

# Minimal Closed Set of Observables in the Theory of Cosmological Perturbations<sup>1</sup>

Mário Novello

Centro Brasileiro de Pesquisas Físicas — Rio de Janeiro

The theory of perturbation of Friedman-Robertson-Walker (FRW) cosmology is analysed exclusively in terms of observable quantities. Although this can be a very complete and general procedure we limit our presentation here to the case of irrotational perturbations for simplicity. We show that the electric part of Weyl conformal tensor  $E$  and the shear  $\Sigma$  constitute the two basic perturbed variables in terms of which all remaining observable quantities can be described. Einstein's equations of General Relativity reduce to a closed set of dynamical system for  $E$  and  $\Sigma$ . The basis for a gauge-invariant Hamiltonian treatment of the Perturbation Theory in the FRW background is then set up.

## 1. Introductory Remarks

It has been a common practice (since Lifshitz's original paper [1]) to start the examination of the Perturbation Theory of Einstein's General Relativity by considering variations of non observable quantities such as  $\delta g_{\mu\nu}$ . The main drawback of this procedure is that it mixes true perturbations and arbitrary (infinitesimal) coordinate transformations. We are then faced with an extra task: the separation of true perturbation terms from a mere coordinate transformation. This is the so called gauge problem of the perturbation theory. A solution for this difficulty was found by many authors (cf. [2], [3], [4], [5], [6], [7], [8]) by looking for gauge-independent combinations which are written in terms of the metric tensor and its derivatives.

The next step would then be to provide from Einstein's equations, that deal with  $\delta g_{\mu\nu}$ , the dynamics of these gauge-independent variables which would then be used to describe physically relevant quantities.

Here we will follow a simpler (and more direct) path, inverting this procedure. That is, we will choose from the beginning, as the basis of our analysis, the gauge-invariant, physically observable quantities<sup>2</sup>. The dynamics for these fundamental quantities will then be analysed

<sup>1</sup>This talk was based on the homonymous paper, by M. Novello, J.M. Salim, M.C. Motta da Silva, S.E. Jorás & R. Klippert, published in *Phys. Rev. D* 51, 2 (1995), 50-461.

<sup>2</sup>Actually, the gauge problem does not even appear into our scheme and it can be completely ignored as far as basic physical quantities of the perturbation theory of FRW geometry are concerned. That is, there exists a dynamical system that can be analysed without references to this problem. However, to make contact with the standard procedure that deals with  $\delta g_{\mu\nu}$ , we will present the evolution of these associated gauge-dependent terms later on.

and any remaining gauge-dependent objects which we usually deal with will be obtained from this fundamental set.

There are basically two fundamental approaches by which the perturbation theory can be elaborated: one of them makes use of the standard Einstein's equations [1] and the other is based on the equivalent quasi-Maxwellian description [4] [9] [10]. In the case of the spatially homogeneous and isotropic FRW cosmological model the vanishing of Weyl conformal tensor suggests that the second approach is more attractive. In this case the variation of Weyl conformal tensor  $\delta W_{\alpha\beta\mu\nu}$  is the basic quantity to be considered, once there is certainly no doubt that  $\delta W_{\alpha\beta\mu\nu}$  is a true perturbation, which can not be achieved by a coordinate transformation. This solves *ab initio* the gauge problem that was pointed out in the first approach.

The crucial point of distinction between these approaches is that the dynamics of the observable quantities, as we shall see, does not require the knowledge of all components of  $\delta g_{\mu\nu}$ .

From a technical standpoint, instead of considering tensorial quantities, one should restrain oneself to scalar ones. There are two ways to implement this:

- Expand the relevant quantities in terms of a complete basis of functions (e.g. the spherical harmonics basis).
- Analyse the invariant geometric quantities one can construct from  $g_{\mu\nu}$  and its derivatives in the Riemannian background structure, that is, examine the 14 Debever invariants.

In any of these ways we shall see that the net result is that there is a set of perturbed quantities which can be divided into "good" quantities (i.e., the ones whose unperturbed counterparts have zero value in the background and, consequently, Stewart's lemma [12] guarantees that the associated perturbed quantity is really a gauge-independent one) and "bad" ones (whose corresponding values in the background are nonzero). One should limit therefore the analysis only to the "good" ones.

This same kind of behaviour seen for the geometrical structure of the model also exists both for the kinematic and dynamic quantities for the matter. Therefore the "good" quantities which constitute the set of variables with which we work should then be chosen from these particular scalars that come from these three structures: geometric, kinematic and dynamic. Does that mean that the present approach effectively avoids the gauge problem?

To answer this question affirmatively one should be able to exhibit a set of "good" variables in such a way that its corresponding dynamics is closed. That is, if we call  $\mathcal{M}_{\{A\}}$  the set of these variables, Einstein's equations should provide the dynamics of each element of  $\mathcal{M}_{\{A\}}$ , depending only on the background evolution quantities (and, eventually, on other elements of  $\mathcal{M}_{\{A\}}$ ). This would exhaust the perturbation problem and we shall show in this paper that this is indeed the case.

What should be learned from this discussion is that one should then understand the gauge problem not as a basic difficulty on the perturbation theory but just as a simple matter of asking a bad question<sup>3</sup>. One could imagine (what has been used a number of times in the literature [4], [5], [8]) that for FRW cosmology the perturbations of its main characteristics (the energy density,  $\delta\rho$  the scalar of curvature  $\delta R$  and the Hubble expansion factor  $\delta\Theta$ ) would be natural quantities to be considered as basic for the perturbation scheme. However, these are not "good" scalars, since they are not zero in the background<sup>4</sup>. We shall see in the next sections which scalars replace these ones.

### 1.1. Synopsis

In this paper we will deal only with observable quantities which are associated to true perturbations of the cosmological FRW geometry as the background. We are interested thus in the variation of the electric part of the Weyl tensor<sup>5</sup>, along with the variation of shear and acceleration, since these quantities are not induced by coordinate transformations.

We will analyse a certain set of "good" scalar quantities which constitute a closed set, i.e., one which provides a complete characterization of the perturbation problem. Let us choose them as the Electric Weyl tensor  $\delta E_{ij}$ , the shear  $\delta\sigma_{ij}$  and the anisotropic stress  $\delta\Pi_{ij}$  characterized by its corresponding magnitudes:

$$\sqrt{\delta E_{ij}\delta E^{ij}}$$

$$\sqrt{\delta\Pi_{ij}\delta\Pi^{ij}}$$

$$\sqrt{\delta\sigma_{ij}\delta\sigma^{ij}}$$

for the geometry, the dynamics and the kinematics respectively. In the case of perturbations allowing for vorticity we should add other invariants containing the Magnetic Weyl tensor  $\delta H_{ij}$ . One should not consider the restriction to the irrotational case as a limitation of this method but instead as an attempt of making our approach clearer and simpler in this paper. The application of this method to the case of vorticity perturbation and gravitational waves will be dealt with in a forthcoming paper.

In Section 2. we will present the complete set of definitions and equations which will be needed in this paper. We include for completeness the 14 Debever invariants and in Section 3. we will

<sup>3</sup> Let us point out that some of the gauge dependent terms are particularly relevant,  $\delta\rho$  among these.

<sup>4</sup> However, as it will be seen in a next section, we can construct associated "good" vector quantities in terms of these scalars.

<sup>5</sup> We will limit ourselves here only to irrotational perturbation of the velocity field. This implies that the corresponding magnetic part of Weyl tensor is absent. To prove this statement one has to use eqs.(7.83) and (7.92).

prove that they are not suitable to produce a fundamental nucleus  $\mathcal{M}_{[A]}$  from which we would describe all our theory. Nevertheless one of the 14 invariants will be included in  $\mathcal{M}_{[A]}$ . In this same section we will characterize the set  $\mathcal{M}_{[A]}$ , whose dynamics will be given in Section 4. A comparison with previous gauge-invariant variables is then established. For completeness the equations for some special remaining gauge-dependent quantities ( $\delta\rho$  and  $\delta\Theta$ ) are also exhibited. We shall see that, under very general circumstances, only the pair of gauge-invariant quantities,  $(E, \Sigma)$  (respectively the electric part of Weyl tensor and the shear), constitutes a closed dynamical system. This suggests the use of the gauge-invariant Hamiltonian treatment for this problem. We shall then lay the foundation for such a treatment, applied to these variables. This exhausts the total problem of perturbation theory in the FRW background<sup>6</sup>. In Section 5, the Fierz-Lanczos potential is analysed in the framework of FRW geometry. We exhibit the perturbation associated to this tensor and its relationship with the fundamental kinematic quantities, shear and acceleration. We end with Section 6., in which some general comments are given and future developments are sketched.

All equations we need (quasi-Maxwellian, the equations of constraint of matter and the equations of evolution and constraints for the kinematical parameters of a generic fluid) are presented in the Appendix.

## 2. Definitions and Notations

Greek indices run into the set  $\{0, 1, 2, 3\}$ . Latin indices run into the set  $\{1, 2, 3\}$ . Weyl conformal tensor is defined by means of the expression

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6}Rg_{\alpha\beta\mu\nu} \quad (2.1)$$

in which

$$g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu} \quad (2.2)$$

and

$$2M_{\alpha\beta\mu\nu} = R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu}. \quad (2.3)$$

<sup>6</sup> However, if one persists in asking questions about the evolution of intrinsically gauge-dependent quantities (as for instance the perturbed density  $\delta\rho$ ), then a gauge must be obviously fixed. We would like to emphasise once again that this does not constitute a drawback of the fundamental theory of perturbation, once — as we shall prove in the subsequent sections — it is possible, in order to solve this problem, to deal with a complete closed system of differential equations: this is precisely what will be done in this paper.

