Part V

THEORETICAL COSMOLOGY
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Chapter 23

Introduction

In these lectures I will present a review of some distinct approaches to describe the global structure of the spacetime. I will limit our analysis to Einstein's General Relativity as a theory of gravity. We will concentrate our discussion to the standard approach (FRW cosmologies) and to specific difficulties of it. Some proposals of solutions to these drawbacks of the standard model will be analysed. Although the present course should be considered as a continuation of my previous lectures on earlier Brazilian Schools of Cosmology it is organized in a self-contained form.

Chapter 24

The Cosmic Fluid

In the Old Standard Cosmological Model [1, 4] the distribution of energy in the Universe is represented by an Ideal Gas. This high degree of simplification was one of the main causes of its success and, in the same way, the origin of its difficulties.

This description, although the Universe is not taken as a static structure, is therefore devoid of a true evolution: its entropy is constant, its homogeneity property has a primordial origin; and above all this, it has a singular origin, a "date of birth ". During all the time in which this model dominated the cosmological scene this oversimplified scheme was not considered to cause a further theoretical discomfort. This was due to its success to provide a framework that made understandable certain astronomical data. The very fact that in this model the Universe starts from a very hot singular origin could be at the basis of an explanation both for the abundance of the observed chemical elements at large and for the necessary conditions for a global thermodynamical equilibrium. We will come back to these generic questions later on.

1 The Ideal Gas

The energy-momentum tensor of an Ideal Gas is represented by:

$$T_{\mu\nu} = \rho V_\mu V_\nu - p h_{\mu\nu} \tag{1}$$

This is the source of the spacetime curvature which is to be identified to a homogeneous and isotropic Friedmann geometry. In terms of the co-moving coordinates, the element of length is taken as

$$ds^2 = dt^2 - A^2(t)[d\chi^2 + \sigma^2(d\theta^2 + sin^2\theta d\varphi^2)] \tag{2}$$

The function σ depends only on the variable χ and the dependence on the cosmical time is concentrated on the function $A(t)$, known as the radius of the Universe. The scalar of curvature R is given by

$$R = 6\left[\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\epsilon}{A^2}\right] \tag{3}$$

in which ϵ measures the curvature of the 3-space. Calling ${}^{(3)}R$ the corresponding Ricci scalar for the 3-geometry, we find:

$${}^{(3)}R = -\frac{2}{\sigma^2}[2\sigma'' + \sigma'^2 - 1] \tag{4}$$

that is

$$\epsilon \equiv \frac{1}{6}{}^{(3)}R$$

Note that $\sigma' \equiv \frac{d\sigma}{d\chi}$.

The evolution of this model is controlled by two equations:

- The conservation of energy.
- Raychaudhuri's equation.

Let us go into the details. The fundamental equation that describes the gravitational process is provided by Einstein's General Relativity (GR), which relates the properties of matter distribution to the curvature of spacetime:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu} \tag{5}$$

From this original form it follows the identity that generalizes in curved space the law of the conservation of energy:

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{6}$$

Let us now consider the kinematics. We define the projector $h_{\mu\nu}$ that is generated in the rest-space H of the comoving observer characterized by the four-velocity V_μ through the expression:

$$h_{\mu\nu} = g_{\mu\nu} - V_\mu V_\nu \tag{7}$$

From this definition the basic properties follow:

- $h_{\mu\nu} = h_{\nu\mu}$
- $h_{\mu\nu}V^\mu = 0$

- $h_{\mu\nu}h^{\nu\lambda} = h_{\mu}{}^{\lambda}$

There is no nongravitational force acting on the fluid. Then it follows that its acceleration vanishes.That is

$$a^{\mu} \doteq V^{\mu}{}_{;\lambda}V^{\lambda} = 0$$

The conservation of the norm of the four-vector V^{μ} yields the orthogonality of a^{μ} and V^{ν}. The hypothesis of isotropy of the fluid is nothing but the condition of vanishing of the shear, defined by the symmetric part, in the rest-space H, of the derivative of the velocity, that is

$$\sigma_{\alpha\beta} \equiv h^{\mu}{}_{(\alpha}h_{\beta)}{}^{\lambda}V_{\mu;\lambda} - \frac{1}{3}\theta h_{\alpha\beta}$$

The quantity θ is called the expansion and it constitutes the divergence of the velocity:

$$\theta \equiv V^{\mu}{}_{;\mu}$$

From this definition it follows that the properties of the shear tensor are:

- symmetry: $\sigma_{\mu\nu} = \sigma_{\nu\mu}$
- trace-free: $\sigma_{\mu\nu}g^{\mu\nu} = 0$
- it belongs to H, that is, it is orthogonal to $V_{\mu} : \sigma_{\mu\nu}V^{\mu} = 0$

Besides the above hypothesis which we have attributed to the Old Standard Model there is one more that allows us to say that the model is vorticity-free. The vorticity tensor is the anti-symmetric complement of the shear, defined by:

$$\omega_{\alpha\beta} \equiv h^{\mu}{}_{[\alpha} h_{\beta]}{}^{\lambda} V_{\mu;\lambda}$$

The vorticity has the corresponding properties:

- anti-symmetry : $\omega_{\mu\nu} = -\omega_{\nu\mu}$
- orthogonality: $\omega_{\mu\nu} V^{\mu} = 0$

The vanishing of the vorticity allows the definition of a global time (sometimes called the cosmical time). In the beginning of modern Cosmology the so called Cosmological Principle was set up, which has precisely the motivation to permit the stablishing of a global separation of the spacetime into space plus time by the assumption of the existence of a three-dimensional spatial hypersurface Σ. This surface separates the world into two complementary parts which we call F (for 'future') and P (for 'past'). The world lines of all existing matter are orthogonal to Σ. We can thus use the proper time of the matter as the label to describe the evolutionary history of the Universe.

The kinematical quantities defined above are nothing but the irreducible parts of the derivative of the velocity field. Indeed, we can write in general:

$$V_{\mu;\beta} = \sigma_{\mu\beta} + \omega_{\mu\beta} + a_{\mu} V_{\beta} + \frac{1}{3}\theta h_{\mu\beta} \tag{8}$$

In the very particular case which we are considering (the Old Standard cosmological model) we set:

$$V_{\mu;\nu} = \frac{1}{3}\theta h_{\mu\nu} \tag{9}$$

We are thus prepared to consider the two fundamental equations that control the dynamics of this simplified Cosmology.

2 The Conservation of Energy

Let us come back to equation (6). Contracting it with V_μ yields:

$$T^{\mu\nu}{}_{;\mu}V_\nu = 0 \tag{10}$$

Using the expression for the Ideal Gas (1) we obtain:

$$\dot{\rho} + (\rho + p)\theta = 0 \tag{11}$$

in which we have used the expression (9); a dot means derivative projected along the velocity field.

3 Raychaudhuri's Equation

We start from the definition of the spacetime curvature. Thus, we set:

$$V_{\alpha;\beta;\gamma} - V_{\alpha;\gamma;\beta} = R_{\alpha\epsilon\beta\gamma}V^\epsilon \tag{12}$$

Multiplying by V^γ:

$$(\dot{V}_{\alpha;\beta}) + V_{\alpha;\gamma}V^\gamma_{;\beta} = R_{\alpha\epsilon\beta\gamma}V^\epsilon V^\gamma$$

Contracting indices to arrive at a scalar equation gives:

$$\dot{\theta} + \frac{1}{3}\theta^2 = R_{\mu\nu}V^\mu V^\nu \tag{13}$$

Using Einstein's equations we can rewrite eq.(13) in the form:

$$\dot{\theta} + \frac{1}{3}\theta^2 + \frac{1}{2}(\rho + 3p) = 0 \tag{14}$$

In the Standard framework the pressure p is provided by some additional hypothesis concerning its thermodynamical behaviour. It is usually accepted that there exists a simple relation between p and the density of energy ρ, e.g.

$$p = \lambda\rho, \tag{15}$$

in which $0 < \lambda < 1$.

The equations of evolution of ρ and θ are written in the form of a Dynamical System, e.g.:

$$\dot{\rho} = F(\rho, \theta) \tag{16}$$

$$\dot{\theta} = G(\rho, \theta) \tag{17}$$

This system will be completely equivalent to Einstein's GR equations if we just add to them the constraint (no time derivative):

$$\rho - \frac{1}{3}\theta^2 - \frac{3\epsilon}{A^2} = 0 \tag{18}$$

We remark that this constraint is preserved throughout the entire evolution due to the above Dynamical System. At this point, the Old Standard Model starts a systematic procedure of trying to find exact solutions of this set of equations. For each solution a specific cosmological geometry is obtained. Before going into this, however, one should explore a little of the elegant form of the Dynamical System and try to examine some generic properties shared by the various classes of solution.

Exercise 1. Analyse the phase space of the system (16)-(17).

Consider the case in which the relation provided by eq.(15) holds. The dynamical system is

$$\dot{\rho} = -(\rho + p)\Theta \equiv F \tag{19}$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\rho + 3p) \equiv G \tag{20}$$

The equilibrium points (that is, the extrema of both equations) are given by the simultaneous zeros of the functions F and G. In the present example the only finite equilibrium point is the origin $(\rho, \Theta) = (0, 0)$. We recognize that this point represents the flat empty Minkowski Universe.

There are three distinct sectors, according to the value of the 3-curvature (see picture 1).

Exercise 2. The introduction of a negative cosmological constant(Λ).

The modified equations are:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\rho + 3p) - \Lambda \tag{21}$$

$$\dot{\rho} = -(\rho + p)\Theta \tag{22}$$

The parabola that separates the distinct sectors assumes the form:

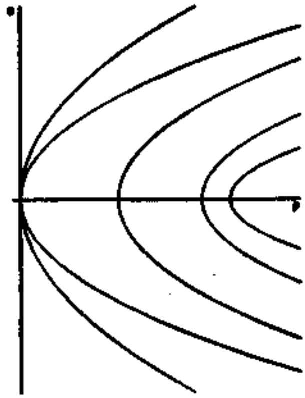

Figure 1: Graph $\theta \times \rho$

$$\rho - \frac{1}{3}\theta^2 - \frac{3\epsilon}{A^2} - \Lambda = 0 \tag{23}$$

The singular point is no more the Minkowski vacuum but it becomes the deSitter vacuum: $(\rho, \Theta) = (-\frac{2}{1+3\lambda}\Lambda, 0)$. Plot the corresponding graph.

4 Equilibrium Thermodynamics of the Ideal Gas

The fundamental law of thermodynamical equilibrium sets

$$TdS = dE + p_{th}dV. \tag{24}$$

The conservation of the entropy in an expanding Universe follows from a direct application of this expression and the eq.(11) of the energy conservation. In the case of electromagnetic radiation a standard use of the equilibrium Thermodynamics yields for some relevant quantities the following expressions, which we leave for the reader to obtain.

The energy density:

$$\rho = \frac{\pi^2}{15}T^4. \tag{25}$$

The pressure:

$$p = \frac{1}{3}\rho. \tag{26}$$

The total number of photons:

$$N = \frac{2}{\pi^2}\zeta(3)VT^3. \tag{27}$$

The free-energy:

$$\mathcal{F} = -\frac{\pi^{\in}}{\Delta\nabla}\mathcal{V}T^{\Delta}. \tag{28}$$

The total entropy:

$$S = -\frac{4\mathcal{F}}{T}. \tag{29}$$

Evolution of the number density of photons $n = \frac{N}{V}$:

$$\dot{n} + n\Theta = 0. \tag{30}$$

In the next chapter we shall see how these expressions are modified in the case the coupling of electromagnetic field with Gravity does not follow strictly the Equivalence Principle (e.g., the non minimal coupling).

5 The Deformed Friedmann Universe

In this section we will consider some examples of viscous fluids. There are various mechanisms that have been examined which induces viscosity. In the early 60's it was [5] proposed that the interaction of neutrino with matter at very dense regime could be treated as a viscous fluid endowed with an additional (non-thermodynamical) part of the pressure. Some authors (see [6] for instance) argued that the creation of particles by the gravitational field can induce viscous effects. In general we represent these situations by setting

$$p = p_{th} + \pi \tag{31}$$

Besides, non-isotropic components can appear, inducing the non-vanishing of the anisotropic pressure $\Pi_{\mu\nu}$.

6 Stokesian Fluid

We call a fluid *Stokesian* if the general pressure (the isotropic and the anisotropic parts) is a functional only of the dilatation tensor[1] Θ^i_j defined by

$$\Theta^i_j \equiv \sigma^i_j + \frac{1}{3}\Theta\delta^i_j.$$

We leave the analysis of the anisotropic pressure to a subsequent section. The introduction of viscosity in the framework of homogeneous and isotropic geometry changes drastically some properties of the standard FRW Universe.

[1]Some authors have generalized this terminology by calling Stokesian any fluid that has a functional dependence between its kinematical (shear, expansion and acceleration) and dynamical parts (general pressure and heat flux).

Example I.

In the isotropic case the pressure is a function of the expansion uniquely

$$\tilde{p} = p_{th} + F(\Theta)$$

A simple example: the quadratic case. We set:

$$\tilde{p} = p - \alpha\Theta - \beta\Theta^2 \tag{32}$$

In this case the functions F and G (cf. eq.(16) and (17) are:

$$F = -(\rho + p)\Theta + \alpha\Theta^2 + \beta\Theta^3 \tag{33}$$

$$G = -\frac{1}{2}\rho - \frac{3}{2}p - \frac{1}{3}\Theta^2 + \frac{3}{2}\alpha\Theta + \frac{3}{2}\beta\Theta^2 \tag{34}$$

The singular points of the system are

- $P_1 = (\Theta_1, \rho_1) = (0, 0)$
- $P_2 = (\Theta_2, \rho_2) = \left(-\frac{3\alpha}{3\beta-\gamma}, \frac{1}{3}\Theta_2^2\right)$

7 Non-Perfect Fluids

The generalization of the equations of motion of all quantities associated to the fluid will be presented later on (see section 2).

