.21}$$

$$\frac{\ddot{s}_1}{s_1} + 3\frac{\dot{a}}{a}\frac{\dot{s}_1}{s_1} + \frac{\dot{s}_1}{s_1}\frac{\dot{s}_2}{s_2} = 0, \tag{8.2.22}$$

$$\frac{\ddot{s}_2}{s_2} + 3\frac{\dot{a}}{a}\frac{\dot{s}_2}{s_2} + \frac{\dot{s}_1}{s_1}\frac{\dot{s}_2}{s_2} = 0, \tag{8.2.23}$$

and the constraint equation

$$3\frac{\dot a}{a}\left(\frac{\dot a}{a}+\frac{\dot s_1}{s_1}+\frac{\dot s_2}{s_2}\right)+\frac{\dot s_1}{s_1}\frac{\dot s_2}{s_2}+\frac{3k}{a^2}-\frac{q_2^2(5-\xi)}{2a^6s_1^2s_2^2}=0. \tag{8.2.24}$$

As in 4- and 5-dimensional cases the nonsingular solutions of the equations (8.2.21-24) exist only for the open models ($k=-1$). Analogously to 5-dimensional case the scale factor of 3-space $a(t)$ in these models decreases monotonously from infinity to its minimal value a_0 and then grows to infinity at $t\to+\infty$, while the scale factors $s_i(t)$, $i=1,\ 2$, of interior space changes monotonously from $s_{i-}=\lim_{t\to-\infty}s_i(t)$ to $s_{i+}=\lim_{t\to\infty}s_i(t)$. The necessary condition for the realization of the dimensional reduction scenario in this case are defined by the following inequalities

$$\frac{3}{a_0^2}+\frac{q_2^2(5-\xi)}{2a_0^6s_{10}s_{20}}>0, \tag{8.2.25}$$

and

$$\dot s_1(0)<0, \dot s_2(0)<0 \tag{8.2.26}$$

It is necessary to note that inequality (8.2.25) is the necessary condition for $\dot s_1$ and $\dot s_2$ to be of the same sign. The time behavior of scale factors $a(t)$, $s_1(t)$ and $s_2(t)$ in this case is qualitatively the same as in 5-dimensional case (Figure 13).

8.3. Integrable Weyl cosmology in theory with non minimal scalar field

In this section we consider cosmological models in gravitation theories with Lagrangian

$$L=R\left(1+\frac{1}{2(n-1)}\varphi^2\right)+\xi\omega^\alpha\omega_\alpha+\eta\varphi^\alpha\varphi_\alpha, \tag{8.3.1}$$

where R is defined by (8.1.3), φ is a real scalar field, $\eta=\pm1$ and $\xi=const$ as above. In the limiting case $\varphi=const$ Lagrangian (8.3.1) coincides with (8.2.1) while in another limiting case $\omega=const$ it coincides with the Lagrangian for the conformal-invariant scalar field.

The substitution of (8.1.3) into (8.3.1) gives after simplification

$$L=\bar R\left(1+\frac{\varphi^2}{2(n-1)}\right)-\varphi\varphi^\alpha\omega_\alpha-\frac{(n-1)(n-2)-4\xi}{4}\omega^\alpha\omega_\alpha-\frac{(n-2)}{8}\varphi^2\omega^\alpha\omega_\alpha+\eta\varphi^\alpha\varphi_\alpha, \tag{8.3.2}$$

where the total derivatives of the scalar fields are omitted.

Variation of (8.3.2) with respect to independent variables $g_{\mu\nu}$, ω and φ yields the equations

$$\left(\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}\right)\left(1+\frac{\varphi^2}{2(n-1)}\right)-\frac{(n-1)(n-2)-4\xi}{4}\left(\omega_{\mu}\omega_{\nu}-\frac{1}{2}g_{\mu\nu}\omega^{\alpha}\omega_{\alpha}\right)-$$
$$\frac{1}{2}\varphi\left(\varphi_{,\mu}\omega_{,\nu}+\varphi_{,\nu}\omega_{,\mu}\right)-\frac{n-2}{8}\varphi^2\left(\omega_{,\mu}\omega_{,\nu}-\frac{1}{2}g_{\mu\nu}\omega^{\alpha}\omega_{\alpha}\right)+\frac{\varphi}{n-1}\left(g_{\mu\nu}\Box\varphi-\varphi_{,\mu\|\nu}\right)+$$
$$\varphi_{,\mu}\varphi_{,\nu}\left(\eta-\frac{1}{n-1}\right)+g_{\mu\nu}\varphi^{\alpha}\varphi_{\alpha}\left(\frac{1}{n-1}-\frac{\eta}{2}\right)=0, \tag{8.3.3}$$
$$\left(\frac{(n-1)(n-2)-4\xi}{2}+\frac{n-2}{4}\varphi^2\right)\Box\omega-\varphi\Box\varphi-\varphi_{,\nu}\varphi_{,}^{\nu}=0, \tag{8.3.4}$$

and

$$\eta\Box\varphi-\left(\Box\omega+\frac{1}{n-1}\tilde{R}-\frac{n-2}{4}\omega_{,}^{\nu}\omega_{,\nu}\right)\varphi=0, \tag{8.3.5}$$

Equation (8.3.5) shows that non-Riemannian nature of space-time geometry in the considered model leads to the effective mass generation for the scalar field φ.

