Part VIII

GENERALIZED QUANTIZATIONS OF GAUGE THEORIES

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Abstract

A generalized version is proposed for the field–antifield formalism. The antibracket operation is defined in arbitrary field–antifield coordinates. The antisymplectic definitions are given for first– and second–class constraints. In the case of second–class constraints the Dirac's antibracket operation is defined. The quantum master equation as well as the hypergauge fixing procedure are formulated in a coordinate–invariant way. The general hypergauge functions are shown to be antisymplectic first–class constraints whose Jacobian matrix determinant is constant on the constraint surface. The BRST–type generalized transformations are defined and the functional integral is shown to be independent of the hypergauge variations admitted.

Section 1

Covariant quantization of gauge–field systems has a long–time history started from the famous works of Feynman [1], Faddeev and Popov [2] and DeWitt [3].

A unique closed approach to the covariant quantization problem has been proposed in work [4] of Batalin and Vilkovisky. These authors have introduced the field–antifield phase space concept as well as the antibracket operation that is an antisymplectic counterpart of the well–known Poisson bracket. Moreover, a nilpotent second–order differential operator has been discovered, that differentiates the antibracket according to the Leubnitz rule. Due to the mentioned property, henceforth we shall refer this remarkable operator as "antisymplectic differential".

The authors of the paper [4] have formulated the general quantization principle to be applied directly to the Lagrangian formalism. The principle requires for the exponential of $i/\hbar$ times quantum action to be annihilated by the antisymplectic differential. Thus the quantum master equation has appeared to acquire its great importance. The corresponding classical master equation requires for the classical master action to commute with itself in the antibracket sense to give zero.

The above–mentioned strategy has been applied successfully to the gauge theories with irreducible open algebras [4] and to the theories with linearly–dependent gauge generators [5], as well. Also the recent developments [6, 7, 8, 9] in secondary–quantized string field theory are substantially based on the BV approach.

Many authors have contributed to develop and apply the field–antifield formalism. For detailed references see the review lecture of Henneaux [10].

The contributions of Zinn–Justin [11], Kallosh [12], de Wit and van Holten [13] had been important to reveal the general status of the classical master equation.

Witten [14] has given a deep geometric interpretation of the quantum master equation.

An Sp(2)–covariant version of the BV formalism has been proposed recently by Batalin, Lavrov and Tyutin [15, 16, 17].

Henneaux [18] has extended the Witten's interpretation to cover the Sp(2)–covariant formulation.

A relation between the Hamiltonian BFV and Lagrangian BV formalisms has been revealed by Grigoryan, Grigoryan and Tyutin [19]. These authors have used a functional counterpart of the operator method proposed originally by Batalin and Fradkin [20].

Independently of the gauge field quantization problem, an invariant geometric description of the symplectic and antisymplectic structures on the Kählerian superspaces has been given by Khudaverdian and Nersessian [21, 22].

Volkov et all [23, 24] studied the antibracket reformulation and quantization of supersymmetric mechanics.

In the present work we undertake further steps in developing the field-antifield formalism.

The first problem is to give a coordinate-invariant formulation to the quantum master equation and to the hypergauge fixing procedure as well.

The second problem is to assign the antifields to the hypergauge Lagrangian multipliers and thus to give start to the hierarchical proliferation process that introduces the hypergauges of higher levels.

Notation and Convention. as is usual, $\varepsilon(A)$ denotes the Grassmann parity of a quantity A. By $\mathrm{rank}||X_{AB}||$ we denote maximal size of the invertible square block of a supermatrix $||X_{AB}||$.

Other notation is clear from the context.

Section 2

Let:

$$\Gamma^A, \quad A = 1, \ldots, 2N, \quad \varepsilon(\Gamma^A) \equiv \varepsilon_A, \tag{2.1}$$

be a set of field-antifield variables:

$$\{\Gamma^A\} = \{\varphi^a, \varphi^*_a | a = 1, \ldots, N, \varepsilon(\varphi^*_a) = \varepsilon(\varphi^a) + 1\}. \tag{2.2}$$

We consider the variables (2.2) to be local coordinates of the corresponding field–antifield phase space $\mathcal{M}$.