8 The Expansion Equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 - a^{\mu}{}_{;\mu} = R_{\mu\nu}V^{\mu}V^{\nu} \tag{35}$$

9 The Shear Equation

$$\begin{aligned} h_{\alpha\mu}h_{\beta\nu}\dot{\sigma}^{\mu\nu} &+ \frac{1}{3}[a^{\lambda}{}_{;\lambda} - \frac{1}{2}\omega^2 - \sigma^2]h_{\alpha\beta} \\ &+ a_{\alpha}a_{\beta} - \frac{1}{2}h_{\alpha\mu}h_{\beta\nu}[a^{\mu;\nu} + a^{\nu;\mu}] + \frac{2}{3}\Theta\sigma_{\alpha\beta} \\ &+ \sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta} - \omega_{\alpha}\omega_{\beta} \\ &= R_{\alpha\epsilon\beta\nu}V^{\epsilon}V^{\nu} - \frac{1}{3}R_{\mu\nu}V^{\mu}V^{\nu}h_{\alpha\beta} \end{aligned} \tag{36}$$

10 The Vorticity Equation

$$\begin{aligned} h^{\alpha\mu}h^{\beta\nu}\dot{\omega}_{\mu\nu} - \frac{1}{2}h^{\alpha\mu}h^{\beta\nu}(a_{\mu;\nu} - a_{\nu;\mu}) + \\ \frac{2}{3}\Theta\omega^{\alpha\beta} + \sigma^{\alpha\mu}\omega_{\mu}{}^{\beta} - \sigma^{\beta\mu}\omega_{\mu}{}^{\alpha} = 0 \end{aligned} \tag{37}$$

In the above expressions I used the definitions:

$$\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu}$$

$$\omega^2 \equiv \omega_{\mu\nu}\omega^{\mu\nu}$$

$$\omega^{\tau} \equiv \frac{1}{2}\eta^{\alpha\beta\rho\tau}\omega_{\alpha\beta}V_{\rho}$$

Chapter 25

Light Propagation

The great majority of all our information on the Universe comes by electromagnetic means. Thus, we must be aware of any eventual misinterpretation of the data due to a theoretical prejudice. The question to which we would like to have a direct answer is: how do the electromagnetic waves behave in a curved spacetime?

1 Electromagnetic Waves

The propagation of the electromagnetic effects can be described through the exam of the geometric optics approximation. In order to study this we must know the effective interaction between electromagnetic and gravitational fields. We review briefly here the state of art of such a coupling.

The interaction between Gravity and any field can be of two types:

- Minimal
- Non Minimal

The minimal coupling states that the interaction is a true local procedure and does not involve any function of the curvature of the spacetime. In order to achieve this coupling one has to proceed as in the flat spacetime, using Maxwell equations and changing simple derivatives into covariant ones. The minimal coupling can thus be considered as the statement of validity of the Equivalence Principle in **latu sensu**.

The Non Minimal coupling can be defined by oposition. The Lagrangean of the interaction depends not only on the electromagnetic field but also on a functional of the curvature. We write the action S in the form:

$$S = \int \sqrt{-g} L d_4(x).$$

2 Minimal Coupling

The above prescription allows the immediate knowledge of the Lagrangean. We have:

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \equiv -\frac{1}{4} F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} \tag{1}$$

It then follows the generalized Maxwell equation:

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{2}$$

3 Non-Minimal Coupling

One of the main criticisms against the dependence of the interaction on the curvature rests on the apparent high degree of arbitrarity it contains, once there is not a unique way to implement such coupling. We shall see however that, as far as our interest keeps limited to Cosmology, this arbitrarity is reduced to an acceptable level. Although we will not consider here the other important case of the influence of the gravitational field in the neighborhood of a star, it seems worth to comment that a similar reduction occurs in this situation.

There are seven possible candidates for the interacting Lagrangean:

$$L_1 = R A_\mu A^\mu \tag{3}$$

$$L_2 = R_{\mu\nu} A^\mu A^\nu \tag{4}$$

$$L_3 = R F_{\mu\nu} F^{\mu\nu} \tag{5}$$

$$L_4 = R F^*_{\mu\nu} F^{\mu\nu} \tag{6}$$

$$L_5 = R_{\mu\nu} F^{\mu\alpha} F_\alpha{}^\nu \tag{7}$$

$$L_6 = R_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu} \tag{8}$$

$$L_7 = R^{\alpha\beta\mu\nu} F^*_{\alpha\beta} F_{\mu\nu} \tag{9}$$

Let us make some comments on them. L_1 and L_2 are not gauge invariant. Although this is not a sound criterion, as far as cosmological processes are concerned, to eliminate these Lagrangeans, we will not consider them further on in this section. The reason is simple: we are interested here in electromagnetic disturbances and these Lagrangeans do not affect the propagation of the waves, as we shall see. Besides, we will not consider expressions L_4 and L_7 either. The reason is the assumed hypothesis that the actual interaction does not violate the flat space property of parity conservation of Electromagnetism. We note, however, that this is a further hypothesis that does not have an experimental support.

We are then left with L_3, L_5 and L_6.

In a Friedmannian background geometry we can reduce these possibilities further. Indeed, once the metric is conformally flat, that is the Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ vanishes, the curvature can be written in terms of the contracted quantities $R_{\alpha\beta}$. We have:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}[R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu}] - \frac{1}{6}Rg_{\alpha\beta\mu\nu} \tag{10}$$

It then follows that L_6 can be written in terms of L_3 and L_5.

4 Electromagnetic Disturbances

We can now turn our attention to the analysis of the evolution of generic electromagnetic shock waves in an arbitrary curved spacetime. We will follow the so-called Hadamard method which we will describe now very briefly. The reader interested in further information may consult the references quoted in the bibliography, in particular the work by Hadamard [7].

Let Σ be a surface characterized by discontinuities of certains derivatives of the electromagnetic potential A_μ. We represent the equation of this surface by

$$\Phi(x) = 0$$

We are not concerned here with gravitational waves, and thus we can consider that the metric and its derivatives of any desired order are continuous through Σ. We represent the discontinuity of any quantity F through Σ by the symbol

$$[F]_\Sigma = \lim_{x^+, x^- \to x} (F(x^+) - F(x^-))$$

We can distinguish different types of processes according to the order of the lowest derivative which is discontinuous. We will call it of first order if the field itself is discontinuous; and second order if the field is continuous but its first derivative is discontinuous. Let us examine here the second order discontinuity.

Following Hadamard we set:

$$\begin{aligned} [A_\lambda]_\Sigma &= 0 \\ [A_{\lambda,\mu}]_\Sigma &= 0 \\ [A_{\lambda,\mu\nu}]_\Sigma &= \varphi_\lambda (K_\mu K_\nu)_{,\rho} + K_\rho (\varphi_\lambda K_\mu)_{,\nu} \\ &\quad + K_\nu (\varphi_{\lambda,\mu} + X_\lambda K_\mu) \end{aligned} \tag{11}$$

In the above formula we used the assumed continuity of the metric to replace a covariant derivative (;) by a simple derivative (,), that is:

$$[A_{\lambda;\mu}]_\Sigma = [A_{\lambda,\mu}]_\Sigma$$

The vector K_μ is the normal to Σ, that is,

$$K_\mu = \nabla_\mu \Phi \tag{12}$$

5 Minimal Waves

Let us use the equation of motion (2) to calculate the evolution of the electromagnetic disturbances. From the above discontinuity of the potential it follows that the field is continuous and its first derivative gives

$$[F_{\mu\nu,\alpha}]_\Sigma = \varphi_\mu K_\nu K_\alpha - \varphi_\nu K_\mu K_\alpha$$

The discontinuity of (2) gives:

$$\varphi_\mu K^\mu K_\lambda - \varphi_\lambda K^2 = 0 \tag{13}$$

That is, either φ_μ is proportional to the propagation vector or K_μ is a null vector. The formula of discontinuity of $F_{\mu\nu}$ gives that in the first case there is no real disturbance, once the proportionality of φ_μ and the wave vector K_μ is nothing but a simple gauge that can always be eliminated.

It then follows that

$$K_\mu K_\nu g^{\mu\nu} = 0 \tag{14}$$

Using this result back into (13) it yields that the discontinuity vector φ_μ is orthogonal to K_μ:

$$\varphi_\mu K^\mu = 0 \tag{15}$$

Using eqs.(12),(14) gives that the vector K_μ satisfies the equation of geodesics:

$$K_{\mu;\nu} K^\nu = 0 \tag{16}$$

or, calling s the affine parameter on the curve tangent to K_μ, we can set equivalently:

$$\frac{DK^\mu}{Ds} = 0$$

Let us define a null complex vector m_μ orthogonal to K_μ. This can be accomplished in terms of normalized spacelike vectors e_μ and f_μ, both orthogonals to K_μ. We set

$$m_\mu = \frac{1}{\sqrt{2}}(e_\mu + i f_\mu) \tag{17}$$

We require further that

$$e_\mu e^\mu = f_\mu f^\mu = -1$$

and also that

$$e_\mu K^\mu = f_\mu K^\mu = e_\mu f^\mu = 0$$

We will propagate this base of vectors such that it is preserved along the wave vector, that is we choose:

$$\frac{De^\mu}{Ds} = \frac{Df^\mu}{Ds} = 0$$

From what we have learned we can write

$$\varphi_\mu = \frac{\sqrt{J}}{\sqrt{2}}(e^{i\Psi} m_\mu + e^{-i\Psi}\overline{m}_\mu) \tag{18}$$

The quantity J is called the amplitude of the disturbance and Ψ is the polarization. The first order Hadamard equations are conditions that are imposed on the behaviour of the electromagnetic discontinuity to be valid on the surface Σ. It only remains to set up the equations describing the evolution of these quantities. This is provided by the compatibility requirement that the derivative of eq.(2) must be continuous through Σ. We then have

$$[F^{\mu\nu,\alpha}{}_{,\nu}]_\Sigma = 0 \tag{19}$$

Using the expressions of the discontinuity of the field we have:

$$\begin{aligned}\varphi_\mu(K^2)_{,\rho} &- (\varphi^\nu K_\nu)_{,\rho}K_\mu + (\varphi_\mu K^\nu)_{,\nu}K_\rho \\ &+ \varphi_{\mu,\nu}K^\nu K_\rho - (\varphi^\nu K_\mu)_{,\nu}K_\rho \\ &- \varphi_{\nu,\mu}K^\nu K_\rho - X_\nu K^\nu K_\mu K_\rho \\ &= 0\end{aligned} \tag{20}$$

This equation will provide us with the evolution of both the amplitude of the disturbance J and the polarization Ψ. For this let us firstly multiply eq.(20) by φ^{μ}. We obtain:

$$(\varphi^{\mu}\varphi_{\mu}K^{\alpha})_{;\alpha} = 0 \tag{21}$$

that is

$$(JK^{\alpha})_{;\alpha} = 0 \tag{22}$$

This is the classical expression for the conservation of the number of photons. It is convenient for subsequent reference to define the associated quantity $\tilde{\varphi}^{\mu}$ by setting:

$$\tilde{\varphi}^{\mu} = \frac{\sqrt{J}}{i\sqrt{2}}(e^{i\Psi}m^{\mu} - e^{-i\Psi}\overline{m}^{\mu}) \tag{23}$$

The properties

$$\varphi_{\mu}\tilde{\varphi}^{\mu} = 0$$

and

$$\tilde{\varphi}^{\mu}\tilde{\varphi}_{\mu} = -J$$

then follow. Multiplying eq.(20) by $\tilde{\varphi}^{\mu}$ yields:

$$\frac{D\Psi}{Ds} = 0 \tag{24}$$

This means that the minimal interaction between gravitational and electromagnetic fields is not able to modify the wave polarization. From what we have demonstrated here we can say that this kind of interaction is characterized by the properties:

- The waves propagate through the null geodesics of the background geometry;
- The polarization vector is perpendicular to the wave vector and is parallel-propagated along the waves;
- The photon number is conserved.

We recognize here the important phenomenon of the red-shift induced by the expansion of the Universe.

Exercise. Show that the wave frequency is modified by the time variation of the radius of the Universe. (See [1]; [8]).

6 Non-Minimal Waves

We are interested here in examining the propagation of the electromagnetic waves in a Friedmann-like geometry. From what we have learned previously it is enough to concentrate our analysis into L_3 and L_5.

Let us consider Lagrangean L_3 (we leave L_5 to the reader). We set

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\lambda}{4}RF^{\mu\nu}F_{\mu\nu} \tag{25}$$

This Lagrangean yields the equation of motion:

$$F^{\mu\nu}{}_{;\nu} = \lambda(RF^{\mu\nu})_{;\nu}. \tag{26}$$

We take the same case as before and consider just the second order discontinuity given by (11).

From eq.(26) we obtain

$$(1-\lambda R)(\varphi_\lambda K^2 - \varphi_\mu K^\mu K_\lambda) = 0, \tag{27}$$

once there is no discontinuity in the field $F_{\mu\nu}$. We arrive then at the same equations for the disturbances on Σ than in the previous case, that is the set given by eqs.(14), (15). We must turn now to the description of the evolution of this perturbation. Following the same procedure as above we take the derivative of (26) and then analyse its discontinuity. We obtain:

$$(1-\lambda R)F^{\mu\nu}{}_{;\nu;\rho} - \lambda F^{\mu\nu}{}_{;\nu}R_{,\rho} - \lambda R_{,\nu}F^{\mu\nu}{}_{;\rho} - \lambda R_{,\nu;\rho}F^{\mu\nu} = 0. \tag{28}$$

The discontinuity of this equation, due to the presence of the scalar of curvature's derivative, contains three new different terms. The term on $R_{,\rho}$ is multiplied by the divergence of $F^{\mu\nu}$. Its discontinuity vanishes on mass shell. There is no contribution coming from the last term on second derivatives of R too, due to the basic condition of second order discontinuity. We are then left with

$$\begin{aligned}(1-\lambda R)\ [\ &(\varphi_\mu K^2)_{;\rho} - (\varphi^\nu K_\nu)_{;\rho}K_\mu \\ &+ (\varphi_\mu K^\nu)_{;\nu}K_\rho + (\varphi_{\mu;\nu}K^\nu)K_\rho - (\varphi^\nu K_\mu)_{;\nu}k_\rho \\ &- \varphi_{\nu;\mu}K^\nu K_\rho - X_\nu K^\nu K_\mu K_\rho] - \lambda R_{,\nu}(\varphi_\mu K^\nu \\ &- \varphi^\nu K_\mu)K_\rho = 0\end{aligned} \tag{29}$$

From here on we proceed exactly as in the case of minimal coupling. We multiply first by φ_μ and then by $\tilde{\varphi}_\mu$. We obtain, respectively:

$$\left[(1-\lambda R)\varphi^2 K^\mu\right]_{;\mu} = 0 \tag{30}$$

and

$$\frac{D\Psi}{Ds} = 0 \tag{31}$$

We can summarize these calculations by stating that the net effect of the nonminimal coupling of electromagnetic disturbances and gravity characterized by Lagrangean L_3 leads to the following consequences:

- The waves propagate through the null geodesics of the background geometry;
- The polarization vector is perpendicular to the wave vector and is parallel-propagated along the waves;
- The photon number is **not** conserved.