We use the completely skewsymmetric Levi-Civita tensor  $\eta_{\alpha\beta\mu\nu}$  to perform the dual operation. The 10 algebraically independent quantities of Weyl tensor can be separated in the corresponding electric and magnetic parts, defined (by analogy with the electromagnetic field) as:

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu} V^\mu V^\nu \quad (2.4)$$

$$H_{\alpha\beta} = -W^*_{\alpha\mu\beta\nu} V^\mu V^\nu. \quad (2.5)$$

From the symmetry properties of Weyl tensor it follows that the dual operation is independent on the pair in which it is applied.

These definitions yield that tensors  $E_{\mu\nu}$  and  $H_{\mu\nu}$  are symmetric, traceless and belong to the tridimensional space, orthogonal to the observer with 4-velocity  $V^\mu$ , that is:

$$\begin{aligned} E_{\mu\nu} &= E_{\nu\mu} \\ E_{\mu\nu} V^\mu &= 0 \\ E_{\mu\nu} g^{\mu\nu} &= 0, \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} H_{\mu\nu} &= H_{\nu\mu} \\ H_{\mu\nu} V^\mu &= 0 \\ H_{\mu\nu} g^{\mu\nu} &= 0. \end{aligned} \quad (2.7)$$

The metric  $g_{\mu\nu}$  and the vector  $V^\mu$  (tangent to a timelike congruence of curves  $\Gamma$ ) induce a projector tensor  $h_{\mu\nu}$  which separates any tensor in terms of quantities defined along  $\Gamma$  plus quantities defined on the 3-dimensional space  $\mathcal{H}$ , orthogonal to  $V^\mu$ . The tensor  $h_{\mu\nu}$ , defined in  $\mathcal{H}$ , is symmetric and a true projector, that is

$$h_{\mu\nu} h^{\nu\lambda} = \delta_\mu^\lambda - V_\mu V^\lambda = h_\mu^\lambda. \quad (2.8)$$

FRW geometry is written in the standard Gaussian coordinate system as

$$ds^2 = dt^2 + g_{ij} dx^i dx^j \quad (2.9)$$

in which  $g_{ij} = -A^2(t)\gamma_{ij}(x^h)$ . The 3-dimensional geometry has constant curvature and thus the corresponding Riemannian tensor  ${}^{(3)}R_{ijM}$  can be written as

$${}^{(3)}R_{ijM} = K\gamma_{ijM}.$$

Covariant derivative in the 4-dimensional space-time will be denoted by the symbol (;) and the 3-dimensional derivative will be denoted by ( $\parallel$ ).

The irreducible components of the covariant derivative of  $V^\alpha$  are given in terms of the expansion scalar ( $\Theta$ ), shear ( $\sigma_{\alpha\beta}$ ), vorticity ( $\omega_{\mu\nu}$ ) and acceleration ( $a_\alpha$ ) by the standard definition:

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} + \omega_{\alpha\beta} + a_\alpha V_\beta \quad (2.10)$$

in which

$$\begin{aligned} \sigma_{\alpha\beta} &= \frac{1}{2}h_{(\alpha}^{\mu} h_{\beta)}^{\nu} V_{\mu\nu} - \frac{1}{3}\Theta h_{\alpha\beta} \\ \Theta &= V^\alpha{}_{;\alpha} \\ \omega_{\alpha\beta} &= \frac{1}{2}h_{[\alpha}^{\mu} h_{\beta]}^{\nu} V_{\mu\nu} \\ a_\alpha &= V_{\alpha;\beta} V^\beta \end{aligned} \quad (2.11)$$

We also define

$$\Theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}. \quad (2.12)$$

Since the original Lifshitz paper [1] it has shown to be useful to develop all perturbed quantities in the spherical harmonics basis. Once we are limiting ourselves to irrotational perturbations, it is enough to our purposes to take into account only the scalar  $Q(x^k)$  (with  $\dot{Q} = 0$ ) and its derived vector and tensor quantities. We have thus

$$\begin{aligned} Q_i &\equiv Q_{;i} \\ Q_{ij} &\equiv Q_{;i;j} \end{aligned} \quad (2.13)$$

where the scalar  $Q$  obeys the eigenvalue equation defined in the 3-dimensional background space by:

$$\nabla^2 Q = mQ \quad (2.14)$$

where  $m$  is the wave number and

$$\nabla^2 Q \equiv \gamma^{ik} Q_{;i|k} = \gamma^{ik} Q_{;i;k} \quad (2.15)$$

where the symbol  $\nabla^2$  denotes the 3-dimensional Laplacian. The traceless operator  $\hat{Q}_{ij}$  is defined as

$$\hat{Q}_{ij} = \frac{1}{m} Q_{;ij} - \frac{1}{3} Q \gamma_{ij} \quad (2.16)$$

and the divergence of  $\hat{Q}_{ij}$  is given by

$$\hat{Q}_{||a}^a = 2 \left( \frac{1}{3} - \frac{K}{m} \right) Q^i. \quad (2.17)$$

We remark that  $Q$  is a 3-dimensional object; therefore indices are raised with  $\gamma^{ij}$ , the 3-space metric.

In [13] the complete 14 algebraically independent invariants constructed with the curvature tensor were presented. Considering that we are using an adimensional metric tensor, we can classify them with respect to dimensionality as follows:

Dimensionality	Invariants
$L^{-3}$	$I_3$
$L^{-4}$	$I_1, I_2, I_6$
$L^{-5}$	$I_2, I_4, I_7, I_9, I_{12}$
$L^{-6}$	$I_8, I_{10}, I_{13}$
$L^{-10}$	$I_{11}, I_{14}$

The expressions for these invariants are:

$$I_1 = W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu}$$

$$I_2 = W_{\alpha\beta}{}^{\rho\sigma} W_{\rho\sigma}{}^{\mu\nu} W_{\mu\nu}{}^{\alpha\beta}$$

$$I_3 = W^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu}^*$$

$$I_4 = W^{\alpha\beta\rho\sigma} W_{\rho\sigma}{}^{\mu\nu} W_{\mu\nu}^{\alpha\beta}$$

$$I_5 = R$$

$$I_6 = C_{\mu\nu} C^{\mu\nu}$$

$$I_7 = C_{\alpha\beta} C^{\beta\mu} C_{\mu}{}^{\alpha}$$

$$I_8 = C_{\alpha\beta} C^{\beta\mu} C_{\mu\lambda} C^{\alpha\lambda}$$

$$I_9 = C_{\mu\nu} D^{\mu\nu}$$

$$I_{10} = D_{\mu\nu} D^{\mu\nu}$$

$$I_{11} = C_{\alpha\beta} D^{\beta\mu} D_{\mu}{}^{\alpha}$$

$$I_{12} = \tilde{D}_{\mu\nu} C^{\mu\nu}$$

$$I_{13} = \tilde{D}_{\mu\nu} D^{\mu\nu}$$

$$I_{14} = \tilde{D}_{\mu\nu} \tilde{D}^{\mu\alpha} C^{\nu}{}_{\alpha}$$

in which we used the following definitions:

$$C_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}$$

$$D_{\mu\nu} \equiv W_{\mu\alpha\nu\beta} C^{\alpha\beta} \tag{2.18}$$

$$\tilde{D}_{\mu\nu} \equiv W_{\mu\alpha\nu\beta}^* C^{\alpha\beta}.$$

### 3. Fundamental Perturbations of FRW Universe

As we observed in the previous section, a complete examination of the perturbation theory should naturally include the analysis of the evolution of the Debever metric invariants associated to FRW geometry.



The only non identically zero invariants of FRW geometry are given by

$$I_5 = (1 - 3\lambda)\rho$$

$$I_6 = \frac{3}{4}(1 + \lambda)^2 \rho^2$$

$$I_7 = -\frac{3}{8}(1 + \lambda)^3 \rho^3$$

$$I_8 = \frac{21}{64}(1 + \lambda)^4 \rho^4$$

in which we used Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu}$$

and the stress-energy tensor is that of a perfect fluid

$$T_{\mu\nu} = (1 + \lambda)\rho V_\mu V_\nu - \lambda\rho g_{\mu\nu}.$$

If we restrict ourselves to the linear perturbation theory, the only invariants which have non identically zero linear perturbation terms are  $I_5$ ,  $I_6$ ,  $I_7$ ,  $I_8$ ,  $I_9$  and  $I_{12}$ . Among these the first four are nonzero in the background and the latter two are zero, since the geometry is conformally flat. This could lead to the conclusion that  $I_9$  and  $I_{12}$  are the "good" scalars to be examined. However, a direct calculation shows that the latter two invariants have zero linear perturbation. Indeed, it follows from FRW geometry that the perturbation of  $I_9$  reduces to

$$\delta I_9 = C^{\mu\nu} C^{\alpha\beta} \delta W_{\mu\nu\alpha\beta}.$$

Then (due to the fact that Weyl tensor is trace-free) the above quantity vanishes identically. This result depends of course on the fact that the source of the background geometry is given by a perfect fluid. In effect we have in this case

$$\delta I_9 = (\rho + p)^2 \left( V^\mu V^\nu - \frac{1}{4}g^{\mu\nu} \right) \left( V^\alpha V^\beta - \frac{1}{4}g^{\alpha\beta} \right) \delta W_{\mu\nu\alpha\beta}$$

which is zero. For the same reasoning  $\delta I_{12}$ , given by

$$\delta I_{12} = C^{\mu\nu} C^{\alpha\beta} \delta W_{\mu\nu\alpha\beta}^*$$

also vanishes.

