Part VI

PERTURBATIONS IN THE EXPANDING UNIVERSE

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Perturbations in the expanding universe

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Chapter 30

Introduction

The organizers of the 7th Brazilian School of Cosmology and Gravitation asked me to give five lectures on perturbations in the expanding universe. This is a very wide field, so I have had to restrict to some topics. I have chosen the generation and evolution of scalar perturbations which are closely related to the observed large scale structure in the universe. Probably scalar perturbations are generated during an early stage of cosmological evolution. The structure of our universe is surely formed by gravitational instability from small perturbations. It seems the best explanation that the simple scalar perturbations surviving from the earliest cosmological stages are the seeds of large scale structure. However, also orientable field structures, textures, could be generated during the cosmological evolution and influence the large scale structure formation. In the following I will not discuss this possibility. I also do not discuss tensorial perturbations, the gravitational waves. However, their contribution to the microwave background fluctuations could be important, so that lower amplitudes of the scalar perturbations would follow from the measured amplitudes of the temperature fluctuations.

Throughout the lectures I have worked within the spatially flat Einstein-de Sitter cosmological model. Inflation predicts that the density parameter is close to one and recent observations on very large scales seem to confirm this prediction.

Due to the observed abundance of light elements and the theory of cosmological nucleosynthesis the baryon density is only a small fraction of the mean cosmological density which is dominated by the density of invisible weak interacting massive particles. Structure formation tells us that these particles must be probably cold. The lightest of the supersymmetric partners of the fermions could be stable and form the cold dark matter, another candidate would be the axions. So the scenario of structure formation in a cold dark matter dominated Einstein-de Sitter universe could be called the standard model which explains many aspects of the observed structure in the universe quite well.

In the first section of this article, I discuss the evolution of the homogeneous isotropic background cosmological model including an early inflationary stage. In the second section I introduce the Newtonian theory of linear gravitational instabilities. Inside of the Hubble radius the Newtonian theory is sufficient for describing the evolution of perturbations in the expanding universe. As a first step for describing nonlinear evolution, the Zeldovich approximation is presented in the third section. In the fourth section I turn to the relativistic theory of perturbations, where the choice of the gauge is important for understanding the physical meaning of the perturbations. The following section describes the generation of perturbations during the inflationary stage of the cosmological evolution. The sixth section is devoted to the observation of perturbations in the universe. Discussing the large scale structure of the universe one can directly relate the microphysics of the early cosmological evolution to the observational results. By this way one can come to conclusions about the mechanisms of inflation. I finish with a more speculative section which briefly shows how an observed excess of large scale structure can be explained by a special inflationary model.

There are some excellent textbooks about cosmology and large scale structure: The Early Universe - Facts and Fiction (Börner 1988), The Early Universe (Kolb and Turner 1990), Particle Physics and Inflationary Cosmology (Linde 1990), The Large-Scale Structure of the Universe (Peebles 1980), The Structure and Evolution of the Universe (Zeldovich and Novikov 1975). Preparing these lectures I have used these textbooks and the review articles of Efstathiou (1990) and Mukhanov et al. (1992). I have used also parts of my papers and especially the literature cited therein. Therefore, the list of references is by no means complete and also not representative.

Chapter 31

Cosmological models

The aim of cosmology is to understand the origin and evolution of the universe. Observational cosmology is concerned with the determination of such general cosmological parameters as the Hubble constant, the age of the universe, the deceleration parameter, the density parameter, the cosmological constant. But observations show us also that the matter in the universe is not distributed homogeneously but clumps into structures from small up to the largest observable scales. Moreover, besides the qualitative description of large scale structure also more and more quantitative results were found during the last decade. Therefore, theoretical predictions of correlations lengths, temperature fluctuations, characteristic lengths and velocities can be directly compared with observations.

Observations indicate an evolution of the universe starting from an extremely high density and temperature. The era, where the density was higher than $\rho_{Pl} \approx 10^{93}\mathrm{gcm}^{-3}$ and the temperature was higher than $T_{Pl} \approx 10^{19}$ GeV is called the Planck era, during which quantum gravity effects are believed to be very important. It is a matter of fact that the standard hot big bang cosmology satisfactorily accounts for the Hubble expansion, the cosmic microwave background radiation, and the cosmological (primordial) nucleosynthesis. However, the standard cosmology fails to account for a number of fundamental cosmological problems, these are the horizon, flatness, homogeneity, and isotropy problems. Moreover, it cannot explain the origin of the primordial cosmological perturbations which lead to the formation of structure in the almost homogeneous early universe. The origin of all these problems is the standard assumption of decelerated expansion throughout the whole age of the universe due to positive energy density and nonnegative pressure. The model of the inflationary universe proposed by Guth (1981) (for recent reviews see e. g. Börner (1988), Linde (1990) and Gottlöber et al. (1992)) solves these problems. Its essential success lies in the possibility of providing a causal mechanism for obtaining small density fluctuations (and maybe gravitational waves) of the order 10^{-4} on the horizon of the observed Friedmann universe.

Modern cosmology began with Friedmann's discovery that General Relativity predicts the expanding universe. This result was later on confirmed by Hubble's observation of the approximately linear relation between the radial velocities of galaxies and their distances. Obviously, the long-range force of gravity plays the deciding role in cosmology. Contrary to the electromagnetic interaction it can not be compensated by negative charges. The first attempt to describe a homogeneous, infinite universe by Newton's gravity law led to the known paradox of a divergent gravitational potential in the case of a finite mean matter density. This paradox was removed by Einstein's General Relativity. In order to solve Einstein's equations one has to make assumptions on the metric. In the simplest case, in cosmology these assumptions are formulated in the cosmological principle. This principle is the hypothesis that the universe is spatially homogeneous and isotropic. Obviously, the planets, stars, galaxies and systems of galaxies are far away from a homogeneous and isotropic matter distribution. Our intuitive ideas of homogeneity and isotropy are mainly based on the observation of the highly isotropic background radiation from which we deduce a homogeneous and isotropic distribution of matter at the time of recombination.

In the following sections of this report we will consider the mechanisms which produce the observed structure starting from the initially nearly homogeneous distribution of matter. For the consideration of cosmological evolution the universe can be assumed to be homogeneous and isotropic. In the language of mathematics this means that the three-dimensional space is maximal symmetric. Consequently, it is a space of constant curvature. In the four-dimensional space-time the scalar curvature of this three-dimensional space is in general time dependent. The components g_{0i} of the metric tensor must vanish, because otherwise a direction would be preferred. (Greek indices run from 0 to 3, Latin indices run from 1 to 3.) For the component g_{00} one gets $g_{00} = 1$ by an appropriate choice of the time coordinate (synchronisation). That means, that the cosmological principle leads to a drastic simplification: The four-dimensional space-time can be described by the Robertson-Walker line element

$$ds^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right\}. \tag{1}$$

Here $a(t)$ denotes the scale factor and $k = 0, \pm 1$ is a constant. We have assumed that the velocity of light is $c = 1$. The scalar curvature of the three-dimensional space is ${}^{(3)}P = 6ka^{-2}$. Therefore, the constant k determines the type of the spatial curvature. The case $k = 0$ corresponds to a flat three-dimensional space (Einstein-de Sitter universe), $k = +1$ is a closed space with the finite volume $2\pi^2a^3$, and $k = -1$ is a space with negative curvature. For the line element (1) the Einstein equations of General Relativity

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi G T_{\alpha\beta} \tag{2}$$

are reduced to one equation for the scale factor $a(t)$. In order to obtain this equation we compute the components of the Ricci tensor in the metric (1):

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{3}$$

$$R_{ik} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right]\delta_{ik}. \tag{4}$$

The mixed components R_{0i} vanish identically. The Ricci scalar reads

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{5}$$

In the homogeneous isotropic universe the matter must be described by scalar quantities which depend only on time. The energy-momentum tensor must have the same form as that of an ideal fluid in the rest frame,

$$T_{\alpha\beta} = (\varepsilon + P)u_\alpha u_\beta - P g_{\alpha\beta}, \tag{6}$$

where $u^o = 1$ and $u^i = 0$. The quantity ε is the energy density of the matter, P is the pressure. Then the 0-0-component of Einstein's equations is the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\varepsilon, \tag{7}$$

and combining it with the trace equation one has

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3P). \tag{8}$$

It is characteristic for this equation that it does not allow a static solution except for $\varepsilon + 3P \equiv 0$. With $\varepsilon + 3P \geq 0$ an expanding universe ($\dot{a} > 0$) must have started its expansion in a singular state ($a = 0$) with formally divergent energy density ($\varepsilon = \infty$). From the Bianchi identity it follows that the divergence of the energy-momentum tensor must be zero ($T^\beta_{\alpha;\beta} = 0$). This condition leads to the equation of motion of the matter

$$\frac{\dot{\varepsilon}}{\varepsilon + P} = -3\frac{\dot{a}}{a}, \tag{9}$$

which is of course already contained in the equations (7) and (8). Knowing a relation between the pressure P and the energy density ε - the equation of state $P(\varepsilon)$ - eq. (9) can be integrated, and it defines the energy density as function of the scale factor $a(t)$. Note, that in general the equation of state contains the temperature T as a second state variable. For our considerations the temperature specifies only the form of the relation $P(\varepsilon)$ (see eq. (10)).

On the one hand we can see baryonic matter in the present universe — namely all the objects which emit the observed radiation and the indirectly seen dark matter (for example interstellar dust). On the other hand, there exists besides the radiation from the stellar type objects a photon gas distributed with high accuracy homogeneously and isotropically, namely the microwave background radiation. Moreover, probably there exist also dark nonbaryonic matter. From the viewpoint of cosmology the baryonic matter in the universe can be assumed to be concentrated in "particles" the inner structure of which is most of the time unimportant for the dynamics of the cosmological evolution. Such "particles" would be the galaxies, which move relatively to the cosmological rest frame with velocities which are small in comparison to the speed of light ($v \approx 10^{-3}$). Therefore, the ratio of the kinetic energy density to the rest mass density is of the order 10^{-6}, i.e., the energy density equals the mass density of baryonic matter. These "particles" interact only gravitationally, the ratio of the pressure of the "gas" of this "particles" to the rest mass density is of the same order of magnitude (10^{-6}), i.e., the pressure can be neglected and the baryonic matter is described as dust-like matter with $P = 0$.

In the present universe the energy density of the observed radiation can be neglected in comparison to the mean density of the baryonic matter. Therefore, radiation does not influence the present dynamics of the cosmological evolution. If the universe has the critical density as predicted by the inflationary cosmological models, the present universe is dominated by nonbaryonic dark matter which obeys also the equation of state $P = 0$. Therefore, from eq. (9) one has $\varepsilon a^3 = \text{const.}$, where ε is the total energy density of the baryonic and dark matter. Then the Friedmann equation (7) can be solved to find the time dependence $a(t)$. For the most interesting case of an Einstein-de Sitter model with a flat three-dimensional space ($k = 0$) we have a power law expansion, $a(t) \propto t^{2/3}$. More general, for a linear equation of state

$$P = (\gamma - 1)\varepsilon, \qquad 0 \leqq \gamma \leqq 2, \tag{10}$$

one obtains from eq. (9) $\varepsilon a^{3\gamma} = \text{const.}$ and for the Einstein-de Sitter model we have $a(t) \propto t^s$ and $s = 2/(3\gamma)$, where $\gamma \neq 0$. Expressions for the time dependence of the scale factor in spatially open and closed models can be given in parametric form. In every case, we obtain a finite age of the universe,

$$t_p = f(q_p) H_p^{-1}, \tag{11}$$

where we use the classical parameters of the cosmological models, the present Hubble parameter $H_p = \dot{a}(t_p)/a(t_p)$, and the deceleration parameter $q_p = -a(t_p)\ddot{a}(t_p)/\dot{a}^2(t_p)$. Both parameters are only known with very limited accuracy. Therefore, we write $H_p = 100$ h $\times$ km s^{-1} Mpc^{-1} with the possible range $0.5 \leqq h \leqq 1$. The deceleration parameter q_p is only known to be positive and certainly not much larger than 1. The function $f(q_p)$ is a monotonically decreasing function of q_p with $f(0) = 1$ and $f(0.5) = s = 2/3$, the latter originates from the power law $t^{2/3}$ of the dust model in the Einstein-de Sitter universe.

The photon gas practically does not interact with the baryonic matter in the present universe. Therefore, the radiation component also fulfils eq. (9) and $\varepsilon_{rad} a^4 = \text{const.}$. Since the energy density of the radiation decreases faster than that of the dust-like matter, one can deduce the existence of the radiation dominated stage of the cosmological evolution in the past. Then we have an equation of state (10) with $\gamma = 4/3$. Approximately at the transition time from the radiation dominated universe to the dust universe the cosmic matter becomes transparent. In this moment the decreasing energy density of the background photons becomes too small to ionize the continuously recombining atoms. During the recombination era the background photons are scattered for the last time. Informations on the spectrum and the isotropy of the microwave background concern this era. To summarize, the standard model of the cosmological evolution asserts that starting with an initial singularity with formal infinite temperature and energy density the homogeneous and isotropic universe expands and cools, where the energy density of the initially dominating radiation decreases faster than that of the baryonic matter, and the universe becomes matter dominated. For a detailed account of physical processes in the standard model see, e.g., Weinberg (1972), or Zeldovich and Novikov (1975/1983).

This standard cosmological model is supported by the observation of the Hubble flow of galaxies and the 3 K background radiation. The main problem of the standard model is the existence of the initial singularity, which is a quite general property of cosmological models in General Relativity. First of all it hints at a breakdown of Einstein's General Relativity in the limit of strong gravitational fields. Moreover, the following two properties of the universe can be understood in the framework of the standard model only as special properties of the initial singularity:

- At present the observed mean density is close to that of the flat Einstein-de Sitter model, i.e., the density parameter $\Omega = \varepsilon/\varepsilon_c$ is close to unity ($\varepsilon_c = 3H^2/(8\pi G)$ is the critical density of the spatially flat Einstein-de Sitter model, $H(t) = \dot{a}/a$ is the time-dependent Hubble parameter). From the Friedmann equation (7) it follows that $\Omega(t) - 1 = k/\dot{a}^2$. In models with $k = \pm 1$ the quantity $|\Omega(t) - 1|$ increases with time. From the presently observed small value of $|\Omega - 1|$ it follows that, for example, at Planck time $|\Omega - 1|_{PL} \lesssim 10^{-60}$. The exact limit depends on the details of the model. This problem is called flatness problem.