8.3.1. Four-dimensional models. In four-dimensional case the equations (8.3.3)-(8.3.5) consist of the constraint equation

$$\left(\frac{\dot{a}}{a}+\frac{k}{a^2}\right)^2\left(3+\frac{\varphi^2}{2}\right)+\frac{\dot{a}}{a}\varphi\dot{\varphi}+\frac{\eta}{2}\dot{\varphi}^2-\frac{1}{2}\varphi\dot{\varphi}\dot{\omega}-\frac{1}{8}\varphi^2\dot{\omega}^2-\frac{3-2\xi}{4}\dot{\omega}^2=0, \tag{8.3.6}$$

and three dynamical equations

$$\left(2+\frac{\varphi^2}{3}\right)\frac{\ddot{a}}{a}+\frac{1}{3}\varphi\ddot{\varphi}+2\left(\frac{\dot{a}}{a}\right)^2\left(2+\frac{\varphi^2}{3}\right)+\frac{5}{3}\frac{\dot{a}}{a}\varphi\dot{\varphi}+\frac{1}{3}\dot{\varphi}^2+\left(2+\frac{\varphi^2}{3}\right)\frac{2k}{a^2}=0, \tag{8.3.7}$$

$$\left(\frac{\varphi^2}{2}-2\xi+3\right)\ddot{\omega}-\varphi\ddot{\varphi}+\left(9-6\xi+\frac{3}{2}\varphi^2\right)\frac{\dot{a}}{a}\dot{\omega}-3\frac{\dot{a}}{a}\varphi\dot{\varphi}-\dot{\varphi}^2=0, \tag{8.3.8}$$

and

$$\eta\ddot{\varphi}-\varphi\ddot{\omega}+2\varphi\frac{\ddot{a}}{a}+3\frac{\dot{a}}{a}\left(\eta\dot{\varphi}-\varphi\dot{\omega}\right)+\frac{1}{2}\varphi\dot{\omega}^2+2\varphi\left(\frac{\dot{a}}{a}\right)^2+\frac{2k}{a^2}\varphi=0. \tag{8.3.9}$$

The coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ in the equations (8.3.7)-(8.3.9) depend both on the parameters ξ, η and on the scalar field φ. The determinant of the matrix of coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ is equal to

$$d=\left(\tfrac{\eta}{2}-\tfrac{2}{3}\right)\varphi^4+\left(2\eta+\tfrac{4\xi}{3}-\tfrac{2\eta\xi}{3}-4\right)\varphi^2+6\eta-4\eta\xi.$$

The points where $d=0$ are the singular points of the system (8.3.7)-(8.3.9). These points are not described by the system (8.3.6)-(8.3.9) because for fixed η and ξ equation $d=const$

defines not more than four fixed values of φ and the system (8.3.6)-(8.3.9) reduces to the first order system. Therefore the initial value of the field φ must be from the open set $d \neq 0$.

For η equation $d = 0$ divide the half-plane (ξ, φ^2), in three regions that will be denoted as A, B and C, while for $\eta = -1$ there are only two regions A and B (figure 14a,b). The behavior of the model depends on the region where the point (ξ, φ_0^2) is situated.

Numerical investigation of equations (8.3.1)-(8.3.3) shows that for the closed ($k = 1$) and flat ($k = 0$) cosmological models only singular solutions exist for any initial conditions. For the open models ($k = -1$) if the pair (ξ, φ_0^2) defines the point in the region B (both for $\eta = 1$ and $\eta = -1$) or C (for $\eta = 1$) than only singular solutions of the equations (8.3.7)-(8.3.9) exist. If the pair (ξ, φ_0^2) defines the point in the region A then solutions may be both regular and singular. The numerical investigation does not permit to find the exact conditions of regularity, but it shows that both regular and singular solutions are stable against finite perturbations of the initial conditions. The typical qualitative behavior of the universe scale factor $a(t)$, Weyl field ω and the matter scalar field φ are shown in figure 15a-c.