The corresponding perturbations for the remaining invariants are given by

$$\delta I_5 = (1 - 3\lambda)\delta\rho$$

$$\delta I_6 = \frac{2}{3}(1 + \lambda)^2 \rho \delta\rho$$

$$\delta I_7 = -\frac{8}{3}(1 + \lambda)^3 \rho^2 \delta\rho$$

$$\delta I_8 = \frac{28}{15}(1 + \lambda)^4 \rho^3 \delta\rho.$$

It follows from these results that the perturbations of these quantities are algebraically related<sup>7</sup>. Besides, once all these scalars have a non-zero background value, they do not belong to the minimum set of good quantities that we are searching for.

Corresponding difficulties occur for the standard kinematical and dynamical variables, that is, the expansion parameter  $\Theta$  and the density of energy  $\rho$  suffer from the same disease.

This is thus the bad choice for the basic variables which we should avoid. Let us now turn our attention to which good variables should be considered as the fundamental ones.

### 3.1. Geometric Perturbation

From the previous section it follows that

$$\sqrt{\delta E_{ij} \delta E^{ij}}$$

is the only quantity that characterizes without ambiguity a true perturbation of the Debever invariants<sup>8</sup>. We need thus to consider only the perturbed  $E_{ij}$  since, as we shall see, any other metric quantity does not belong to the "good" basic nucleus needed for a complete knowledge of the true perturbations. We then set the expansion of this tensor in terms of the spherical harmonic basis

$$\delta E_{ij} = E(t) \hat{Q}_{ij}(x^h). \quad (3.19)$$

Thus  $E(t)$  is the geometric quantity whose dynamics we are looking for.

<sup>7</sup> One can write these invariants in a pure geometrical way without using Einstein's equations. This does not modify our argument.

<sup>8</sup> This is a consequence of the vanishing of the perturbation of the magnetic part of Weyl tensor (cf. above).

### 3.2. Kinematical Perturbations

We restrict our considerations only to linear perturbation terms. The normalization of the 4-velocity yields that the variation of the time component of the perturbed velocity is related to the variation of the (0-0) component of the metric tensor, that is:

$$\delta V_0 = \frac{1}{2} \delta g_{00}. \quad (3.20)$$

The corresponding contravariant quantities are related as follows:

$$\delta V^0 = \frac{1}{2} \delta g^{00} = -\delta V_0. \quad (3.21)$$

The expansion of the perturbations of the 4-velocity in terms of the spherical harmonic basis is<sup>9</sup>

$$\begin{aligned} \delta V_0 &= \frac{1}{3} \beta(t) Q(x^i) + \frac{1}{3} Y(t) \\ \delta V_k &= V(t) Q_k(x^i). \end{aligned} \quad (3.22)$$

For the acceleration we set

$$\delta a_k = \Psi(t) Q_k(x^i). \quad (3.23)$$

For the shear

$$\delta \sigma_{ij} = \Sigma(t) \hat{Q}_{ij}(x^k) \quad (3.24)$$

and for the expansion we set

$$\delta \Theta = H(t) Q(x^i) + Z(t) \quad (3.25)$$

where  $Y(t)$  and  $Z(t)$  are homogeneous terms that are not true perturbations.

Let us point out that, once we are limiting ourselves to the analysis of true perturbed quantities, the important kinematical variable whose dynamics we need to examine is only  $\Sigma(t)$ , since the other gauge-invariant quantity  $\Psi$  is a function of  $\Sigma$  (and  $E$ ), as we shall see ( $\beta$  is just a matter of choice of the coordinate system).

<sup>9</sup>The vorticity is of course zero, since we are limiting ourselves to the irrotational case.

### 3.3. Matter Perturbation

Since we are considering a background geometry in which there is a state equation relating the pressure and the energy density, i.e.  $p = \lambda\rho$ , we will consider the standard procedure that accepts the preservation of this state equation under arbitrary perturbations. Besides, our frame is such that there is no heat flux. Thus the general form of the perturbed energy-momentum tensor is given by

$$\delta T_{\mu\nu} = (1 + \lambda) \delta(\rho V_\mu V_\nu) - \lambda \delta(\rho g_{\mu\nu}) + \delta \Pi_{\mu\nu}. \quad (3.26)$$

We write  $\delta\rho$  in terms of the scalar basis as:

$$\delta\rho = N(t) Q(x^i) + \mu(t) \quad (3.27)$$

in which the homogeneous term  $\mu(t)$  is not a true perturbation<sup>10</sup>.

According to causal thermodynamics the evolution equation of the anisotropic pressure is related to the shear through [14]

$$\tau \dot{\Pi}_{ij} + \Pi_{ij} = \xi \sigma_{ij} \quad (3.28)$$

in which  $\tau$  is the relaxation parameter and  $\xi$  is the viscosity parameter. For simplicity of this present treatment we will limit ourselves to the case in which  $\tau$  can be neglected and  $\xi$  is a constant<sup>11</sup>; eq.(3.28) then gives

$$\Pi_{ij} = \xi \sigma_{ij} \quad (3.29)$$

and the associated perturbed equation is:

$$\delta \Pi_{ij} = \xi \delta \sigma_{ij}. \quad (3.30)$$

Following the same reasoning as before,  $\delta \Pi_{ij}$  is the matter quantity that should enter in the complete system of differential equations which describes the perturbation evolution. One should also be interested in the dynamics of  $\delta\rho$  although it is not a fundamental part of the basic system of equations. We will examine its evolution later on.

The "good" set  $\mathcal{M}_{[A]}$  has therefore three elements:  $\delta E_{ij}$ ,  $\delta\sigma_{ij}$  and  $\delta \Pi_{ij}$ . But, since  $\delta \Pi_{ij}$  is written in terms of  $\delta\sigma_{ij}$ , the set  $\mathcal{M}_{[A]}$  which will be considered reduces to:

$$\mathcal{M}_{[A]} = \{\delta E_{ij}, \delta\sigma_{ij}\}.$$

So much for definitions. Let us then turn to the analysis of the dynamics.

<sup>10</sup>We will set  $Y = Z = \mu = 0$ , since these homogeneous terms are just a matter of choice of the coordinate system. Nevertheless we are not interested in examining pure gauge quantities such as  $Y$ ,  $Z$  and  $\mu$ .

<sup>11</sup>In the general case  $\xi$  and  $\tau$  are functions of the equilibrium variables, for instance  $\rho$  and the temperature  $T$  and, since both variations  $\delta \Pi_{ij}$  and  $\delta\sigma_{ij}$  are expanded in terms of the traceless tensor  $\hat{Q}_{ij}$ , it follows that the above relation does not restrain the kind of fluid we are examining. However, if we consider  $\xi$  as time-dependent, the quantity  $\delta \Pi_{ij}$  must be included in the fundamental set  $\mathcal{M}_{[A]}$ .

#### 4. Dynamics

In this section we will show that  $E(t)$  and  $\Sigma(t)$  constitute the fundamental pair of variables in terms of which all the dynamics for the perturbed FRW geometry is given, that is,  $\mathcal{M}_{[A]} = \{E(t), \Sigma(t)\}$  is the minimal closed set of observables in the perturbation theory of FRW which characterizes and determines completely the spectrum of perturbations. Indeed, the evolution equations for these two quantities (which come from Einstein's equations) generate a dynamical system involving only  $E$  and  $\Sigma$  (and background quantities) which, when solved, contains all the necessary information for a complete description of all remaining perturbed quantities of FRW geometry. Such a conclusion does not seem to have been noticed in the past.

We remark that we will limit ourselves only to the examination of the perturbed quantities that are relevant for the complete knowledge of the system. These equations are the quasi-Maxwellian equations of gravitation and the evolution equations for the kinematical quantities. In [6] and [15] this system of equations was presented and analysed; we will list them in the Appendix for completeness.

##### 4.1. The Perturbed Equation for the Shear

The perturbed equation for the shear eq.(7.98) is written as:

$$\begin{aligned} h_{\alpha}^{\mu} h_{\beta}^{\nu} (\delta\sigma_{\mu\nu})^{\circ} &+ \frac{2}{3}\Theta \delta\sigma_{\alpha\beta} + \frac{1}{3}h_{\alpha\beta} \delta a^{\lambda}_{;\lambda} \\ &- \frac{1}{2}h_{\alpha}^{\mu} h_{\beta}^{\nu} [\delta a_{\mu\nu} + \delta a_{\nu\mu}] \\ &= \delta M_{\alpha\beta} \end{aligned} \quad (4.31)$$

where

$$M_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu} V^{\mu} V^{\nu} - \frac{1}{3}R_{\mu\nu} V^{\mu} V^{\nu} h_{\alpha\beta}. \quad (4.32)$$

Using the above spherical harmonics expansion and eq.(3.30), eq.(4.31) reduces to:

$$\dot{\Sigma} = -E - \frac{1}{2}\xi \Sigma + m \Psi. \quad (4.33)$$

##### 4.2. The Perturbed Equation for $E_{ij}$

The perturbed equation for the electric part of the Weyl tensor is given in the Appendix. Using the above spherical harmonics expansion and eq.(3.30) one obtains:

$$\begin{aligned} \dot{E} &= -\frac{(1+\lambda)}{2}\rho \Sigma - \left(\frac{\Theta}{3} + \frac{\xi}{2}\right) E \\ &- \frac{\xi}{2} \left(\frac{\xi}{2} + \frac{\Theta}{3}\right) \Sigma + \frac{m}{2} \xi \Psi. \end{aligned} \quad (4.34)$$

This suggests that  $E$  and  $\Sigma$  may be considered as canonically conjugated variables. We shall see later on that this is indeed the case.