- The distance traversed by signals during the time interval $t_1 - t_o$ in a space-time described by the line element (1) is

$$d = a(t_1) \int_{r_o}^{r_1} \frac{dr}{(1 - kr^2)^{1/2}} = a(t_1) \int_{t_o}^{t_1} \frac{dt'}{a(t')}. \tag{12}$$

Since there is a finite time from the moment of singularity $t_o = 0$, eq. (12) defines a particle horizon (Rindler 1956). For a power law of the expansion, $a(t) \propto t^s$ with $s < 1$, one gets $d_H \equiv a(t) \int_0^t dt'/a(t') = t/(1-s)$. Only events inside this horizon can influence a given point in the space-time. On the other hand the observed high isotropy of the background radiation tells us, that at the moment of recombination the universe was homogeneous and isotropic over distances much larger than the horizon. This problem is called horizon problem. Note, that even if $d_H(t)$ diverges for $s \geq 1$, the temporal Hubble radius H^{-1} determines the maximal length which can be influenced by physical processes during a certain time interval $\tau = (t_1 - t_o)$. This led to the notion of the 'horizon' H^{-1} as a causality bound on coherent processes being of influence in the early universe.

The existence and the smallness of the horizon (12) represents the main obstacle for theories of the origin of structures in the universe. For example, the horizon at the epoch of recombination corresponds to a comoving scale of presently 100 Mpc. There are structures in the universe as planets, stars, galaxies and galaxy clusters up to the largest superclusters and voids, ranging up to this scale. According to the standard model with a power law of the scale factor $a(t)$, no causal origin for them seems possible operating in the early evolution of the universe.

The inflationary models are characterized by an early stage of accelerated expansion due to an effective negative pressure (cp. eq.(8)). The inflationary scenario solves not only the horizon and flatness problem but delivers also the primordial perturbations for structure formation. The origin of density fluctuations (and gravitational waves) are quantum fluctuations of the inflation driving field becoming enlarged to macroscopic scales and growing beyond the cosmological horizon. The inflationary models differ in having different driving mechanisms for inflation and in solving the graceful exit problem.

Driving mechanisms could be renormalization corrections to gravity (Starobinsky 1980) or scalar fields as in the "old" (Guth 1981), "new" (Linde 1982, Albrecht and Steinhardt 1982), "chaotic" (Linde 1983), "extended" (La and Steinhardt 1989) or "natural" (Adams et al. 1993) inflation. The end of inflation is reached in the simplest inflationary models if $\varphi \to 0$, $V(\varphi_0) = 0$. Then via scalar field oscillations and corresponding oscillations in the metrical perturbations particles will be created the interaction of which leads to the hot universe. Another exit is the sudden end of inflation due to rolling in another direction in the twodimensional potential ("waterfall"). Note that the same behaviour may lead under certain conditions to double inflation (Gottlöber et al. 1991, Amendola et al. 1991). Also a soft first order phase transition may end the inflation.

For the following discussion of the cosmological perturbations we will illustrate the basic ideas of inflation driven by a scalar field with a potential $V(\varphi)$. The time evolution of the scale factor $a(t)$ of a Robertson-Walker metric (1.1) with the curvature k can be obtained by solving the Friedmann equation (1.7). In the simplest case of de Sitter inflation a constant vacuum energy density $\varepsilon = \rho_V$ dominates the r.h.s of this equation, and for spatially flat models we find an exponentially rising scale factor,

$$a = a_0 \exp(Ht) , \quad H \equiv \dot{a}/a = \text{const.} \tag{13}$$

A natural generalisation of the exponential expansion law is a slowly varying energy density in eq. (1.7), $|\dot{\varepsilon}|/\varepsilon \ll H$, which leads to a quasi-de Sitter stage with an almost constant Hubble parameter, $|\dot{H}| \ll H^2$. It was Guth's (1981) insight that this stage may solve the problems mentioned above if it lasts long enough (at least 60 e-0 of $a(t)$).

At first we take the energy density of a coherent scalar field with some potential $V(\varphi)$ in the Friedmann equation,

$$\varepsilon = \frac{1}{2}\dot{\varphi}^2 + V(\varphi). \tag{14}$$

and the field equation of the scalar field,

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \tag{15}$$

where $V'(\varphi) = dV/d\varphi$. Under the condition of a slowly changing scalar field we can neglect the kinetic energy in eq. (14) and the second time derivative in eq. (15). Then the e-fold expansion of spatially flat models is given by the rate of change of the scalar field,

$$\ln\frac{a}{a_0} = -8\pi G\int_{\varphi_0}^{\varphi} d\tilde{\varphi}\,\frac{V(\tilde{\varphi})}{V'(\tilde{\varphi})}. \tag{16}$$

Simple models employ potentials $V = m^2\varphi^2/2$ or $V = \lambda\varphi^4/4$ for which the integral in eq. (16) can be solved 0. For the massive scalar field the time behaviour of the scale factor is given 0 by

$$\ln\frac{a}{a_0} = H_0(t-t_0) - \frac{m^2}{6}(t-t_0)^2 \tag{17}$$

This exponential expansion was first noticed by Parker and Fulling (1973) in a numerical solution of a bouncing spatially closed model. It was then derived analytically by Starobinsky (1978) and discussed by Linde (1983) as the simplest example of an inflationary model which leads to a natural end of inflation if $|\dot{H}|$ is of the order of H^2 and scalar field oscillation cause oscillations around the average behaviour of the scale factor $a(t) \propto t^{2/3}$ (known as solution for dust-like matter). The coupling to other physical fields besides the inflaton field causes particle creation and the transition of the dust-like to a radiation dominated Friedmann-model. From the inflationary models with a high temperature phase transition this process is called reheating since at this time, after the supercooling stage during the exponential expansion again a high temperature is reached (for details see Linde (1990)).

As a natural extension of this model one can consider several noninteracting scalar fields as source of gravitation (Starobinsky 1985, Polarski and Starobinsky 1992). Then the energy density is given by

$$\varepsilon = \sum_n \left(\frac{1}{2}\dot{\varphi}_n^2 + V_n(\varphi_n)\right) \tag{18}$$

and each of the scalar fields φ_n fulfils the equation of motion (15). The exponentially increasing scale factor during the quasi-de Sitter stage is given by

$$\ln\frac{a}{a_o} = -8\pi G\sum_n\int^{\varphi_n}\frac{V_n(\tilde{\varphi}_n)}{V_n'(\tilde{\varphi}_n)}d\tilde{\varphi}_n \tag{19}$$

Depending on the path in the corresponding phase space the combined action of two (or more) scalar fields may lead to double (or multiple) inflation. A first inflationary stage could be also driven by vacuum polarisation effects. Describing these effects with a higher order term in the gravitational Lagrangian the resulting theory is conformal equivalent to General Relativity with a scalar field as source. Within the higher order theory a scalar field as source may lead to double inflation (Gottlöber et al. 1991).

If consecutive inflationary stages exist between which the universe expands according to the Friedmann law then the scale invariance of the de Sitter stage is broken (by introduction of a typical length into the theory — the horizon at the moment of transition from one inflation to the other). Due to the second inflation this broken scale invariance could be mapped on observable scales of the universe. Observing the perturbation spectrum at different scales one can obtain informations about the early inflationary stage of the universe (Gottlöber and Mücket 1993, Gottlöber et al. 1993).

Chapter 32

The linear theory of gravitational instabilities in Newtonian theory

The problem of gravitational instability of a homogeneous static medium against small perturbations of the hydrodynamical equilibrium was considered for the first time by Jeans (1902). Within the framework of General Relativity Lifshitz (1946) calculated the evolution of small fluctuations in an expanding Friedmann universe. Bonnor (1957) obtained the corresponding results in Newtonian cosmology. The problem has been extensively treated in many standard monographs in cosmology (Zeldovich and Novikov 1975/1983, Weinberg 1972, Peebles 1980). Here we want to describe briefly the solution of the problem within Newtonian theory. We can use Newtonian theory in Friedmann cosmology if the sizes of the perturbations are much smaller than the horizon. Due to the predictions of the inflationary scenario we assume in the following that the universe is spatially flat.

The actual state of the homogeneous matter distribution in the universe should be determined by the mean density $\varrho(t)$, the pressure $P(t)$, the scale factor $a(t)$ (which describes the expansion), the entropy $S(t)$, and the expansion velocity $H(t) = \dot{a}/a$, i.e., the Hubble parameter. In spatially flat universe these quantities are connected by the following equations (cp. eq. 7 and 9)

$$\dot{\varrho} + 3H(\varrho + P) = 0, \tag{1}$$

$$H^2 = \frac{8\pi}{3} G\varrho, \tag{2}$$

$$P = P(\varrho, S). \tag{3}$$

For an adiabatically expanding universe the entropy S is approximately constant (after reheating) and the pressure is a function of the density only, i.e., $P = P(\varrho)$. The perturbations of the equilibrium solution are determined by the perturbed particle velocities $\boldsymbol{w}$ with respect to the local Hubble flow, i.e., to the peculiar velocities $\boldsymbol{u} - Hx$, by the perturbations of the density ϱ, the gravitational potential φ and, depending on the special kind of perturbations, of the entropy S. Neglecting entropy perturbations, the ansatz for the perturbed quantities should have the following form:

$$\varrho = \bar{\varrho}\,(1 + \delta) \tag{4}$$

$$\boldsymbol{u} = H\boldsymbol{x} + \boldsymbol{w}, \tag{5}$$

$$\varphi = \bar{\varphi} + \Phi, \tag{6}$$

$$P = \bar{P}(t) = \bar{P} + \frac{\partial P}{\partial \varrho}\bar{\varrho}\delta, \tag{7}$$

where $\partial P/\partial \varrho = b^2$ is the sound velocity with respect to the unperturbed medium. The unperturbated quantities $\bar{\varrho}$, $\bar{P}$, and $\bar{\varphi}$ depend only on time. The proper velocity $\boldsymbol{u}$ is the sum of the uniform Hubble flow $H\boldsymbol{x}$ and the peculiar velocity $\boldsymbol{w}$.

The perturbed quantities depend, in general, on the spatial coordinates, too. Therefore, they must satisfy the complete set of hydrodynamical equations, i.e., the equation of continuity, Euler's equations, and the Poisson equation must be fulfilled.

$$\frac{\partial \varrho}{\partial t} + \nabla(\varrho u) = 0, \tag{8}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\boldsymbol{u} + \frac{1}{\varrho}\nabla P + \nabla\varphi = 0, \tag{9}$$

$$\Delta\varphi = 4\pi G\varrho. \tag{10}$$

Inserting eqs. (5)-(7) we find

$$\frac{\partial\delta}{\partial t} + \nabla \boldsymbol{w} + H\boldsymbol{x}\nabla\delta + \delta\nabla\boldsymbol{w} + \boldsymbol{w}\nabla\delta = 0, \tag{11}$$

$$\frac{\partial \boldsymbol{w}}{\partial t} + H(\boldsymbol{x}\nabla)\boldsymbol{w} + H\boldsymbol{w} + b^2\nabla\delta + \nabla\Phi + (\boldsymbol{w}\nabla)\boldsymbol{w} - b^2\delta\nabla\delta = 0, \tag{12}$$

$$\Delta\Phi = 4\pi G\bar{\varrho}\delta. \tag{13}$$

Following the arguments of the classical stability analysis and the principles of the perturbation theory, the nonlinear terms with respect to the perturbation quantities can be neglected in comparison with the linear ones while the fluctuations are sufficiently small. This conclusion concerns the last two terms in eqs. (11) and (12). After these terms have been omitted, the eqs. (11) and (12) obtain the form

$$\dot{\delta} + \nabla\boldsymbol{w} + H\boldsymbol{x}\nabla\delta = 0, \tag{14}$$

$$\dot{\boldsymbol{w}} + H(\boldsymbol{x}\nabla)\boldsymbol{w} + H\boldsymbol{w} + b^2\nabla\delta + \nabla\Phi = 0. \tag{15}$$

The unperturbed quantities $H, \bar{\varrho}$, and $\bar{P}$ should be determined by help of the eqs. (1) - (7). The universe homogeneously in the average can be realized by density perturbations satisfying periodic boundary conditions. The perturbations can be decomposed into Fourier components,

$$\delta = \sum_{\boldsymbol{k}} \delta_k e^{i\boldsymbol{k}\boldsymbol{x}}, \quad \boldsymbol{w} = \sum_{\boldsymbol{k}} \boldsymbol{w}_k e^{i\boldsymbol{k}\boldsymbol{x}}, \quad \Phi = \sum_{\boldsymbol{k}} \Phi_k e^{i\boldsymbol{k}\boldsymbol{x}}, \tag{16}$$

where $\lambda = 2\pi/k$ is the length of the wave vector and the $\delta_k, \boldsymbol{w}_k$, and Φ_k are functions depending only on time t. Using the decomposition (16), the equations (13-15) obtain the following form:

$$\dot{\delta}_k + i\boldsymbol{k}\boldsymbol{w}_k = 0, \tag{17}$$

$$\dot{\boldsymbol{w}}_k + H\boldsymbol{w}_k + i\boldsymbol{k}b^2\delta_k + i\boldsymbol{k}\Phi_k = 0, \tag{18}$$

$$-k^2\Phi_k = 4\pi G\bar{\varrho}\delta_k. \tag{19}$$

Note, that in the linear approximation the spatial distribution of the density fluctuations is phase shifted with respect to the velocity distribution by $-\pi/2$. We have taken into consideration that the length as well as all distances between particles in an expanding universe are growing with time proportional to the scale factor $a(t)$. The equations (17)-(19) show that for each mode k the same equations are valid. Resolving (17) with regard to w_k we compute the time derivative $\dot{w}_k$ and substitute these quantities in (18) by the obtained expressions. By the help of (19) we substitute Φ_k in (18) and we get for each mode k the equation

$$\ddot{\delta}_k + 2H\dot{\delta}_k + k^2\Theta_k\delta_k = 0, \tag{20}$$

with

$$\Theta_k = b^2 - 4\pi G\bar{\varrho}/k^2 \tag{21}$$

Let us consider for a moment the static case (i.e., $a(t) =$ const. and $H = 0$), which was discussed by Jeans (1902). We find

$$\delta_k \propto e^{t/\tau} \tag{22}$$

with $\bar{\varrho} =$ const. and $\tau^{-1} = \pm(-k^2\Theta_k)^{1/2}$. For a static background the equation (22) describes the time evolution of the instability. The sign of the expression under the square root defines whether the solution is stable and small perturbations lead to small oscillations around the equilibrium state ($\tau^{-2} < 0$) or becomes unstable ($\tau^{-2} > 0$) and formally the δ_k will grow up unlimited. Also in an expanding background the sign of Θ_k decides about stability or instability of the considered hydrodynamical state of the medium against perturbations of the typical length $\lambda = 2\pi/k$ (Jeans criterion).