Let $E^{AB}(\Gamma)$ be a nondegenerate odd contravariant metric with the adjoint–antisymmetry property:

$$\varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1, \quad E^{AB} = -E^{BA}(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}, \tag{2.3}$$

while $F_A(\Gamma)$, $\varepsilon(F_A) = \varepsilon_A$, be a "connection" field.

Let us introduce the following generating operator [4]:

$$\Delta \equiv \frac{1}{2}(-1)^{\varepsilon_A}(\partial_A + F_A)E^{AB}\partial_B, \quad \varepsilon(\Delta) = 1, \tag{2.4}$$

to be nilpotent:

$$\Delta^2 = 0 \tag{2.5}$$

This nilpotency condition gives immediately:

$$\Delta(-1)^{\varepsilon_C}(\partial_C + F_C)E^{CD} = 0, \tag{2.6}$$

[4] In fact, the operator (2.4) is nothing other but the general form of a second–order differential operator whithout the derivativeless term. From this viewpoint even the adjoint–antisymmetry property (2.3) is not to be imposed imperatively. When being nonzero, the adjoint–symmetric part of E^{AB} can be absorbed into redefinition of the "connection" field F.

$$\partial_A F_B - \partial_B F_A (-1)^{\varepsilon_A \varepsilon_B} = 0, \tag{2.7}$$

$$(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)} E^{AD} \partial_D E^{BC} + \text{cycle}(A, B, C) = 0. \tag{2.8}$$

The equation (2.7) gives locally:

$$F_A = \partial_A \ln M(\Gamma), \tag{2.9}$$

so that the equation (2.6) determines, in fact, the scalar density $M(\Gamma)$:

$$\Delta (-1)^{\varepsilon_C} M^{-1} \partial_C M E^{CD} = 0, \tag{2.10}$$

$$\Delta = \frac{1}{2} (-1)^{\varepsilon_A} M^{-1} \partial_A M E^{AB} \partial_B. \tag{2.11}$$

An invariant measure $d\mu(\Gamma)$ on the phase space $\mathcal{M}$:

$$d\mu(\Gamma) \equiv M(\Gamma) d\Gamma \tag{2.12}$$

is naturally associated with the density $M(\Gamma)$.

In its own turn, the cyclic equation (2.8) is nothing other but the anti-symplecticity property of the metric E^{AB}. This property allows one to introduce naturally the antibracket operation:

$$(A,B) \equiv A\overleftarrow{\partial_C} E^{CD} \overrightarrow{\partial_D} B \tag{2.13}$$

with the following algebraic properties [25, 4]:

$$\varepsilon\left((A,B)\right) = \varepsilon(A) + \varepsilon(B) + 1, \tag{2.14}$$

$$(A,B) = -(B,A)(-1)^{(\varepsilon(A)+1)(\varepsilon(B)+1)}, \tag{2.15}$$

$$(A,BC) = (A,B)C + B(A,C)(-1)^{(\varepsilon(A)+1)\varepsilon(B)}, \tag{2.16}$$

$$((A,B),C)(-1)^{(\varepsilon(A)+1)(\varepsilon(C)+1)} + \mathrm{cycle}(A,B,C) = 0. \tag{2.17}$$

Besides, the formula [26]:

$$\Delta(A,B) = (\Delta A,B) + (A,\Delta B)(-1)^{\varepsilon(A)+1}. \tag{2.18}$$

represents the property of the antibracket operation with respect to applying the operator Δ.

On the other hand, applying the operator Δ to the ordinary product AB, we have:

$$\Delta(AB) = (\Delta A)B + (A,B)(-1)^{\varepsilon(A)} + A(\Delta B)(-1)^{\varepsilon(A)}. \tag{2.19}$$

The formula (2.18) shows that the operator Δ differentiates the antibracket (A,B) according to the Leubnitz rule. Due to this remarkable property, the nilpotent operator Δ is called "antisymplectic differential".

Having the antibracket operation at our disposal, we can introduce naturally a formal counterpart of the well–known Dirac's terminology in order to define constraints to be of the first or second class.