Equations (4.33) and (4.34) contain three variables:  $E$ ,  $\Sigma$  and  $\Psi$ . We will now show that using the conservation law for the matter we can eliminate  $\Psi$  in all cases, except when  $(1+\lambda) = 0$ . We will return to this particular (vacuum) case in a later section.

The proof is the following. Projecting the conservation equation of the energy-momentum tensor in the 3-space, that is

$$T^{\mu\nu}{}_{;\nu} h_{\mu}{}^{\lambda} = 0 \quad (4.35)$$

and using the perturbed quantities this equation gives:

$$(1+\lambda)\rho \delta a_k - \lambda(\delta\rho)_{,k} + \lambda\dot{\rho} \delta V_k + \delta\Pi_k{}^i{}_{;i} = 0. \quad (4.36)$$

Using the decomposition in the spherical harmonics basis we obtain

$$(1+\lambda)\rho\Psi = \lambda[N - \dot{\rho}V] + 2\xi\left(\frac{1}{3} - \frac{K}{m}\right) A^{-2} \Sigma. \quad (4.37)$$

Now comes a remarkable result: the right hand side of eq.(4.37) can be expressed in terms of the variables  $E$  and  $\Sigma$  only (since we are analysing here the case where  $(1+\lambda)$  does not vanish). Indeed, from the equation of divergence of the electric tensor (see Appendix), we find

$$N - \dot{\rho}V = \left(1 - \frac{3K}{m}\right) \xi \Sigma A^{-2} - 2\left(1 - \frac{3K}{m}\right) A^{-2} E. \quad (4.38)$$

Combining these two equations we find that  $\Psi$  is given in terms of the background quantities and the basic perturbed terms  $E$  and  $\Sigma$ :

$$(1+\lambda)\rho\Psi = 2\left(1 - \frac{3K}{m}\right) A^{-2} [-\lambda E + \frac{1}{2}\lambda\xi\Sigma + \frac{1}{3}\xi\Sigma]. \quad (4.39)$$

Thus the whole set of perturbed equations reduces, for the variables  $E$  and  $\Sigma$ , to a time-dependent dynamical system:

$$\begin{aligned} \dot{\Sigma} &= F_1(\Sigma, E) \\ \dot{E} &= F_2(\Sigma, E) \end{aligned} \quad (4.40)$$

with

$$F_1 \equiv -E - \frac{1}{2}\xi\Sigma + m\Psi$$

and

$$F_2 \equiv -\left(\frac{1}{3}\Theta + \frac{1}{2}\xi\right) E \\ - \left(\frac{1}{4}\xi^2 + \frac{(1+\lambda)}{2}\rho + \frac{1}{6}\xi\Theta\right) \Sigma \\ + \frac{m}{2}\xi\Psi$$

in which  $\Psi$  is given in terms of  $E$  and  $\Sigma$  by eq.(4.39).

### 4.3. Comparison with Previous Gauge-Invariant Variables

FRW cosmology is characterized by the homogeneity of the fundamental variables that specify its kinematics (the expansion factor  $\Theta$ ), its dynamics (the energy density  $\rho$ ) and its associated geometry (the scalar of curvature  $R$ ). This means that these three quantities depend only on the global time  $t$ , characterized by the hypersurfaces of homogeneity. We can thus use this fact to define in a trivial way 3-tensor associated quantities, which vanish in this geometry, and look for its corresponding non-identically vanishing perturbation. The simplest way to do this is just to let  $U$  be a homogeneous variable (in the present case, it can be any one of the quantities  $\rho$ ,  $\Theta$  or  $R$ ), that is  $U = U(t)$ . Then use the 3-gradient operator  ${}^{(3)}\nabla_\mu$  defined by

$${}^{(3)}\nabla_\mu \equiv h_\mu^\lambda \nabla_\lambda \quad (4.41)$$

to produce the desired associated variable

$$U_\mu = h_\mu^\lambda \nabla_\lambda U. \quad (4.42)$$

In [16] these quantities were discussed and its associated evolution analysed. In the present section we will exhibit the relation of these variables to our fundamental ones. We shall see that under the conditions of our analysis<sup>12</sup> these quantities are functionals of our basic variables ( $E$  and  $\Sigma$ ) and the background ones.

### 4.4. The Matter Variable $\chi_i$

It seems useful to define the fractional gradient of the energy density  $\chi_\alpha$  as [16]

$$\chi_\alpha \equiv \frac{1}{\rho} {}^{(3)}\nabla_\alpha \rho. \quad (4.43)$$

<sup>12</sup>We remind the reader that we restrain here our examination to irrotational perturbation. The formulas which we obtain are thus simpler. However the method of our analysis is not restrictive and the study of generic cases can be obtained through the same lines.

Such quantity  $\chi_\alpha$  is nothing but a combination of the acceleration and the divergence of the anisotropic stress. Indeed, from eq.(4.36) it follows (in the frame in which there is no heat flux)

$$\delta\chi_i = \frac{(1+\lambda)}{\lambda} \delta a_i + \frac{1}{\lambda\rho} \delta\Pi_i{}^\beta{}_{;\beta} \quad (4.44)$$

From what we have learned above it follows that this quantity can be reduced to a functional of the basic quantities of perturbation, that is  $\Sigma$  and  $E$ , yielding

$$\delta\chi_i = -2 \left(1 - \frac{3K}{m}\right) \frac{1}{\rho A^2} \left(E - \frac{\xi}{2}\Sigma\right) Q_i. \quad (4.45)$$

#### 4.5. The Kinematical Variable $\eta_i$

The only non-vanishing quantity of the kinematics of the cosmic background fluid is the (Hubble) expansion factor  $\Theta$ . This allows us to define the quantity  $\eta_\alpha$  as:

$$\eta_\alpha = h_\alpha{}^\beta \Theta_{;\beta}. \quad (4.46)$$

Using the constraint relation eq.(7.90) we can relate this quantity to the basic ones:

$$\delta\eta_i = -\frac{\Sigma}{A^2} \left(1 - \frac{3K}{m}\right) Q_i. \quad (4.47)$$

#### 4.6. The Geometrical Variable $\tau$

We can choose the scalar of curvature  $R$  which depends only on the cosmical time  $t$  like  $\rho$  and  $\Theta$  to be the  $U$ -geometrical variable. However it seems more appealing to use a combined expression  $\tau$  involving  $R$ ,  $\rho$  and  $\Theta$  given by

$$\tau = R + (1 + 3\lambda)\rho - \frac{2}{3}\Theta^2. \quad (4.48)$$

In the unperturbed FRW background this quantity is defined in terms of the curvature scalar of the 3-dimensional space and the scale factor  $A(t)$ :

$$\frac{{}^{(3)}R}{A^2}$$

We define then the new associated variable  $\tau_\alpha$  as

$$\tau_\alpha = h_\alpha{}^\beta \tau_{;\beta}. \quad (4.49)$$



This quantity  $\tau_a$  vanishes in the background. Its perturbation can be written in terms of the previous variations, since Einstein's equations give

$$\tau = 2 \left( \rho - \frac{1}{3} \Theta^2 \right).$$

We can thus, without any information loss, limit all our analysis to the fundamental variables. Nevertheless, just for completeness, let us exhibit the evolution equations for some gauge-dependent variables.

#### 4.7. Perturbed Equations for $\rho$ and $\Theta$

From eq.(7.103) and using the decomposition of the perturbed energy density in the scalar basis (eq.(3.27)) we obtain the equation of evolution for  $\delta\rho$  as:

$$\dot{N} - \frac{1}{2} \beta \dot{\rho} + (1 + \lambda) \Theta N + (1 + \lambda) \rho H = 0. \quad (4.50)$$

Applying the same procedure for the perturbed Raychaudhuri equation (eq.(7.100)) and using the decomposition eq.(3.25) we obtain

$$\dot{H} - \frac{1}{2} \beta \dot{\Theta} + \frac{2}{3} \Theta H + \frac{m}{A^2} \Psi + \frac{(1 + 3\lambda)}{2} N = 0. \quad (4.51)$$

To solve these two equations we need to fix the gauge ( $\beta(t)$ ) and to use the values for  $E$  and  $\Sigma$  which were obtained from the fundamental closed system found in the previous section (eqs.(4.40)). All the remaining geometrical and kinematical quantities can be likewise obtained. This exhausts completely our analysis of the irrotational perturbations of FRW universe.

#### 4.8. The Singular Case $(1 + \lambda) = 0$ : The Perturbations of De Sitter Universe

We have seen that all the system of reduction to the variables  $\Sigma$  and  $E$  was based on the possibility of writing the acceleration in terms of  $E$  and  $\Sigma$ . This was possible in all cases, except in the special one in which  $(1 + \lambda) = 0$ . Although no known fluid exists with such negative pressure, the fact that the vacuum admits such an interpretation has led to the identification of the cosmological constant with this fluid. It is therefore worthwhile to examine this case in the same way as it was done for the previous sections.