If at some instant of time the configuration is unstable against perturbations with λ_1 then it is unstable against all perturbations of coherence lengths $\lambda_2 > \lambda_1$, too. For very large λ, i.e., in the limiting case $k \to 0$, it follows

$$\ddot{\delta}_k + 2H\dot{\delta}_k - 4\pi G\bar{\varrho}\delta_k = 0. \tag{23}$$

Note, that this equation is fulfilled generally if pressure can be neglected ($b = P = 0$). The scale factor of an Einstein-de Sitter universe filled with dust increases like $a(t) \propto t^{2/3}$, and we obtain from (23) for the decaying and the growing modes

$$\delta_k \propto t^{-1} \quad \text{and} \quad \delta_k \propto t^{2/3}. \tag{24}$$

In the expanding background the linear perturbations grow as a power-law, i.e., much slowlier than in the static case (22).

We will now consider a universe which contains mainly relativistic particles. These particles are smoothly distributed with $\bar{\varrho}_r \propto a^{-4}$ because any density perturbation would be erased. Then the expansion of the background in eq. (23) is described by

$$H^2 = \frac{8\pi}{3} G\bar{\varrho}_r(1+\xi), \tag{25}$$

where we have introduced the relation $\xi = \bar{\varrho}_m/\bar{\varrho}_r$ between the density of nonrelativistic and of relativistic (i.e. radiation) matter. Then $\xi \propto a$ and we can transform eq. (23) to

$$\delta_k'' + \frac{2+3\xi}{2\xi(1+\xi)}\delta_k' - \frac{3\delta_k}{2\xi(1+\xi)} = 0, \tag{26}$$

where δ_k' denotes $d\delta_k/d\xi$. The solution for the growing mode of the nonrelativistic matter is now

$$\delta_k \propto 1 + 3\xi/2. \tag{27}$$

Thus fluctuations of the nonrelativistic matter do not grow if the evolution of the universe is dominated by relativistic particles ($\xi \ll 1$).

Let us return to the Jeans criterion. For a fixed background density it exists a critical coherence length λ_J (the Jeans length), which separates the stable region ($\lambda < \lambda_J$) from the unstable region ($\lambda > \lambda_J$).

$$\lambda_J = b\left(\frac{\pi}{G\bar{\varrho}}\right)^{1/2} \approx bt_f. \tag{28}$$

If the scale size of the perturbation 0 the Jeans length gravity becomes important and the pressure can be neglected. For very small perturbations only the pressure gradient is important. These perturbations oscillate as acoustic waves. The quantity t_f is of the order of the free-fall-time of a homogeneous spherical matter distribution. It is also of the order of magnitude of the time interval during which a spatially flat universe has been expanded from $\varrho = \infty$ to $\varrho = \bar{\varrho}$. Hence, the Jeans length is nearly equal to the distance which a sound signal is spreading during a time interval of the order of the free-fall-time. These results remain also valid in an expanding universe with decreasing mean density.

In a homogeneous medium the Jeans length defines a mass enclosed in a cube of length λ_J, the Jeans mass

$$M_J = \left(\frac{\lambda_J}{2}\right)^3 \bar{\varrho}. \tag{29}$$

Both the Jeans length λ_J and the density ϱ depend on time. The Jeans length increases with the free-fall-time, but the sound velocity decreases during the cosmological expansion. It depends on the relation of matter and pressure. Before recombination electrons and photons are tightly coupled by Thomson scattering. Thus, matter and radiation act as a single fluid. If the temperature T is smaller than the mass of the baryons the pressure of the baryons can be neglected. Hence, $P = P_{rad} = \varepsilon_{rad}/3$. Sufficiently far before the recombination era the radiation dominates over the matter density. Thus, matter density can be also neglected and $b = c/\sqrt{3}$. The energy density of the radiation decreases faster than the energy density of matter and at the time $t = t_{eq}$ both are equal. Thus, the matter density also influences the sound velocity.

The entropy per baryon is defined by the number of photons per baryon. Hence, the entropy perturbations can be written in the form

$$\frac{\delta S}{S} = \frac{3}{4}\frac{\delta \varepsilon_{rad}}{\varepsilon_{rad}} - \frac{\delta \varepsilon_{mat}}{\varepsilon_{mat}} = 0, \tag{30}$$

where we have assumed adiabatic perturbations. Taking into account $\delta\varepsilon = \delta\varepsilon_{rad} + \delta\varepsilon_{mat}$ we obtain for the sound velocity

$$b^2 = \frac{c^2}{3}\frac{1}{1 + 3\varepsilon_{mat}/(4\varepsilon_{rad}}). \tag{31}$$

At $t \ll t_{eq}$ the sound velocity is constant ($b = c/\sqrt{3}$), the Jeans length increases with time according to (28) and the scale factor is $a \propto t^{1/2}$. Thus, the Jeans length λ_J (28) grows like

$$\lambda_J \approx ct \propto a^2 \propto (1+z)^{-2}. \tag{32}$$

The Jeans length is of the order of the horizon ct. The Jeans mass depends on the matter density (without radiation). If today the mean matter density is ϱ_0 then the corresponding mean density at time t characterized by the cosmological redshift z is given by $\bar{\varrho} = \bar{\varrho}_0(1+z)^3$ and

$$M_J = \bar{\varrho}\left(\frac{\lambda_J}{2}\right)^3 \propto (1+z)^{-3}, \tag{33}$$

Throughout the radiation dominated epoch the Jeans mass increased with time and reached its maximum value at the transition to the matter dominated universe t_{eq}. When matter becomes important ($t \gg t_{eq}$) the velocity of sound decreases

$$b^2 \approx \left(\frac{2c}{3}\right)^2 \frac{\varepsilon_{rad}}{\varepsilon_{mat}} \propto a^{-1}. \tag{34}$$

Now the scale factor grows like $a \propto t^{2/3}$. Thus, the Jeans mass is

$$M_J = \bar{\varrho}\left(\frac{\lambda_J}{2}\right)^3 \approx \text{const}. \tag{35}$$

The Jeans mass was remaining constant until the recombination (for $\Omega = 1$ the transition to the matter dominated universe takes place before recombination). The maximum Jeans mass is of the order of magnitude

$$M_1 = M_{J,max} \approx 10^{17} M_\odot. \tag{36}$$

That means that the perturbations on scales smaller than the scales corresponding to superclusters oscillate.

Immediately after the recombination epoch the pressure of the neutral gas becomes dominant. The adiabatic sound speed of the monoatomic gas is

$$b = \left(\frac{5kT}{3m_p}\right)^{1/2} \tag{37}$$

At recombination ($T \approx 3000$K) the Jeans mass decreases by several orders of magnitude to

$$M_2 \approx 2 \cdot 10^6 M_\odot. \tag{38}$$

It is in this moment of the order of the mass of globular clusters.

Taking into consideration the dissipative processes before and throughout the recombination era one can derive a further characteristic mass (Silk 1968). The photons are coupled to electrons by Thomson scattering, but the photons and electrons are not perfectly coupled. Due to the imperfect coupling perturbations on small scales are damped. This photon viscosity leads to dissipation of all adiabatic fluctuations on mass scales smaller than M_3. Let us derive an order-of-magnitude estimate of this mass. The mean free time for Thomson scattering is $t_T = 1/(\sigma_T n_e c)$, where σ is the Thomson cross section and n_e is the electron density. The photons perform random walks. The diffusion time for a distance λ is $t_{diff} = \lambda^2/(c^2 t_T)$. If this time is smaller than the expansion time the sound waves of wavelength λ can be damped by photon diffusion before recombination. Thus, at recombination the damping length is

$$\lambda_D \simeq \left(\frac{ct_{rec}}{\sigma_T n_e}\right)^{1/2} \tag{39}$$

and the mass within a volume of λ_D is

$$M_3 \approx 10^{12} M_\odot. \tag{40}$$

Note, that this is only a rough estimate of the damping scale. For the exact calculations one has to solve numerically the Boltzmann equation which leads to a little bit larger mass (e.g. Peebles 1981).

Let us mention also the possibility of entropy fluctuations. Their evolution during the radiation era defines a further characteristic mass. Entropy fluctuations are isothermal ones and are defined by a local fluctuation of the baryon-to-photon ratio. They are remaining constant until the recombination epoch. With respect to the matter density, instability occurs but does not grow essentially. The growth is suppressed by the "friction" of the moving matter against radiation. As a consequence, the initial spectrum of entropy fluctuations remains almost unchanged up to recombination. Only those fluctuations with coherence lengths smaller than a critical one will be damped out entirely. The mass corresponding to this critical length is the lower mass limit $M_4 \approx 15 M_\odot$. Note, that this limit is much smaller than the Silk mass (40). Therefore, in principle, they could lead to small scale structures shortly after recombination.

Fluctuations with smaller mass content have been smeared out until the recombination. Hence, four mass scales are at hand which characterize the dominant processes up to recombination. For detailed discussion of the damping of fluctuations in the universe see e.g. Weinberg (1972), Zeldovich and Novikov (1975), Peebles (1980), Börner (1988), Efstathiou (1990).

We have derived the equations under the simplifying supposition that the cosmological medium as a mixture of various kinds of matter can be described by one equation of state only. Actually, the cosmic medium is a multi-component medium, the components of which should be described in principle by different equations of state. According to its actual equations of state the perturbations of each component exhibit different time behaviours. Density perturbations occurring in the components over a fixed coherence length are interacting gravitationally. This implies certain consequences for the instability behaviour of the perturbations belonging to the single components. It is a matter of fact that galaxies and clusters of galaxies contain matter that is hidden from direct observation, the dark matter. From nucleosynthesis it follows that this material cannot be composed only of baryons. Candidates for the dark matter component in the universe are the neutrino of finite rest mass (HDM) or more massive weakly interacting particles (CDM).

The linear theory of density perturbations in an universe containing a neutrino component and baryonic matter has been analyzed by Wasserman (1981). The influence of the components relative motion upon the gravitational instability was investigated in the Newtonian limit by Polyachenko and Fridman (1981) and Grishchuk and Zeldovich (1981). Fargion (1983) and Soloveva and Nurgaliev (1985) have treated the corresponding problems for a multi-component medium in the expanding universe. In Newtonian approximation they obtained asymptotic expressions for the solutions as $k \to 0$ and k and $a(t)$ are the quantities introduced above. Assuming an equation of state P_i and a sound velocity $b_i^2 = dP_i/d\varrho_i$ Fargion (1983) and Soloveva and Starobinsky (1985) established exact solutions for the evolution of small density perturbations in a two-component medium against the background of an expanding Friedmann model. A systematic approach to the exact solutions for the evolution of small density perturbations of a two-component medium against the background of an expanding Friedmann universe in the Newtonian approximation is given by Mathai et al. 1988 (see also Gottlöber et al. 1990).

Chapter 33

The Zeldovich approximation

Already the linear theory of instability evolution yields important results, e.g. the characteristic evolution time can be determined with sufficient accuracy. Indeed, the linear stage of instability growth takes the main part of the whole time evolution. On the other hand, at this point we cannot yet say anything about the nonlinear evolution stages. However, the situation is not so hopeless as it seems at first sight. The nonlinear phase and, if it exists, the final stage of condensing objects can be investigated by help of methods which give a qualitative answer on the question for further evolution, i.e., after $\delta \approx 1$. There are mainly three methods:

- One can try to find the critical or singular points of the solutions, respectively, of the corresponding differential equations, and to look for asymptotic representations near these points without the knowledge of the explicit solutions. The mentioned critical points are distinguished among others because in their environment the qualitatively, physically significant modifications in the behaviour of the corresponding solutions take place. Hence this method, if it can be applied, is especially suitable to understand the qualitative behaviour of some systems or processes.

- In some cases it is possible to find an approximation method by means of which the solutions obtained in linear approximation can be extrapolated to the nonlinear stage. Such a method allows to describe the transition to the nonlinear behaviour. The further application to later stages may be restricted by the method itself.

- Special suppositions about geometry and initial conditions can lead to exact solutions. Especially, this concerns perturbations with spherical symmetry. In this case the influence of neighbouring perturbations vanishes.

Although the three approaches seem to be very different, naturally they are connected. Especially this becomes evident for considerations concerning the behaviour of dustlike matter, i.e., collisionless matter, ($P = 0$). It has been already mentioned above, that matter with $P = 0$ represents a special case, which is for a number of equations of state an asymptotic one. The case $P = 0$ describes the purely gravitational interaction between the particles and the kinematical properties determined by the initial conditions imposed on the particles at some instant of time. It is a quite realistic description while the particle number density is sufficient small, respectively, the gravitational interaction dominates over the other ones for some time. Of course, this cannot be a suitable approximation if one wishes to describe the eventually occurring new state of stable equilibrium between pressure forces and gravitation. Such state is usually accompanied by a transition to a new equation of state.

The method sketched above under the first point goes back to the ideas of Poincaré (1889) and Bendixson (1901). It can be considered as the analysis of the trajectory of the state vector throughout the phase space of the system under investigation. This analysis ,i.e., general properties, classification and methods, has been well elaborated at the present time. The methods presuppose in most cases the possibility to define those terms as phase space and related ones for the system under consideration. This might be a considerable restriction in some cases of problems. The analysis leads to the determination of the equilibrium states (i.e., the stationary points) of the system and to an investigation of the stability character of these points. Mathematically this is equivalent to the qualitative analysis of a system of ordinary nonlinear differential equations. For dynamical systems powerful methods of investigation are well elaborated (e.g. Bogoyavlenski 1985).