By definition, the functions:

$$G_i(\Gamma), \quad i = 1,\ldots,K, \quad \varepsilon(G_i) \equiv \varepsilon_i. \tag{2.20}$$

are called first–class constraints if the antibracket involution relations hold:

$$(G_i, G_j) = G_k U_{ij}^k, \tag{2.21}$$

where the structural coefficients $U_{ij}^k(\Gamma)$ possess the properties:

$$\varepsilon(U_{ij}^k) = \varepsilon_i + \varepsilon_j + \varepsilon_k + 1, \quad U_{ij}^k = -U_{ji}^k(-1)^{(\varepsilon_i+1)(\varepsilon_j+1)}. \tag{2.22}$$

Alternatively, the functions:

$$\Theta_\alpha(\Gamma), \quad \alpha = 1,\ldots,2L, \quad \varepsilon(\Theta_\alpha) \equiv \varepsilon_\alpha. \tag{2.23}$$

are called second–class constraints if their antibracket matrix:

$$Q_{\alpha\beta}(\Gamma) \equiv (\Theta_\alpha(\Gamma), \Theta_\beta(\Gamma)) \tag{2.24}$$

is nondegenerate:

$$\exists Q^{\alpha\beta} : \quad Q_{\alpha\beta}Q^{\beta\gamma} = \delta_\alpha{}^\gamma, \tag{2.25}$$

so that we have:

$$\varepsilon(Q^{\alpha\beta}) = \varepsilon_\alpha + \varepsilon_\beta + 1, \quad Q^{\alpha\beta} = -Q^{\beta\alpha}(-1)^{\varepsilon_\alpha\varepsilon_\beta}. \tag{2.26}$$

If the condition (2.25) is satisfied, then one can define the following antisymplectic counterpart of the well–known Dirac's bracket:

$$(A, B)_{(D)} \equiv (A, B) - (A, \Theta_\alpha)Q^{\alpha\beta}(\Theta_\beta, B). \tag{2.27}$$

It can be checked directly that the Dirac's anti–bracket (2.27) possesses all the algebraic properties (2.14)-(2.17).

Having the anti–bracket (2.27), one can define naturally the Dirac's antisymplectic differential:

$$\Delta_{(D)} \equiv \frac{1}{2}(-1)^{\varepsilon_A}M^{-1}_{(D)}\partial_A M_{(D)}E^{AB}_{(D)}\partial_B, \quad E^{AB}_{(D)} \equiv (\Gamma^A, \Gamma^B)_{(D)}, \tag{2.28}$$

to be nilpotent:

$$\Delta_{(D)}^2 = 0, \tag{2.29}$$

that gives the following equation for the density $M_{(D)}(\Gamma)$:

$$\Delta_{(D)}(-1)^{\varepsilon_A} M_{(D)}^{-1} \partial_A M_{(D)} E_{(D)}^{AB} = 0. \tag{2.30}$$

The corresponding Dirac's measure has the form:

$$d\mu_{(D)}(\Gamma) \equiv M_{(D)}(\Gamma)\delta(\Theta)d\Gamma. \tag{2.31}$$

Of course, the Dirac–type counterparts of the equations (2.18), (2.19) hold true for the antibracket (2.27) and differential (2.28).

Contrary to the standard symplectic formalism, in the present, antisymplectic, case one cannot express the measure densities $M(\Gamma)$ or $M_{(D)}(\Gamma)$ explicitly (i.e. algebraically) in terms of the metric E^{AB} or, respectively, of the metric $E_{(D)}^{AB}$ and the matrix $Q_{\alpha\beta}$. To a considerable extent, this is because of the fact that the standard superdeterminant concept does not work for odd supermatrices such as the ones E^{AB} or $Q_{\alpha\beta}$.

Section 3

The antisymplectic differential concept plays the key role when constructing the general form for the Lagrangian functional integral. In fact, the basic idea of the BV–approach is to involve an initially–given gauge theory into the universal hypertheory whose hypergauge generators are always nilpotent at the classical hyperextremals. To define the above–mentioned hypertheory in a most natural and effective way, one should require for the exponential of $i/\hbar$ times quantum action to be annihilated by the antisymplectic differential Δ. Thus we arrive at the quantum master equation.

To define the functional integral to be nondegenerate, one needs hypergauge fixing. If the hypergauge conditions are imposed directly on the basic field–antifield variables (2.1), then, by definition, the functional integral is called "of the first level".

However, it will be shown in the next Section that the hypergauge Lagrangian multipliers of the first level can be assigned their own antifields, which appear to be new hypergauge variables, and thus require new, second–level, hypergauge conditions for themselves. In that case, by definition, the functional integral is called "of the second level", and so on.