At this point it must be remarked that, contrarily from all the previously studied cases, perturbations of this fluid must necessarily contain contributions which come from the heat flux or the anisotropic pressure. Indeed, if we take both of these quantities as vanishing, then the set of perturbed equations implies that all equations are trivially satisfied, since all perturbative

quantities vanish, except for the cases where  $\delta p = \bar{\lambda} \delta \rho$ , with  $\bar{\lambda} = 0$ , and  $\lambda + 1 = 0$ . We will analyse these cases below.

When  $\delta p = \bar{\lambda} \delta \rho$ , for  $\bar{\lambda} = 0$ , the system is stable. Indeed, we obtain for the electric part of Weyl tensor, in the case that  $\Theta$  is constant in the background, the following expression:

$$E(t) = E_0 e^{-\frac{2}{3}t}.$$

The other case of interest is the one in which the condition  $(1 + \lambda) = 0$  is preserved throughout the perturbation. Looking at eq.(4.50) it follows that, from the fact that  $\dot{\rho} = 0$  and reminding the reader that  $(1 + \lambda) = 0$ , temporal variation of the energy density exists only if we take into account the perturbed fluid with heat flux. We then write

$$q_i = q(t) Q_i(x^b).$$

Equation (7.103) gives

$$\dot{N} = \frac{m}{A^2} q. \quad (4.52)$$

The projected conserved equation gives (see eq.(7.104)):

$$\dot{q} + \Theta q + N = \frac{2\xi}{3A^2} \left(1 - \frac{3K}{m}\right) \Sigma. \quad (4.53)$$

The evolution equation for the electric part of Weyl tensor gives:

$$\dot{S} + \frac{\Theta}{3} S = -\frac{m}{2} q \quad (4.54)$$

in which we used the definition

$$S \equiv E - \frac{1}{2} \xi \Sigma.$$

Finally, from the equation that gives the divergence of  $E_{ij}$ , we have the constraint

$$\frac{2}{A^2} \left(1 - \frac{3K}{m}\right) S = -(N + \Theta q). \quad (4.55)$$

The evolution equation for the shear provides the value of the acceleration  $\Psi$ . Equations (4.53)-(4.55) constitute thus a complete system for the variables  $E$ ,  $\Sigma$  and  $q$ . This completes the general explicitly gauge-invariant scheme that we presented here even in the singular case  $(1 + \lambda) = 0$ . Notwithstanding, just as an additional comment, it would be interesting to consider the perturbation scheme in the framework of Lanczos potential. This will be done in a later section.

## 4.9. Hamiltonian Treatment

The examination of the perturbations in FRW cosmology, which we analysed above, admits a Hamiltonian formulation that is worth considering here [17]. In this vein, the variables  $E$  and  $\Sigma$ , analysed in the previous section, are the ones that must be employed to obtain such a formulation. From the evolution equations for  $\Sigma$  and  $E$  (eq.(4.40)) it follows that they are not canonically conjugated for arbitrary geometries of the background.

The natural step would be to define canonically conjugated variables  $Q$  and  $P$  as a linear functional of  $\Sigma$  and  $E$  as<sup>13</sup>:

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \alpha & \eta \\ \delta & \beta \end{bmatrix} \begin{bmatrix} \Sigma \\ E \end{bmatrix}. \quad (4.56)$$

It should be expected that functionals of the background geometry would appear in the construction of the canonical variables in the functions  $\alpha$ ,  $\beta$ ,  $\eta$  and  $\delta$ . It seems worth to remark that this matrix is univocally defined up to canonical transformations. We can thus use this fact to choose  $\eta$  and  $\delta$  as zero; we shall use this choice in order to simplify our analysis.

The Hamiltonian  $\mathcal{H}$  which provides the dynamics of the pair  $(Q, P)$  is obtained from the evolution equations of  $E$  and  $\Sigma$  (4.40). The condition for the existence of such a Hamiltonian is given by the equation

$$\begin{aligned} \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} - \xi - \frac{1}{3} \Theta \\ + \frac{2m\xi}{3(1+\lambda)\rho A^2} \left(1 - \frac{3K}{m}\right) = 0. \end{aligned} \quad (4.57)$$

It then follows that the Hamiltonian which provides the dynamics of our problem takes the form

$$\mathcal{H} = \frac{h_1}{2} Q^2 + \frac{h_2}{2} P^2 + 2 h_3 P Q \quad (4.58)$$

where  $h_1$ ,  $h_2$  and  $h_3$  are defined as

$$\begin{aligned} h_1 \equiv \frac{\beta}{\alpha} \left\{ \frac{(1+\lambda)}{2} \rho + \frac{\xi}{2} \left( \frac{\xi}{2} + \frac{\Theta}{3} \right) \right. \\ \left. - \frac{m\xi^2}{(1+\lambda)\rho A^2} \left(1 - \frac{3K}{m}\right) \left( \frac{\lambda}{2} + \frac{1}{3} \right) \right\} \end{aligned} \quad (4.59)$$

$$h_3 \equiv -\frac{\alpha}{\beta} \left\{ 1 + \frac{2m\lambda}{(1+\lambda)\rho A^2} \left(1 - \frac{3K}{m}\right) \right\} \quad (4.60)$$

<sup>13</sup>The attentive reader should notice that in this subsection the quantity  $Q$  shall not be confused with the previous scalar basis.

$$h_3 \equiv \frac{\Theta}{3} - \frac{\beta}{\beta} + \frac{\xi}{4} \left\{ 1 + \frac{2m\lambda}{(1+\lambda)\rho A^2} \left( 1 - \frac{3K}{m} \right) \right\}. \quad (4.61)$$

Let us consider the case in which  $\xi = 0$ , that is, there is no anisotropic pressure. The case where  $\xi$  does not vanish presents some interesting peculiarities which will be left to a forthcoming paper.

We will choose  $\beta = A$  and take  $\alpha$  as given by eq.(4.57). We then define the canonical variables  $Q$  and  $P$  by setting

$$Q = \Sigma$$

$$P = A E.$$

It then follows that  $\mathcal{H}$  is given by

$$\mathcal{H} = -\Delta^2(t) P^2 + \gamma^2(t) Q^2 \quad (4.62)$$

where  $\gamma(t)$  and  $\Delta(t)$  are given in terms of the energy density of the background  $\rho$ , the scale factor  $A(t)$  and the wave number  $m$  as:

$$\gamma^2(t) \equiv \left( \frac{1+\lambda}{4} \right) \rho A \quad (4.63)$$

$$\Delta^2(t) \equiv \frac{1}{2A} \left( 1 + \frac{2m\lambda}{(1+\lambda)\rho A^2} \left( 1 - \frac{3K}{m} \right) \right).$$

Let us make two comments here: first of all, the fact that the system is not conservative (which means  $\dot{\mathcal{H}}$  is not zero) is a consequence of the fact that the ground state of this theory ( $Q = P = 0$ ) corresponds not to Minkowski flat space-time but to FRW expanding universe. The second remark is that the same applies to the non-positivity of the Hamiltonian; this is also a consequence of the non-vanishing of the curvature of the fundamental state. The system which we are analysing is not closed and so momentum and energy can be pumped from the background.

We notice that the Hamiltonian structure obtained in terms of the variables  $E$  and  $\Sigma$  is completely gauge-invariant and, as such, deserves an ulterior analysis, which we will make elsewhere. We would like only to exhibit an example where this pumping effect can be easily recognized; this will be achieved by applying the Hamiltonian treatment to a static model of the universe.

### Einstein's Static Universe

In this case the expansion vanishes and consequently  $\gamma(t)$  and  $\Delta(t)$  become constant. The above Hamiltonian reduces thus to:

$$\mathcal{H} = -\frac{1}{2\mu^2} P^2 + \frac{1}{2} \omega^2 Q^2 \quad (4.64)$$

where  $\mu$  and  $\omega$  are obtained from eq.(4.63) for  $\rho$  and  $A$  constant.

This is nothing but the oscillator Hamiltonian with an imaginary mass. We recover then the well known result of the instability of Einstein's universe.

## 5. Fierz-Lanczos Potential

As it was remarked in a previous section, perturbations of conformally flat spacetimes do not need<sup>14</sup> the complete knowledge of all components of the perturbed metric tensor  $\delta g_{\mu\nu}$ , although they certainly need to take into account the Weyl conformal tensor, since all the observable information we need is contained in it (namely,  $\delta E_{ij}$  and  $\delta H_{ij}$ ).

Let us note at this point that the tensor  $W_{\alpha\beta\mu\nu}$  can be expressed in terms of the 3-index Fierz-Lanczos potential tensor, [18], [19], that we will denote by  $L_{\alpha\beta\mu}$ , and which deserves a careful analysis. Indeed, one could consider  $\delta L_{\alpha\beta\mu}$  as the good object for studying linear perturbation theory, since as we shall see it combines both  $\delta\Sigma_{ij}$  and  $\delta a_k$  (which are alternative variables to describe  $\delta E_{ij}$ ).

Before going into the perturbation-related details let us summarise here some definitions and properties of  $L_{\alpha\beta\mu}$ , since the literature has very few papers on this matter<sup>15</sup>.