Thus, unless we are able to observe the effects of the variation of the photon number in an expanding Universe, there is no way of deciding between the above two forms of coupling, as far as the electromagnetic field is treated as a test-field, without taking into account its perturbation on the background geometry.

This result entails immediately the question: how to treat this mechanism of variation of the photon number? We will examine this phenomenon in the next section.

Comment. The consequences of the interaction controlled by L_3 cannot be observed in the neighborhood of a star, e.g. our Sun. Its effects can be felt only in Cosmology.

Exercise. What are the modifications on the above conclusions if we take into account Lagrangean L_5 ?

7 Creation of Photons in an Expanding Universe

The minimal coupling of Electrodynamics and Gravity implies that the total number of photons N is a conserved quantity in an expanding FRW Universe. We have shown this in the previous section in a classical framework. This result is not maintained in the non-minimal case. Indeed, let us set $N = nV$, in which n is the density of photons in the Universe.

In the minimal case the conservation of the total photon number is represented by the equation of evolution

$$\dot{n} + n\Theta = 0. \tag{32}$$

This result is no more true for the non-minimal coupling. Let us restrict our analysis, in this section, to the case controlled by Lagrangean L_3.

Exercise. From the equation of evolution of electromagnetic disturbances given in the precedent section, evaluate the dependence of the total photon number for L_3 in a FRW cosmological background.

Answer:

A direct calculation gives

$$\dot{n}(1 - \lambda R) + n\Theta = \lambda n(\dot{R} + R\Theta) \tag{33}$$

The natural framework to analyse the possibility of photon production by the gravitational field associated with an expanding spatially homogeneous Universe is the attempt to quantify this process. To this end, the standard strategy was to put into use the quantum evolution for the photon vacuum. Nevertheless, keeping in tune with the preceding classical developments, and from what we have presented in the previous chapter, we shall pursue here a statistical analysis, through the exam of the influence of a variation of the number of photons upon the thermodynamical behaviour of a cosmic radiation field, in order to compare the resulting picture with the actual, observed characteristics of the Microwave Background Radiation (MBR).

In Condensed Matter Physics, changes in the number of elementary components of a given reaction are usually taken into account by the introduction of a chemical potential term. Now, the interaction of photons with matter is characterised precisely by a null value of the chemical potential, once an indeterminate number of photons can be arbitrarily emitted or absorbed, in principle, in any chosen reaction. In spite of this well-known fact, one could legitimately wonder whether the absence of a chemical potential is actually a necessary condition, or even a valid conjecture, in the case of the photon-Gravity interaction.

With respect to this question, the standard hypothesis of the minimal coupling (HMC) approach provides an unequivocal answer: once the concept of a vanishing chemical potential for the photon holds locally, then a straightforward use of the equivalence principle would extend its validity to any circunstances whatever.

However, as we saw in the precedent section, a strict attachment to empirical criteria leads to a more general conclusion, since the ensemble of observational data presently available on electromagnetic processes in gravitational fields does not suffice to establish the HMC as the only type of admissible coupling between Gravity and all other fields (see the next chapter for another support on this). Therefore, we will analyse here the alternative hypothesis of direct coupling to the curvature (HDCC).

8 Thermodynamics of a Photon Gas

The statistical distribution function of a boson gas in equilibrium endowed with a chemical potential μ is given by

$$dN_\omega = \frac{1}{e^{[\frac{\omega-\mu}{T}-1]}} \tag{34}$$

The photonic chemical potential may be conveniently split into two independent parts:

$$\mu = \mu_0(P,T) + \Delta\mu \tag{35}$$

where $\Delta\mu$ is the gravitationally-induced component and $\mu_0(P,T)$ is the flat-space contribution which, in view of the arguments quoted above, vanishes. In order to proceed one must consider the question: what is the form of the functional dependence of $\Delta\mu$ on the curvature? We will limit our analysis here to the particular case in which the background curvature is described by a FRW geometry. Besides, we will treat photons as test particles, that is, we will neglect their contribution to the total energy-momentum tensor. Such hypothesis requires that throughout the history of the Universe — even at primordial epochs of great compression — the energy density ρ_γ of the photons must have remained very small. We shall see that this *regularisation* of the photon energy density, in the case of FRW models, follows as a natural consequence of photon number non-conservation.

Due to the spatial homogeneity of the FRW model, $\Delta\mu$ shall depend only on cosmic time. It seems reasonable to suppose that $\Delta\mu$ may be written as a combination of powers of the unique curvature parameter available, the expansion factor Θ. Thus we set

$$\Delta\mu = -b^2\Theta \tag{36}$$

where b is a constant (we absorb the λ factor on it) and the minus sign arises both from the bosonic nature of photons and from the fact that we presently live in an expanding era ($\Theta > 0$). In this way the arrow of time provided by the cosmic expansion coincides with the thermodynamical arrow, as we shall see below.

Exercise. Show that the inclusion of higher powers of Θ does not qualitatively affect the results we will obtain in this section.

From the Lagrangean L_3 it follows that photons travel along null geodesics. Also, as in the standard model, the frequency ω varies as A^{-1}. (Show this). The temperature T behaves in the same manner. (Is there other possibility?)

Our task now is to exhibit the thermodynamical quantities for such a gas in equilibrium. Let us do this, as an example, for the thermodynamical potential $\Omega = -PV$.

We have

$$\Omega = \frac{VT}{\pi^2}\int_0^\infty \omega^2 ln[1 - e^{-\frac{(\omega-\mu)}{T}}]d\omega \tag{37}$$

in which μ is given by eqs.(35, 36).

Calling

$$\beta \equiv -\frac{\mu}{T} = b^2\frac{\Theta}{T}$$

we obtain (show!)

$$\Omega = \frac{-2}{\pi^2} V T^4 \sum_{m=1}^{\infty} \frac{e^{-m\beta}}{m^4} \tag{38}$$

Infinite series of this kind will appear frequently in this problem. We will present some of its properties in the subsequent section. Here let us give the results of the application of the present model for the photon properties in a curved background and compare it with the standard conjecture (see the precedent section).

The energy density:

$$\rho = \frac{6}{\pi^2} T^4 \sum_{1}^{\infty} \frac{e^{-m\beta}}{m^4}. \tag{39}$$

The pressure:

$$p = \frac{1}{3}\rho. \tag{40}$$

The total number of photons:

$$N = \frac{2}{\pi^2} V T^3 \sum_{1}^{\infty} \frac{e^{-m\beta}}{m^3} \tag{41}$$

The free-energy:

$$\mathcal{F} = -\frac{2}{\pi^2} V T^4 \sum_{m=1}^{\infty} \frac{e^{-m\beta}}{m^4} + N\mu. \tag{42}$$

The total entropy:

$$S = -\frac{4\mathcal{F}}{T} + 3\frac{N\mu}{T} \tag{43}$$

Evolution of the photons density $n = \frac{N}{V}$:

$$\dot{n} + n\Theta = -\frac{2b^2}{\pi^2}T^2 R_{00}\sum_1^\infty \frac{e^{-m\beta}}{m^2} \tag{44}$$

Let us see what are the consequences of these expressions in the standard FRW background.

Let us take the scale factor as $A(t) = A_0 t^q$, where the parameter q varies in the range $0 < q < 1$. This expression seems to be in good agreement with current observational data, including the evidence regarding primordial cosmic abundances of the chemical elements. Accordingly, the functional dependence of β with respect to cosmic time goes as $\beta \sim t^{q-1}$. Thus, for very long time intervals ($t \to \infty$), the factor β becomes extremely small, and we obtain

$$\lim_{\beta\to 0}\sum_{m=1}^\infty \frac{e^{-m\beta}}{m^4} = \zeta(4).$$

Therefore the thermodynamical potential in this limit condition approaches the value

$$\Omega_\infty = -\frac{1}{45}\pi^2 V T^4$$

Hence, we see that the distribution generated in the non-minimal coupling case converges assymptotically to a black-body spectrum.

Exercise. Try to evaluate, from the above property, a limit imposed by the observed MBR (micro-wave background radiation).

9 Some Mathematics[1]

Let us define the quantity $L(\beta, s)$ by the series

$$L(\beta, s) = \sum_{m=1}^{\infty} \frac{e^{-m\beta}}{m^s} \tag{45}$$

There are some correlated expressions which are of importance to deal with these series. We present some particular results which have been shown in the quoted paper [9]. We have

$$\sum_{m=1}^{\infty} \frac{\sinh(m\beta)}{m^{2N+2}} = \sum_{a=0}^{\infty *} \frac{\beta^{2a+1}}{(2a+1)!}\zeta(2N-a+1) + \frac{\beta^{2N+1}}{(2N+1)!}[\gamma + \psi(2N+2) - ln\beta], \tag{46}$$

for $N = 0, 1, 2, \ldots$.

The asterisk above the summation symbol means that the value $a = N$ is excluded. The reason is that in this case the Riemann zeta function reduces to the divergent expression $\zeta(1)$. We have

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

The process of regularisation proposed by [9] gives precisely the extra term involving the Euler number γ, by the use of an integral expression for the series (46). The additional term is the result of the contribution to the integral, in the complex plane, of the residue of the pole $\zeta(1)$. We can also evaluate the series

[1]See [9] for more detailed calculations of the series presented in this section

$$\sum_{m=1}^{\infty} \frac{\cosh(m\beta)}{m^{2N+1}} = \sum_{a=0}^{\infty *} \frac{\beta^{2a}}{(2a)!}\zeta(2N-a+1) + \frac{\beta^{2N}}{(2N)!}[\gamma+\psi(2N+1)-ln\beta], \tag{47}$$

for $N = 0, 1, 2, \ldots$.

In these expressions $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1} dx \tag{48}$$

Some especial values which are useful for our analysis here are:

$$\zeta(0) = -\frac{1}{2}$$

$$\zeta(1) = \infty$$

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(3) = 1202$$

$$\zeta(4) = \frac{\pi^4}{90}$$

We note that the function $\psi(Z)$, which appeared in the above expressions, is the so-called psi or digamma function

$$\psi(Z) = \frac{d}{dZ}[ln\Gamma(Z)]$$

which for integer values of the arguments gives

$$\psi(1) = -\gamma$$

$$\psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m}$$

where γ is the Euler number.

The recurrence formula needed to obtain some special values of interest is given by

$$\psi(Z+1) = \psi(Z) + \frac{1}{Z}$$

Using these properties we can obtain some formulas of interest for this section. We have:

$$\sum_{m=1}^{\infty} \frac{\sinh(m\beta)}{m^4} = \frac{1}{6}\beta^3(\frac{11}{6} - ln\beta) + \sum_{a=0}^{\infty *} \frac{\beta^{2a+1}}{(2a+1)!}\zeta(3-2a)$$

$$\sum_{m=1}^{\infty} \frac{\cosh(m\beta)}{m^4} = \frac{1}{90}\pi^4 + \frac{1}{2}\zeta(3)\beta^2 - \frac{1}{48}\beta^4$$

$$\sum_{m=1}^{\infty} \frac{\sinh(m\beta)}{m^3} = \frac{1}{6}\pi^2\beta - \frac{1}{12}\beta^3$$

$$\sum_{m=1}^{\infty} \frac{\cosh(m\beta)}{m^3} = \frac{3}{4}\beta^2 - \frac{1}{2}\beta^2 ln\beta + \sum_{a=0}^{\infty *} \frac{\beta^{2a}}{(2a)!}\zeta(3-2a).$$

Chapter 26

The Scalar Field

The description by the Old Standard Model of the distribution of the energy content in the Universe by means of an Ideal Gas was a consequence of the high degree of simplification for the representation of the existing matter concentrated in the form of the Galactic Fluid, Radiation, Photons, Neutrinos, etc. During the last decade a lot of interest was devoted to the scalar field. Curiously, this was not a consequence of any observation of the presence of this field in the actual process of cosmic evolution. In fact the importance of this field came from speculative theories which allow the application to the Universe of some models of elementary particles. In any case, the theory acts as a model for more realistic configurations. Thus it seems worth to spend some time to analyse these proposals.

1 The Gravitational Interaction of the Scalar Field Φ

The first question we must solve is precisely the one we have found in the case of propagation of electromagnetic effects, that is: What is the way the scalar field Φ

couples to Gravity? Contrary to the previous case of the vector field, in which the great majority accepted the minimal coupling as the good one, in the present case there is not a consensus. There are at least three reasons for this almost unanimity for the previous case:

- The identification of the propagation of electromagnetic waves through null geodesics and the (false) belief that only minimal coupling provides for such behaviour;
- The (undue) uses of the Equivalence Principle as a generator of physical laws;
- The conservation of Maxwell's conformal invariance.

We shall see now how these arguments appear in the case of the scalar field.

Once there is not a single direct/indirect observation concerning the propagation of Φ in a curved spacetime, we cannot invoke the first argument. We are thus left with the other two criteria. Unfortunately, in the case of the Φ field, their application does not yield a unique proposal, as we shall see.

2 Minimal Coupling

The Lagrangean for this case is given by:

$$L = \frac{1}{2}\Phi_{,\mu}\Phi_{,\nu}g^{\mu\nu} - V(\Phi) \tag{1}$$

The potential V contains not only the matter term but any self-coupling that might be present. The equation of motion is given by:

$$\Box\Phi + \frac{\delta V}{\delta\Phi} = 0. \tag{2}$$

The energy-momentum tensor is defined by:

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-g}L}{\delta g^{\mu\nu}}.$$

It then follows for $T_{\mu\nu}(\Phi)$ in the minimal coupling case the form:

$$T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}(\Phi_{,\lambda}\Phi^{,\lambda} - 2V). \tag{3}$$

The simplest form of the potential V containing a non-linear term is provided by the model

$$V = \frac{1}{2}M^2\Phi^2 + \frac{1}{4}\lambda\Phi^4, \tag{4}$$

in which the constant λ is assumed to be positive.