So, we propose the following basic formula for the general Lagrangian functional integral of the first level:

$$Z = \int \exp\{\frac{i}{\hbar}[W(\Gamma) + G_a(\Gamma)\pi^a]\} d\pi d\mu(\Gamma), \tag{3.1}$$

where the quantum action $W(\Gamma)$ satisfies the master equation:

$$\Delta \exp\{\frac{i}{\hbar}W\} = 0, \tag{3.2}$$

or equivalently:

$$\frac{1}{2}(W, W) = i\hbar \Delta W, \tag{3.3}$$

while the hypergauge functions:

$$G_a(\Gamma), \quad a = 1, \ldots, N, \quad \varepsilon(G_a) = \varepsilon_a, \tag{3.4}$$

are subjected to the conditions:

$$(G_a, G_b) = G_c U^c_{ab}, \tag{3.5}$$

$$\Delta G_a - U^b_{ba}(-1)^{\varepsilon_b} = G_b V^b_a, \tag{3.6}$$

$$V^a_a = G_a \tilde{G}^a, \tag{3.7}$$

and should remove a gauge degeneracy of the action $W(\Gamma)$.

as for the structure functions:

$$U^c_{ab} = -U^c_{ba}(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}, \quad V^b_a, \quad \tilde{G}^a, \tag{3.8}$$

they are subjected to the compatibility conditions of the equations (3.5)-(3.7), only.

The equations (3.5) have the form of the antibracket involution relations (2.21). Thereby the admitted hypergauge functions G_a appear to be, in fact, first–class constraints. Moreover, these constraints are restricted by the additional equations (3.6), (3.7) that control the Jacobian matrix determinant of the functions G_a. Henceforth we shall refer the equations (3.5)-(3.7) as "unimodular involution relations".

It is a crucial circumstance that the total set of equations (3.2), (3.5)-(3.7) provides for the functional integral (3.1) to be invariant under the BRST-type transformations:

$$\delta\Gamma^A = (\Gamma^A, -W + G_a\pi^a)\mu, \tag{3.9}$$

$$\delta\pi^a = (-U^a_{bc}\pi^c\pi^b(-1)^{\varepsilon_b} + 2i\hbar V^a_b\pi^b + 2(i\hbar)^2\tilde{G}^a)\mu, \tag{3.10}$$

where μ is a Fermionic parameter.

Choosing the parameter μ to be the function:

$$\mu = \frac{i}{2\hbar}\delta\Psi(\Gamma), \tag{3.11}$$

that satisfies the equations:

$$i\hbar\Delta\delta\Psi = G_a\delta K^a, \quad \Delta(G_a\delta K^a) = 0, \tag{3.12}$$

and making the additional variations:

$$\delta\Gamma^A = \frac{1}{2}(\Gamma^A, \delta\Psi), \quad \delta\pi^a = \delta K^a, \tag{3.13}$$

one generates the following effective change of the hypergauge functions G_a alone:

$$G_a(\Gamma) \quad \rightarrow \quad G_a(\Gamma + (\Gamma, \delta\Psi)), \tag{3.14}$$

in the functional integral (3.1).

Thus, it is proven formally that the functional integral (3.1) does not depend on the hypergauge variations of the canonical form:

$$\delta G_a = (G_a, \delta\Psi). \tag{3.15}$$

The variations (3.15) certainly retain the form of the unimodular involution relations (3.5)-(3.7), but the most general hypergauge variations with the mentioned property are of the form: $\delta G_a = (G_a, \delta\Psi) + G_b \delta\Lambda^b_a$. Hence the variation (3.15) induce the most general actual changes admitted for the hypergauge surface $G_a = 0$. Thus the canonical hypergauge variations (3.15) are shown to be quite sufficient for our purposes (see also Eq.(3.34)).