The universe scale factor $a(t)$ in the typical nonsingular solution evolves from infinity at $t = -\infty$ to its minimal value $a_0 = a(0)$ and then grows to infinity at $t \to \infty$ (figure 15a). Both scalar fields, the Weyl field ω and the field φ evolves between two limiting values: from $\omega_- = \lim_{t\to-\infty}\omega(t)$ and $\varphi_- = \lim_{t\to-\infty}\varphi$ to $\omega_+ = \lim_{t\to\infty}\omega(t)$ and $\varphi_+ = \lim_{t\to\infty}\varphi(t)$. The difference in the evolution of these fields is that the field ω evolves monotonously (figure 15b) while the field φ near $t = 0$ (i. e. near the minimum of $a(t)$) may have several intermediate extrema with one absolute maximum if $\eta = 1$ (figure 15c) or absolute minimum if $\eta = -1$. As $\varphi(t)$ for big $|t|$ tends asymptotically to constants, the model evolves asymptotically as an empty Weyl cosmological model that is considered in section 8.2.1. It is necessary to note also that the evolution of the universe scale factor $a(t)$ has a small time-asymmetry in comparison with the case of the empty space. This asymmetry is a result of non symmetrical evolution of the matter field φ because the field equations (8.3.7)-(8.3.9) contain both φ and $\dot{\varphi}$.

8.3.2. 5-dimensional models. In 5-dimensional case equations (8.3.3)-(8.3.5) after simplification become

$$3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)+\frac{k}{a^2}\right\}\left\{1+\frac{\varphi^2}{8}\right\}+\frac{\varphi\dot{\varphi}}{4}\left(\frac{3\dot{a}}{a}+\frac{\dot{s}}{s}\right)+\frac{\eta\dot{\varphi}^2}{2}-\frac{\varphi\dot{\varphi}\dot{\omega}}{2}+$$
$$+\frac{\dot{\omega}^2}{2}\left(\xi-3-\frac{3\varphi^2}{8}\right)=0, \tag{8.3.10}$$

$$\left\{\frac{\ddot{a}}{a}-\frac{\ddot{s}}{s}+2\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{s}}{s}\right)+\frac{2k}{a^2}\right\}\left\{1+\frac{\varphi^2}{8}\right\}+\frac{\varphi\dot{\varphi}}{4}\left(\frac{\dot{a}}{a}-\frac{\dot{s}}{s}\right)=0, \tag{8.3.11}$$

$$3\left\{1+\frac{\varphi^2}{8}\right\}\left\{\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\left(2\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)+\frac{2k}{a^2}\right\}+\frac{\varphi\ddot{\varphi}}{4}+\frac{\varphi\dot{\varphi}}{2}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{2s}+\frac{\dot{\varphi}}{2\varphi}\right)=0, \tag{8.3.12}$$

$$\ddot{\omega}\left(\frac{3\varphi^2}{4}-2\xi+6\right)-\varphi\ddot{\varphi}+\dot{\omega}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)\left(3\frac{\varphi^2}{4}-2\xi+6\right)-\varphi\dot{\varphi}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{s}+\frac{\dot{\varphi}}{\varphi}\right)=0, \qquad (8.3.13)$$

and

$$\frac{\varphi}{2}\left(\frac{3\ddot{a}}{a}+\frac{\ddot{s}}{s}\right)+\eta\ddot{\varphi}-\varphi\ddot{\omega}+\varphi\dot{\omega}\left(\frac{3\dot{\omega}}{4}-3\frac{\dot{a}}{a}-\frac{\dot{s}}{s}\right)+\eta\dot{\varphi}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)+\frac{3\varphi\dot{a}}{2a}\left(\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)$$
$$+\frac{3k\varphi}{2a^2}=0. \qquad (8.3.14)$$

The equation (8.3.1) is the constraint that must be satisfied by the initial conditions and the equations (8.3.11-14) are the dynamical. The determinant of the matrix of the coefficients before $\ddot{a}/a$, $\ddot{s}/s$, $\ddot{\omega}$ and $\ddot{\varphi}$ in the dynamical equations (8.3.11-14) is equal to

$$d=\left(\tfrac{9}{256}\eta-\tfrac{3}{32}\right)\varphi^6+\left(\tfrac{27}{32}\eta+\tfrac{1}{8}\xi-\tfrac{3}{32}\eta\xi-\tfrac{3}{8}\right)\varphi^4+\left(\tfrac{27}{4}\eta+\xi-\tfrac{3}{2}\eta\xi-6\right)\varphi^2+18\eta-6\eta\xi.$$

The qualitative features of function $d(\xi,\eta,\varphi)$ are the same as in 4-dimensional case: for $\eta=1$ equation $d=0$ divides the half-plane $(\xi,\varphi^2>0)$ in three regions that are denoted as A, B and C, while for $\eta=-1$ there are only two regions A and B (figure 16a,b). The behavior of the model depends on the region where the point (ξ,φ_0^2) is situated.