We note that the trace of $T_{\mu\nu}$ does not vanish in the limit $V = 0$. Indeed, we have

$$T = -\Phi_{,\lambda}\Phi^{,\lambda} + 4V. \tag{5}$$

This is a peculiar situation, which is not common among the others massless fields (photon, neutrino). Besides, the above equation of motion for the massless Φ field is not conformally invariant, a property that in the microworld seems to be a very widespread symmetry. This led to the idea that, as far as the interaction of Φ to gravity is concerned, one has to abandon the minimal coupling hypothesis.

3 Conformal Coupling: a Feasible Example of Non-Minimal Interaction

In order to produce an equation for the scalar field which is conformal-invariant in the massless limit we have to use a direct coupling with the curvature of the spacetime. For this we need to review some results of the conformal mapping. Let $g_{\mu\nu}$ represent an arbitrary Riemannian metric. We define the transformed geometry by setting

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{6}$$

Then the corresponding changes follow:

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \tag{7}$$

$$\begin{aligned}\tilde{\Gamma}^{\alpha}_{\mu\nu} &= \Gamma^{\alpha}_{\mu\nu} \\ &\quad + \Omega^{-1}\{\Omega_{,\mu}\delta^{\alpha}_{\nu} + \Omega_{,\nu}\delta^{\alpha}_{\mu} - g_{\mu\nu}g^{\alpha\epsilon}\Omega_{,\epsilon}\}\end{aligned} \tag{8}$$

$$\tilde{R} = \Omega^{-2}(R + 6\frac{\Box\Omega}{\Omega}) \tag{9}$$

$$\tilde{W}^{\alpha}_{\beta\mu\nu} = W^{\alpha}{}_{\beta\mu\nu} \tag{10}$$

These expressions allow us to combine the conformal map with the transformation on the scalar field $\tilde{\Phi} = \Omega^{-1}\Phi$ and go into the modified non minimal Lagrangean:

$$L = \frac{1}{2}\Phi_{,\mu}\Phi_{,\nu}g^{\mu\nu} - V(\Phi) - \frac{1}{12}R\Phi^2. \tag{11}$$

The equation of motion is then given by:

$$\Box\Phi + \frac{\delta V}{\delta\Phi} + \frac{1}{6}R\Phi = 0. \tag{12}$$

Exercise. Show that, in the limit $V = 0$, eq.(12) is invariant by the conformal transformation on both the geometry and the scalar field as defined above.

The energy-momentum tensor $T^c_{\mu\nu}$ in this case is:

$$\begin{aligned}T^c_{\mu\nu} &= T_{\mu\nu} - \frac{1}{6}\Phi^2\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) \\ &\quad + \frac{1}{6}(\Box\Phi^2 g_{\mu\nu} - \Phi^2{}_{,\mu;\nu}),\end{aligned} \tag{13}$$

in which $T_{\mu\nu}$ is the minimal tensor (3). The corresponding equation for the metric tensor is

$$(1 - \frac{1}{6}\Phi^2)\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = -T^c_{\mu\nu}. \tag{14}$$

Taking the trace of this modified Einstein's equation gives:

$$T^c = T + \frac{1}{6}R\Phi^2 + \Box\Phi^2. \tag{15}$$

Using the equation of motion for Φ and the above form of V we arrive at the simple expression:

$$T^c = M^2\Phi^2. \tag{16}$$

We thus see that the nonminimal coupling restores the lost property of the scalar massless field.

4 The Fundamental Solution

In the framework of the spontaneous symmetry breaking mechanism [10] the constant solution $\Phi = \Phi_0$ appears as a very fundamental one. Its importance is related to the behaviour of its associated energy. In the conventional case of a free field and in the absence of any sort of self-coupling the field cannot admit a completely homogeneous solution but the one which has vanishing energy. This trivial solution is generally associated to the ground state of the theory. However, when the field exhibits some special sort of non-linearity, then it happens that the existence of another constant solution besides the trivial one may be possible. The interest on this situation appears very naturally in the gravitational framework. The reason is simple: the energy-momentum tensor associated to this solution is undistinguishable of the one produced by the cosmological constant. A very interesting property appears: the energy-momentum tensor assumes precisely the same form irrespectly of the way the Φ field couples to Gravity. Let us prove this.

In the minimal coupling case we find for the fundamental solution (besides the trivial one):

$$\Phi_0^2 = -\frac{M^2}{\lambda}. \tag{17}$$

Thus, it follows

$$\begin{aligned} T_{\mu\nu} &= V(\Phi_0) g_{\mu\nu} \\ &= \frac{1}{4} M^2 \Phi_0^2 g_{\mu\nu}. \end{aligned} \tag{18}$$

In the non minimal case (conformal coupling) we find:

$$\Phi_0^2 = -\frac{M^2}{\lambda + \frac{M^2}{6}}. \tag{19}$$

Thus,

$$\begin{aligned} T^{c}_{\mu\nu} &= \frac{1}{\left(1 - \frac{\Phi_0^2}{6}\right)} V(\Phi_0) g_{\mu\nu} \\ &= \frac{1}{4} M^2 \Phi_0^2 g_{\mu\nu}. \end{aligned} \tag{20}$$

We conclude that the fundamental state, which extremizes the potential, can be identified to the classical cosmical vacuum introduced by *a priori* reasons by Einstein in the birth of Modern Cosmology (1917). This simple result is at the basis of the most fashionable cosmical scenario of the 80's, the so called **inflationary** model.

Exercise. Show that the energy distribution of the fundamental solution can be interpreted in terms of an Ideal Gas, with the equation of state provided by $p + \rho = 0$.

Exercise. Examine the scalar field theory with a potential V given by

$$V = \frac{1}{2}\mu^2\Phi^2 + \mu^4\sqrt{1 - \frac{1}{\mu^2}\Phi^2(\frac{T}{T_c} - 1)}$$

in which T represents the medium equilibrium temperature. The existence of T_c acts as a sort of self-regulator quantity. Analyse the thermodynamical properties of this system.

It is possible to generalize the preceding theory and set up a scalar analogy of the Born-Infeld Electrodynamics. What will be the cosmological consequences of such a theory? Does this theory provide a singular origin of the Universe?

5 Structure of Spacetime

At the heart of the Old Standard Model we find the statement that the creation of the world was a non-accessible explosive event. The Universe starts its expansion from an extremely condensed configuration and it is immediately projected in a Friedmannian state controlled by Einstein's equations. This model is a naif — although direct — consequence of the application of General Relativity. Latter on, more sophisticated arguments were considered and even the role of the classical structure that we call **spacetime** was examined. Before this, however, a less ambitious program went into the reexam of the Riemannian nature of spacetime.

In the classical GR this condition is an *a priori*. We were first conducted to the abandonment of the Euclidean nature of ST in favour of a pseudo-Euclidean one in the Special Theory of Relativity; and then, from this, in a smooth way, to the general Riemannian framework of GR.

The various suggestions to derive the actual structure of the spacetime's geometry from first principles failed. In the last years there has been a renewing of interest on ancient ideas concerning not only this geometrical nature, but even other apparently sound properties, as its number of dimensions. In some Gravity textbooks one finds arguments set forth by Palatini, which intend to derive the Riemannian nature from a variational principle [11]. This method, however, does not provide an absolute answer but depends basically on the way matter couples to gravity. This seems to be little noticed in the literature. Thus, we will present here this method as a good exercise on the various distinct consequences of the gravity-matter interaction. Indeed, let us take the situation in which matter is represented by a scalar field Φ. In the minimal coupling interaction the Lagrangean takes the form:

$$L = R + \frac{1}{2}\Phi_{,\mu}\Phi^{,\mu} - V(\Phi). \tag{21}$$

Independent variation of $g_{\mu\nu}$ and of the connection $\Gamma^{\mu}_{\alpha\beta}$ yield respectively Einstein's equations and the Riemannian identification of the connection to the Christoffel symbol.

Let us turn now to the case in which matter is non-minimally coupled to gravity. We follow the same example as above and treat matter in terms of a scalar field. We set

$$L = R + \frac{1}{2}\Phi_{\mu}\Phi^{\mu} - V(\Phi) + \lambda R\Phi^{2}. \tag{22}$$

Let us call $\Psi \equiv (1 + \lambda\Phi^2)$, for convenience. The structure of the geometry is controlled by that part of the Lagrangean which contains terms proportional to R. We have

$$\delta \int \sqrt{-g}\Psi R = \int \sqrt{-g}\Psi \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \int \sqrt{-g}\Psi^2 g^{\mu\nu}\delta R_{\mu\nu}. \tag{23}$$

We are interested in the variation of the connection that controls the geometry. We have

$$\int \sqrt{-g}\Psi^2 g^{\mu\nu}\delta R_{\mu\nu} = 0.$$

Thus, a straightforward calculation gives the desired equation:

$$(\sqrt{-g}\Psi^2 g^{\mu\nu})_{;\lambda} = 0. \tag{24}$$

A simple manipulation of this equation gives

$$g_{\mu\nu;\lambda} = -2\frac{\Psi_{,\lambda}}{\Psi}g_{\mu\nu}. \tag{25}$$

We arrive thus to the conclusion that the riemannian structure is obtained from the variational priciple only in the very particular case in which matter couples minimally to gravity. In the case of nonminimally coupling it follows that the geometry is characterized by eq.(25) which is a particular case of a Weyl geometry. Once we will consider this structure in a subsequent section, it seems worth to spend some time in its further caracterization.

6 Weyl Integrable Spacetime: WIST

In the early days when the implementation of Einstein's idea of treating gravity phenomena as intimately related to specific aspects of the geometry of spacetime was a growing program, some physicist went through the idea of generalyzing this program to other parts of Physics. As a consequence, a series of tentatives of modifying Riemann geometry appeared. Although the subsequent history of all these proposals has failed in its original goal, some of them left a powerful mathematical tool which has been considered in other contexts. Weyl's geometry is one of these. Originally this geometry was created to turn Electrodynamics into a geometrical scheme. Soon, however, it was concluded that this proposal was not a successful one. Thus, its program as a unified scheme was abandoned. However, this geometrical scheme appeared since then in other contexts, one of which, a cosmological scenario [12], will be examined further later on.

Weyl geometry is defined by the non conservation of the length in a parallel transport. This is exhibited by the non vanishing of the covariant derivative of the metric tensor:

$$g_{\mu\nu;\lambda} = W_\lambda g_{\mu\nu}. \tag{26}$$

It then follows that its affine connection is dependent not only on the metric itself but also on the vector W_λ. We find:

$$\Gamma^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} - \frac{1}{2}(W_\mu \delta^\alpha_\nu + W_\nu \delta^\alpha_\mu - W^\alpha g_{\mu\nu}). \tag{27}$$

Given this Weyl connection it is straightforward to write Weyllian expressions for the geometrical quantities with the use of the corresponding Riemannian formula. The covariant derivative of a vector Q^μ reads, for instance:

$$Q^{\alpha}{}_{;\mu} = \nabla_{\mu}Q^{\alpha} - \frac{1}{2}[W_{\mu}Q^{\alpha} + W_{\nu}Q^{\nu}\delta^{\alpha}_{\mu} - Q_{\mu}W^{\alpha}]. \tag{28}$$

In this expression the symbol ∇_{μ} represents the covariant derivative using the Christoffel symbol of the associated Riemannian metric.

Exercise. Evaluate the curvature tensor of Weyl's geometry in terms of the Riemannian curvature and the vector W_{μ}.

Exercise. Evaluate the contracted curvature tensor.

Answer:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \nabla_{\nu}W_{\mu} - \frac{1}{2}W_{\mu}W_{\nu} - \frac{1}{2}g_{\mu\nu}[\nabla_{\alpha}W^{\alpha} - W^{\alpha}W_{\alpha}] \tag{29}$$

7 The First and the Second Clock Effects

Equation (26) implies that the length is not preserved by a parallel transportation in Weyl manifold. Indeed, it follows

$$dL = LW_{\alpha}dx^{\alpha} \tag{30}$$

This property has some consequences which imply, in the general case, observational difficulties. The simplest way to see this is precisely to follow Einstein's criticism to the general Weyl geometry. Let us consider the case in which two identical clocks are synchronized at a given point P of the space-time. According to General Relativity, if these two clocks travel to another point Q through distinct paths, gravitational efects may cause them to lose their synchronization. This is the so-called *first clock effect*. In Weyl spaces, due to the distinct variation of the units of measure along the two different paths, the discrepancy between time measurements units at Q adds a suplementary contribution to the loss of synchronization. This is called the *second clock effect.* In the case of closed circuits such additional synchronization loss would disagree with observations.

There is only one way to escape from this difficulty: to impose that the clocks measure the same value in a closed path. That is, we must impose

$$\oint dL = 0. \tag{31}$$

Thus, Stoke's theorem implies that Weyl's vector is a gradient:

$$W_\mu = \partial_\mu W \tag{32}$$

In this case the Weyl geometry reduces to a WIST, that is, to a Weyl Integrable SpaceTime. The characterization of a Wist depends only on an unique function W. We will associate this function W to a matter field and examine the dynamics in a simple case.

Exercise. Show that a WIST is conformally related to a Riemann space.

Exercise. Show that the scalar of curvature R of a Wist is related to the quantities of the associated Riemannian geometry by the expression:

$$R = \hat{R} - 3\Box W + \frac{3}{2} W_{,\mu} W^{,\mu} \tag{33}$$

Exercise. Assume the hypothesis that it is possible to foliate the structure of the spacetime. Suppose then that on a certain hypersurface Σ_0 the covariant derivative of the metric tensor vanishes

$$g_{\mu\nu;\alpha} = 0.$$

This corresponds to a Riemannian configuration on Σ_0. Question: what can be said about the value of $g_{\mu\nu;\alpha}$ on another hypersurface Σ_1?

This exercise reveals the so-called Structural Problem of the Spacetime. We shall come back to this latter on.