Let us note that the basic equations (2.10), (3.2) admit a natural arbitrariness for their solutions. First, let us consider the equation (2.10) for the measure density $M(\Gamma)$. Let $M(\Gamma)$ be a solution to this equation. Then the function $M(\Gamma)J(\Gamma)$ satisfies the same equation if the function $J(\Gamma)$ possesses the property:

$$\Delta\sqrt{J} = 0. \tag{3.16}$$

Next, let us change the measure density in the functional integral (3.1), as well as in the equations (3.2), (3.6), according to the rule:

$$M(\Gamma) \quad \rightarrow \quad M(\Gamma)J(\Gamma), \tag{3.17}$$

where the function $J(\Gamma)$ satisfies the equation (3.16). It can be checked directly that the change (3.17) induces the following transformation for the solution W to the equation (3.2):

$$W(\Gamma) \quad \rightarrow \quad W(\Gamma) - \frac{\hbar}{i}\ln\sqrt{J(\Gamma)}. \tag{3.18}$$

To compensate the changes (3.17), (3.18) in the functional integral (3.1), the hypergauge δ–function $\delta(G)$ should behave as:

$$\delta(G) \quad \rightarrow \quad \delta(G)(\sqrt{J})^{-1}. \tag{3.19}$$

By making use of the unimodular involution relations (3.5)-(3.7), one can confirm that the hypergauge δ–function changes under the transformation (3.17) just according to the rule (3.19).

So, we have studied all the required conditions for the hypergauge functions G_a and thus we have found the unimodular involution relations (3.5)-(3.7). We have established also that the natural arbitrariness of the measure density $M(\Gamma)$ can be absorbed certainly into the change (3.36) of the quantum action $W(\Gamma)$, whereas its classical part remains unchanged.

Let us demonstrate explicitly that the standard version of the BV–formalism [4] follows directly from the general functional representation (3.1) if one chooses the Darboux coordinates:

$$E^{AB} = \begin{pmatrix} 0 & \delta^{ab} \\ -\delta^{ab} & 0 \end{pmatrix}, \quad (A,B) = A(\overleftarrow{\partial_a}\overrightarrow{\partial^a_*} - \overleftarrow{\partial^a_*}\overrightarrow{\partial_a})B, \tag{3.20}$$

and the trivial measure density:

$$M = 1, \quad \Delta = (-1)^{\varepsilon(\varphi^a)}\partial_a\partial^a_*. \tag{3.21}$$

Let the hypergauge functions G_a be explicitly solvable with respect to the antifield variables φ^*_a:

$$G_a(\varphi,\varphi^*) = (\varphi^*_b - f_b(\varphi))\Lambda^b_a(\varphi,\varphi^*), \tag{3.22}$$

where $\Lambda^b_a(\varphi,\varphi^*)$ is an even nondegenerate matrix,

$$\varepsilon(\Lambda^b_a) = \bar{\varepsilon}_a + \bar{\varepsilon}_b, \tag{3.23}$$

where:

$$\bar{\varepsilon}_a = \varepsilon_a + 1 = \varepsilon(\varphi^*_a) + 1 = \varepsilon(\varphi^a). \tag{3.24}$$

Substituting the ansatz (3.22) into the antibracket involution relations (3.5), we have at $\varphi^* = f(\varphi)$:

$$(\varphi^*_a - f_a(\varphi), \varphi^*_b - f_b(\varphi)) = 0, \tag{3.25}$$

that gives locally:

$$f_a(\varphi) = \partial_a \Psi(\varphi), \quad \varepsilon(\Psi) = 1. \tag{3.26}$$

Let us substitute this solution into the ansatz (3.22) and then expand the result near the hypersurface $\varphi^*_a = \partial_a \Psi(\varphi)$:

$$\begin{aligned} G_a(\varphi, \varphi^*) &= (\varphi^*_b - \partial_b \Psi(\varphi)) \Lambda^b_a(\varphi, \varphi^*) = \\ &= (\varphi^*_b - \partial_b \Psi(\varphi)) \Lambda^b_a(\varphi) + \tfrac{1}{2}(\varphi^*_c - \partial_c \Psi(\varphi))(\varphi^*_b - \partial_b \Psi(\varphi)) \Lambda^{bc}_a(\varphi) + \ldots, \end{aligned} \tag{3.27}$$

where:

$$\Lambda^b_a(\varphi) \equiv \Lambda^b_a(\varphi, \varphi^* = \partial \Psi(\varphi)), \tag{3.28}$$

$$\Lambda_a^{bc}(\varphi) = \Lambda_a^{cb}(\varphi)(-1)^{(\bar{\varepsilon}_b+1)(\bar{\varepsilon}_c+1)}. \tag{3.29}$$