Numerical investigation of equations (8.3.10-14) shows that as well as in the previous 4-dimensional case only singular solutions exist at any initial conditions for the closed ($k=1$) and flat ($k=0$) cosmological models. For the open models ($k=-1$) if the pair (ξ,φ_0^2) defines the point in the region B (both for $\eta=1$ and $\eta=-1$) or C (for $\eta=1$) than only singular solutions of the equations (8.3.10-14) exist, while if the pair (ξ,φ_0^2) defines the point in the region A than the solution may be both regular and singular. The regularity of solutions depends on the constants of integration that may be considered as the initial conditions at $t=0$. It was found that the regularity of solutions depends mainly on the signs of $\dot{s}(0)$, $\dot{\omega}(0)$ and $\dot{\varphi}(0)$. Their possible combinations that give nonsingular solutions of equations (8.3.11-14) are represented in table 2. The last column of this table shows the general direction of the interior space evolution by means of the signs of the difference $\Delta=s_+-s_-$, where $s_\pm=\lim_{t\to\pm\infty}s(t)$.

Table 2.

Conditions of the solutions regularity and the direction of $s(t)$ evolution

$\operatorname{sign}\dot{s}(0)$	$\operatorname{sign}\dot{\omega}(0)$	$\operatorname{sign}\dot{\varphi}(0)$	$\operatorname{sign}(s_+ - s_-)$
-1	-1	0	-1
-1	+1	0	-1
-1	-1	+1	-1
-1	+1	-1	-1
0	+1	0	-1
0	-1	0	+1
+1	+1	-1	+1
+1	-1	0	+1
+1	+1	0	+1
+1	-1	+1	+1

The typical behavior of the nonsingular solution of the equations (8.3.11-14) for $\eta = 1$ is shown at the figures 17a-d for the case $\Delta \leq 0$, i. e. for the contracting interior space.

In general nonsingular solution the radius of the universe changes monotonously from infinity at $t = -\infty$ to minimal value a_0 and then grows to infinity (figure 17a), while the radius of the internal space starts from $s_- = \lim_{t\to-\infty} s(t)$, passes through several (one or two) intermediate extrema, that are situated near minimum of $a(t)$ and may be absent in some cases, and then changes to $s_+ = \lim_{t\to\infty} s(t)$ (figure 17b). Note that s_+ and s_- may be of the same or different order. The field φ evolves analogously to 4-dimensional case (figure 17c). Note that the extremal points of the functions $a(t)$, $s(t)$ and $\varphi(t)$ do not coincide with each other in general case and the function $a(t)$ is time asymmetrical especially near its minimum. Finally the Weyl field ω changes monotonously between two limiting values: $\omega_- = \lim_{t\to-\infty}\omega(t)$ and $\omega_+ = \lim_{t\to\infty}\omega(t)$ (figure 17d). In the case $\eta = -1$ the model evolves as above but the extremal points of the field φ change type: minimum becomes maximum and vice versa.

8.4. Concluding remarks

We have considered the qualitative evolution of multidimensional cosmological models based on the integrable Weyl geometry both in vacuum space-time and in the presence of nonminimal scalar field. The existence of nonsingular solutions of field equations for open cosmological models that realized the dimensional reduction scenario was demonstrated. It was shown that in multidimensional case the evolution of the scale factor of the universe $a(t)$ becomes time-asymmetric unlike the four-dimensional case. We have shown also that all nonsingular cosmological models considered above have some common features. In particular the evolution of the universe scale factor (radius) $a(t)$ for big $|t|$ is asymptotically linear. Further in all nonsingular models Weyl scalar field $\omega(t)$ as well as the matter field $\varphi(t)$ in the models with nonminimal coupling tend asymptotically to constants. So the models tend to the pure Einsteinian models of the corresponding dimensions and the change of the collapse era into expansion one may be considered as a cosmological phase transition induced by the transition of scalar fields $\omega(t)$ and $\varphi(t)$ from one stationary state $\omega = \omega_-$ and $\varphi = \varphi_-$ into another stationary state $\omega = \omega_+$ and $\varphi = \varphi_+$. At the late stages of the universe evolution the fields $\omega(t)$ and $\varphi(t)$ are unobservable.

There are several qualitative differences between the vacuum models and the models with nonminimal scalar field. First of all in vacuum models the existence of cosmological singularity depends only on the parameters of the theory while in the case of nonminimal scalar field it depends on the initial conditions also. Secondly, in the models with nonminimal scalar field the evolution of the internal space scale factor $s(t)$ may be nonmonotonous. In the typical scenario one of the limiting values of $s(t)$ at $t = \pm\infty$ is much smaller than another but in several models both limiting values of internal radius $s(t)$ may be arbitrary small and it become finite only near minimum of the universe scale factor $a(t)$.