8 Dynamics for a WIST

We will examine here the consequences of changes in Einstein's General Relativity in a cosmical context in order to take into account the structural problem [12]. The simplest theory is given by the action

$$S = \int \sqrt{-g}[R + mW^{\alpha}{}_{;\alpha} + nW^{\mu}W_{\mu}] \tag{34}$$

The curvature R in a WIST can be written

$$R = \hat{R} + \frac{3}{2}W^{\alpha}W_{\alpha} - 3\nabla_{\mu}W^{\mu} \tag{35}$$

Then, up to a total divergence the action (34) takes the form

$$S = \int \sqrt{-g}[R + \xi W^{\mu}W_{\mu}] \tag{36}$$

and ξ is a dimensionless parameter.

Variation of the independent quantities $g_{\mu\nu}$ and W yield

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} + \nabla_\nu W_{,\mu} = 0 \tag{37}$$

and consequently

$$\Box W = 0. \tag{38}$$

This last equation takes the form

$$\frac{1}{\sqrt{-g}}[\sqrt{-g}W_{,\alpha}g^{\alpha\beta}]_{,\beta} = 0 \tag{39}$$

We can rewrite the equations of evolution in terms of the associated Riemannian quantities:

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}g_{\mu\nu} - \lambda^2 W_\mu W_\nu + \frac{\lambda^2}{2}W_\alpha W^\alpha g_{\mu\nu} = 0 \tag{40}$$

and

$$\Box W = 0. \tag{41}$$

in which [1] we defined $\lambda^2 = \frac{4\xi-3}{2}$. This equation is equivalent to Einstein equation in which the WIST field W is identified to a matter term. In the next section we will show that it can be interpreted as a *stiff* matter state.

9 The Cosmic Model

The simplest example of a WIST in a cosmological context is provided by the following model. We will limit our analysis to the case of spatially homogeneous

[1] We will analyse here only the case in which $4\xi - 3 > 0$. We leave to the reader the exam of the alternative case in which this quantity is negative.

FRW-type cosmologies, described by the line element eq.(2). Then the WIST field W is a function of the cosmic time only: $W = W(t)$. The Weyl vector W_α becomes

$$W_\alpha = \dot{W}\delta^0_\alpha \tag{42}$$

where the dot denotes simple differentiation with respect to the time. In this case the equation of motion (40) is similar to Einstein's General Relativity and the source term is identified with a perfect fluid. Indeed, we have

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}g_{\mu\nu} = -T_{\mu\nu}(W) \tag{43}$$

in which

$$T_{\mu\nu} = (\rho_W + p_W)V_\mu V_\nu - p_W g_{\mu\nu} \tag{44}$$

The density of energy and the pressure have the values [2]

$$\rho_W = p_W = \frac{-1}{2}\lambda^2\dot{W}^2 \tag{45}$$

In these expressions the co-moving four-velocity is $V^\mu = \delta^\mu_0$.

Using the condition of spatial homogeneity into eq.(41) yields a first integral for the WIST function:

$$\dot{W} = \gamma A^{-3} \tag{46}$$

where γ is a constant. The remainig equations for the radius of the Universe are:

[2]The interpretation of the WIST as a matter term in the framework of General Relativity appears somewhat strange, since it produces possible negative values for the corresponding energy. Thus, if one is not able to admit this it remains just one thing to be done: to come back to the geometrical description and do no changes to the Einstein's version of the WIST scheme.

$$\dot{A}^2 + \epsilon + \frac{1}{6}\lambda^2(\dot{W}A)^2 = 0 \tag{47}$$

and

$$2A\ddot{A} + \dot{A}^2 + \epsilon - \frac{1}{2}\lambda^2(\dot{W}A)^2 = 0. \tag{48}$$

where $\epsilon = (0, +1, -1)$ is the 3-curvature parameter. From these equations it follows that if $(3 - 4\xi) = -\lambda^2 < 0$ an open Universe is obtained ($\epsilon = -1$). Combining the above equations supplies the fundamental dynamical equation

$$\dot{a}^2 = 1 - \left(\frac{A_0}{A}\right)^4 \tag{49}$$

in which $A_0 = constant = (\frac{1}{6}\gamma^2\lambda^2)^{\frac{1}{4}}$.

Let us examine now some properties of this model, namely:

- The age of the Universe
- The accelerated Universe
- The cosmological structural problem

9.1 The Eternal Universe

From the evolution of the scale factor it follows that the minimum possible value of the radius of the Universe is the value $A(t = 0) = A_0 \neq 0$. There is thus a previous collapsing era (for $t < 0$). The Universe has no singular point. The evolution of the Universe begins at the infinitely remote past due to the unstability of a spatially infinite, empty Minkowski space-time. This can be seen from a direct inspection of the above equations of evolution or from the analytical expression for $A(t)$.

Exercise. Solve the equation for the radius $A(t)$.

Answer: The solution is given in terms of the elliptic functions

$$F(\alpha, \frac{\sqrt{2}}{2}) = \int_0^\alpha \left(1 - \frac{1}{2}\sin^2 z\right)^{-\frac{1}{2}} dz$$

and

$$E(\alpha, \frac{\sqrt{2}}{2}) = \int_0^\alpha \left(1 - \frac{1}{2}\sin^2 z\right)^{\frac{1}{2}} dz$$

by the implicit expression

$$t = A_0 \left[\frac{\sqrt{2}}{2}F(\alpha, \frac{\sqrt{2}}{2}) - \sqrt{2}E(\alpha, \frac{\sqrt{2}}{2}) + \sin\beta(\cos\beta)^{-\frac{1}{2}}\right] \tag{50}$$

in which $\alpha = \arccos\left(\frac{A_0}{A}\right)$ and $\beta = \arccos\left(\frac{A_0}{A}\right)^2$.

9.2 The Accelerated Universe

In the course of the everlasting collapse, the Universe is always accelerated (or *inflationary*). Indeed, from the equations of motion we have

$$\ddot{A} = \frac{2}{A}\left(\frac{A_0}{A}\right)^4 > 0.$$

The Universe, in this model, starts to evolve due to Weylian perturbations of an empty Minkowski space-time; thus, the most remote image of the cosmic history is that of a collapsing primordial Universe of infinite radius. Throughout this collapsing era the cosmic evolution is driven by the energy of the WIST field $W(t)$ which provides for the acceleration of the model. In fact, were the Universe always dominated by the W-energy only, it would accelerate forever. However, in

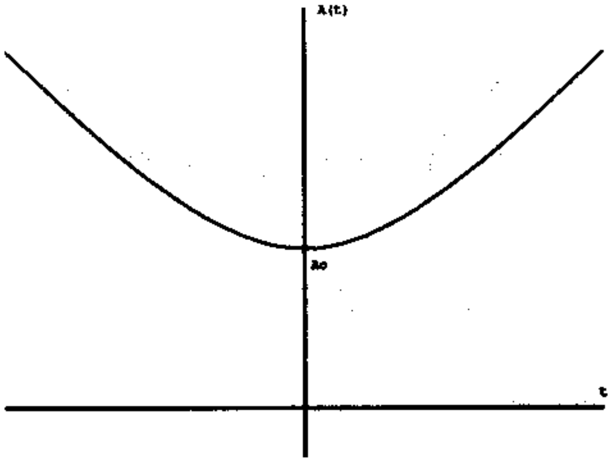

Figure 1: Qualitative behavior of the scale factor $A(t)$ according to the implicit solution Eq. (50). Given that $|\dot{A}| < 1$, the angle of inclination of the curve is always less than $\pi/4$.

the neighborhood of the maximally condensed epoch a significant amount of matter may come to appear, therefore implying important modifications of the evolution of the geometry in the ensuing expanding era.

9.3 The Hubble Parameter

Let us examine the correlate behaviour of the expansion factor or Hubble constant $H \equiv \frac{\dot{A}}{A}$. We have

$$H = \frac{1}{A}\left[1 - \left(\frac{A_0}{A}\right)^4\right]^{\frac{1}{2}}.$$

In distinction of the old standard cosmological model, here we have a sort of *Big, although not infinite, Bang.* The Hubble parameter H is always a smooth function of the cosmic time t and does not diverge at the origin of the expanding era; quite on the contrary, it vanishes at $t = 0$. The corresponding evolution of the cosmos may be outlined as follows: the Universe stays for a long period in a phase of slow adiabatic contraction, until parameter H attains the minimum value $H_m = -3^{-\frac{3}{4}}\sqrt{2}\frac{1}{A_0}$. Then an abrupt transition occurs: a fast compression turns into a fast expansion until H attains the maximum value $H_M = -H_m$; afterwards the expansion proceeds in an adiabatic slow pace again. The time lapse between the two extrema is in fact very short, once the value of the scale factor at both occasions, $A(t_-) = A(t_+) = 3^{\frac{1}{4}}A_0$, keeps very close to the minimum A_0.

9.4 The Structural Problem

For very large times the scale factor behaves as $A \sim t$. Thus, assymptotically, the geometrical configuration assumes a Riemannian character (once $\dot{W} \to 0$) in the

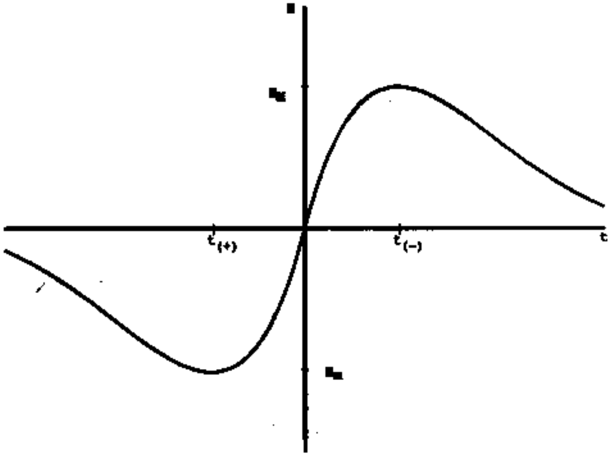

Figure 2: The Hubble parameter $H = A^{-1}(1 - A_0/A)^4)^{1/2}$. Here, $H_M = H(t_{(+)}) = (3^{3/4}\sqrt{2}/A_0) = -H_m = -H(t_{(-)})$, where $t_{(\pm)}$ corresponds to $A_{(\pm)} = 3^{1/4}A_0$.

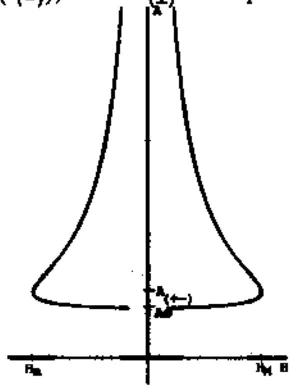

Figure 3: Behavior of the Hubble parameter with respect to the scale factor $A(t)$. Here, $A(t_{(\pm)}) = 3^{1/4}A_0$.

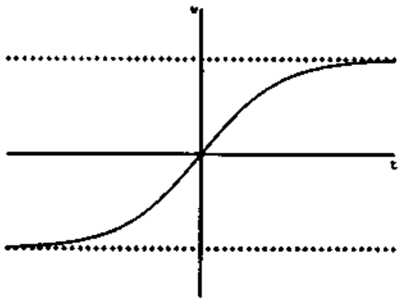

Figure 4: Behavior of the WIST function $\omega(t)$.

form of a flat Minkowski space (in Milne's coordinate system). In consequence, in the present model the evolution of the Cosmos may be assigned to a primordial unstability of Minkowski space, at the remote past, against Weylian perturbations of the Riemann structure. In order to know how the modification of the geometry occurs in the evolutionary scenario we must turn our attention to the behaviour of the structure's control function W. Using the results of the equations (46), (50) we get

$$W = \frac{\gamma}{2A_0^2} \arccos \left(\frac{A_0}{A} \right)^2 \tag{51}$$

The behaviour of $W(t)$ is portrayed in the picture 9.4. We note that when $A \to \infty$, that is for large times, the structural function tends to a constant: $W \to \pm\frac{\gamma\pi}{4A_0^2}$.

The important quantity, as far as the structural question is concerned, is not W but its time-derivative $\dot{W}$. Once this function $\dot{W}$ has a strong peak in the neighborhood of the minimum radius A_0 (see fig. 9.4, there occurs in this region the greatest deviation from the Riemannian configuration. In this sense, a sort of

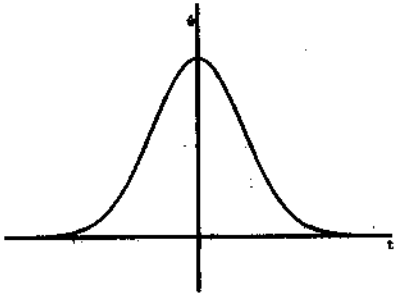

Figure 5: Behavior of the WIST function $\dot{\omega} = \gamma/A^3$.

structural *phase transition* takes place when the Universe approaches its maximally condensed state. From the behaviour of the associated *energy* of the WIST *fluid* it follows that it hinders a further collapse to a singularity and reverses the cosmic evolution into an expansion. In other words, the variations of measuring scales are responsible for the reversion of the collapsing process.

9.5 WISTons and Anti-WISTons

In the derivation of the above solution of the WIST *structural* function W no attention was paid to the sign of the constant γ. The only information we have about γ is that $\gamma^2 = \frac{6A_0^4}{\lambda^2}$. Therefore γ in fact admits both a positive and a negative value:

$$\gamma^{(\pm)} = \pm\sqrt{6}A_0^2\lambda^{-1} \tag{52}$$

Hence, the corresponding solutions

$$W^{(\pm)} = W_0^{(\pm)} \arccos\left[\frac{A_0}{A}\right]^2 \tag{53}$$

in which $W_0^{(\pm)} = \pm\sqrt{\frac{3}{2}}\lambda^{-1}$. Thus the amplitude of the solutions depends exclusively on the dimensionless parameter ξ.

Remark, however, that the energy density ρ_W of the equivalent *stiff matter* state associated to the WIST field is the same in both cases

$$\rho_W = -3\left[\frac{A_0}{A}\right]^4 \frac{1}{A^2}$$

Thus, in spite of the fact that the pairs of WIST functions $(W^{(+)}, \dot{W}^{(+)})$ and $(W^{(-)}, \dot{W}^{(-)})$ have different characteristics, they induce the same type of cosmological evolution:the solution given by eq.(50) does not distinguish which pair is the source of Einstein's equations. Their only distinction is connected to lenght variations, once according to the WIST interpretation $\Delta L^{\pm} = L\dot{W}^{\pm}\Delta t$.