In addition, if a system is depending on a number of parameters then it might happen that for certain parameter values the system degenerates and goes over into a qualitatively new behaviour. The states of degeneration or, with other words, the singular states of a system decide between different regions of states wherein the system alters continuously with changing parameters. Similarly to the qualitative analysis of the stationary points of ordinary differential equations, these states of degeneration or singular points can be classified, too.

A typical singularity (degeneration) for the propagation of perturbations is the formation of so-called caustics. For such propagation phenomena the time t itself plays the role of a parameter. During time evolution all "parameter values" will be entered automatically. In particular, the 0 of caustics is a quite general phenomenon in a collisionless medium. All transformations

$$r = x + v_0(x)t, \tag{1}$$

which transmit a particle from an initial position x into the position $r(x,t)$, after some time t_1 lead to singularities like caustics if $v_0(x) \neq$ const., i.e., the initial velocity field is a function of the particle position. If the initial density is equal to $\varrho_0(x)$ at $t = 0$ then it alters with time being a function $\varrho(x,t)$. In the neighbourhood of the particle x the density can be computed as

$$\varrho(x,t) = \varrho_0 J^{-1}(\partial r/\partial x) = \varrho_0/(1 + v_0'(x)t), \tag{2}$$

where $J(\partial r/\partial x)$ denotes the Jacobian of the transformation (1). In the direction defined by the negative gradient $v_0'(x) < 0$ an instant of time $t = t_1$ exists at which $1 + v_0'(x)t_1 = 0$ and the density becomes infinite. This characterizes the moment at which particles are concentrated in the neighbourhood of x (infinitely close to x). A caustic forms at x. At the distance ε apart from the caustic the density is of the order of magnitude $\varepsilon^{-1/2}$. For the higher dimensional cases this property will be conserved. Only the dimension of the surfaces at which the density becomes infinite changes correspondingly.

These results remain also valid in the case if a field of forces with potential character is present and the distribution of the initial particle velocities is potential-like. In this case the description of the singularity of the transformation (1) is equivalent to the determination of the corresponding caustics. This is the subject of the theory of the Lagrangian singularities.

The above mentioned suppositions concerning the origin of caustics are fulfilled for the evolution of adiabatic density perturbations in the universe. This approach was developed and applied to the problem of the extrapolation of the linear theory into the nonlinear stage by Zeldovich (1970). The perturbation amplitudes of density become of the order unity if the linear dimensions of perturbation regions are already essentially smaller than the Hubble radius ct and much smaller than the curvature radius of the expanding universe. Therefore, the Newtonian physics is entirely sufficient to describe the evolution of instabilities.

The idea is to write the Eulerian coordinate $\boldsymbol{r}$ as function of the comoving coordinate $\boldsymbol{s}$, where this transformation describes the perturbations starting from an smooth initial distribution (see Arnold et al. 1982). For the undisturbed solution of an expanding Friedmann universe we can write

$$\boldsymbol{r} = a(t)\boldsymbol{s}, \tag{3}$$

where the scale factor $a(t)$ is normalized in such a manner that at some instant of time t_0 the unperturbed position of the particle is determined by $\boldsymbol{s} = \boldsymbol{r}(t_0)$, i.e., $a(t_0) = 1$. Assuming background perturbations one is led to the ansatz

$$\boldsymbol{r} = a(t)\boldsymbol{s} + \boldsymbol{F}(\boldsymbol{s}, t). \tag{4}$$

In the case of small growing fluctuations $\boldsymbol{F}(s, t)$ can be written as the product

$$\boldsymbol{F}(\boldsymbol{s}, t) = b(t)\chi(\boldsymbol{s}), \tag{5}$$

because for $P = 0$ each increasing density perturbation grows proportional to the initial fluctuation, i.e., $\delta \propto b(t)\chi(r/a)$. Equations (4) and (5) are of the type of the transformation (1). The velocity field of adiabatic density perturbations is potential-like (rot $\boldsymbol{u} \equiv 0$). Because of

$$\boldsymbol{w} = \frac{d\boldsymbol{r}}{dt} - H\boldsymbol{r} \propto \boldsymbol{\chi}(\boldsymbol{s}), \tag{6}$$

it follows rot $\boldsymbol{\chi} = 0$. From the linear approach $b(t)$ is a known function, namely $b(t) \propto a(t)\delta_0(t)$. The approximation consists in the application of the equations (5) and (4) with the known functions $a(t)$ and $b(t)$ to the nonlinear stage.

The density evolution depending on time t can be computed straightforward. To this aim one has to compute the transformation of the volume of a mass element according to (2). Equation (4) represents a coordinate transformation $\boldsymbol{s} \rightarrow \boldsymbol{r}$, and it is valid

$$dV_r \approx dV_s J(\partial\boldsymbol{r}/\partial\boldsymbol{s}), \tag{7}$$

where $\boldsymbol{J}(\partial\boldsymbol{r}/\partial\boldsymbol{x}) = \boldsymbol{J}$ denotes as above the Jacobian of the transformation (4). Hence, it is

$$\varrho = \frac{dm}{dV_r} = \frac{dm}{dV_s}\boldsymbol{J}^{-1} = \varrho_0 a^3 \boldsymbol{J}^{-1}. \tag{8}$$

Let us assume, that the mean density is $\overline{\varrho} = \varrho_0 a^3$ and expand the Jacobian to first order. We find

$$\frac{\delta\varrho}{\varrho} = -\frac{b}{a}\nabla_s\boldsymbol{\chi}. \tag{9}$$

With an initial perturbation

$$\boldsymbol{\chi}(\boldsymbol{s}) = \sum_k \frac{i\boldsymbol{k}}{k^2} A_k \exp(i\boldsymbol{k}\boldsymbol{s}) \tag{10}$$

and the growing mode in an Einstein-de Sitter universe

$$\frac{b}{a} = \delta_k = (t/t_0)^{2/3} \tag{11}$$

the linear perturbation is

$$\frac{\delta\varrho}{\varrho} = (t/t_0)^{2/3} \sum_k A_k \exp(i\boldsymbol{k}\boldsymbol{s}). \tag{12}$$

Zeldovich's great insight was to point out that (3) does not only describe the linear theory but might also provide a first approximation to nonlinear behaviour of $\delta\varrho/\varrho$.

The Jacobian has the explicit form

$$a^{-3}\boldsymbol{J}(\partial\boldsymbol{r}/\partial\boldsymbol{s}) = \begin{vmatrix} 1+\frac{b}{a}\frac{\partial x_1}{\partial s_1} & \frac{b}{a}\frac{\partial x_1}{\partial s_2} & \frac{b}{a}\frac{\partial x_1}{\partial s_3} \\ \frac{b}{a}\frac{\partial x_2}{\partial s_1} & 1+\frac{b}{a}\frac{\partial x_2}{\partial s_2} & \frac{b}{a}\frac{\partial x_2}{\partial s_3} \\ \frac{b}{a}\frac{\partial x_3}{\partial s_1} & \frac{b}{a}\frac{\partial x_3}{\partial s_2} & 1+\frac{b}{a}\frac{\partial x_3}{\partial s_2} \end{vmatrix} \tag{13}$$

Because of rot $\boldsymbol{\chi} = 0$, it exists a transformation diagonalizing the matrix connected with (13) for each fluid element. Therefore, it can be written

$$\varrho = \frac{\bar{\varrho}}{\left(1-\frac{b}{a}\lambda_1\right)\left(1-\frac{b}{a}\lambda_2\right)\left(1-\frac{b}{a}\lambda_3\right)}. \tag{14}$$

The quantities λ_i are functions depending only on the coordinate s and on $b/a = f(t)$. The quantities λ_1, λ_2, and λ_3 give the deformation with respect to the main axes of the tensor of deformation $\tau_{ik} = \partial x_i/\partial s_k$. They are functions of the tensor components τ_{ik}. In general, it is $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and it exists a largest value $\lambda_{max} = \max(\lambda_i)$. If $\lambda_{max} > 0$ the quantity $\lambda_{max} b/a$ increases with time until it reaches the value unity. At this instant ϱ becomes infinite for the particle corresponding to the coordinate s. The infinite spatial density occurs because of a one-dimensional 0 in the direction of λ_{max}. In the result, at this position an extremely flat particle configuration forms which is comparable with a "pancake". A caustic has been formed. The surface density remains finite (a finite mass is distributed over a finite surface). This circumstance serves as the main argument in favour of applying this approximation up to the occurrence of the caustic. Indeed, the pancake causes a finite gravitational field. Hence the error due to the further application of the approximation remains finite even in the extreme case $\varrho \to \infty$.

The advantage of this approach consists in the fact that the procedure uses as the starting point the exact results of the linear stage, i.e., the approach is asymptotically exact. On the other hand it describes the transition to the nonlinear stage and represents a sufficient good approximation and qualitatively right description of the behaviour in the nonlinear regime (Doroshkevich and Shandarin 1974), i.e., the one-dimensional character of the collapse with crossing particle trajectories and the formation of an extremely flat matter configuration with very large density. In addition, the obtained solution is exact for perturbations depending only on one Cartesian coordinate (Zeldovich 1970).

The theory of the formation of "pancakes" in the expanding universe has been comprehensively worked out (Shandarin et al. 1983) as well as the further development of the singularities of the transformation (1). The motions in the medium lead to further pancakes . The firstly originated ones change their form and should interact. For the analysis of all formations and possible growth the complicated mathematical apparatus of the theory of the Lagrangian singularities is necessary (Arnold 1980).

During the further evolution singularities occur with different orders of magnitude with respect to the density. The density near the various singularity types behaves on the surface of the caustics as $\varepsilon^{-1/2}$, along singular lines and knots as $\varepsilon^{-2/3}$ or $\varepsilon^{-3/4}$ or ε^{-1}. The corresponding analysis and a comparison with the large-scale matter distribution was carried out by Doroshkevich et al. (1983).

The above described theory of the formation of singularities in the matter distribution like caustics concerns perturbations in dust-like matter or perturbations with sufficient large wave length. More realistic matter leads to the formation of shock waves in the pancake region. By comparison with the observations the occurring pancakes will be preferently identified with the superclusters of galaxies, i.e., actually with the large-scale matter configurations. For those configurations the applied approximation can be justified at best. The process of structure formation does not end with the formation of pancakes. Objects on smaller mass scales can appear due to fragmentation of the pancakes. For this the driving mechanisms are gravitational and thermal instabilities in the heated gas between the shock fronts of the pancake. After the formation of the caustics, respectively of the pancakes, the velocities of the latter ones surely deviates from the corresponding Hubble velocities. Possibly it also exists a velocity field for the pancake distribution which differs from the Hubble flow. Taking into account the gravitational interaction between the pancakes this should lead to clustering during the further evolution.

The described mechanism of the formation of pancakes leads to further consequences. The transformation (1), respectively the transformations (4) and (5), causes not only the occurrence of caustics, i.e., a matter concentration in certain regions, but also an evacuation of space regions large compared to the regions of matter concentration. From (2) we get formally for the region with $v_0'(x) > 0$ that the density tends to zero for $t \to \infty$. The originally almost homogeneous matter distribution goes over to a distribution with vanishing density in almost all of the space and to some rare space regions with highly contracted matter. Considering only inertial motion during time evolution the caustics are decaying and will form again at other places. Correspondingly, for the "empty" regions some temporary matter inflow takes place. Taking into consideration the gravitational interaction the highly inhomogeneous matter distribution (concentration on caustics and empty regions) will become stable.

In order to study the formation of galaxies and clusters of galaxies a non-linear theory of gravitational clustering is necessary. The best description of this complex evolution is provided by n-body calculations. During the last two decades a lot of different codes were developed which describe the gravitational clustering (see e.g. Hockney and Eastwood 1988). Several authors compared the results of n-body calculations with the Zeldovich approximation. The Zeldovich approximation describes the qualitative behaviour correctly, but the motion of particles is not stopped in the (formal) singularity. Improvements of the Zeldovich approximation take into account viscosity which suppresses particles crossing (Burger's approximation). For an recent review of the Zeldovich approximation and its extensions see Shandarin and Zeldovich (1989). Higher order corrections to the Zeldovich approximation were discussed by Buchert (1992).

Chapter 34

The relativistic theory of perturbations

In section 1 we have discussed homogeneous isotropic cosmological models, where all quantities depend only on time. In section 2 we have investigated the evolution of one single mode of perturbation within Newtonian theory. Now we will consider small perturbations of the metric tensor and the matter and describe their evolution within General Relativity. We shall start with a general discussion of the problem of describing perturbations in an invariant way and then turn to the evolution of classical perturbations during inflation where they can be connected to quantum fluctuations.

The background model is the homogeneous isotropic Friedmann universe. Within this section we will specify to the flat Robertson-Walker background metric with $k = 0$, because curvature terms may be important only at the very beginning of inflation. We write the metric as

$$ds^2 = (\overline{g}_{\alpha\beta} + \delta g_{\alpha\beta})dx^\alpha dx^\beta, \tag{1}$$

where $\overline{g}_{\alpha\beta}$ is the background metric tensor of eq. (1) with $k = 0$. The small perturbation of the metric $\delta g_{\alpha\beta}$ are coupled to small perturbations of the matter.