We have discussed here only the models with the one- or two- dimensional interior space because if interior space has dimension $d \geq 3$ and direct product topology of torus on several spheres then the models have the same qualitative features as considered above. In particular, the nonsingular solutions exist only for toroidal interior space topology.

The models considered above show that the real geometrical structure of space-time may have a non-Riemaniann nature but the universe may evolve in such a way that its non-Riemaniann nature is essential only near $t = 0$ and become unobservable at late stages of the evolution. Therefore, the consideration of generalized geometrical structures in multidimensional cosmology may be of a considerable interest. In particular, the models considered above may be generalized in the following manner. First of all, both Weyl scalar field $\omega(t)$ and matter field $\varphi(t)$ may be massive and have nonlinear potential. Secondly, the possible influence of the cosmological term Λ must be considered also. At last, the term $R\varphi^2/2(n-1)$ in the lagrangian (8.3.1) may have negative sign. One may suppose that in this case nonsingular solutions of the field equations may be obtained not only for open models, but for closed and flat models also. These possibilities will be considered elsewhere.

9. Exact Solutions in Integrable Weyl Geometry in Multidimensional Cosmology [152-153]

9.1. Introduction

Here we continue to study multidimensional models in integrable Weyl geometry started in section 8. We stress that the gravitational field in a Weyl-integrable space-time (WIST) is determined by the tensor g_{AB} and the scalar ω, just as in scalar-tensor theories (STT) of gravity. The difference between these two cases is determined by Eq. (8.1.1). Namely, both in STT and in WIST there is a conformal gauge in which test particles move along geodesics; however, in WIST, unlike

STT, even in this frame the motion in general depends on both the metric and the scalar field. Thus, a gravitation theory on the basis of WIST is in general not a special case of STT due to a nonminimal coupling between the matter and the scalar field.

However, field equations in STT and WIST-based theories in many cases coincide, in particular, for all vacuum space-times.

The description of cosmological models in STT is often reduced to that of Einsteinian cosmologies with scalar fields. The latter were considered by many authors [21,50] in both 4-dimensional and (4+d)-dimensional space-times.

In this section we consider the evolution of multidimensional cosmological models based on integrable Weyl geometry with finding exact solutions for some simplest cases of empty spaces. The main characteristic features of the solutions are illustrated graphically. Keeping in mind the possible applications of the results to the description of quantum stages of the universe evolution we also consider WIST with the Euclidean signature.

9.2. Model

As is the case with STT, the gravitational field Lagrangian may in general contain various invariant combinations of g_{AB} and ω. Let us restrict ourselves to Lagrangians which are (a) linear in the scalar curvature and (b) quadratic in ω_A. Then the general form of the Lagrangian satisfying (a) and (b) is

$$L = A(\omega)R + B(\omega)\omega^A\omega_A - 2\Lambda(\omega) + L_m \tag{9.2.1}$$

where R is the Weyl scalar curvature corresponding to the connection (8.1.1), A, B and Λ are arbitrary functions and L_m is the nongravitational matter Lagrangian.

Using the expression (8.1.2) for R in terms of the Riemannian curvature $\bar{R}$ corresponding to the metric g_{AB}, the conformal mapping well-known in STT [154], modified for D dimensions [78,152]:

$$g_{MN} = A^{-2/(D-2)}\bar{g}_{MN}. \tag{9.2.2}$$

and omitting a total divergence, we obtain the following form of the Lagrangian:

$$\bar{L} = A(\omega)\bar{R} + F(\omega)\bar{g}^{AB}\omega_A\omega_B + A^{-D/(D-2)}[-2\Lambda(\omega) + L_m] \tag{9.2.3}$$

where

$$F(\omega) = \frac{1}{A(\omega)^2}\Big[A(\omega)B(\omega) - (D-1)A(\omega)\Big(A_\omega + \frac{D-2}{4}\Big) + \frac{D-1}{D-2}A_\omega^2\Big]. \tag{9.2.4}$$

Let us consider vacuum cosmological models with the following structure of the space-time W_D:

$$W_D = \mathrm{R} \times M_1 \times \ldots \times M_n; \qquad \dim M_i = N_i; \tag{9.2.5}$$

where the subspaces M_i are assumed to be maximally symmetric. The component R corresponds to the time τ; besides, we assume $\omega = \omega(\tau)$. Thus, the effective Riemannian metric is written in the form

$$d\bar{s}^2 = \bar{g}_{AB}dx^A dx^B = e^{2\gamma(\tau)}d\tau^2 - \sum_{i=1}^{n} e^{2\beta_i(\tau)}ds_i^2 \tag{9.2.6}$$

where ds_i^2 are τ-independent metrics of the N_i-dimensional spaces of constant curvatures K_i; with no loss of generality one can put $K_i = 0, \ \pm 1$.