9.6 Time Reversal

The above system is invariant with respect to the time reversal operation

$$t \rightarrow -t,$$

if W^+ is concurrently mapped into W^- and reciprocally. In this sense, the WIST instanton-like functions W^+ and W^- may be called **Wiston and Anti-Wiston** solutions, respectively, since an anti-Wiston may be described as a Wiston running backwards in time.

The above solutions of $W(t)$ reveals an instanton behaviour typical of non-linear theories of self-interacting scalar fields. Thé root of such non-linearity is related to the fact that W is taken as the actual source of the curvature of the metric structure, which in turn modifies the D'Alembertian operator $\Box$ due to the introduction of W-dependent terms. Let us exhibit this situation by means of a change of variable. Define the new variable $\sigma(t) = \dot{W}(t)$. From the dynamics of the Wistons it follows:

$$\dot{\sigma} + 3\gamma A^{-4}\dot{A} = 0 \tag{54}$$

and

$$A^3 - \gamma\sigma^{-1} = 0. \tag{55}$$

Taking σ to represent a generalized coordinate associated to a one-particle dynamical system, we obtain the conservation equation

$$\frac{1}{2}(\dot{\sigma})^2 + V(\sigma) = 0, \tag{56}$$

in which the associate potential $V(\sigma)$ is given by

$$V(\sigma) = \frac{3\lambda^2}{4}\left[\sigma^4 - b^2\sigma^{\frac{4}{3}}\right] \tag{57}$$

with $b^2 = 6\lambda^{-2}\gamma^{\frac{-2}{3}}$. The evolution of the field σ is equivalent to a unit mass particle moving in a potential with vanishing total energy.

Exercise. From the above characterization of a Wiston, we can set that the ***density of the number of wistons*** is provided by the association

$$\rho_W \sim n^2$$

show that the total *number* of Wistons N is conserved through the evolution of the Universe. Hint: Use the equation of the Wistons to obtain $\dot{n} + \Theta n = 0$, in which n is the density of particles.

9.7 Creation of the Universe

In the present model the Universe passed a previous collapsing era which was reversed by the *Weylization* of the spacetime geometry. This solution resembles the fate of a cosmos controlled, in Einstein's General Relativity, by a cosmological constant. Indeed, let us come back to the basic equation

$$\dot{A}^2 + \epsilon = \frac{1}{3}\rho_V A^2 \tag{58}$$

in which ρ_V is the vacuum density of energy. Then, in the case of closed section ($\epsilon = 1$), we obtain the classical de Sitter solution

$$A = \frac{1}{H}\cosh(Ht). \tag{59}$$

where $\frac{1}{3}\rho_V = H^2$. This solution represents a Universe which passes continuously from a collapsing to an expanding phase. The responsible for such well-behaved (i. e. non-singular) behaviour is precisely the energy of the vacuum. Besides such simple classical characterization it is possible to present the expanding era classically disconnected from the contracting part of the de Sitter geometry [13]. The starting point of this is the observation that there exists a precise relation between the quantum tunneling effect in ordinary Quantum Mechanics and the existence of non-trivial solutions of the classical equations of motion in the analytical continuation for imaginary times [14]. Thus, the extension of the Friedmann's equation (58) in this case yields

$$\dot{A}^2 = 1 - \frac{1}{3}\rho_V A^2 \tag{60}$$

and the corresponding solution

$$A = \frac{1}{H}\cos(Ht). \tag{61}$$

Let us come back to the classical solution (59) and interpret it as the description of a point like particle which position is provided by the one-dimensional coordinate $x(t) \equiv A(t)$. There exists a barrier at the point $A = A_0$ which the particle cannot penetrate due to the presence of the potential H^2A^2. However, in the quantum regime the particle acquires a non-vanishing probability to overcome the barrier. The classical deSitter solution $cosh(t)$ can be matched to the tunneling solution at the point $A(t = o) = \frac{1}{H}$. This allows the interpretation of an Universe being created in the *primordial* state $A(t = 0) = \frac{1}{H}$ from quantum tunneling. In other words, the quantum creation mechanism which allows the scale factor to become zero (cf. eq.(61)), acts as a substitute of the previous collapsing phase.

This is the so-called deSitter instanton solution. The corresponding primordial quantum state is called *Nothing.*

Exercise. Investigate the possibility of interpreting the Winston solution as an instanton. Could we identify the process of Weylization in terms of quantum tunneling?

Comments.

Some authors argued that the Euclideanization of spacetime is nothing but a mathematical trick to deal with quantum process. The real structure of the world should be identified to a Lorentzian manifold. The previous exercise suggest the modification of the contest Euclides X Lorentz to the alternative: Riemann X Weyl.

This question is part of a larger program in which stochastic Quantum Mechanics is related to the Weyl structure of the background geometry. Compare with [15].

9.8 Cosmogony

In the Old Standard Cosmological Model the question of the origin of matter was related to the inaccessible primordial explosion. Thus, there is not in a strict sense a Cosmogony in this model. In the case of models with no singularity one must face the problem of matter creation. In this section we will analyse this question in the particular case of a WIST configuration.

There are many possible ways to deal with the non conservation of matter in the framework of WIST theory. Here we will present a simple model for Cosmology.

Following the standard prescription based on the Minimum Coupling Principle let us assume that in the WIST scenario the energy-momentum tensor $T_{\mu\nu}$ satisfies the relation

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{62}$$

where the semi-colon denotes covariant differentiation in a Weyl manifold. In the particular case of a Riemann configuration, i. e., when the WIST field $W(t)$ vanishes, this expression reduces to the usual conservation law of General Relativity. Thus, eq.(62) should be considered as a specific model of the interaction between matter and the field W.

Let us follow our previous method and see what consequences such an interaction process has in the associated Riemannian spacetime. We have

$$T^{\mu\nu}{}_{;\nu} = \nabla_\nu T^{\mu\nu} - 3W_\nu T^{\mu\nu} + \frac{1}{2}W^\mu T \tag{63}$$

We can interpret this equation as if it represents the non-conservation of matter produced by the decay of the W-field in the Riemannian background.

In the case of a relativistic fluid (say, photons) in a Friedmann-Robertson-Walker geometry eq.(62) yields

$$\dot{\rho} + 4H\rho - 3\dot{W}\rho = 0 \tag{64}$$

We can thus interpret the WIST *energy* function $\dot{W}$ as the time-dependent total decay width Γ_W of the associated bosonic field W into photons. Integrating this equation we arrive at

$$\rho = \rho_0 A^{-4} e^{3W} \tag{65}$$

where the symbol (0) denotes the value of small fluctuation of matter density ocurred at some ocasion in the past.

We can now use the particular solution for the field W which we obtained in previous section to get some conclusion on the cosmogonic scenario that we are considering.

A remarkable consequence of the introduction of *dissipative* effects induced by the WIST character of the spacetime background is the exponential dependence of matter properties on the behaviour of the WIST field $W(t)$.

The reader will find no difficulty to prove the following description concerning an Universe driven by a Wiston[3]

According to the solution eq.(51) it follows that any fluctuation $(\Delta\Psi)_0$ experienced by a given matter field Ψ at the remote past is strongly damped in the course of the collapsing phase ($t < 0$); then there occurs a transition from supression to stimulation around $t = 0$, and a equally strong amplification begins as the

[3]For an anti-Wiston Universe this account should be inverted.

expanding phase takes place $t > 0$. This production mechanism, however, saturates very rapidly, and for later times $t \gg 0$ it becomes insignificant. Thus, in distinction of other eternal, bouncing cosmologies, the infinite span of the contracting phase in the present model does not imply a boundless matter-energy production.

Due to the exponential damping of any primeval irregularity, only fluctuations taking place near $t = 0$ do care for the subsequent evolution; but these fluctuations are exponentially amplified for a short period, so as to allow for arbitrarily large amounts of matter and entropy to be created. This period of intense creation is equivalent to a non-equilibrium process; notwhithstanding this fact, after the amplification mechanism has been shut down one might expect the WIST field declining contribution to the source of Einstein's equations to be rapidly outmatched by the newly produced matter content. In this way, the primordial *stiff matter* state associated to the *Big, but not infinite, Bang* described here could be straightforwardly continued to a standard sequence of radiation-dominated and matter-dominated phases.

The operation of this amplification mechanism also provides a fresh perspective with which the standard baryon assymetry problem may be envisaged. The prevalence of matter (e.g., baryons) against anti-matter in the observed Universe — as well as the observed ratio of entropy per baryon — is not explained in standard Cosmology except with the use of fine-tuned initial conditions. In the present scenario, on the other hand, an eventual baryon excess fluctuation $\Delta N_0 = N_B - N_{\bar{B}}$ taking place shortly after the stage of maximum contraction at $t = 0$ may be exponentially increased up to a convenient amount, since in this case we have

$$\Delta N_B = \Delta N_0 e^{2W}.$$

10 The Cosmic Evolution

We can synthesize all the preceding considerations as follows.

Due to Weylian scale fluctuations a primordial empty Minkowski spacetime begins to collapse at a remote past. This collapsing phase of indefinite duration is driven by the WIST field $W(t)$, whose effects are thermodinamically equivalent to a *stiff matter* state of a perfect fluid with energy density given by $\rho_W \sim A^{-6}$ (see eq.(45)). Throughout the collapse, the Universe is accelerated or *inflationary.* Any eventual matter-energy fluctuation is exponentially supressed in the course of the entire collapsing phase. This resembles the *memory loss* of certain models of inflationary scenarios. The collapse proceeds adiabatically in a very slow pace until a stage of greatest condensation — corresponding to the minimum A_0 of the cosmic radius — is approached. In fact, in the neighborhood of this maximally condensed stage the contraction is accelerated to an acme and then decreases suddenly, reverting to an expansion when the minimum radius A_0 is attained.

In analogy to the quantum creation models, the infinite collapsing phase of the present scenario may be associated to the propagation of a Weyl instanton or **Wiston** in an Euclideanized, classically forbidden region. According to this alternative interpretation, the Universe (as a classical entity) emerged from *Nothing*, endowed with a minimum radius in a *stiff matter* state characterized by the absence of a matter content (e.g., baryons and leptons), except for small fluctuations. However, as the Universe begins to expand, a non-adiabatical amplification mechanism starts to operate, driven by the energy of the WIST field $W(t)$, in such a way that matter-energy fluctuations may come to be converted, in an exponential rate, into large amounts of particles and radiation. An eventual baryon excess may be amplified in the same fashion. This **Big, but not infinite, Bang** stage lasts for a very short period, once the energy of the produced material soon dominates the energy of the WIST field; in this way the Universe enters in the Friedmannian radiation-dominated and matter-dominated regimes.

Chapter 27

Isotropization

In the Old Standard Model the Universe is postulated to be generated in a very symmetric isotropic state to conform with actual observation. However, it has been argued that it should be a true progress in our understanding if one could derive such state from less symmetric configurations. Since the end of the 60's many attempts in this direction have been undertaken. We can divide these proposals into three classes, according to the emphasis on the method. They are:

- Geometric
- Thermodynamic
- Λ-Dependence

We will describe generically their basic properties and consider some examples to clarify their meanings.

1 Geometric Analysis of Isotropization

The best example of this approach is certainly the proposal made by Lifshitz and his Russian collaborators, mainly Khalatnikhov and Belinskii [16]. At the first moment, this method received a severe criticism, due to the misunderstanding related to the problem of the real existence of a true singularity in the general solution of Einstein's equations. Indeed, LKB were originally interested in the exam of the generic behaviour of the geometry near the cosmological singularity. They comment, in the first version of this work, that it could be possible that the singularity exhibited by Friedmann's model was not a common property of any geometry admitted by General Relativity (GR). This assertion was wrong, as these authors soon recognized, and had just a partial importance in this research. However, this initial sin was never forgiven. At those days, the Physics community faced the dictatorship of the so-called *singularity theorems* which intend to demonstrate that, for any physically acceptable source for the gravitational field, an unavoidable true singularity occurs in the corresponding geometry of the Universe. Nowadays, as it occurs generally in human society, this period of absolutism has passed. However it was responsible by the largely spread attitude of rejection towards the entire LKB program.

Here, we do not consider details of this proposal. The reason is just this: during the IIth Brazilian School of Cosmology and Gravitation, held in João Pessoa, we had professor Evgenii Lifshitz with us. His lectures were published in the proccedings of that School [1]. Subsequently, during the IIIth Brazilian School of Cosmology and Gravitation we had the presence of Khalatnikhov and Belinskii which provided further developments of this program. I will limit thus this section to present roughly its main lines of investigation and conclusions.

The method was christened "oscillatory regime of approach to the singularity ".

[1] These lectures were published in Portuguese [4].

They intended to examine the analytical properties of the generic solution of Einstein's equations near the singularity and, as a sub-product, to show that Friedmann's Universe had a previous, less symmetric phase. They limit their main considerations to the exam of global (i.e., non local) singularities. LKB show that, in the case we can treat the source of gravity as a perfect fluid, then the *gravitational energy* becomes more important than the matter in the singularity's neighborhood. This means that matter does not direct the evolution of the geometry near the singularity: the geometry sustains itself, and if we consider the running backwards of the time, the behaviour of the geometry towards back to the singularity is controlled solely by the geometry. This, of course, is possible only because GR provides a set of nonlinear equations for the metric.

Exercise: Examine under what conditions one can neglect the matter influence on the evolution of the cosmological metric. Hint:use the metric in the form

$$ds^2 = dt^2 - (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta) dx^\alpha dx^\beta \tag{1}$$

in which $a = t^{p_l}$, $b = t^{p_m}$ and $c = t^{p_m}$. The quantities p_i are constants and the vectors $l_\alpha, m_\alpha, n_\alpha$ constitute a basis of the 3-dim. space. Show that near the origin $t = 0$ the matter terms behaves as a lower power of the time t than the curvature terms. (This proof, like all others claims from LKB in this section, can be found in the references, in particular at the proceedings of the IIth Brazilian School of Cosmology and Gravitation). See, however, the arguments in [17] on some special cases in which matter is not unimportant near a singularity.