### 5.1. Basic Properties

In any 4-dimensional Riemannian geometry there exists a 3-index tensor  $L_{\alpha\beta\mu}$  which has the following symmetries:

$$L_{\alpha\beta\mu} + L_{\beta\alpha\mu} = 0 \quad (5.65)$$

$$L_{\alpha\beta\mu} + L_{\beta\mu\alpha} + L_{\mu\alpha\beta} = 0. \quad (5.66)$$

With such  $L_{\alpha\beta\mu}$  we may write the Weyl tensor in form of a homogeneous expression in the potential expression, that is

$$\begin{aligned} W_{\alpha\beta\mu\nu} &= L_{\alpha\beta[\mu\nu]} + L_{\mu\nu[\alpha\beta]} + \\ &+ \frac{1}{2} [L_{(\alpha\nu)}g_{\beta\mu} + L_{(\beta\mu)}g_{\alpha\nu} - L_{(\alpha\mu)}g_{\beta\nu} - L_{(\beta\nu)}g_{\alpha\mu}] + \\ &+ \frac{2}{3} L^{\sigma\lambda}{}_{\sigma;\lambda} g_{\alpha\beta\mu\nu} \end{aligned} \quad (5.67)$$

<sup>14</sup>The reader should refer to the above quoted gauge problem which has been widely discussed in the literature (see the references given in the Introduction).

<sup>15</sup>This tensor was introduced in the 30's to provide, in a similar way as the symmetric tensor  $\varphi_{\mu\nu}$  does — in a more used approach — an alternative description of spin-2 field in Minkowski background. In the 60's Lanczos rediscovered it — without recognising he was dealing with the same object — as a Lagrange multiplier in order to obtain the Bianchi identities in the context of Einstein's General Relativity. However only recently [20], [21] a complete analysis of Fierz-Lanczos object was undertaken and it was discovered that its generic (Fierz) version describes not only one but two spin-2 fields. The restriction to just a single spin-2 field is usually called the Lanczos tensor. We will limit all our considerations here to this restricted quantity.

where

$$L_{\alpha\mu} \equiv L_{\alpha}{}^{\sigma}{}_{\mu\sigma} - L_{\alpha\mu}$$

and

$$L_{\alpha} \equiv L_{\alpha}{}^{\sigma}{}_{\sigma}$$

Let us point out that, due to the above symmetry properties, eqs.(5.65) and (5.66), Lanczos tensor has 20 degrees of freedom. Since Weyl tensor has only 10 independent components, it follows that there is a gauge symmetry involved. This gauge symmetry can be separated into two classes:

$$\Delta^{(1)}L_{\alpha\beta\mu} = M_{\alpha} g_{\beta\mu} - M_{\beta} g_{\alpha\mu} \quad (5.68)$$

$$\begin{aligned} \Delta^{(2)}L_{\alpha\beta\mu} &= W_{\alpha\beta;\mu} - \frac{1}{2}W_{\mu\alpha\beta} + \frac{1}{2}W_{\mu\beta;\alpha} \\ &+ \frac{1}{2}g_{\mu\alpha} W_{\beta}{}^{\lambda}{}_{;\lambda} - \frac{1}{2}g_{\mu\beta} W_{\alpha}{}^{\lambda}{}_{;\lambda} \end{aligned} \quad (5.69)$$

in which the vector  $M_{\alpha}$  and the antisymmetric tensor  $W_{\alpha\beta}$  are arbitrary quantities.

## 5.2. Lanczos Tensor for FRW Geometry

The fact that Friedmann-Robertson-Walker geometry is conformally flat implies that the associated Lanczos potential is nothing but a gauge. That is, we can write the Lanczos potential for FRW geometry as

$$\begin{aligned} L_{\alpha\beta\mu} &= N_{\alpha} g_{\beta\mu} - N_{\beta} g_{\alpha\mu} + F_{\alpha\beta\mu} - \frac{1}{2}F_{\mu\alpha\beta} \\ &+ \frac{1}{2}F_{\mu\beta;\alpha} + \frac{1}{2}g_{\mu\alpha} F_{\beta}{}^{\lambda}{}_{;\lambda} - \frac{1}{2}g_{\mu\beta} F_{\alpha}{}^{\lambda}{}_{;\lambda} \end{aligned} \quad (5.70)$$

for the arbitrary vector  $N_{\alpha}$  and the antisymmetric tensor  $F_{\alpha\beta}$ .

## 5.3. Perturbed Fierz-Lanczos Tensor

In the case we are examining in this paper (irrotational perturbations) the perturbed Weyl tensor reduces to the form

$$\delta W_{\alpha\beta\mu\nu} = (\eta_{\alpha\beta\gamma\epsilon} \eta_{\mu\nu\lambda\rho} - g_{\alpha\beta\gamma\epsilon} g_{\mu\nu\lambda\rho}) V^{\gamma} V^{\lambda} \delta E^{\epsilon\rho} \quad (5.71)$$

since the magnetic part of Weyl tensor remains zero in this case.

It then follows that the perturbed electric tensor is given in terms of Lanczos potential as:

$$\begin{aligned}
 -\delta E_{ij} &= \delta L_{0i[0;j]} + \delta L_{0j[0;i]} - \frac{1}{2}\delta L_{(00)}\gamma_{ij} \\
 &- \frac{1}{2}\delta L_{(ij)} + \frac{2}{3}\delta L^{\sigma\lambda}{}_{\sigma;j}\gamma_{ij}.
 \end{aligned} \tag{5.72}$$

Although the  $L_{\alpha\beta\mu}$  tensor is not a unique well defined object (since it has the gauge freedom we discussed above) we can use some theorems (see [22], [23]) that enable one to write  $L_{\alpha\beta\mu}$  in terms of the associated kinematic quantities of a given congruence of curves present in the associated Riemannian manifold. Following these theorems and choosing the case of irrotational perturbed matter it follows that  $\delta L_{\alpha\beta\mu}$  (the perturbed tensor of FRW background) is given by

$$\delta L_{\alpha\beta\mu} = \delta\sigma_{\mu[\alpha}V_{\beta]} + F(t)\delta a_{[\alpha}V_{\beta]}V_{\mu} \tag{5.73}$$

where

$$F(t) = 1 - \frac{1}{m} \frac{\Sigma}{\Psi} \left( \frac{2}{3}\Theta + \frac{1}{2}\xi \right). \tag{5.74}$$

In other words, the only non identically zero components of  $\delta L_{\alpha\beta\mu}$  are:

$$\delta L_{000} = -F(t)\Psi Q_0 \tag{5.75}$$

and

$$\delta L_{0ij} = -\Sigma(t)\hat{Q}_{ij} \tag{5.76}$$

that coincides with the previous results.

From what we have learned in the previous section, we can conclude that this is not a univocal expression, that is, eqs.(5.75) and (5.76) are obtained by a specific gauge choice.

Let us apply the above gauge transformation to the present case. In the first gauge, eq.(5.68), we decompose vector  $M_\alpha$  in the spatial harmonics (scalar and vector):

$$M_0 = M^{(1)}(t) Q(x) \tag{5.77}$$

$$M_i = M^{(2)}(t) Q_i(x) \tag{5.78}$$

and in the second gauge, eq.(5.69), we have

$$W_{0i} = W^{(1)}(t) Q_i(x) \tag{5.79}$$

and

$$W_{ij} = -\frac{1}{A^2} \varepsilon_{ijk} W^{(2)}(t) Q^k(x). \quad (5.80)$$

To sum up, asking what is the Lanczos tensor for the perturbed FRW geometry is one of those questions (like the one about the perturbed tensor  $\delta g_{\mu\nu}$ ) that should be avoided, since this quantity is gauge-dependent. A good question to be asked should be — as we remarked before: What is the perturbation of Weyl tensor? This was precisely the motivation of the previous section.

## 6. Conclusions

In this paper we have shown that the electric part of the Weyl tensor  $E$  and the shear  $\Sigma$  can be taken as the two basic quantities which describe the evolution of all perturbed quantities of FRW universe in the irrotational case. For rotational perturbations the magnetic part of Weyl tensor,  $H$ , and the vorticity  $\Omega$  should be also included in the set  $\mathcal{M}_{[A]}$ . The proof of this remark will be presented in a forthcoming paper.

We used the quasi-Maxwellian system of equations, which is equivalent to Einstein's equations, but is more convenient to treat perturbations in conformally flat universes, e.g., FRW cosmology. We showed that it is possible to reduce all the dynamics to a pair of equations in  $\Sigma$  and  $E$ , providing a dynamic planar system. A reparametrization of these variables allows us then to establish a gauge-invariant Hamiltonian treatment for this class of perturbation.

This suggests a natural way of quantization, in which  $Q$  and  $P$  become operators of a Hilbert space. A simple look into the Hamiltonian suggests that this quantization will give rise to single mode squeeze states [17]. It is not difficult to see that introducing the pair  $(\Omega, H)$  will generate double mode squeeze states. This analysis is now under development.

## 7. Appendix — Quasi-Maxwellian Equations

We list below the quasi-Maxwellian equations of gravity<sup>16</sup>. They are obtained from Bianchi identities as true dynamical equations which describe the propagation of gravitational disturbances. Making use of Einstein's equations and the definition of Weyl tensor, Bianchi identities can be written in an equivalent form as

$$\begin{aligned} W^{\alpha\beta\mu\nu}{}_{;\nu} &= \frac{1}{2} R^{[\mu\alpha;\beta]} - \frac{1}{12} g^{\mu(\alpha} R^{;\beta]} \\ &= -\frac{1}{2} T^{\mu(\alpha;\beta]} + \frac{1}{6} g^{\mu(\alpha} T^{;\beta]}. \end{aligned}$$

<sup>16</sup> See, for instance, Ellis [11].