To the first order in $(\varphi^* - \partial\Psi(\varphi))$ the involution relations (3.5) give:

$$\Lambda_a^c(\varphi)\overleftarrow{\partial}_d\Lambda_b^d(\varphi) - \Lambda_b^c(\varphi)\overleftarrow{\partial}_d\Lambda_a^d(\varphi)(-1)^{\bar{\varepsilon}_a\bar{\varepsilon}_b} = \Lambda_d^c(\varphi)U_{ab}^d(\varphi), \tag{3.30}$$

where:

$$U_{ab}^c(\varphi) \equiv U_{ab}^c(\varphi, \varphi^* = \partial\Psi(\varphi)). \tag{3.31}$$

In its own turn, the equation (3.6) gives at $\varphi^* = \partial\Psi(\varphi)$:

$$(-1)^{\bar{\varepsilon}_b}\partial_b\Lambda_a^b(\varphi) = -U_{ba}^b(\varphi)(-1)^{\bar{\varepsilon}_b}. \tag{3.32}$$

It should be noted here that even the second term in (3.27) does not contribute to ΔG_a at $\varphi^* = \partial\Psi(\varphi)$ because of the adjoint-symmetry property (3.29).

It follows from the equations (3.30), (3.32) that:

$$\partial_a \ln\det\Lambda(\varphi) = 0, \tag{3.33}$$

and hence:

$$\det\left(\partial_* G(\varphi, \varphi^*)\right)\big|_{\varphi^*=\partial\Psi(\varphi)} = \text{const}. \tag{3.34}$$

This constancy property is an explicit example how the equation (3.6) controls the Jacobian matrix determinant of the hypergauge functions G_a, and that is why the equations (3.5)-(3.7) are called "the unimodular involution relations".

Due to the property (3.34), we find:

$$\delta(G) = \text{const} \cdot \delta\left(\varphi^* - \partial\Psi(\varphi)\right), \tag{3.35}$$

that is the standard BV–gauge.

Finally, we obtain the standard functional integral [4]:

$$Z_{standard} = \int \exp\{\frac{i}{\hbar} W\left(\varphi, \varphi^* = \partial\Psi(\varphi)\right)\} d\varphi, \tag{3.36}$$

$$\overrightarrow{\partial^a_*} \exp\{\frac{i}{\hbar} W(\varphi, \varphi^*)\} \overleftarrow{\partial_a} = 0. \tag{3.37}$$

So, we have shown that the standard BV ansatz (3.36) follows from the general functional representation (3.1), being the Darboux coordinates and trivial measure density chosen to work with.

On the other hand, the proposed functional integral (3.1) possesses, by construction, a quite invariant and symmetric form.

Section 4

Above we have formulated the unimodular involution relations (3.5)-(3.7) for the hypergauge functions G_a. It is a remarkable circumstance that these relations can be generated by means of a unique supermechanism that synthesizes in itself the

characteristic features of the Hamiltonian and Lagrangian gauge–algebra–generating equations.

In its own turn, the above–mentioned supermechanism will be shown below to generate an effective action of the general functional integral of the second level. Thus we shall make actually the first step in hierarchical proliferation process that converts successively the hypergauge Lagrangian multipliers into anticanonical pairs by assigning a new antifield to each of the preceding–stage Lagrangian multipliers.

To begin with, let us assign an antifield to each of the initial Lagrangian multipliers:

$$\pi^a, \varepsilon(\pi^a) = \varepsilon_a, \quad \rightarrow \quad \pi^*_a, \varepsilon(\pi^*_a) = \varepsilon_a + 1. \tag{4.1}$$

Let:

$$\Gamma'^{A'} \equiv \begin{pmatrix} \Gamma^A \\ \begin{pmatrix} \pi^a \\ \pi^*_a \end{pmatrix} \end{pmatrix}, \quad \varepsilon(\Gamma'^{A'}) \equiv \varepsilon'_{A'}, \quad A' = 1, \ldots, 4N, \tag{4.2}$$

be an extended set of the field–antifield variables.