From the above remark it then follows that our interest must turn to the cosmological vacuum solution of Einstein's equations. The simplest metric exhibiting such property is Kasner geometry. It is a Bianchi type-I geometry [18], which can be written in Euclidean coordinates (x, y, z) in the form (1) for the specific values:

$$l_\alpha = \delta_\alpha^1 \tag{2}$$

$$m_\alpha = \delta_\alpha^2 \tag{3}$$

$$n_\alpha = \delta_\alpha^3 \tag{4}$$

Besides, Einstein's equations imply that the parameters p_i must satisfy the two following conditions:

$$p_1 + p_2 + p_3 = 1 \tag{5}$$

$$p_1^2 + p_2^2 + p_3^2 = 1 \tag{6}$$

This solution is spatially homogeneous and anisotropic. We note that Kasner's geometry exhibits the property that the axis x, y, z expands and contracts through the power law t^{p_i} that is submitted to conditions (6), that is: one axis contracts when the other two expand. LKB argues that the generic behaviour of the cosmological metric near the singularity must be obtained by a straightforward generalisation of Kasner geometry (cf. (1)). After finding such a geometry one has to examine the process of evolution which goes beyond the vicinity of the singularity to the ulterior phase of Friedmann's Universe. This process evolves by alternation of the axis of expansion/contraction. Then comes the most controversial claim of LKB: this evolution occurs in a spontaneously stochastic way. If this should be the case, then the Universe would lose any memory of its initial condition (including any arbitrary existing primordial anisotropy) which, for some physicists, is a very desirable property. Unfortunately an undoubted proof of this assertion is still missing.

2 Thermodynamic Treatment — I

The Old Standard Model treats the matter content of the Universe as an adiabatic fluid. The Gibbs relation between the total energy $E = \rho V$, the total Entropy S, the thermodynamical pressure p_{th} and the volume V for an equilibrium temperature T is:

$$TdS = dE + p_{th}dV \tag{7}$$

Thus, for the time derivative:

$$T\dot{S} = [\dot{\rho} + (\rho + p_{th})\theta]V \tag{8}$$

The most general form of the energy-momentum tensor is:

$$T_{\mu\nu} = \rho V_{\mu}V_{\nu} - ph_{\mu\nu} + Q_{(\mu}V_{\nu)} + \Pi_{\mu\nu} \tag{9}$$

in which Q_{μ} is the heat-flux and $\Pi_{\mu\nu}$ is the anisotropic pressure, which has the following properties:

- $\Pi_{\mu\nu}V^{\mu} = 0$
- $\Pi_{\mu\nu} = \Pi_{\nu\mu}$
- $\Pi_{\mu\nu}g^{\mu\nu} = 0$

Using the projected equation of conservation of $T_{\mu\nu}$ in the V^{μ} direction yields:

$$\dot{\rho} + (\rho + p)\theta + \dot{Q}_\mu V^\mu + Q^\mu{}_{;\mu} - \Pi_{\mu\nu}\sigma^{\mu\nu} = 0 \tag{10}$$

For the Ideal Gas both quantities Q_μ and $\Pi_{\mu\nu}$ vanish. Then, using (11) into (8) gives $\dot{S} = 0$. This is the situation in the Old Standard Model: the evolution of the Universe proceeds in an adiabatic way. This simplification should not be invoked ulteriorly to produce a real *cosmological problem*, on the origin of the actual high value of the total entropy of the Universe. However this will be precisely the case, as we comment elsewhere in this text.

In [5] the consequences of assuming the possible existence of a huge quantity of massless, almost non-interacting neutrinos in a highly dense past era of the Universe is examined. The viscosity produced by neutrino scattering induces the presence of an anisotropic pressure. Combining (8) and (10) it follows, in the absence of heat flux, in order to conform with the second law of Thermodynamics:

$$\Pi_{\mu\nu}\sigma^{\mu\nu} > 0 \tag{11}$$

In [5] Misner considers that, at the very hot epoch of the primitive Universe, the energy content could be approximated by a linear Stokesian fluid. He then sets:

$$\Pi_{\mu\nu} = \eta\sigma_{\mu\nu} \tag{12}$$

The main idea behind the above hypothesis is that, in some sort, the neutrino pressure could counterbalance any *primordial* anisotropy. However, it was remarked by [19] that this linear approximation was not enough to smooth out the Universe. The analysis of this phenomenon depends on the study of the evolution of the shear, which is obtained from (12), through the same procedure as we did above for Raychaudhury's equation.

Exercise: Obtain the equation of evolution of the shear and the generalized equation for the expansion parameter.

Answer:

1. The equation for θ:

$$\dot{\theta} + \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 - a^{\mu}{}_{;\mu} + \dot{a}_{\mu}V^{\mu} + a^{\mu}a_{\mu} = R_{\mu\nu}V^{\mu}V^{\nu} \tag{13}$$

In the above expression the following definitions were used: $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega^2 \equiv \omega_{\mu\nu}\omega^{\mu\nu}$.

2. The equation for $\sigma_{\mu\nu}$:

$$\begin{aligned} h_{\alpha\mu}h_{\beta\nu}\dot{\sigma}^{\mu\nu} &+ \frac{1}{3}[a^{\lambda}{}_{;\lambda} - \frac{1}{2}\omega^2 - \sigma^2]h_{\alpha\beta} \\ &+ a_{\alpha}a_{\beta} - \frac{1}{2}h_{\alpha\mu}h_{\beta\nu}[a^{\mu;\nu} + a^{\nu;\mu}] \\ &+ \frac{2}{3}\Theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta} - \omega_{\alpha}\omega_{\beta} \\ &= R_{\alpha\epsilon\beta\nu}V^{\epsilon}V^{\nu} - \frac{1}{3}R_{\mu\nu}V^{\mu}V^{\nu}h_{\alpha\beta} \end{aligned} \tag{14}$$

Obs: we use the definition

$$\omega^{\tau} \equiv \frac{1}{2}\eta^{\alpha\beta\rho\tau}\omega_{\alpha\beta}V_{\rho}$$

Note that the shear's evolution depends not only on matter (through Einstein's equations), but also on the Weyl tensor $W_{\alpha\beta\mu\nu}$. [2]

A further generalisation of the above scheme would lead us to examine more general Stokesian fluids. From the above remarks and the generic properties of the

[2]Some authors, [22], have speculated on the origin of the vanishing of Weyl tensor in our Universe.

kinematical quantities we can write, for the traceless anisotropic pressure a series in powers of Θ and $\sigma_{\mu\nu}$. Let us limit ourselves here to the polynomial containing terms up to the third order:

$$\Pi_{\mu\nu} = (\alpha_0 + \alpha_1\Theta + \beta\sigma^2)\sigma_{\mu\nu} + \delta\sigma_{\mu\alpha}\sigma^{\alpha}{}_{\nu} - \frac{1}{3}\sigma^{\alpha\beta}\sigma_{\alpha\beta}h_{\mu\nu} \tag{15}$$

We shall see later on that, at least for the treatment of the phenomenon of phase transition, the additional terms of higher orders have a minor importance.[3]

3 Changing of Observer

The decomposition of equation (9) gives the reducible quantities for the energy-momentum tensor for a given observer endowed with the velocity V_μ. This means thus that the corresponding quantities ρ, p, q_μ and $\Pi_{\mu\nu}$ are observer-dependent. In this section we will exhibit the modifications that arise in these quantities when we deal with another observer, $\tilde{V}_\mu$.

Let us write the new frame characterized by $\tilde{V}_\mu$ in terms of the original four-velocity V_μ and a spacelike vector e_μ, which is orthogonal to V_μ and normalized:

$$V_\mu\, e^\mu = 0$$

$$e_\mu\, e^\mu = -1$$

We thus write for the normalized $\tilde{V}_\mu$

$$\tilde{V}^\mu = cosh\gamma\; V^\mu + sinh\gamma\; e^\mu \tag{16}$$

[3]Remark however that in case of bi-axial fluid terms of higher orders must be taken into account [25].

Let us now evaluate each one of the irreducible components of $T_{\mu\nu}$ in the new frame. We have:

$$T_{\mu\nu} = (\rho + p)\, V_\mu\, V_\nu - p\, g_{\mu\nu} + q_{(\mu}\, V_{\nu)} + \Pi_{\mu\nu}$$
$$= (\tilde{\rho} + \tilde{p})\, \tilde{V}_\mu\, \tilde{V}_\nu - \tilde{p}\, g_{\mu\nu} + \tilde{q}_{(\mu}\, \tilde{V}_{\nu)} + \tilde{\Pi}_{\mu\nu} \tag{17}$$

(i)The Density of Energy ($\tilde{\rho}$):

$$\tilde{\rho} = T_{\mu\nu}\, \tilde{V}^\mu\, \tilde{V}^\nu \tag{18}$$

Then it follows

$$\tilde{\rho} = cosh^2\gamma\, (\rho + p) - p + 2\, sinh\gamma\, cosh\gamma\, q_\mu\, e^\mu + sinh^2\gamma\, \Pi_{\mu\nu}\, e^\mu\, e^\nu \tag{19}$$

(ii)The Pressure ($\tilde{p}$):

$$\tilde{p} = p + \frac{1}{3}\, (\tilde{\rho} - \rho) \tag{20}$$

(iii)The Heat Flux ($\tilde{q}_\mu$):

$$\tilde{q}_\mu = cosh\gamma\, q_\mu + V_\mu\, [cosh\gamma\, (\rho - \tilde{\rho}) + sinh\gamma\, q_\nu\, e^\nu] - e_\mu\, sinh\gamma\, (\tilde{\rho} + p) + sinh\gamma\, \Pi_{\mu\nu}\, e^\nu \tag{21}$$

(iv)The Anisotropic Pressure:

$$\tilde{\Pi}_{\mu\nu} = \tag{22}$$

Exercise: Show that for a perfect fluid ($q_\mu = 0 = \Pi_{\mu\nu}$) and such that an equation of state is provided by $p = \lambda\rho$, the corresponding equation $\tilde{p} = \tilde{\lambda}\tilde{\rho}$ is such that

$$\tilde{\lambda} = \frac{[\lambda + \frac{1}{3}\, sinh^2\gamma\,(1+\lambda)]}{[1 + sinh^2\gamma\,(1+\lambda)]} \tag{23}$$

Let us note that the only case in which the equation of state is preserved is $\lambda = \tilde{\lambda} = -1$. This is nothing but the well-known vacuum frame-invariance.

4 Thermodynamic Treatment II — Phase Transition

In the 80's physicists developed the idea that in the super-dense era of the primitive Universe a cosmological phase-transition could occur. This was thought in the framework of field theory and could materialize the desire of a unified treatment of the interactions, at least in a determined epoch of the Universe. Indeed, the Standard Model assumes the existence of a very hot primordial phase. A direct consequence of this is that, at this extremely condensed period, we cannot treat the processes of interacting particles in a completely empty spacetime at zero temperature. In that epoch the matter and radiation are so intensively connected in all domains of spacetime that we are induced to treat all interacting process, in analogy to Thermodynamics, as if they were immersed in a thermal bath at a given (time-dependent) temperature T.

Thus, the natural framework to describe these changing phenomena concerning interacting particles is the Finite-Temperature QFT. The most striking result concerning the applications of this theory to the Universe is precisely the presence of a possible similar phenomenon of phase transition. The reader interested in details of this subject should consult the references [10], [20], [21]. Before going into this matter it seems worth to make a short *resumé* of Landau's phase transition theory. We remind the reader that we will not give a complete treatment of this theory, but just a very condensed review of its main properties.

4.1 Landau's Phase Transition Theory

We start by the definition of an order parameter, which we call η. Different states of a fluid are characterized by the distinct values of η. We choose the following characterization:

- $\eta = 0$ represents a symmetric phase
- $\eta \neq 0$ represents a less symmetric phase

Thus $\eta = 0$ is the less ordered state. The theory argues that it makes sense to develop the free energy as a series of powers on the parameter η. We thus set:

$$F(p, T, \eta) = F_0 + \alpha\eta + \frac{A}{2}\eta^2 + \frac{B}{3}\eta^3 + \frac{C}{4}\eta^4 + \ldots \tag{24}$$

The quantity F_0 represents the remaining part of the free energy that is independent of the order parameter. The minima of the function F, with respect to η, represent the most favourable states of the fluid. In order to allow the fluid to have access to the state of equilibrium in the maximal disordered case we have to impose a priori that the expansion (24) starts at least on η^2, that is, we set $\alpha = 0$. Now a further hypothesis is made concerning the dependence of the coefficients A, B, C, ... on the temperature T. In principle this dependence must be provided by a theory. In the absence of this we may take a phenomenological short-cut and assume, following Landau [23], that only A depends on T, the other coefficients are taken to be constants. We set:

$$A = a_0^2(T - T_c) \tag{25}$$

The analysis of the extremum (minima) of the free energy yields the result:

- If $T > T_c$ the most favourable state is for $\eta = 0$
- If $T < T_c$ the most favourable state is for $\eta \neq 0$

It then follows that the temperature is the parameter that controls the actual phase of the fluid. Note that the most disordered phase ($\eta = 0$) occurs for high values of the temperature. Thus, the increase of temperature points in the same direction as the evolution of the system with the corresponding increase of entropy. See below, however, for a special case in which these arrows (T and S) do not coincide.

4.2 Phase Transition in QFT

Let us come back to the scalar field theory submitted to the potential (4). One can provide a simple example of the spontaneous symmetry mechanism breaking by considering that the mass term has the "wrong " sign. Indeed, let us set $M^2 = -\mu^2 < 0$. If we then look for the equilibrium points of this theory we find, by extremizing V, that there are two solutions:

- $\Phi_v = 0$
- $\Phi_0^2 = \frac{\mu^2}{\lambda}$

This mechanism takes a very important place in the modern Unification Program of the Fundamental Interactions [10]. Our interest here is to examine the modifications imposed on this mechanism in a non stationary cosmological background. The net consequence of the thermal bath (cf. above) provided by the cosmical environment at the very condensed phase of the Universe is the introduction of Temperature-dependent terms in the self-interacting potential V (see [24] and references therein). The free energy F can be approximated (for high temperatures: $T \gg M$):

$$F = V_{(T=0)} + \frac{1}{8}\lambda T^2 \Phi^2 + \dots \tag{26}$$

The theory contains again the possibility of two minima:

- $\Phi_v = 0$
- $\Phi_0^2 = \frac{1}{4}(T_c^2 - T^2)$

in which the critical temperature is defined as $T_c \equiv \frac{2\mu}{\sqrt{\lambda}}$. A new phenomenon then occurs: the existence of the minimum Φ_0 depends now on the surrounding matter's temperature. Indeed, for high temperatures $T > T_c$, the null solution is the only possible state for the system. However, as the temperature drops below the critical value T_c, a symmetry breaking solution occurs.