Making use of the freedom to choose the time coordinate τ, let us introduce the harmonic time by putting

$$\gamma = \sum_{i=1}^{n} N_i\beta_i. \tag{9.2.7}$$

Then the Ricci tensor for $\bar{g}_{AB}$ has the following nonzero components:

$$\begin{aligned} \bar{R}^{\tau}_{\tau} &= e^{-2\gamma}\Big(\ddot{\gamma} - \dot{\gamma}^2 + \sum_{i=1}^{n} N_i\dot{\beta}_i^2\Big), \\ \bar{R}^{m_i}_{n_i} &= \delta^{m_i}_{n_i}\Big[e^{-2\gamma}\ddot{\beta}_i + (N_i - 1)K_i e^{-2\beta_i}\Big] \end{aligned} \tag{9.2.8}$$

where the indices m_i, n_i belong to the subspace M_i.

The field equations take an especially simple form under the additional condition $\Lambda \equiv 0$:

$$\bar{R}_{MN} + F(\omega)\omega_M\omega_N = 0, \tag{9.2.9}$$

$$2\nabla_M[F(\omega)\omega^M] - F_\omega\omega^M\omega_M = 0. \tag{9.2.10}$$

9.3. Solutions

They can be integrated completely under one of the above assumptions: (i) if all the subspaces M_i are Ricci-flat and (ii) if one of M_i (for instance, M_1) is a space of nonzero constant curvature (K_1). Indeed, putting $K_i = 0$ $(i > 1)$, we obtain:

$$(F\dot{\omega}^2)^{\cdot} = 0 \quad \Rightarrow \quad F\dot{\omega}^2 = S = \mathrm{const}; \tag{9.3.1}$$

$$\ddot{\beta}_i = 0 \quad \Rightarrow \quad \beta_i = \beta_{i0} + h_i\tau, \qquad i > 1; \tag{9.3.2}$$

$$\ddot{\gamma} - \ddot{\beta}_1 = -K_1 d^2 e^{2\gamma - 2\beta_1} \tag{9.3.3}$$

where $d+1 = N_1 = \dim M_1$. The equation (9.3.3) leads to different results for different K_1: for $K_1 = 0$ (case (i)) Eq. (9.3.2) may be regarded to include $i = 1$; for $K_1 \neq 0$ (case (ii)) we get:

$$e^{\beta_1-\gamma} = \frac{d}{k}\cosh k\tau, \qquad k>0 \qquad (K_1=+1), \tag{9.3.4}$$

$$e^{\beta_1-\gamma} = d\cdot s(k,\tau) \equiv \begin{cases} (d/k)\sinh k\tau, & k>0, \\ d\cdot\tau, & k=0, \\ (d/k)\sin k\tau, & k<0, \end{cases} \qquad (K_1=-1) \tag{9.3.5}$$

where k = const and another integration constant is elimintaed by a particular choice of the origin of τ. Lastly, a combination of components of (9.2.9) representing the temporal component of the Einstein equations (the initial data equation) leads to the following relation among the integration constants:

$$\left(\sum_{i=1}^{n} N_i h_i\right)^2 - \sum_{i=1}^{n} N_i h_i^2 = S, \qquad K_1 = 0; \tag{9.3.6}$$

$$\frac{d+1}{d}k^2 \operatorname{sign} k = \frac{1}{d}\left(\sum_{i=2}^{n} N_i h_i\right)^2 + \sum_{i=2}^{n} N_i h_i^2 + S, \qquad K_1 \neq 0. \tag{9.3.7}$$

Thus, the set of equations (9.2.9-10) has been integrated in quadratures.

As the original functions $A(\omega)$ and $B(\omega)$ and hence $F(\omega)$ are arbitrary, it is difficult to describe the physical properties of the models in a general form. Therefore, here we would like to restrict ourselves to some simple special cases.

Thus, we will assume $A \equiv 1$ while $B(\omega)$ remains arbitrary, so that the metrics $\bar{g}_{AB}$ and g_{AB} coincide.

9.4. Special Cases

As the first step consider 4-dimensional homogeneous isotropic cosmologies. For this purpose we must put $n = 1$, $d = 2$, $\beta_1 \equiv \beta(\tau)$. The condition that τ is a harmonic coordinate takes the form $\gamma = 3\beta$ and for the scale factor we get:

$$e^{2\beta} = a^2(\tau) = \begin{cases} 1/2s(k,\tau), & K_1 = 1, \\ e^{k\tau}, & K_1 = 0, \\ 1/2\cosh k\tau, & K_1 = -1, \end{cases} \tag{9.4.1}$$

where $s(k,\tau)$ is defined by (9.3.5) and the physical time is determined by the integral $t = \pm\int e^{\gamma(\tau)} d\tau$. The constant k is connected with the "scalar charge" S according to (9.3.6), (9.3.7) where one should substitute $h_i = 0$ $(i > 1)$ and $h_1 = k/2$:

$$2S = \begin{cases} 3k^2 \operatorname{sign} k, & K_1 = \pm 1, \\ 3k^2, & K_1 = 0. \end{cases} \tag{9.4.2}$$