The general problem which arises is that of the physical interpretation of the perturbations. To this end one has to define a specific hypersurface of simultaneity. Only observers within the horizon can exchange signals and consequently define a physical hypersurface of simultaneity. Choosing a hypersurface of simultaneity without physical constraints (for example outside the horizon) one can always reach that the perturbation vanishes in that specific coordinate system. Under a transformation

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha(x), \tag{2}$$

where the $\xi^\alpha(x)$ are small quantities, the metric changes to

$$\tilde{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x) - 2\xi_{(\alpha;\beta)}, \tag{3}$$

where () denotes symmetrization. An arbitrary scalar quantity $f(x)$ changes as

$$\tilde{f}(x) = f(x) - \xi^\alpha \frac{\partial f}{\partial x^\alpha}. \tag{4}$$

Let us consider, for example, a scalar field $\varphi(x) = \overline{\varphi}(t) + \delta\varphi(x)$ which changes to

$$\tilde{\varphi}(x) = \overline{\varphi}(t) - \xi^0(x)\dot{\overline{\varphi}}(t) + \delta\varphi(x). \tag{5}$$

By an appropriate choice of $\xi^0(x)$ one obtains an unperturbated field $\tilde{\varphi}(x) = \overline{\varphi}(t)$. Therefore, to distinguish between physically relevant perturbations and pure coordinate effects one has to construct gauge-invariant quantities which describe the perturbations. At first, let us introduce the conformal time $d\eta = a^{-1}dt$. Thus, the background metric of the Einstein-de Sitter model reads

$$\begin{aligned} ds^2 &= \bar{g}_{\alpha\beta}dx^\alpha dx^\beta \\ &= a^2(\eta)(d\eta^2 - \delta_{ik}dx^i dx^k) \end{aligned} \tag{6}$$

According to their behaviour under coordinate transformations in the background metric we distinguish between scalar, vector, and tensor perturbations. Within the spatial flat metric of the Einstein-de Sitter model we can construct these perturbations in the following way (in spatially curved metrics one has to use the corresponding covariant derivatives). Scalar metric perturbations are constructed by the four scalar functions f, b, g, and e, which are functions of the four space-time coordinates

$$\delta g_{\mu\nu}^{(scalar)} = a^2(\eta) \begin{pmatrix} 2f & -b_{,i} \\ -b_{,i} & 2(g\delta_{ik} - e_{,ik}) \end{pmatrix}, \tag{7}$$

where "," means the partial spatial derivative.Vector perturbations are constructed from the divergence free vectors S_i and F_i

$$\delta g_{\mu\nu}^{(vector)} = -a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,k} + F_{k,i} \end{pmatrix}. \tag{8}$$

Tensor perturbations are described by the tensor h_{ik}, from which the scalar (trace) and vectorial parts are extracted. Thus,

$$\delta g_{\mu\nu}^{(tensor)} = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ik} \end{pmatrix}, \tag{9}$$

where $(h_i^i = h_{ik}^{;k} = 0)$.

Scalar metric perturbations can be connected with the perturbations of the energy density or to perturbations of a scalar field. Tensor metric perturbations are gravitational waves. Gravitational waves can be created during the inflationary stage and they may be important in understanding the fluctuations of the microwave background (Steinhardt et al. 1993). Scalar perturbations created during the inflationary stage are believed to be the basic ingredient for the formation of structure. Therefore, we will concentrate in the following only on scalar perturbations (for a complete treatment of gauge-invariant cosmological perturbations, see Bardeen 1980)

Generally, the transformation (2) does not preserve the type of perturbations (but tensorial perturbations are invariant under these transformations). The scalar nature of perturbations is preserved if the conformal time and the spatial coordinates transform like

$$\begin{aligned} \tilde{x}^0 &= \tilde{\eta} = \eta + \xi^0(x), \\ \tilde{x}^i &= x^i + \delta^{ik}\xi_{,k}, \end{aligned} \tag{10}$$

where $\xi^0(x)$ and $\xi_{,k}$ are independent functions. The metric changes according to (3). Thus,

$$\begin{aligned}\tilde{f} &= f - \frac{a'}{a}\xi^0 - \xi^{0\prime} \\ \tilde{g} &= g + \frac{a'}{a}\xi^0 \\ \tilde{b} &= b + \xi^0 - \xi' \\ \tilde{e} &= e + \xi,\end{aligned} \tag{11}$$

where $a' = da/d\eta$. Following Bardeen (1980) we can construct the following gauge invariant variables:

$$\begin{aligned}\Phi_A &= f - \frac{1}{a}\left((b - e')a\right)' \\ \Phi_H &= -g + \frac{a'}{a}(b - e').\end{aligned} \tag{12}$$

Note, that any combination of these variables is again gauge invariant.

Within a particular gauge the gauge-dependent quantities as the scalar field perturbation (or energy density perturbations in general) have their physical sense by comparing them with gauge-independent quantities introduced above. Within the horizon they have, of course, their sense also by comparing them directly with well defined physical measurable quantities. For example, describing perturbations outside the horizon within a fixed gauge and connecting these perturbations to the physical perturbations generated by physical mechanisms inside the horizon these perturbations have a well defined physical sense and pure gauge solutions can be excluded. Let us now consider two often used gauges, the synchronous gauge and the longitudinal gauge.

Setting $f = b = 0$ we achieve the synchronous gauge $\delta g_{0\alpha} = 0$ which was widely discussed by Lifshitz (1946) (see also Landau and Lifshitz 1979). In their notation the perturbated metric reads

$$ds^2 = dt^2 - a^2(t)\left(\delta_{ik} + \lambda(t)P_{ik} + \mu(t)Q_{ik}\right)dx^i dx^k, \tag{13}$$

where the P_{ik} and Q_{ik} are constructed from a scalar function Q which is directly connected with the perturbation in the density. The synchronous gauge leaves a residual gauge freedom with

$$\xi^0 = \xi^0(r), \quad \xi^i = \xi^0{}_{,i} \int^t \frac{dt'}{a^2} + C^i(r). \tag{14}$$

For a growing mode of scalar perturbations (which is only of physical interest), the gauge freedom (14) may be used to represent the perturbed metric in the following simple approximate form

$$ds^2 = dt^2 - a^2(t)(1 + h(x))\delta_{ik} dx^i dx^k. \tag{15}$$

For a scale of perturbations λ well outside the Hubble radius H^{-1} corrections to δg^{ik} are of the order $h(1/H\lambda)^2$. This particular gauge (defined for the growing mode only) was called by Starobinsky (1982) ultra-synchronous. It has no remaining gauge-freedom, so that $h(x)$ is a gauge-invariant quantity.

An other often used gauge is the longitudinal gauge $\delta g_{00} = 2\Phi$, $\delta g_{0i} = 0$, and $\delta g_{ik} = 2\Psi a^2(t)\delta_{ik}$. Then the perturbated metric reads

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Psi)\delta_{ik} dx^i dx^k. \tag{16}$$

The quantity Φ is the gravitational potential, whereas Ψ describes the intrinsic curvature (the unperturbated background model has vanishing curvature). Setting $b = e = 0$ we find in this gauge $\Phi = \Phi_A$ and $\Psi = -\Phi_H$, where Φ_A and Φ_H are the gauge-invariant perturbations defined in (12). Contrary to the synchronous gauge there is no remaining gauge freedom in the longitudinal gauge. Indeed, by the condition $b = e = 0$ we conclude from (11) that ξ and ξ^0 are uniquely fixed. If the metric perturbations are assumed to be bounded at spatial infinity, then $\Phi = \Psi$ for the system under consideration. This follows from the nondiagonal spatial component of the perturbations of the Ricci tensor (eq. (20), note that the nondiagonal spatial perturbations of the energy momentum tensor vanish). In that case the scalar perturbations are described by only one free function.

Well outside the horizon, there is a simple relation between the quantities $h(x)$ (eq. (15)) and $\Phi = \Psi$ (eq. (16)).

$$\Phi = -\frac{1}{2}h + \frac{\dot{a}}{2a^2}h \int^t a dt' \tag{17}$$

In the following we will use the longitudinal gauge. Using (16) we calculate the perturbations of the Ricci tensor:

$$\delta R_{00} = 3\ddot{\Psi} + 6\frac{\dot{a}}{a}\dot{\Psi} + 3\frac{\dot{a}}{a}\dot{\Phi} + \Delta\Phi \tag{18}$$

$$\delta R_{ik} = -\left(a^2\ddot{\Psi} + a\dot{a}\dot{\Phi} + 6a\dot{a}\dot{\Psi} + (2\ddot{a}a + 4\dot{a}^2)(\Phi + \Psi) + a^2\Delta\Psi\right)\delta_{ik} + \Psi_{,ik} - \Phi_{,ik} \tag{19}$$

$$\delta R_{0i} = 2\left(\Psi_{,0i} + \frac{\dot{a}}{a}\Phi_{,i}\right). \tag{20}$$

Thus, the perturbation to the Ricci scalar is

$$\delta R = 6\ddot{\Psi} + 2\Delta\Phi - 4\Delta\Psi - 2R\Phi + 6H\dot{\Phi} + 24H\dot{\Psi} \tag{21}$$

On the right hand side of Einstein's equations we have to insert the perturbations of the energy-momentum tensor. If the universe is dominated by hydrodynamical matter we use eq. (6). The perturbations of the energy-momentum tensor δT_α^β can be expressed in terms of the perturbations of the energy density $\delta\varepsilon$, of the pressure δP and of the velocity δu_i. We obtain

$$\delta T_0^0 = \delta\varepsilon, \tag{22}$$

$$\delta T_i^k = -\delta_i^k \ \delta P, \tag{23}$$

$$\delta T_i^0 = (\bar{\varepsilon} + \bar{P})\delta u_i. \tag{24}$$

As in section 2 we neglect the pressure of the baryons and assume adiabatic perturbations according to eq. (30). Therefore, the sound velocity b is given by eq. (31). From the absence of nondiagonal spatial perturbations in the energy-momentum tensor eq.(24) we conclude

$$\Phi = \Psi. \tag{25}$$

Thus, the first order correction to the Friedmann equation (i.e. to the (00)-component of the field equation) are

$$\Delta\Phi - 3H\dot{\Phi} - 3H^2\Phi = 4\pi G\delta\varepsilon \tag{26}$$

The $0i$- and the $i{=}k$ components lead to

$$\left(\dot{\Phi} + H\Phi\right)_{,i} = 4\pi G(\varepsilon + P)\delta u_i, \tag{27}$$

$$\ddot{\Phi} + 4H\dot{\Phi} + 2\dot{H}\Phi + 3H^2\Phi = 4\pi G\delta P \tag{28}$$

$$= 4\pi G b^2\delta\varepsilon. \tag{29}$$

Combining these equations we find

$$\ddot{\Phi} + (4 + 3b^2)H\dot{\Phi} + (2\dot{H} + 3H^2(1 + b^2))\Phi - b^2\Delta\Phi = 0. \tag{30}$$

With the quantity $w = P/\varepsilon = \gamma - 1$ (eq. (10)) the velocity of sound (eq. (31) is $b^2 = 4w/(3(1 + w))$. Then the term in front of Φ in eq. (30) is

$$2\dot{H} + 3H^2(1 + b^2) \propto \frac{w(3w - 1)}{(1 + w)^2} \tag{31}$$

¿From eqs. (30) and (31) we conclude that Φ = const is the solution for the matter dominated universe. In the radiation dominated universe it is the solution for modes outside the horizon where the spatial derivatives can be neglected. Finally we find from eq. (26)

$$\frac{\delta\varepsilon}{\varepsilon} = -2\Phi = \text{const.} \tag{32}$$

Introducing the quantity ζ

$$\zeta = \frac{2}{3}\frac{H^{-1}\dot{\Phi} + \Phi}{1 + w} + \Phi \tag{33}$$

and assuming adiabatic perturbations on scales larger than the Hubble radius eq. (30) can be rewritten in the form (Lyth 1985)

$$\dot{\zeta} = 0. \tag{34}$$

Thus, ζ is a conserved quantity. At the moment of transition from the radiation to matter dominated universe w changes from 1/3 to 0. Thus, the constant perturbations of the radiation dominated stage decrease at $t \approx t_{eq}$ by a factor 9/10 and remain again constant up to horizon crossing.

Chapter 35

The generation of perturbations during inflation

During the quasi de Sitter stage the energy density is dominated by the energy density of the field which drives inflation. As discussed in section 1 this is in the simplest case a scalar field. In order to investigate perturbations during the inflationary stage we have to consider the perturbation $\delta\varepsilon$ of the density of the coherent scalar field (14) which is given by the perturbation $\delta\varphi(x)$ of the scalar field. Thus, the perturbation of the 00-component of the energy-momentum tensor is

$$\delta\varepsilon = \delta T_{00} = \dot{\varphi}\delta\dot{\varphi} - \dot{\varphi}^2\Phi + \frac{dV}{d\varphi}\delta\varphi. \tag{1}$$

The perturbations of the other components of the energy-momentum tensor read

$$\delta T_{ik} = \left(\dot{\varphi}^2\Phi - \dot{\varphi}\delta\dot{\varphi} + \frac{dV}{d\varphi}\delta\varphi\right)\delta_{ik} \tag{2}$$

$$\delta T_{0i} = \dot{\varphi}\delta\varphi_{,i}. \tag{3}$$

Inserting the perturbations into the equations of General Relativity one finds the first order correction to the Friedmann equation in the form

$$\Delta\Psi - 3\frac{\dot{a}}{a}\dot{\Psi} - 3\left(\frac{\dot{a}}{a}\right)^2\Phi = \frac{4\pi}{M_{Pl}^2}\left(-\dot{\varphi}^2\Phi + \dot{\varphi}\delta\dot{\varphi} + \frac{dV}{d\varphi}\delta\varphi\right) \tag{4}$$

The $0i$-component gives

$$\left(\dot{\Psi} + \frac{\dot{a}}{a}\Phi - \frac{4\pi}{M_{Pl}^2}\dot{\varphi}\delta\varphi\right)_{,i} = 0 \tag{5}$$

and the non-diagonal spatial components of the gravitational field equations read

$$(\Phi - \Psi)_{,ik} = 0. \tag{6}$$

Finally, we find from the equation of motion of the scalar field (15)

$$\left(\Box + \frac{d^2V}{d\varphi^2}\right)\delta\varphi = (\dot{\Phi} + 3\dot{\Psi})\dot{\varphi} - 2\Phi\frac{dV}{d\varphi}. \tag{7}$$

Let us now Fourier transform the scalar quantities Φ, Ψ, and $\delta\varphi$. Due to the homogeneous background a separation into time and spatial dependent parts in the perturbation equations is possible. During inflation scalar spatial harmonics are simple plane waves, because curvature can be neglected. Following the notation in most of the literature we now omit the index of the Fourier components Φ_k, Ψ_k, and $\delta\varphi_k$. That means, the quantities $\Phi(t)$, $\Psi(t)$ and $\delta\varphi(t)$ now depend only on time. The equations (4 - 6) simplify by the condition $\Phi = \Psi$ coming from eq. (6). Therefore, scalar perturbations are described in this case by only one independent gauge-invariant gravitational potential.