Using the decomposition of Weyl tensor in terms of  $E_{\alpha\beta}$  and  $H_{\alpha\beta}$  (see Section 2.) and projecting appropriately, Einstein's equations can be written in a form which is similar to Maxwell's equations. There are 4 independent projections for the divergence of Weyl tensor, namely:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} V_{\beta} V_{\mu} h_{\alpha}{}^{\sigma}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu} \eta^{\sigma\lambda}{}_{\alpha\beta} V_{\mu} V_{\lambda}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu} h_{\mu}{}^{(\sigma} \eta^{\tau)\lambda}{}_{\alpha\beta} V_{\lambda}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu} V_{\beta} h_{\mu}{}^{(\sigma} h_{\tau)\alpha}$$

The unperturbed quasi-Maxwellian equations are thus given by:

$$\begin{aligned} h^{\alpha\sigma} h^{\lambda\tau} E_{\alpha\lambda;\tau} + \eta^{\sigma}{}_{\beta\mu\nu} V^{\beta} H^{\nu\lambda} \sigma^{\mu}{}_{\lambda} + 3H^{\sigma\nu} \omega_{\nu} \\ = \frac{1}{3} h^{\alpha\sigma} \rho_{\alpha} + \frac{\Theta}{3} q^{\sigma} - \frac{1}{2} (\sigma^{\sigma}{}_{\nu} - 3\omega^{\sigma}{}_{\nu}) q^{\nu} \\ + \frac{1}{2} \pi^{\sigma\mu} a_{\mu} + \frac{1}{2} h^{\alpha\sigma} \pi_{\alpha}{}^{\nu}{}_{;\nu} \end{aligned} \quad (7.81)$$

$$\begin{aligned} h^{\alpha\sigma} h^{\lambda\tau} H_{\alpha\lambda;\tau} - \eta^{\sigma}{}_{\beta\mu\nu} V^{\beta} E^{\nu\lambda} \sigma^{\mu}{}_{\lambda} - 3E^{\sigma\nu} \omega_{\nu} \\ = (\rho + p) \omega^{\sigma} - \frac{1}{2} \eta^{\sigma\alpha\beta\lambda} V_{\lambda} q_{\alpha;\beta} \\ + \frac{1}{2} \eta^{\sigma\alpha\beta\lambda} (\sigma_{\mu\beta} + \omega_{\mu\beta}) \pi^{\mu}{}_{\alpha} V_{\lambda} \end{aligned} \quad (7.82)$$

$$\begin{aligned} h_{\mu}{}^{\sigma} h_{\nu}{}^{\lambda} \dot{H}^{\mu\nu} + \Theta H^{\sigma\lambda} - \frac{1}{2} H_{\nu}{}^{(\sigma} h^{\lambda)\mu} V^{\mu\nu} \\ + \eta^{\lambda\mu\nu\tau} \eta^{\sigma\tau\alpha} V_{\mu} V_{\tau} H_{\alpha\tau} \Theta_{\nu\beta} \\ - \alpha_{\alpha} E_{\beta}{}^{(\lambda} \eta^{\sigma)\tau\alpha\beta} V_{\tau} \\ + \frac{1}{2} E_{\beta}{}^{\mu}{}_{;\mu} h_{\mu}{}^{(\sigma} \eta^{\lambda)\tau\alpha\beta} V_{\tau} \\ = -\frac{3}{4} q^{(\sigma} \omega^{\lambda)} + \frac{1}{2} h^{\sigma\lambda} q^{\mu} \omega_{\mu} \\ + \frac{1}{4} \sigma_{\beta}{}^{(\sigma} \eta^{\lambda)\alpha\beta\mu} V_{\mu} q_{\alpha} \\ + \frac{1}{4} h^{\sigma}{}^{(\lambda} \eta^{\lambda)\alpha\beta\mu} V_{\mu} \pi_{\nu\alpha\beta} \end{aligned} \quad (7.83)$$

$$\begin{aligned} h_{\mu}{}^{\sigma} h_{\nu}{}^{\lambda} \dot{E}^{\mu\nu} + \Theta E^{\sigma\lambda} - \frac{1}{2} E_{\nu}{}^{(\sigma} h^{\lambda)\mu} V^{\mu\nu} \\ + \eta^{\lambda\mu\nu\tau} \eta^{\sigma\tau\alpha} V_{\mu} V_{\tau} E_{\alpha\tau} \Theta_{\nu\beta} + \alpha_{\alpha} H_{\beta}{}^{(\lambda} \eta^{\sigma)\tau\alpha\beta} V_{\tau} \\ - \frac{1}{2} H_{\beta}{}^{\mu}{}_{;\mu} h_{\mu}{}^{(\sigma} \eta^{\lambda)\tau\alpha\beta} V_{\tau} \\ = \frac{1}{6} h^{\sigma\lambda} (q^{\mu}{}_{;\mu} - q^{\sigma} a_{\mu} - \pi^{\mu\nu} \sigma_{\mu\nu}) \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2}(\rho + p)\sigma^{\alpha\lambda} + \frac{1}{2}\dot{q}(\epsilon^{\alpha\lambda}) \\
& - \frac{1}{4}h^{\mu(\epsilon}h^{\lambda)\alpha}q_{\mu\alpha} + \frac{1}{2}h_{\alpha\epsilon}h_{\mu}{}^{\lambda}\dot{\pi}^{\alpha\mu} \\
& + \frac{1}{4}\pi_{\beta}{}^{\epsilon}\sigma^{\lambda\beta} - \frac{1}{4}\pi_{\beta}{}^{\epsilon}\omega^{\lambda\beta} + \frac{1}{6}\Theta\pi^{\epsilon\lambda}.
\end{aligned} \tag{7.84}$$

The contracted Bianchi identities and Einstein's equations give the conservation law

$$T^{\mu\nu}{}_{;\nu} = 0.$$

Projecting it both in the parallel and the orthogonal subspaces we obtain:

$$T^{\mu\nu}{}_{;\nu}V_{\mu} = 0$$

$$T^{\mu\nu}{}_{;\nu}h_{\mu}{}^{\alpha} = 0$$

which give the following equations:

$$\dot{\rho} + (\rho + p)\Theta + \dot{q}^{\mu}V_{\mu} + q^{\alpha}{}_{;\alpha} - \pi^{\mu\nu}\Theta_{\mu\nu} = 0 \tag{7.85}$$

$$\begin{aligned}
(\rho + p)a_{\alpha} & - p_{;\mu}h^{\mu}{}_{\alpha} + \dot{q}_{\mu}h^{\mu}{}_{\alpha} + \Theta q_{\alpha} \\
& + q^{\nu}\Theta_{\alpha\nu} + \dot{q}^{\nu}\omega_{\alpha\nu} + \pi_{\alpha}{}^{\nu}{}_{;\nu} + \pi^{\mu\nu}\Theta_{\mu\nu}V_{\alpha} = 0
\end{aligned} \tag{7.86}$$

and, from the definition of Riemann curvature tensor

$$V_{\mu\alpha;\beta} - V_{\mu\beta;\alpha} = R_{\mu\alpha\beta}{}^{\nu}V_{\nu}$$

we obtain the equations of motion for the unperturbed kinematical quantities as:

$$\dot{\Theta} + \frac{\Theta^2}{3} + 2\sigma^2 + 2\omega^2 - a^{\alpha}{}_{;\alpha} = R_{\mu\nu}V^{\mu}V^{\nu} \tag{7.87}$$

$$\begin{aligned}
h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}\dot{\sigma}_{\mu\nu} & + \frac{1}{3}h_{\alpha\beta}(-2\omega^2 - 2\sigma^2 + a^{\lambda}{}_{;\lambda}) + a_{\alpha}a_{\beta} \\
& - \frac{1}{2}h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}(a_{\mu\nu} + a_{\nu\mu}) + \frac{2}{3}\Theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta} + \omega_{\alpha\mu}\omega^{\mu}{}_{\beta} \\
& = R_{\alpha\beta\gamma\nu}V^{\gamma}V^{\nu} - \frac{1}{3}R_{\mu\nu}V^{\mu}V^{\nu}h_{\alpha\beta}
\end{aligned} \tag{7.88}$$

$$\begin{aligned}
h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}\dot{\omega}_{\mu\nu} & - \frac{1}{2}h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}(a_{\mu\nu} - a_{\nu\mu}) + \frac{2}{3}\Theta\omega_{\alpha\beta} \\
& + \sigma_{\alpha\mu}\omega^{\mu}{}_{\beta} - \sigma_{\beta\mu}\omega^{\mu}{}_{\alpha} = 0.
\end{aligned} \tag{7.89}$$

We also obtain from the definition of  $R_{\alpha\beta\mu\nu}$  three constraint equations:

$$\frac{2}{3}\Theta_{,\mu}h^{\mu\lambda} - (\sigma^{\alpha}_{\gamma} + \omega^{\alpha}_{\gamma})_{;\mu}h^{\gamma\lambda} - a^{\nu}(\sigma_{\lambda\nu} + \omega_{\lambda\nu}) = R_{\mu\nu}V^{\mu}h^{\nu\lambda} \quad (7.90)$$

$$\omega^{\alpha}_{\mu;\nu} + 2\omega^{\alpha} a_{\alpha} = 0 \quad (7.91)$$

$$-\frac{1}{2}h_{\nu}{}^{\alpha}h_{\lambda}{}^{\beta}\eta_{\alpha}{}^{\rho\gamma\nu}V_{\nu}(\sigma_{\alpha\beta} + \omega_{\alpha\beta})_{;\gamma} + a_{(\gamma}\omega_{\lambda)} = H_{\gamma\lambda}. \quad (7.92)$$