Let us define the extended antisymplectic metric, antisymplectic differential and antibrackets as follows:

$$E'^{A'B'} \equiv \begin{pmatrix} E^{AB} & 0 \\ 0 & \begin{pmatrix} 0 & \delta^{ab} \\ -\delta^{ab} & 0 \end{pmatrix} \end{pmatrix}, \tag{4.3}$$

$$\Delta' \equiv \frac{1}{2}(-1)^{\epsilon'_{A'}} M^{-1} \partial'_{A'} M E'^{A'B'} \partial'_{B'}, \tag{4.4}$$

$$(A, B)' \equiv A \overleftarrow{\partial'_{C'}} E'^{C'D'} \overrightarrow{\partial'_{D'}} B. \tag{4.5}$$

The measure density $M(\Gamma)$ remains unchanged, so that the extended measure is:

$$d\mu'(\Gamma') \equiv d\pi d\pi^* d\mu(\Gamma). \tag{4.6}$$

The extensions (4.3)-(4.6) obviously retain all the above–mentioned formal properties of the antisymplectic differential and antibracket.

It is relevant at this stage to define the Planck parity $\mathrm{Pl}(A)$:

$$\mathrm{Pl}(AB) = \mathrm{Pl}(A) + \mathrm{Pl}(B), \quad \mathrm{Pl}(\hbar) \equiv 1, \tag{4.7}$$

$$\mathrm{Pl}(\Gamma^A) = 0, \quad \mathrm{Pl}(\pi^a) = -\mathrm{Pl}(\pi^*_a) = 1. \tag{4.8}$$

Next, let us consider the quantum master equation in its extended version:

$$\Delta' \exp\{\frac{i}{\hbar} W'(\Gamma')\} = 0, \tag{4.9}$$

under the extra conditions:

$$\mathrm{Pl}\,(W'(\Gamma')) = 1, \quad W'(\Gamma')|_{\pi^*=0} = G_a(\Gamma)\pi^a, \tag{4.10}$$

where $G_a(\Gamma)$ are the first–level hypergauge functions considered above.

Let us seek for a solution to the problem (4.9), (4.10) in the form of $\hbar$–power series expansion:

$$W'(\Gamma') = \Omega + i\hbar\Xi + (i\hbar)^2\tilde{\Omega} + \dots \,. \tag{4.11}$$

Then we find the following equations for the functions Ω, Ξ, $\tilde{\Omega}$:

$$(\Omega, \Omega)' = 0, \quad \mathrm{Pl}(\Omega) = 1, \quad \Omega|_{\pi^*=0} = G_a\pi^a, \tag{4.13}$$

$$(\Omega, \Xi)' = \Delta'\Omega, \quad \mathrm{Pl}(\Xi) = 0, \quad \Xi|_{\pi^*=0} = 0, \tag{4.14}$$

$$(\Omega, \tilde{\Omega})' = \Delta'\Xi - \frac{1}{2}(\Xi, \Xi)', \quad \mathrm{Pl}(\tilde{\Omega}) = -1. \tag{4.15}$$

In their own turn, these equations can be solved in the form of π, π^*–power series expansions:

$$\Omega = G_a\pi^a - \frac{1}{2}\pi^*_c U^c_{ab}\pi^b\pi^a(-1)^{\varepsilon_a} + \ldots, \tag{4.16}$$

$$\Xi = \pi^*_a V^a_b \pi^b + \frac{1}{4}\pi^*_b\pi^*_a V^{ab}_{cd}\pi^d\pi^c(-1)^{(\varepsilon_b+\varepsilon_c)} + \ldots, \tag{4.17}$$

$$\tilde{\Omega} = \pi^*_a\tilde{G}^a + \frac{1}{2}\pi^*_b\pi^*_a\tilde{U}^{ab}_c\pi^c(-1)^{\varepsilon_b} + \ldots\ . \tag{4.18}$$

We state that to the lowest order in π, π^* the equations (4.13)-(4.15) give exactly the unimodular involution relations (3.5)-(3.7), whereas to higher π, π^*–orders one obtains all the compatibility conditions for these relations. Thus the equations (4.13)–(4.15) appear to be generating ones for the unimodular involution relations.