In the equations (4, 7) the Δ-operator must be replaced by $-k^2/a^2$, which is the inverse physical wave length of the considered perturbation $\lambda_{phys} = 2\pi k^{-1}a$. During inflation the scale factor $a(t)$ increases exponentially. At first sight, the deciding role of the relation $\lambda_{phys}/\lambda_{hor} = Ha/k$ is clear: completely different solutions of the perturbation equations can be expected in the short wave limit $(k/aH \gg 1)$ and the long wave limit $(k/aH \ll 1)$. This makes clear the sense of the physical horizon $\lambda_{hor} = H^{-1}$. Note, that during the de Sitter stage, H^{-1} is equal to the event horizon, whereas within a pure Friedmann universe H^{-1} is approximately equal to the particle horizon. Therefore, if one says that in the inflationary models perturbations can "leave the horizon" during inflation and "reenter it" during the Friedmann stage, then one has in mind that physical horizon H^{-1}. Of course, each cosmological model has only one well defined 0 horizon,

$$l_{Hor} = a\int_0^t \frac{dt'}{a} \tag{8}$$

which can be "crossed" only once.

Combining the perturbation equations (4) and (7) we find the wave equation

$$\ddot{\Phi}+\left(H-2\frac{\ddot{\varphi}}{\dot{\varphi}}\right)\dot{\Phi}+\left(2\dot{H}-2\frac{\ddot{\varphi}}{\dot{\varphi}}H+\frac{k^2}{a^2}\right)\Phi=0 \tag{9}$$

of the gravitational potential Φ. Note, that this equation does not contain 0 the potential $V(\varphi)$ of the scalar field. Introducing the quantity ζ (Lyth 1985)

$$\zeta=\frac{2}{3}\frac{H^{-1}\dot{\Phi}+\Phi}{1+w}+\Phi \tag{10}$$

we can rewrite eq. (9) in the following form

$$\frac{3}{2}\dot{\zeta}H(1+w)+\frac{k^2}{a^2}\Phi=0 \tag{11}$$

where $w=p/\varepsilon=-1-2\dot{H}/(3H^2)$. In eq. (9, 11) the term k^2/a^2 decreases during inflation exponentially. Far outside the horizon $k/(aH)\ll 1$ the last term can be neglected. Thus ζ is a conserved quantity on those scales.

Let us solve eq. (9) within the short wave limit, $k^2/a^2\gg H^2\gg \dot{H}$. Then after a transformation to conformal time $d\eta=dt/a$ eq.(9) describes oscillations $\Phi\propto e^{ik\eta}$ and in WKB-approximation damped oscillations, which read in the original time

$$\Phi\propto\frac{e^{ik\int dt/a}}{a}. \tag{12}$$

¿From eq. (5) it can be seen, that the perturbation of the scalar field $\delta\varphi$ also show damped oscillations.

In the long wave limit of the quasi-de Sitter stage $k^2/a^2\ll H^2(\mid\ddot{\Phi}\mid\ll\mid H\dot{\Phi}\mid)$ the potential Φ is a slowly increasing function,

$$\Phi\propto H^{-2}. \tag{13}$$

Due to the exponentially increasing scale factor the transition from one approximate solution to the other is very fast. Therefore, the constant in eq. (13)

can be determined by matching eq. (13) to eq. (12) at $H_k = k/a$. The gravitational potential Φ increases proportional to H^{-2} up to the end of inflation. Then the scalar field φ oscillates with the characteristic frequency ω 0 the minimum of the potential $V(\Phi)$ (ω depends on the specific shape of the potential, for example, $\omega = m$ if $V(\Phi) = \frac{1}{2}m^2\Phi^2$). Thus, $\dot{H} \sim H^2 \sim \omega^2$ and in first approximation $\Phi =$ const. Small oscillations with frequency ω are superimposed. The value of Φ after inflation can be determined by matching (12) and (13) at horizon crossing. Therefore, the perturbations Φ after inflation strongly depend on the amplitudes of the oscillations described by eq. (12). Due to the oscillations they depend also on the phase in the moment of transition to the regime (13). These phases are assumed to be distributed at random. Thus, $\langle \Phi \rangle = 0$ and $\langle \Phi^2 \rangle$ describes the perturbation completely.

In order to determine $\langle \Phi^2 \rangle$ in dependence on the wave number k we have to take into consideration how the constant in the short wave limit (12) depends on the wave number k. Due to the $0i$-component of the perturbation equation the gravitational potential Φ is directly coupled to the perturbation of the scalar field $\delta\varphi$ (eq. (5)). The basic idea in determining the perturbation spectrum is now that the inevitable quantum fluctuations of the scalar field $\delta\varphi$ within the quasi-de Sitter stage have well defined amplitudes. These amplitudes and consequently the constant in eq. (12) can be calculated straightforward (see Mukhanov et al. (1992) for a general discussion of quantum fluctuations and their connection to classical perturbations).

So let us quantize the scalar field in the curved space time as described by Birrel and Davies (1982). The scalar field operator $\delta\hat{\varphi}$ can be represented in the form

$$\delta\hat{\varphi} = \frac{1}{(2\pi)^{3/2}} \int d^3k [\hat{a}_k u_k + \hat{a}_k^+ u_k^*], \tag{14}$$

where $\hat{a}_k$ and $\hat{a}_k^+$ are the usual annihilation and creation operators and

$$u_k = \frac{\chi_k(\eta)}{a} e^{ikx}. \tag{15}$$

Let us solve the Klein-Gordon equation in the flat background metric using conformal time, $ds^2 = a^2(\eta)(d\eta^2 - d\vec{r}^{\,2})$. Let us further consider in first approximation the limit $k^2/a^2 \gg dV/d\varphi$, i.e. we treat $\delta\varphi$ as a massless scalar field and assume that the Hubble parameter is constant (de Sitter universe). Then the solution of the wave equation reads

$$u_k = -(1/2)H\eta\,(\pi\eta)^{1/2}H^{(2)}_{3/2}(k\eta), \tag{16}$$

where $H^{(2)}_{3/2}(k\eta)$ denotes the Hankel function

$$H^{(2)}_{3/2}(x) = -\left(\frac{2}{\pi x}\right)^{1/2}\left(1+\frac{1}{ix}\right)e^{-ix}. \tag{17}$$

The short and long wave limits of the solution (16) are

$$u_k = (2k)^{-1/2}H\eta e^{-ik\eta}, \quad k\eta \to \infty, \tag{18}$$

$$u_k = Hk^{-1}(2k)^{-1/2}, \qquad k\eta \to 0, \tag{19}$$

respectively. The solutions (19) and (19) correspond to the two limiting cases of a physical wave length much smaller than the horizon ($\lambda_{phys} \ll H^{-1}$) and much larger than the horizon ($\lambda_{phys} \gg H^{-1}$). Since $\lambda_{phys} = 2\pi a/k$ is an exponentially increasing function the transition from one limit to the other is very rapid. The constant factors in eq. (16) were introduced to normalize the scalar product

$$(u_k,\, u_{k'}) = -i\int u_k \overleftrightarrow{\partial_t}\, u^*_{k'} d^3x = \delta^{(3)}(k-k') \tag{20}$$

We calculate the vacuum expectation value $\langle\delta\varphi^2\rangle$ from the u_k

$$\langle\delta\varphi^2\rangle = \frac{1}{(2\pi)^3}\int |\, u_k^2 \,|\, d^3k = \frac{1}{(2\pi)^3}\int \frac{d^3k_{phys}}{2k_{phys}}\left(1+\frac{H^2}{k^2_{phys}}\right). \tag{21}$$

The first term in eq.(21) leads to the usual vacuum expectation value in the Minkowski space ($H = 0$). It must be removed by renormalization. However, the second term is typical for the 0 stage. It diverges but remains finite in the quasi-de Sitter stage of the inflationary scenario. This term is connected to the metrical 0.

Comparing eq. (21) with the Fourier transform of the classical perturbation of the scalar field used above we conclude that

$$\langle \delta\varphi_k \rangle = 0, \quad \langle \delta\varphi_k^2 \rangle = \frac{H^2}{2k^3}, \tag{22}$$

where H must be taken at the moment of horizon crossing $a = k/H$. Finally we have to determine via eq. (5) the dependence of the gravitational potential on the wave number k. We find

$$\langle \Phi^2 k^3 \rangle = \text{const} \left(\frac{H^2}{|\dot{\varphi}|} \right)_k, \tag{23}$$

where the quantity in the r.h.s. should be taken in the moment of the first Hubble radius crossing, too. Its dependence on k is very weak, so that the well known approximately flat (Harrison-Zeldovich) spectrum arises. No physical process can change this spectrum during the further evolution outside the horizon. The first quantitatively correct expressions for these perturbations were presented independently by Hawking (1982), Starobinsky (1982) and Guth and Pi (1982). The perturbations described can be considered as random Gaussian classical variables with zero average and dispersion given by eq. (23) (e. g. Starobinsky 1988a).

In more complicated models, the perturbation spectrum has to be determined numerically. For models with two inflationary stages driven by a scalar field and higher order corrections to the gravitational Lagrangian double inflation may occur. In this case, an additional length scale in the perturbation spectrum arises (see section 7). An approximate spectrum was calculated for double inflation driven by two scalar fields (Polarski and Starobinsky 1992).

Chapter 36

Observing perturbations in the universe

On every map of the galaxy distribution we can see that galaxies form structures in the universe. These structures have been studied by means of the correlation function. The two-point correlation function of galaxies is defined by the joint probability of finding a galaxy in both the volumes dV_1 and dV_2 at separation r_{12}

$$dP = n_g^2[1 + \xi(r_{12})]dV_1 dV_2, \tag{1}$$

where n_g is the mean number density of galaxies. Empirically the galaxy correlation function is found to be

$$\xi(r) = \left(\frac{r}{r_g}\right)^{-1.8}, \tag{2}$$

where the correlation radius of spiral galaxies is $r_g = 5h^{-1}$ Mpc. This relation holds true in the region 1 Mpc $\leq r \leq$ 20 Mpc. A similar functional form, but with different correlation radii, is found for correlations of elliptical galaxies and clusters of galaxies (for a review see Bahcall 1988). To compare theoretical predictions with the observations we have to connect the correlation function (and other observational data) to the cosmological perturbations.

A first step is to specify the nature of these perturbations. As in section 1, we describe the spatially variable density field by means of the function $\delta(x, t)$

$$\rho(\boldsymbol{x}, t) = \bar{\rho}(t)\,(1 + \delta(\boldsymbol{x}, t)) \tag{3}$$

and Fourier transform it:

$$\delta(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \delta_k \exp(i\boldsymbol{k}\boldsymbol{x}) \tag{4}$$

Usually it is assumed that the phases of the density perturbations are randomly distributed and the density fluctuation obey Gaussian statistics. The probability function is given by

$$dP = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta_k^2}{2\sigma^2}\right) d\delta_k \tag{5}$$

where σ^2 is the variance of the density field. Then all statistical properties of the density field are described by one function, the power spectrum

$$P(k) = \langle \mid \delta_k \mid^2 \rangle \tag{6}$$

On the other hand the correlation function $\xi(\boldsymbol{r})$ and the power spectrum form a Fourier transform pair (cp. Peebles 1980 for a detailed derivation):

$$\langle \mid \delta_k \mid^2 \rangle = \frac{1}{V} \int d^3 r \xi(r) e^{i\boldsymbol{k}\boldsymbol{r}} \tag{7}$$

$$\xi(r) = \frac{V}{(2\pi)^3} \int d^3 k P(k) e^{i\boldsymbol{k}\boldsymbol{r}}. \tag{8}$$

Since the power spectrum depends only on k one can integrate the angular part

$$\xi(r) = \frac{V}{2\pi^2} \int dk k^2 P(k) j_0(kr), \tag{9}$$

where $j_0(kr) = \sin(kr)/(kr)$ is the zeroth order spherical Bessel function. The density fluctuations can be characterized by their *rms* (root mean square) value $\langle \mid \delta(\boldsymbol{x},t) \mid^2 \rangle$. Note, that $\langle \mid \delta(\boldsymbol{x},t) \mid \rangle = 0$ by definition. The *rms* values are connected with the power spectrum by

$$\langle \mid \delta(x,t) \mid^2 \rangle = \frac{V}{(2\pi)^3} \int d^3 k P(k). \tag{10}$$

Gaussian random fields are a very special class of distributions. Their properties can be studied knowing the power spectrum (6) (for a comprehensive discussion see Bardeen et al. 1986). Most authors believe that during inflation perturbations with Gaussian statistics are created. Non-Gaussian perturbations may be created in some nonstandard inflationary scenarios or by topological defects in the early universe. From observations non-Gaussian density perturbations are still allowed (Luo and Schramm 1993), but there is no physical reason to choose a special non-Gaussian density perturbation for the description of the cosmological matter distribution. Therefore, we will concentrate in the following only on Gaussian perturbations.

The next step is to choice the spectrum of perturbations. In the simple cosmological models no fundamental length exists. Therefore, the first choice of the spectrum would be a featureless power spectrum

$$\mid \delta_k^2 \mid \propto k^n. \tag{11}$$

However, recent observations show that there must be more power on larger scales. One explanation for this observational fact may be the existence of a characteristic scale in the primordial perturbation spectrum as mention in the previous section.

Harrison (1970) and Zeldovich (1972) have argued that the density fluctuations averaged over the scale $\lambda \sim ct$ are fixed

$$\left\langle \left(\frac{\delta\rho}{\rho} \right)^2 \right\rangle_{k=1/(ct)} = \epsilon^2 \tag{12}$$

with $\epsilon < 1$. Assuming a scale free spectrum (11) the variance of the density fluctuations (10) at horizon scale $a(t)x_h = ct$ is

$$\left\langle \left(\frac{\delta\rho}{\rho} \right)^2 \right\rangle_{ct} = \frac{V}{(2\pi)^3} \int^{1/x_h} d^3k P(k) \propto x_h^{-(3+n)}. \tag{13}$$

Multiplying with the solution for the growing mode eq. (24) for wavelength which crosses the horizon at $t > t_{eq}$ we find finally $\langle(\delta\rho/\rho)^2\rangle \propto t^{4/3-(n+3)/3}$. Thus, only if the exponent in the power spectrum (11) is $n = 1$, the variance of the density field is scale independent at the Hubble radius ct in accordance with eq. (12). This spectrum is called Harrison-Zeldovich spectrum.