It is easy to obtain that in the case of a spherical world ($K_1 = 1$) the values $\tau = \pm\infty$ correspond to finite times t_1 and t_2 at which $a = 0$ (the initial and final singularities). For a flat world ($K_1 = 0$) at $k \neq 0$ and a hyperbolic one ($K_1 = -1$) at $k > 0$ an initial or final singularity is observed at infinite τ. In the special case $K_1 = -1$, $k = 0$ we obtain the Milne vacuum model which is known to describe a domain in flat space-time (in this case $S = 0$, so that the scalar field is trivial).

Lastly, in the case $K_1 = -1$, $k < 0$ we see that the limits $\tau \to 0$, $\pi/|k|$ correspond to $t \to \pm\infty$; the scale factor $a(t)$ decreases in an asymptotically linear manner in the remote past ($t \to -\infty$), reaches a minimum at $\tau = \pi/2|k|$ and grows in an asymptotically linear manner at $t \to \infty$ while the scalar field ω changes monotonically from one limiting value ω_- at $t \to -\infty$ to another limiting value ω_+ at $t \to +\infty$. The model is time-symmetric with respect to the maximum contraction instant. The tipical shape of the function $a(t)$ for this case is shown in Fig. 18.

By (9.4.2) a necessary condition for the existence of nonsingular solutions is the restriction $F < 0$ on the function (9.2.4), (as in this case $S < 0$), or, in terms of the initial function $B(\omega)$: $B < 3/2$.

These results confirm those of Ref. [140].

Consider now the metric $\overline{g}_{AB}$ for $n = 2$: let $a(t) \equiv e^{\beta_1(\tau)}$ be the scale factor of the ordinary physical space ($N_1 = 3$), while $b(t) \equiv e^{\beta_2(\tau)}$ that of the internal space ($N_2 = N$).

In the case $K_1 = 0$ (spatially flat models) we obtain:

$$d\overline{s}^2 = e^{2(3h_1+Nh_2)\tau} d\tau^2 - e^{2h_1\tau} ds_1^2 - e^{2h_2\tau} ds_2^2 \tag{9.4.3}$$

where with no loss of generality the scales in M_1 and M_2 are chosen so that $\beta_{10} = \beta_{20} = 0$. Herewith

$$6(h_1 + Nh_2/2)^2 = N(N + 1/2) + S \tag{9.4.4}$$

In the special case $3h_1 + Nh_2 = 0$ the time coordinate τ is synchronous, in other words, physical. The metric (9.4.3) is nonsingular at finite τ and describes an exponential expansion (inflation) of one of the spaces (e.g., the physical one, M_1) and a simultaneous exponential contraction of the other, M_2, since h_1 and h_2 have different signs. However, by (9.4.4) and (9.3.1)

$$S = F\dot{\omega}^2 = -h_1^2(2N + 1)/N < 0. \tag{9.4.5}$$

So, a necessary condition for the existence of the special solution (9.4.3) is the restriction

$$B(\omega) < (D - 1)(D - 2)/4, \tag{9.4.6}$$

more general than $B < 3/2$ for the 4-dimensional case.

In the more general case $3h_1 + Nh_2 = H \neq 0$ a transition to the physical time $dt = e^{H\tau}d\tau$ leads to the metric

$$d\bar{s}^2 = dt^2 - t^{2h_1/H}ds_1^2 - t^{2h_2/H}ds_2^2 \tag{9.4.7}$$

which is singular at $t = 0$ if at least one of the constants h_1 or h_2 is nonzero. At $h_1 = h_2 = 0$ the metric is static and (9.4.5) implies that either $\dot{\omega} = 0$ (the solution is trivial), or $F \equiv 0$, a special choice of B such that $\omega(\tau)$ has no dynamics.

For a spherical world ($K_1 = 1$) the metric is

$$d\bar{s}^2 = \frac{e^{-Nh\tau}}{2\cosh k\tau}\left[\frac{d\tau^2}{4\cosh^2 k\tau} - ds_1^2\right] - e^{2h\tau}ds_2^2 \tag{9.4.8}$$

where ds_1^2 is the line element on a unit sphere. A consideration like that as for $K_1 = 1$ leads to the following conclusions:

(a) The model behavior is classified by the values of the constant $h = h_2$ as compared with $k > 0$. The physical time $t = \pm\int e^{\gamma(\tau)}d\tau$ varies either within a finite segment $[t_1,\ t_2]$ (if $|Nh| < 3k$), or within a semi-infinite range (if $|Nh| \geq 3k$).