These results constitute a set of 12 equations which will be used to describe the evolution of small perturbations in FRW background. Writing all the perturbed quantities in the form

$$X_{(\text{perturbed})} = X_{(\text{background})} + \delta X$$

and after straightforward manipulations one finally obtains [6] the perturbed equations from the set of equations (7.81)-(7.92) as:

$$\begin{aligned} (\delta E^{\mu\nu})^{\alpha} h_{\mu}{}^{\alpha} h_{\nu}{}^{\beta} &+ \Theta(\delta E^{\alpha\beta}) - \frac{1}{2}(\delta E_{\nu}{}^{(\alpha} h^{\beta)})_{\mu} V^{\mu\nu} \\ &+ \frac{\Theta}{3}\eta^{\rho\mu\nu}\eta^{\alpha\gamma\lambda}V_{\mu}V_{\nu}(\delta E_{\alpha\lambda})h_{\gamma\nu} \\ &- \frac{1}{2}(\delta H\lambda^{\mu})_{;\gamma}h_{\mu}{}^{(\alpha}\eta^{\beta)\gamma\lambda}V_{\nu} \\ &= -\frac{1}{2}(\rho + p)(\delta\sigma^{\alpha\beta}) \\ &+ \frac{1}{6}h^{\alpha\beta}(\delta q^{\mu})_{;\alpha} - \frac{1}{4}h^{\mu(\alpha}h^{\beta)\nu}(\delta q_{\mu})_{;\nu} \\ &+ \frac{1}{2}h^{\alpha\alpha}h^{\beta\nu}(\delta\Pi_{\mu\nu})^{\alpha} + \frac{1}{6}\Theta(\delta\Pi^{\alpha\beta}) \end{aligned} \quad (7.93)$$

$$\begin{aligned} (\delta H^{\mu\nu})^{\alpha} h_{\mu}{}^{\alpha} h_{\nu}{}^{\beta} &+ \Theta(\delta H^{\alpha\beta}) - \frac{1}{2}(\delta H_{\nu}{}^{(\alpha} h^{\beta)})_{\mu} V^{\mu\nu} \\ &+ \frac{\Theta}{3}\eta^{\rho\mu\nu}\eta^{\alpha\lambda\gamma}V_{\mu}V_{\nu}(\delta H_{\alpha\gamma})h_{\lambda\nu} \\ &- \frac{1}{2}(\delta E\lambda^{\mu})_{;\gamma}h_{\mu}{}^{(\alpha}\eta^{\beta)\gamma\lambda}V_{\nu} \\ &= \frac{1}{4}h^{\nu(\alpha}\eta^{\beta)\gamma\tau\mu}V_{\mu}(\delta\Pi_{\nu\alpha})_{;\gamma} \end{aligned} \quad (7.94)$$

$$(\delta H_{\alpha\mu})_{;\nu}h^{\alpha\alpha}h^{\mu\nu} = (\rho + p)(\delta\omega^{\alpha}) - \frac{1}{2}\eta^{\alpha\beta\mu}V_{\mu}(\delta q_{\alpha})_{;\beta} \quad (7.95)$$

$$\begin{aligned} (\delta E_{\alpha\mu})_{;\nu}h^{\alpha\nu}h^{\mu\nu} &= \frac{1}{3}(\delta\rho)_{;\alpha}h^{\alpha\nu} - \frac{1}{3}\dot{\rho}(\delta V^{\alpha}) \\ &- \frac{1}{3}\rho_{;\alpha}(\delta V^{\alpha})V^{\alpha} \\ &+ \frac{1}{2}h^{\alpha}{}_{\alpha}(\delta\Pi^{\alpha\nu})_{;\mu} + \frac{\Theta}{3}(\delta q^{\alpha}) \end{aligned} \quad (7.96)$$

$$(\delta\Theta)^{\circ} + \dot{\Theta}(\delta V^{\circ}) + \frac{2}{3}\Theta(\delta\Theta) - (\delta a^{\alpha})_{;\alpha} = -\frac{(1+3\lambda)}{2}(\delta\rho) \quad (7.97)$$

$$\begin{aligned} (\delta\sigma_{\mu\nu})^{\circ} + \frac{1}{3}h_{\mu\nu}(\delta a^{\alpha})_{;\alpha} - \frac{1}{2}(\delta a_{(\alpha);\beta})h_{\mu}^{\alpha}h_{\nu}^{\beta} \\ + \frac{2}{3}\Theta(\delta\sigma_{\mu\nu}) = -(\delta E_{\mu\nu}) - \frac{1}{2}(\delta\Pi_{\mu\nu}) \end{aligned} \quad (7.98)$$

$$(\delta\omega^{\mu})^{\circ} + \frac{2}{3}\Theta(\delta\omega^{\mu}) = \frac{1}{2}\eta^{\alpha\mu\rho\gamma}(\delta a_{\rho})_{;\gamma}V_{\alpha} \quad (7.99)$$

$$\begin{aligned} \frac{2}{3}(\delta\Theta)_{;\lambda}h^{\lambda\mu} - \frac{2}{3}\dot{\Theta}(\delta V_{\mu}) + \frac{2}{3}\dot{\Theta}(\delta V^{\circ})\delta_{\mu}^{\circ} \\ - (\delta\sigma^{\alpha\beta} + \delta\omega^{\alpha\beta})_{;\alpha}h^{\beta\mu} = -(\delta q_{\mu}) \end{aligned} \quad (7.100)$$

$$(\delta\omega^{\alpha})_{;\alpha} = 0 \quad (7.101)$$

$$(\delta H_{\mu\nu}) = -\frac{1}{2}h^{\alpha}{}_{(\mu}h^{\beta}{}_{\nu)}((\delta\sigma_{\alpha\gamma})_{;\lambda} + (\delta\omega_{\alpha\gamma})_{;\lambda})\eta_{\beta}{}^{\gamma\lambda}V_{\alpha} \quad (7.102)$$

$$(\delta\rho)^{\circ} + \dot{\rho}(\delta V^{\circ}) + \Theta(\delta\rho + \delta p) + (\rho + p)(\delta\Theta) + (\delta q^{\alpha})_{;\alpha} = 0 \quad (7.103)$$

$$\begin{aligned} \dot{p}(\delta V_{\mu}) + p_{;\alpha}(\delta V^{\alpha})\delta_{\mu}^{\circ} - (\delta p)_{;\beta}h^{\beta\mu} + (\rho + p)(\delta a_{\mu}) \\ + h_{\mu\alpha}(\delta q^{\alpha})^{\circ} + \frac{4}{3}\Theta(\delta q_{\mu}) + h_{\mu\alpha}(\delta\pi^{\alpha\beta})_{;\beta} = 0. \end{aligned} \quad (7.104)$$

## Bibliography

- [1] E. M. Lifshitz and I.M. Khalatnikov, *Adv.Phys.* **12**, (1963), 185.
- [2] J. Bardeen, *Phys. Rev. D* **22**, 8 (1980), 1882.
- [3] B. J. T. Jones, *Rev. Mod. Phys.* **48**, 1 (1976), 107.
- [4] S. W. Hawking, *Ap. J.* **145** (1966), 544.
- [5] D. W. Olson, *Phys. Rev. D* **14**, (1976), 327.
- [6] J. C. Hwang and E.T. Vishniac, *Ap. J.* **353**, (1990), 1-20.
- [7] R. Brandenberger, R. Khan and W. H. Press, *Phys. Rev. D* **28**, (1983), 1809.
- [8] V. F. Mukhanov, H. A. Feldmann and R. H. Brandenberger, *Phys. Rep.* **215**, (1992).
- [9] P. Jordan, J. Ehlers and R. Sachs, *Akad. Wiss. Lt. Meins Abh. Math. Nat. Kl. 1*, (1961), 3.
- [10] M. Novello, and J. M. Salim, *Fund. of Cosmic Phys.* **8**, (1983), 201-342.
- [11] G.F.R.Ellis, in *General Relativity and Cosmology*, Proceedings of the International School of Physics "Enrico Fermi", Course XLVII (Academic, London, 1971), p.104.
- [12] J. M. Stewart and M. Walker, *Proc. R. Soc. London*, **A341**, (1974), 49-74.
- [13] R. Debever, *Cahiers de Physique* **168**, (1964).
- [14] W. Israel, *An. Phys.* **100**, (1976), 310-331.
- [15] M. Novello, J. M. Salim and H. Heintschmann in *J. Leite Lopes Festschrift*, 1988, Ed. S. Joffily.
- [16] G. F. R. Ellis and M. Bruni, *Phys.Rev. D* **40**, 6 (1989), 1804.
- [17] L. P. Grishchuk and Yu V. Sidorov, *Phys. Rev. D*, **42**, (1990), 3413.
- [18] M. Fierz, *Helv. Phys. Acta* **12**, (1939), 379.
- [19] C. Lanczos, *Rev. Mod. Phys.* **34**, (1962), 379.
- [20] M. Novello and N. Pinto-Neto, *Fortschrift der Physik* **40**, (1992), 173.
- [21] M. Novello and N. Pinto-Neto, *Fortschrift der Physik* **40**, (1992), 195.
- [22] M. Novello and A. L. Velloso, *Gen. Rel. Grav.* **19**, (1987), 1251.
- [23] G. Ares de Parga, O. Chavoya, J.L. López Bonilla, *J. Math. Phys.* **30**, (1989), 1294.