Finally we have to specify the type of the perturbations. We distinguish adiabatic perturbations

$$\left(\frac{\delta\rho}{\rho}\right)_{rad} = \frac{4}{3}\left(\frac{\delta\rho}{\rho}\right)_{matter} \tag{14}$$

isothermal perturbations

$$\delta\rho_{matter} \neq 0, \quad \delta\rho_{rad} = 0 \tag{15}$$

and isocurvature perturbations

$$\delta\rho_{matter} + \delta\rho_{rad} = 0. \tag{16}$$

As already mentioned adiabatic fluctuations are a natural result of an early inflationary stage in the cosmological evolution. However, under special circumstances it is possible to generate also entropy perturbations with a scale invariant spectrum. As an example one could consider an universe which contains radiation including relativistic matter and axions. At t_{QCD} the axions acquire masses which leads (in the simple model) to fluctuations in the axion density. Due to energy conservation these fluctuations must be compensated by fluctuations in the radiation and relativistic matter (for details see Efstathiou and Bond (1986)).

Often it is useful to calculate the variance of mass within a certain volume. With the function $\phi(r)$ we can select the volume. The most common selection functions are the 'top-hat' type

$$\phi = 1 \quad \text{at} \quad r \leq R,$$

$$\phi = 0 \quad \text{at} \quad r > R, \tag{17}$$

and the Gaussian type

$$\phi(r) \propto \exp\left(-\frac{r^2}{2R^2}\right). \tag{18}$$

Averaging the fluctuation $M - \bar{\rho}r^3$ we find the variance of mass

$$\left\langle \left(\frac{\delta M}{M}\right)^2 \right\rangle = \int d^3kP(k) \mid W(k) \mid^2, \tag{19}$$

where $W(k)$ is the window function

$$W(k) = \frac{\int_0^\infty dr\ r^2\phi(r)j_0(kr)}{\int_0^\infty dr\ r^2\phi(r)}. \tag{20}$$

For the sharp top-hat window (17) it is given by

$$W(k) = \frac{3}{R^3}\int_0^R dr\ r^2 j_0(kr) = \frac{3}{(kR)}[\sin(kR) - kR\cos(kR)] \tag{21}$$

whereas for the Gaussian window it is

$$W(k) = \sqrt{\frac{2}{\pi}}\frac{1}{R^3}\int_0^\infty dr\ r^2 j_0(kr)\exp\left(-\frac{r^2}{2R^2}\right) = \exp\left(-\frac{(kR)^2}{2}\right). \tag{22}$$

The window functions decrease 0 if $kR \gg 1$. Thus in the Fourier representation they filter out the high frequencies corresponding to scales inside the considered volume. That means by calculating the mass fluctuations we do not take into account any substructures.

The cosmic microwave background radiation was detected by Penzias and Wilson (1965). In the standard model of a hot Friedmann universe the temperature of this black body radiation is independent of the direction of the measurement, but from a theoretical point of view one could expect that due to the seeds of the today observed large scale structure small fluctuations must be imprinted on the isotropic radiation. More than 25 years later these primordial perturbations could be detected (Smoot et al. 1992).

The anisotropies of the temperature of the microwave background radiation $\Delta T/T$ are a direct test of the primordial fluctuations in the linear regime. Nevertheless, the result depends also crucially on the evolution of the fluctuations before recombination and on the recombination physics. This evolution is influenced by the composition of matter. In this section we are mainly interested in the influence of the primordial perturbation spectrum on the microwave background temperature fluctuations. Therefore, we will consider only the case that the nonbaryonic dark matter consists of heavy ($m > 1$ GeV) weakly interacting particles (CDM model).

The fluctuations of the microwave background temperature can be represented as a sum of the Sachs-Wolfe effect, the Doppler effect, and the contribution coming from non-simultaneous recombination. A general gauge-invariant representations is (Traschen and Eardley 1986)

$$\begin{aligned}\frac{\Delta T}{T} &= \frac{\Delta T_{SW}}{T} + \frac{\Delta T_{Dop}}{T} + \frac{\Delta \rho_\gamma}{\rho_\gamma} \\ &= -\frac{1}{2}\int_{\eta_{rec}}^{\eta_0} a^4(\bar{h}_{\mu\nu,0} - 2\bar{h}_{0\nu,\mu})k^\mu k^\nu \, d\eta + \\ &\quad \left(v^i_{rad} e_i + \frac{1}{4}\left(\frac{\delta\varepsilon}{\varepsilon}\right)_{rad}\right)_{\eta=\eta_{rec}} , \end{aligned} \tag{23}$$

where the perturbations of the metric are $\bar{h}_{\mu\nu} = h_{\mu\nu}/a^2$, k_μ denotes the four-momentum of the photon and e_i a spatial unit vector in the direction of the light ray, along which the integrand is to be evaluated. Radiation and baryons are coupled up to recombination and have the velocity v_{rad}, fluctuations $\delta\varepsilon_{rad}$ of the energy density of the radiation lead to recombination at different moments. The three mechanisms included into eq.(23) are responsible for the microwave background fluctuations at different angular scales. In the case of a flat perturbation spectrum the Sachs-Wolfe effect leads to approximately constant temperature fluctuations at angular scales larger than 2° (this corresponds to $400h^{-1}$ Mpc). With decreasing scales, the

Doppler effect becomes important first and the perturbations in the radiation energy density later.

Let us now connect the fluctuations in the background temperature to the perturbation spectrum eq. (6). We are mainly interested in large scale perturbations which are influenced only by the primordial perturbation spectrum, i.e. $P(k) = Ak$ for the Harrison-Zeldovich spectrum. Therefore, we take into account only the Sachs-Wolfe effect in eq. (23). In the spatial flat model the integrand in eq. (23) tends to zero so that the result depends only on the perturbation at the source (the contribution from the potential at the place of the observer is omitted). Thus the temperature fluctuation is given alone by the gravitational redshift of the photons which leave the potential perturbation inside which they were last scattered.

The temperature pattern on the sky can be naturally expressed in terms of the normalized spherical harmonics $Y_{lm}(\theta, \phi)$,

$$\frac{\Delta T}{T} = \sum_{l,m} a_{lm}\, Y_{lm}(\theta, \varphi). \tag{24}$$

Predictions for the multipole moments in eq.(24) can be made by taking the Fourier component of eq. (23), projecting out a given multipole and integrating over the wave vector. The mean square expectation values C_l of the coefficients a_{lm} define a power spectrum. (One has to take into account that the cross terms vanish due to the δ-correlations of the perturbations.) The result is (Peebles 1982)

$$C_l = \langle a_{lm}^2 \rangle = \frac{H^4}{2\pi c^4} \int_0^\infty \frac{dk}{k^2} P(k) j_l^2(kr), \tag{25}$$

where $r = 2c/H$. For the Harrison-Zeldovich spectrum the integral is

$$C_l = \frac{H^4}{2\pi c^4} \int_0^\infty \frac{dx}{x} j_l^2(x) = \frac{AH^4}{4l(l+1)\pi c^4}, \tag{26}$$

where A denotes the normalisation constant of the power spectrum. This constant can be determined from measurements of large scale temperature fluctuations as done by COBE. The functions C_l are related to the temperature autocorrelation function $\xi_T(\theta)$ by

$$\begin{aligned}\xi_T(\theta) &= \left\langle \frac{\Delta T}{T}(\vec{r_1})\ \frac{\Delta T}{T}(\vec{r_2}) \right\rangle \\ &= \frac{1}{4\pi}\sum_{l\geq 2}(2l+1)C_l P_l(\cos\theta),\end{aligned} \tag{27}$$

where θ denotes the angle between $\vec{r_1}$ and $\vec{r_2}$. The *rms* temperature fluctuations are

$$\xi_T(0) = \frac{1}{4\pi}\sum_{l\geq 2}(2l+1)C_l. \tag{28}$$

In order to confront the theoretical predictions with observations one has to take into account the special configuration of the experiment. Let us consider a two-beam experiment which measures the mean-square temperature difference between the two beams of width θ_s and of separation angle θ. Then the theoretical expected *rms* value is

$$\begin{aligned}\sigma_T^2(\theta,\theta_s) &= \left\langle \left(\frac{\Delta T}{T}\right)^2 \right\rangle \\ &= \frac{\langle (T(x)-T(x+\theta))(T(x+\theta_s)-T(x+\theta_s+\theta))\rangle}{\langle T^2\rangle} \\ &= 2\xi(\theta_s)-\xi(\theta+\theta_s)-\xi(\theta-\theta_s),\end{aligned} \tag{29}$$

which reduces for $\theta_s = 0$ to

$$\sigma_T^2 = 2(\xi_T(0)-\xi_T(\theta)). \tag{30}$$

For two Gaussian beams Bond et al. (1991) have proposed to express $\sigma_T^2(\theta_s,\theta)$ as

$$\sigma_T^2 = \frac{1}{2\pi}\sum_{l\geq 2}(2l+1)C_l[1-P_l(\cos\theta)]\exp(-\theta_s^2 l^2). \tag{31}$$

More general filter functions $f_l(\theta)$ for various experiments are shown in Bond et al. (1991) and Efstathiou (1991), where the expected *rms* amplitudes are given by

$$\sigma_T^2 = \frac{1}{2\pi}\sum_{l\geq 2}(2l+1)C_l f_l(\theta). \quad (32)$$

Like the spatial window functions introduced above, these functions $f_l(\theta)$ filter out the multipole range which is effectively measured by the experiment under consideration.

During inflation also gravitational waves may be generated. In the large scale limit these primordial gravitational waves lead to corrections in the temperature fluctuations if the index of the perturbation spectrum is $n \neq 1$ (Crittenden et al. 1993).

An accurate computation of temperature fluctuations on all scales requires a numerical solution of the collisional Boltzmann equation for the photons (Peebles and Yu 1970, Bond and Efstathiou 1984). However, intermediate angular scales can be well described also by approximation formulas(Starobinsky 1988b, Starobinsky and Sahni 1984). This method was used to calculate the temperature fluctuations resulting from primordial perturbation spectra with broken scale invariance (Gottlöber and Mücket 1993).

Once we have determined the normalisation constant in the spectrum $P(k)$, we can calculate properties of the linear density and velocity fields in the universe. To do this we have to specify also the exponent in the simple scale free spectra $P(k) \propto k$. On the other hand, we have shown in the previous section how primordial fluctuations of the potential are created during the inflationary stage. These fluctuations are coupled to density fluctuations. In the second section we have discussed the growing modes of perturbations and characteristic scales. In linear perturbation theory, these modes evolve independently. Thus, we can define a transfer function $T(k)$ which connects the spectra at different moments of time.

Starting with the Poisson equation

$$\Delta\Phi = -4\pi G a^2 \delta\rho, \quad (33)$$

where $\Delta\Phi = -k^2\Phi$ and using the relation between the mean density of the Einstein-de Sitter universe and the Hubble parameter we find (cp eq. (6)

$$P(k) = \frac{4}{9}\frac{k}{(H/a)^4}\langle\Phi^2 k^3\rangle. \tag{34}$$

Thus, we get $P(k) \propto k$ according to eq. (23). As mentioned above, $P(k)$ has to be multiplied by a transfer function which describes the evolution of the spectrum,

$$P(k, t_0) = T^2(k)P(k, t_i). \tag{35}$$

The transfer function is generally normalized to give $T(k) = 1$ for $k \to 0$. The calculations of the transfer function are based on numerical integrations of the perturbation equations. They depend on the type of the fluctuation and the composition of matter in the universe. The linear transfer function has been 0 by several authors (e.g. Peebles 1982, Bond and Efstathiou 1984, Bardeen et al. 1986, Holtzman 1989). Let us assume that the present matter density in the universe is dominated by weakly interacting massive particles with masses greater than 1 GeV, so that they can be considered as nonrelativistic during the time under consideration. Then we can give qualitative arguments for the behaviour of the transfer function on very large and on very small scales. While the scale of the fluctuations is greater than the Hubble radius ct no physical effect can change the spectrum. When the perturbations enter the Hubble radius the baryons together with the photons begin to oscillate. During the radiation dominated era the amplitudes of the perturbations in the CDM component are effectively frozen according to eq.(27). When the universe becomes matter dominated these perturbations grow once again according to eq. (24). After recombination the baryons are decoupled from the photons and fall into the CDM fluctuations. Therefore, the transfer function contains one characteristic length, the Hubble radius at the time of the transition from radiation dominated to matter dominated evolution of the universe. Today this corresponds to

$$\lambda_{eq} \approx 10h^{-2}\ \mathrm{Mpc}. \tag{36}$$

On larger scales which enter the Hubble radius in the matter dominated epoch the initial perturbation spectrum is not changed. Smaller scales are frozen after crossing the horizon up to t_{eq}. However, scales outside the horizon still grow $\propto \eta^2$ up to $k\eta \approx 1$. Thus, we conclude that the spectrum $P(k)$ changes at approximately k_{eq} by an exponent of minus 4. Therefore, the asymptotic behaviour of the spectrum is

$$P(k) \propto k, \qquad k \to 0,$$

$$P(k) \propto k^{-3}, \qquad k \to \infty. \tag{37}$$

(38)

These qualitative considerations are confirmed by numerical simulations.

Chapter 37

Beyond the standard model

The large scale structure of the universe has its origin in small perturbations of the density of matter, which are created during an early stage of cosmological evolution. These perturbations are assumed to be Gaussian. Thus, they can be described completely by a power spectrum $P(k)$. Inflation provides a quite simple mechanism for producing perturbations in a causal way by quantum fluctuations. Due to the exponential expansion, these perturbations are inflated to those large scales characteristic for the evolution of cosmic structures. It became clear that in the simplest inflationary models Gaussian perturbations with a Harrison-Zeldovich spectrum are created. The amplitudes of these perturbations can be calculated straightforwardly (see section 5).

The inflationary cosmological models predict a density parameter $\Omega_{tot} = 1$. On the other hand, by cosmological nucleosynthesis the density of baryons is restricted to be $0.01 < \Omega_{bar} < 0.12$ for $0.4 < h < 1$ (Walker et al. 1991). Recent observations have shown that Ω_{tot} may be of the order of unity as inflation predicts (Rowan-Robinson et al. 1990, Strauss et al. 1992). Assuming $\Omega_{tot} = 1$, most of the matter in the universe must be invisible nonbaryonic matter. This matter consists of weakly interacting massive particles. Depending on the mass of these particles (which defines the moment when they became nonrelativistic) the dark matter is called hot (HDM) or cold (CDM).