Now, let us consider the proposed general form of the Lagrangian functional integral of the second level:

$$Z' = \int \exp\{\frac{i}{\hbar}[W(\Gamma) + W'(\Gamma') + G'_a(\Gamma')\pi'^a]\}d\pi' d\mu'(\Gamma'), \tag{4.19}$$

where: the first-level quantum action $W(\Gamma)$ satisfies the quantum master equation (3.2); the second–level quantum action $W'(\Gamma')$ is defined above to be a solution of the problem (4.9), (4.10);

$$G'_a(\Gamma'), \quad \varepsilon(G'_a) \equiv \varepsilon'_a, \quad a = 1, \ldots, N, \tag{4.20}$$

are new, second–level, hypergauge functions that satisfy the following unimodular involution relations of the second level:

$$(G'_a, G'_b)' = G'_c U'^c_{ab}, \tag{4.21}$$

$$\frac{i}{\hbar}(W, G'_a)' + \Delta' G'_a - U'^b_{ba}(-1)^{\epsilon'_b} = G'_b V'^b_a, \tag{4.22}$$

$$V'^a_a = G'_a \tilde{G}'^a, \tag{4.23}$$

Besides, the matrix:

$$\| \frac{\partial G'_a}{\partial \pi^*_b} \|_{G'=0} \tag{4.24}$$

is supposed to be even and nondegenerate, so that the gauge equations $G'_a = 0$ are solvable with respect to the antifield variables π^*_a. We normalize the matrix (4.24) by requiring for its superdeterminant to take the value 1 at $\pi^a = 0$, $G_a = 0$.

The functional integral (4.19) is invariant under the following BRST–type transformations:

$$\delta \Gamma'^{A'} = (\Gamma'^{A'}, W - W' + G'_a \pi'^a)' \mu', \tag{4.25}$$

$$\delta \pi'^a = [-U'^a_{bc} \pi'^c \pi'^b (-1)^{\epsilon'_b} + 2i\hbar V'^a_b \pi'^b + 2(i\hbar)^2 \tilde{G}'^a] \mu'. \tag{4.26}$$

Choosing the fermionic parameter μ' to be the function:

$$\mu' = \frac{i}{2\hbar}\delta\Psi'(\Gamma') \tag{4.27}$$

that satisfies the equations:

$$i\hbar\Delta'\delta\Psi' - (W, \delta\Psi')' = G'_a\delta K'^a, \tag{4.28}$$

$$i\hbar\Delta' G'_a\delta K'^a - (W, G'_a\delta K'^a)' = 0, \tag{4.29}$$

and making the additional variations:

$$\delta\Gamma'^{A'} = \frac{1}{2}(\Gamma'^{A'}, \delta\Psi')', \quad \delta\pi'^a = \delta K'^a, \tag{4.30}$$

one generates in the functional integral (4.19) canonical change of the hypergauge functions G'_a alone:

$$G'_a(\Gamma') \quad \rightarrow \quad G'_a(\Gamma' + (\Gamma', \delta\Psi')'). \tag{4.31}$$

Thus it is proven formally that the functional integral (4.19) does not depend on the hypergauge variations (4.31). The same as in the first–level case (3.15), the hypergauge variations (4.31) induce the most general actual changes admitted by the relations (4.21)-(4.23) for the second–level hypergauge surface $G'_a = 0$.

Choosing the trivial gauge:

$$G'_a = \pi^*_a, \tag{4.32}$$

and using the second condition (4.10),one returns to the first–level functional integral (3.1):

$$Z' = Z \tag{4.33}$$

Let us compare the general structure of the functional representations (3.1) and (4.19). While in the expression (3.1) the variables π^a are nothing other but usual Lagrangian multipliers to the first–level hypergauge functions G_a, in the expression (4.19) these field variables acquire the corresponding antifields π^*_a, and these anticanonical pairs appear to be working as a set of "ghost" variables with respect to the second–level hypergauge G'_a. It is a remarkable fact that the classical "ghost" action Ω, defined by the equations (4.13), (4.16), possesses exactly the structure of the Hamiltonian BFV-generator, whereas the first–level hypergauge functions G_a play the role of the initially–given first–class constraints. In their own turn, the new variables π'_a in (4.19) are usual Lagrangian multipliers to the new, second–level, hypergauge functions G'_a.

The above–considered proliferation process can be continued by induction for an arbitrary number of steps. At each step the former Lagrangian multipliers acquire their antifields and become ghost variables with respect to the hypergauges of the present stage. At each stage there is no dependence of the functional integral on the admissible variations of each hypergauge function entered.

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