(b) At any finite boundary of the range of t at least one of the scale factors $a(t)$ or $b(t)$ vanishes, i.e., a singularity takes place.

(c) At $t \to \pm\infty$ either $a \to 0,\ b \to \infty$, or conversely, $a \to \infty,\ b \to 0$.

The value $S = -F\dot{\omega}^2$ is determined at $K_1 = \pm 1$ from

$$3k^2 \operatorname{sign} k = N(N+2)h^2 + 2S. \tag{9.4.9}$$

For hyperbolic models ($K_1 = -1$) the metric has the form

$$d\bar{s}^2 = \frac{e^{-Nh\tau}}{2s(k,\tau)}\left[\frac{d\tau^2}{4s^2(k,\tau)} - ds_1^2\right] - e^{2h\tau}ds_2^2 \tag{9.4.10}$$

(the same as (9.4.8) but the function $\cosh k\tau$ is replaced by $s(k,\tau)$ defined in (9.3.5). Preserving generality, let us assume $\tau > 0$.

The model behavior may be briefly described as follows:

(a) At $k > 0$, $Nh \le -3k$ or $k = 0$, $h < 0$ the physical time $t = \pm \int e^{\gamma(\tau)} d\tau$ ranges from $-\infty$ to $+\infty$. The factor $b(t) = e^{h\tau}$ varies from a finite value at $\tau = 0$ ($t = -\infty$) to zero at $\tau \to \infty$ ($t \to \infty$). The factor $a(t)$ describes a power-law contraction from infinity (at $t \to -\infty$) to a regular minimum and an infinite (in general, power-law) expansion at $t \to \infty$. There is no singularity at finite t.

(b) At $k \ge 0$, $Nh > 3k$ the model is singular at finite t corresponding to $\tau \to \infty$. In the special case $h = k = 0$ we come again to the Milne model supplemented with the space M_2 with a constant scale factor.

(c) At $k < 0$ the time t ranges again from $-\infty$ to $+\infty$. The factor $a(t)$ behaves as it did in item (a), however, its variation at $t \to \pm\infty$ is linear (but in general with unequal slopes at the two asymptotics). The factor $b(t)$ changes monotonically between two finite boundary values. The typical time dependence of the scale factors $a(t)$ and $b(t)$ in this case is shown in Fig. 19.

It is necessary to note that, unlike the 4-dimensional models, the nonsingular multidimensional ones with $h \ne 0$ exhibit a time-asymmetric behavior of $a(t)$.

It is seen in a straightforward way that in all the nonsingular models the requirement (9.4.6) is imposed on $B(\omega)$, which, as it could be formulated in general relativity, means the negative scalar field energy density.

Some properties of the above models have been discovered in numerical calculations for a number of special cases with $D = 5$ and $D = 6$ ([125] and section 8).

9.5. Euclidean Solutions

Keeping in mind possible applications of our models to quantum stages of the universe evolution, let us continue them to the Euclidean sector. For this purpose let us replace the metric (9.2.6) by a slightly more general one

$$d\bar{s}^2 = \bar{g}_{AB} dx^A dx^B = e^{2\gamma(\tau)} d\tau^2 + \sum_{i=1}^{n} \varepsilon_i e^{2\beta_i(\tau)} ds_i^2 \qquad (9.5.1)$$

where $\varepsilon_i = \pm 1$. Then in Eqs.(9.2.8) and consequently in the field equations the only change is that K_i are replaced by $\varepsilon_i K_i$. If we put, as before, $K_i = 0$ for $i \ge 2$, the equations depend only on $\varepsilon_1 K_1$. That means that the evolution of the Lorentzian open model ($K_1 = -1$, $\varepsilon_1 = -1$) coincides with that of the Euclidean closed model ($K_1 = 1$, $\varepsilon_1 = 1$) and vice versa, and the evolution of models with a flat 3-space ($K_1 = 0$) does not depend on the metric signature. In particular, the nonsingular Lorentzian model with an open 3-space, whose exterior and interior scale factors are shown in Figs. 19a,b, corresponds to the Euclidean four-dimensional wormhole $S^3 \times R^1$.

In conclusion, we have seen that many of the multidimensional Weyl cosmologies with flat additional spaces are nonsingular: there are special flat-space models with eternally increasing or decreasing scale factors (such models are absent in 4 dimensions) and there are more general hyperbolic models with a cosmological bounce (generalising the 4-dimensional ones [140]) which realize the dimensional reduction scenario. It has been shown that in the multidimensional case the evolution of the scale factor of the universe $a(t)$ becomes time-asymmetric, unlike the 4-dimensional case. In particular the evolution of $a(t)$ for big $|t|$ is asymptotically linear.

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