Assuming a Harrison-Zeldovich spectrum of perturbations, and introducing a biasing parameter b, the CDM model was able to describe very successfully the formation of structure up to scales of approximately 20 Mpc. Most of cosmologists have considered the CDM model as the standard model of structure formation.

However, resent observations on larger scales indicate more power than predicted by this model. In particular, the Mach number test of Ostriker and Suto (1990), the large-scale clustering analysis of Efstathiou et al. (1990a) and Loveday et al. (1992) and the observed angular correlation function of the deep APM galaxy survey (Maddox et al. 1990) disagree with the standard CDM model. Note that "more power" on large scales means that more structures are observed than predicted if the biasing parameter is fixed and the perturbation spectrum is normalized at 8 Mpc. Normalizing the spectrum at very large scales by the COBE measurement of the quadrupole of the cosmic microwave background fluctuations one should keep in mind that observations on small scales show less power than predicted by the theory.

The transfer function $T(k)$ describes the evolution of the primordial perturbation spectrum (see section 6). Within the CDM model the spectrum exhibits the limiting behaviour $P(k) \propto k$ at $k \rightarrow 0$ and $P(k) \propto k^{-3}$ at $k \rightarrow \infty$. The maximum of $P(k)$ defines a characteristic scale, which is related to the Hubble radius at the moment of the transition from radiation dominated to matter dominated evolution of the universe. The exact behaviour of $P(k)$ near the maximum must be calculated numerically. For the CDM model the corresponding transfer function was calculated by Bond and Efstathiou (1984). The observations mentioned above indicate that one more characteristic scale exists in the power spectrum $P(k)$. This scale divides the part with more structure from that with less structure.

There are several possibilities for modelling such a scale. Obviously, one can introduce it either for the primordial perturbation spectrum or for the transfer function. In the first case one has to consider more complicated inflationary models, whereas in the second case one has to introduce an additional component of the dark matter. There are attempts to fit the observational data by models with mixed dark matter (MDM). The best fit of observational data is reached in this model if approximately 70% of the nonbaryonic dark matter consists of very heavy weak interacting particles (CDM) and approximately 30% consists of one species of massive neutrinos (HDM) (Klypin et al. 1993, Jing et al. 1994). Another widely discussed possibility is to introduce a cosmological constant so that $\Omega + \lambda = 1$ (e.g. Efstathiou et al. 1990b). We will discuss in this section a model which leads to a primordial perturbation spectrum of Harrison-Zeldovich type only in the limit of very large and very small scales. At intermediate scales it shows a break. By an appropriate choice of the parameters of the model, the break lies on observable scales (Gottlöber et al. 1993).

It is well known that both vacuum polarisation effects and a scalar field may lead to inflationary stages in the evolution of the early universe. The combined action of these effects may lead to two consecutive inflationary stages. Following the treatment of Gottlöber et al. (1991), we shall discuss here a cosmological model including vacuum polarisation effects which are described by an additional R^2-term in the gravitational action and a coherent massive scalar field. Thus we start with the Lagrangian

$$L = \frac{1}{16\pi G}\left(-R + \frac{1}{6M^2}R^2\right) + \frac{1}{2}\left(\varphi_{,\mu}\varphi^{,\mu} - m^2\varphi^2\right), \tag{1}$$

where $1/(6M^2)$ is a coupling constant, R denotes the Ricci scalar and φ the scalar field with the mass m. Both masses are assumed to satisfy the conditions $M \ll m_{Pl}$, $m \ll m_{Pl}$.

The question of whether the consecutive inflationary stages are separated by an intermediate stage of power law expansion is crucial for the existence of a break in the spectrum. In the model described by the Lagrangian eq.(1), the case of two really disconnected inflationary stages takes place only if the mass of the scalar particles is small in comparison with the inverse coupling constant, i.e. $m \ll M$, and if vacuum polarisation dominates initially, i.e. the energy density of the scalar field is small compared to the Hubble parameter during the first stage of inflation. Otherwise the combined action of the scalar field and vacuum polarisation leads to a single quasi-de Sitter stage with a quasi-flat perturbation spectrum.

The variation of the Lagrangian eq.(1) leads to the generalized Friedmann equation

$$H^2 + \frac{1}{M^2}(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H}) = \dot{\varphi}^2 + m^2\varphi^2, \tag{2}$$

and to the equation of motion of the scalar field

$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0. \tag{3}$$

Inflationary stages driven solely by the R^2 term or the massive scalar field are characterized by the conditions $|\dot{H}| \ll H^2$ and $|\ddot{\varphi}| \ll H|\dot{\varphi}|$. Then eqs. (2)–(3) become simply and can be integrated. The solution has the following parametric form where the role of the parameter is played by the scalar field itself ($M \neq m\sqrt{2}$):

$$H^2 = H_1^2\left(\frac{\varphi}{\varphi_1}\right)^{M^2/m^2} - \frac{M^2\varphi^2}{2 - M^2/m^2}\left(1 - \left(\frac{\varphi}{\varphi_1}\right)^{M^2/m^2-2}\right), \tag{4}$$

$$t - t_1 = -3\int_{\varphi_1}^{z} d\varphi\, \frac{H(\varphi)}{m^2\varphi}, \tag{5}$$

$$\ln(a/a_1) = 3\frac{\varphi_1^2 - \varphi^2}{2} + 3\frac{H_1^2 - H^2}{M^2}, \tag{6}$$

where a_1, H_1 and z_1 denote the values of a, H and z at the moment $t = t_1$ when inflation begins ($H_1 \gg M, m$; $z_1 \gg 1$).

To calculate the spectrum of perturbations we take into account small scalar perturbations of the metric of the Einstein-de Sitter model (see eq. 16). The perturbations of the scalar field are denoted by $\delta\varphi = \varphi - \varphi^{(o)}$, and the perturbations of the Ricci scalar by $\delta R = R - R^{(o)}$, where $\varphi^{(o)}$ and $R^{(o)}$ are the solutions of the background equations. We decompose the perturbations and the potentials in plane waves ($\delta\varphi = \delta\varphi_k \exp(ikr)$ etc.).

By linearization of the field equations we find

$$\delta\ddot{R} + 3H\delta\dot{R} + (k^2/a^2 + M^2)\delta R = (2\Phi M^2)(-R + 6(\dot{\varphi}^2 - 2m^2\varphi^2)) + (\dot{\Phi} + 3\dot{\Psi})\dot{R}$$

$$+12M^2(\dot\varphi\delta\dot\varphi - \dot\varphi^2\Phi - 2m^2\varphi\delta\varphi), \tag{7}$$

$$\delta R = 6\ddot\Psi + 2(k^2/a^2)(2\Psi - \Phi) + 12(\dot H + 2H^2)\Phi$$
$$+6H\dot\Phi + 24H\dot\Psi, \tag{8}$$

$$\Phi = \Psi - \delta R/(R - 3M^2), \tag{9}$$

$$\delta\ddot\varphi + 3H\delta\dot\varphi + (k^2/a^2 + m^2)\delta\varphi = (\dot\Phi + 3\dot\Psi)\dot\varphi - 2m^2\varphi\Phi \tag{10}$$

$$3\dot\varphi\delta\varphi = (1 - R/3M^2)(\dot\Psi + H\Phi)$$
$$+(\delta\dot R - \Phi\dot R - H\delta R)/6M^2. \tag{11}$$

The indices for the k-modes of the perturbations are omitted. The solutions of the linear perturbation equations (7-11) show damped oscillations. Following the general description of section 5 we calculate the amplitudes of these oscillations depending on the wave number k. Then one has to solve the equations (7-11) assuming in the short-wave limit damped oscillations with these amplitudes, i.e. $\sqrt{1/2k}\,/a$ for $\delta\varphi$ and $\sqrt{192\pi G(1 - 3M^2/R)/2k}\,MH/a$ for δR. Performing the numerical integrations for different values of the wave number k and taking the mean over the phases one finds the perturbation spectrum

$$\langle \Phi k^{3/2} \rangle = \begin{cases} A[\log(\frac{k_b}{k})]^\gamma + B[\log(\frac{k_1}{k_b})]^\beta, & k \le k_b \\ B[\log(\frac{k_1}{k})]^\beta, & k > k_b \end{cases} \tag{12}$$

The constants in eq.(12) are related to the masses m and M in eq.(1). They must be determined by observations.

The perturbations of the density lead to temperature fluctuations of the microwave background radiation. First positive measurements of the CMB fluctuations were obtained in the COBE experiment (Smooth et al. 1992). The predicted variance of the COBE experiment $\sigma_T^2(\theta_{FWHM})$ can be expressed as

$$\sigma_T^2(\theta_{FWHM}) = \frac{1}{4\pi}\sum_{l\ge 2}(2l+1)C_l \exp(-l(l+1)\theta_s^2), \tag{13}$$

where $\theta_s = \theta_{FWHM}/2\sqrt{\ln 4}$ is the angle characterizing the smearing due to the finite beam size. With the measured $\sqrt{\sigma^2(10^\circ)} = 1.1 \times 10^{-5}(1 \pm 0.17)$ the normalization constant in the spectrum Φ_k is determined. The COBE result fixes the power at very large scale, i.e. it fixes the value of A in eq. (12), which depends mainly on the coupling parameter in eq. (1).

Provided the power spectrum is given by the spectrum of potential perturbations and the transfer function (see eq. (6)), then the variance σ of the mass fluctuation $(\delta M/M)^2$ in a sphere of radius R can be calculated (eq. (19) and (21). Thus σ_0^2 is the variance of the general mass-density distribution. However the galaxy distribution, i.e. the distribution of the visible matter, might not follow the general mass-density field. It is more probable that the galaxies have been formed only at the peaks of the mass-density. Therefore, a common bias factor for correction between the variance σ_0^2 and the variance of the galaxy distribution must be introduced,

$$\sigma_g^2(r) = b_g^2 \sigma_0^2(r). \tag{14}$$

¿From observations it follows that $\sigma_g^2(8h^{-1}$ Mpc) =1. Thus,

$$\sigma_0^2(8h^{-1}\mathrm{Mpc}) = b_g^{-1}, \tag{15}$$

i.e. we define the bias factor using the COBE normalized spectrum. The bias factors for various spectra depends mainly on the power of the spectrum on smaller scales. Therefore, assuming a bias factor of approximately 2, as indicated by other observations, the parameter B in eq. (12) will be restricted.

However, one should note that that the bias factor should not be taken too seriously as distinguishing evidence for or against a model. There are indications that the bias factor depends on the kind of objects under consideration (optical galaxies, IRAS galaxies, clusters of galaxies). However, spectra are certainly not satisfactory if they lead to $b > 3$. Besides the bias factor, consideration of peculiar bulk velocities and the Mach number may serve as criteria for distinguishing between various initial perturbation spectra (cp. Gottlöber et al. 1993).

The most powerful test of different spectra is the counts-in-cell analysis.

Let us compare the predicted mass variances $\sigma(l)$ at the scale l with the counts-in-cells analysis of large scale clustering in the newly completed Stromlo-APM galaxy redshift survey (Loveday et al., 1992). The $\sigma(l)$ is related to the two-point galaxy correlation function according to

$$\sigma^2(l) = \frac{1}{V^2} \int_{V=l^3} \xi(r_{12}) dV_1 dV_2 \tag{16}$$

and in terms of the density perturbation spectrum $P(k)$ it can be written

$$\sigma^2(l) = \frac{1}{2\pi^2} \int_0^\infty P(k) k^2 W_1(kl) dk \tag{17}$$

where

$$W_1(\xi) = 8 \int_0^1 dx \int_0^1 dy \int_0^1 dz (1-x)(1-y)(1-z) \frac{\sin(\xi\sqrt{x^2+z^2+z^2})}{\xi\sqrt{x^2+z^2+z^2}} \tag{18}$$

is the appropriate window function. Comparing the variances calculated with three different power spectra (i. e. different parameters in eq. (12)) one tests the spectra over a wide range of k values.

Finally one can calculate the excess-power introduced by Wright et al. 1992. By definition, the excess power

$$EP = 3.6 \frac{\sigma(25h^{-1}\mathrm{Mpc})}{\sigma(8h^{-1}\mathrm{Mpc})} \tag{19}$$

is equal to unity for the flat standard CDM model. It is obvious that EP does not depend on the normalization of the spectrum and it might give some indication on the shape of the initial spectrum. From observations it follows that $EP = 1.3 \pm 0.15$ (Wright et al. 1992).

On the basis of different spectra we have compared theoretical predictions with observational data (for more details see Gottlöber et al. 1993). The best fit to the observations is reached if the spectrum has a break at approximately 7 Mpc and a height of the step $\Delta = 3$. This corresponds to the parameters k_b

$= 0.1$, $k_1 = e^{60}$, $A = 788$, $B = 60.6$, and $\gamma = 0.5$ and $\beta = 0.66$ in eq. (12). The biasing parameter introduced for giving $\sigma(8h^{-1}$ Mpc$) = 1$ is $b = 2.7$. The perturbation spectrum $P(k) = kf^2(k)T(k)$ in this parametric form coincides very well with numerical solution for $P(k)$. A spectrum with a higher step ($\Delta \approx 5$) would lead to similar features as the MDM and $\Omega + \lambda$ models. However, the models with values $\Delta \geq 4$ are ruled out mainly by the tests including large scale bulk flow velocity observations. Thus the latter will certainly lead to a crucial test in future. The parameter space allowing for fitting all considered observational data includes also models with a break at a scale slightly less than 7 Mpc. For those models the biasing parameter ranges about 2.5, which is in better agreement with recent calculations (see e.g. Gelb 1994). Due to the 1.5σ errors in the COBE data the remaining parameter space for the fit is effectively larger than here obtained. On the other hand there is some correlation between the step height in the spectra and the break scale. Therefore, the permitted region in the parameter space remains very tight. Further observations, e.g. measurements of the anisotropy of the microwave background radiation at small angular scales, could become a crucial test for the models going beyond the standard model.

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