Exercise: Compare the entropy of both states.

It then follows that as the Universe expands, and the temperature diminishes, the system consisting of the scalar field gets into a less symmetric configuration. When the system *sits* on its state Φ_0 , its corresponding distribution of energy is equivalent to a cosmic fluid with equation of state $p + \rho = 0$, typical of a cosmological constant Λ, given by (see eq.(3)):

$$\Lambda = -\frac{1}{64}\lambda(T^2 - T_c^2)^2 \tag{27}$$

In this case, if there is no other competitive source of curvature, the Universe can jump into a deSitter type of geometry. Remark that this occurs only under the very stringent hypothesis that the scalar system is homogeneous.

One further remark: In the Old Standard Cosmological Model there is a simple dependence of the temperature T with the cosmical time t through the function $A(t)$, the *radius* of the Universe: $T \sim A^{-1}$. In the special case in which $A \sim t^n$ it then follows that we can write $T \sim \Theta^n$. In general, even for other more general functions $A(t)$, the temperature still is a regular function of Θ. One can thus use the expansion Θ as the true parameter that controls the phase transition mechanism. In a subsequent section we shall find precisely this situation when dealing with viscous fluids.

4.3 Liquid Crystal

In the previous examples we have considered in this section, we dealt with an order parameter (e.g., η) that is a scalar. This, of course, is not imposed by any theory but just a phenomenological choice. Let us now turn our interest to a case in which this parameter must have a tensorial character. This is the case when we are concerned with the distinction between gas/liquids and, on the other hand, crystals. The first ones have no internal privileged direction: their properties are the same, irrespectively of the direction we choose to observe them. This means that they have the highest symmetry of isotropy. On the other hand, non-amorphous solids (crystals) have a lower symmetry: they are in, general, anisotropic. The interesting case in which an anisotropic crystal can cohabit with the isotropic liquid phase of the same substance is called the liquid crystalline state of the matter. The distinction between the two phases is not made by a scalar quantity, as the associated order parameter has a tensorial character. Once we are interested in the bulk properties of the fluid[4] we must look for a macroscopic order parameter[5]. This is the standard Landau's treatment. Once we have in mind the future application of the present theory in Cosmology, let us try to unify the variables employed to examine these processes. This means that we will choose the anisotropy tensor $\sigma_{\mu\nu}$ as the order parameter [6]. The shear tensor belongs to the orthogonal H space (see Section 24) of the comoving observer. Thus we will treat σ_{ij} as a three-dimensional object which we parametrize as [25]

$$\begin{pmatrix} -\frac{1}{2}(x+y) & 0 & 0 \\ 0 & -\frac{1}{2}(x-y) & 0 \\ 0 & 0 & x \end{pmatrix}. \tag{28}$$

The case in which none of the parameters x, y vanish represents the general bi-axial phase. For didactical reasons we will restrict ourselves here to the consideration of the simpler case in which the fluid exhibits a plane of isotropy. We choose this by setting $y = 0$. The free energy F can thus be developed in powers of the order parameter as in (15). Once F is a scalar, and σ_{ij} is trace-free, it follows (note that $Tr\sigma^2 = \frac{3}{2}x^2$ and $Tr\sigma^3 = \frac{3}{4}x^3$):

$$F = F_0(P,T) + \frac{3}{4}Ax^2 + \frac{1}{4}Bx^3 + \frac{9}{16}Cx^4 + ... \tag{29}$$

[4]We remark that we use the term **fluid** in a generic way: it can be identified to any of the above state of matter

[5]In the cosmological treatment, this means that we will not take into account local specificities. This microscopic (e.g., local) treatment can be of importance when examining the evolution of inhomogeneities.

[6]We note that any functional of the shear can be taken, equivalently, as the order parameter.

There are two possible extrema:

- $x = 0$
- $3Cx_0^2 + Bx_0 + 2A = 0$

Following the de Gennes-Landau's scheme we set parameters B, C, to be constants and $A = a^2(T - T^*)$. Then, the extremum x_0 occurs only for temperatures in the domain

$$T < T^* + \frac{B^2}{24a^2C} \equiv T^+$$

We note that there is an interval for the temperature in which both states — which, from now on we will represent by I (for the isotropic case, $x = 0$) and U (for the uni-axial anisotropic solution x_0)— coexist.

Exercise. Show that the liquid crystal admits the following four configurations:

- $T > T^+$. The most favourable state is I;
- $T_U < T < T^+$. The most favourable state is I but there is a local minimum corresponding to a small anisotropy;
- $T^* < T < T_U$. The most favourable state is U but there is a local minimum corresponding to an isotropic (less favourable) phase;
- $T < T^*$. Corresponds to the phase U.

in which we have defined

1. $T^{+} \equiv T^{*} + \frac{1}{24}\frac{B^2}{a^2C}$;

2. $T_U \equiv T^{*} + \frac{1}{27}\frac{B^2}{a^2C}$.

Exercise. Show that the variation of the entropy $S = -\frac{\partial F}{\partial T}$, during the transition is given by $\Delta S = \frac{1}{27}\frac{a^2B^2}{c^2}$.

In the above expansion of the free energy (29) a term linear in the order parameter does not exist. The reason is simple: $\sigma_{\mu\nu}$ is trace-free. Besides, this is a necessary condition to allow the existence of the isotropic phase. We should note, however, that the influence of an external field can modify drastically this situation. Indeed, let us consider a typical diamagnetic liquid crystal and look into the net effects of the application of an external magnetic field on it. The free energy gains an additional interacting term, which depends on the magnetic field $H - \mu$ and on the oreder parameter. Once F is a scalar, in the first order we have:

$$\Delta F = nH_\mu H_\nu \sigma^{\mu\nu} \tag{30}$$

which can change drastically the behaviour of the fluid [25].

Exercise. Examine the influence of an external magnetic field in the phenomenon of phase transition when the free energy takes the form

$$F = F_0 + nH_\mu H_\nu \sigma^{\mu\nu} + ATr\sigma^2 + BTr\sigma^3 + C(Tr\sigma^2)^2.$$

The above interacting term ΔF can be written in a more general form, in terms of the energy-momentum tensor $T_{\mu\nu}$ of the electromagnetic field. Indeed, we have in general:

$$T_{\mu\nu} = \rho_{em} V_\mu V_\nu - \frac{1}{3}\rho h_{\mu\nu} + \Pi_{\mu\nu}. \tag{31}$$

in which the anisotropic tensor is given by

$$\Pi_{\mu\nu} = -H_\mu H_\nu + \frac{1}{3}H^2 h_{\mu\nu} \tag{32}$$

and $H^2 \equiv H^\mu H_\mu$. Using the properties of the projector operator and of the shear it follows

$$H_\mu H_\nu \sigma^{\mu\nu} = \Pi_{\mu\nu}\sigma^{\mu\nu} = T_{\mu\nu}\sigma^{\mu\nu}. \tag{33}$$

In these expressions we used the decomposition of the electromagnetic field $F_{\mu\nu}$ in terms of the comoving fluid velocity V^μ as in the previous sections, to write

$$F_{\mu\nu} = -V_\mu E_\nu + V_\nu E_\mu + \eta_{\mu\nu}^{\alpha\beta} V_\alpha H_\beta.$$

Finally, we can write the above interacting term of the influence of the magnetic field in the liquid crystal under the general form:

$$\Delta F = -n T_{\mu\nu}\sigma^{\mu\nu} \tag{34}$$

This form is very suggestive to translate the interaction into a geometrical language. Using Einstein's equations, we have:

$$\begin{aligned} \Delta F &= n(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu})\sigma^{\mu\nu} \\ &= n R_{\mu\nu}\sigma^{\mu\nu} \end{aligned} \tag{35}$$

in which the second equality comes from the fact that the shear is traceless.

4.4 The Cosmic Fluid as a Liquid Crystal: the Isotropization

In the Old Standard Cosmological Model the matter content of the Universe was treated as a fluid in its state of maximal symmetry: an isotropic gas (or a liquid) which creates a correspondingly simple geometry, the conformally flat Friedmann Universe. Here we will analyse the model in which, at least for a certain epoch of its evolution, in its highly condensed stage, the cosmic matter is identified to a more realistic configuration: a viscous fluid in a less symmetrical state, generating an anisotropic Universe. We assume that there is no heat flux and the tensor of pressure is given as in eq.(15):

$$\Pi_{\mu\nu} = [-a^2(\Theta - \Theta^*) + \beta\sigma^2]\sigma_{\mu\nu} + \delta[\sigma_{\mu\alpha}\sigma_\nu{}^\alpha - \frac{1}{3}\sigma_{\alpha\beta}\sigma^{\alpha\beta}h_{\mu\nu}],$$

in which we have redefined the constants by setting, for convenience, $\alpha_0 \equiv a^2\Theta^*$ and $\alpha_1 \equiv -a^2$. The question we are considering is precisely this: to find a way which the Universe could have used to dissipate this anisotropy. A natural process to do this is to apply what we have learned from the previous sections. The shear, thus, will be treated again as the order parameter. The free energy of the self-gravitating cosmic fluid is given by

$$F = F_0 + m^2[-\frac{3}{2}a^2(\Theta - \Theta^*)\Sigma^2 + \frac{3}{4}\delta\Sigma^3 + \frac{9}{4}\beta\Sigma^4]. \qquad (36)$$

We have assumed the existence of planar (uni-axial) anisotropy.[7] The planar anisotropy allows us to set the matrix of shear as being given by

$$\begin{pmatrix} -\frac{1}{2}\Sigma & 0 & 0 \\ 0 & -\frac{1}{2}\Sigma & 0 \\ 0 & 0 & \Sigma \end{pmatrix} \qquad (37)$$

[7] In investigating more general cases of bi-axial fluids one can follow the same procedure. Note, however, that in this case there is a necessity to take into account more terms in the series of F: at least of sixth power of the order parameter (see [25]).

The search for the equilibrium states of the fluid reduces to the examination of the minima of F, with respect to Σ. We note that, as we have already discussed previously, we find here that the distinct equilibrium phases of the fluid, characterized by the order parameter Σ, are controlled by the expansion factor Θ. The system admits two equilibrium points:

- $\Sigma = 0$
- Σ_0

which satisfy the equation:

$$3\beta\Sigma_0^2 + \frac{3}{4}\delta\Sigma_0 - a^2(\Theta - \Theta^*) = 0. \tag{38}$$

There is a non-null solution only if

$$\Theta > \Theta^* - \frac{3}{64}\frac{\delta^2}{a^2\beta} \equiv \Theta_c.$$

We thus see that the possibility of the existence of an anisotropic phase depends on the value of the *Hubble constant* Θ. In the case there is a value $\Sigma_0 \neq 0$, it will be a minimum if it satisfies the inequality

$$9\beta\Sigma_0^2 + \frac{3}{2}\delta\Sigma_0 - a^2(\Theta - \Theta^*) > 0. \tag{39}$$

It the follows that if $\Theta < \Theta^*$ the isotropic solution is not stable. There exists a domain for Θ in which both states coexist. In this case

$$\Theta < \Theta^* - \frac{1}{24}\frac{\delta^2}{a^2\beta} \equiv \Theta_t.$$

We can then summarize all this information by discriminating the relation between the actual value of Θ and the critical values Θ_c and Θ_t.

- $\Theta < \Theta_c$: The most favourable state is I;
- $\Theta_c < \Theta < \Theta_t$: The most favourable state is I but there is a local minimum U corresponding to a small anisotropy;
- $\Theta_t < \Theta < \Theta^*$: The most favourable state is U but there is a local minimum corresponding to an isotropic (less favourable) phase;
- $\Theta^* < \Theta$: Corresponds to the isotropic phase I.

The non-ideal self gravitating cosmic fluid can thus provide an internal mechanism which Nature could have used to attain the isotropic state through the stimulation of a continuous, second order, phase transition. We remark that in our actual Universe the isotropic phase occurs in a latter period for lower temperatures. This is not the common situation in ordinary matter, in which a phase transition to a more disordered state is accompanied with an increase of temperature (cf. above). Nevertheless, this situation is not forbidden by any physical law and, besides, there do exist some materials which evolve in this peculiar direction like, for instance, some two-components organic thermotropic systems [25].

Exercise. Examine the general case in which the pressure is given by

$$\Pi_{\mu\nu} = f_1 h_{\mu\nu} + f_2 \sigma_{\mu\nu} + f_3 \sigma_{\mu\alpha} \sigma_\nu{}^\alpha,$$

in which the scalar coefficients f_i are given by the series

$$f_i = f_{i0} + f_{i1} I_1 + [f_{i2} I_1^2 + f'_{i2} I_2] + [f_{i3} I_1^3 + f'_{i3} I_1 I_2 + f''_{i3} I_3] + \ldots$$

in which I_k are given by the canonical invariants of the matrix $\hat{\theta}$ of components $\theta^{\mu}_{\nu} = \sigma^{\mu}_{\nu} + \frac{1}{3}\Theta h^{\mu}_{\nu}$, that is:

- $I_1 = Tr\hat{\theta} = \Theta$.
- $I_2 = Tr\hat{\theta}^2 = \sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{3}\Theta^2$.
- $I_3 = Tr\hat{\theta}^3 = \sigma_{\mu\nu}\sigma^{\mu\alpha}\sigma_{\alpha}{}^{\nu} + \Theta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{9}\Theta^3$.

It remains to analyse the consequences of the above mechanism to the production of entropy. From the regular functional dependence of the expansion Θ on T, through the geometry, we can evaluate the change of entropy during the phase transition:

$$\Delta S \sim \frac{\partial \Theta}{\partial T}\Sigma_0^2. \tag{40}$$

The term on $\frac{\partial \Theta}{\partial T}$ in the Old Standard Model contributes with a (small) positive power of T. In general it will provide a certain regular function of T, which does not vanish in the neighborhood of Θ_t. We are particularly interested in the other term that contains the dependence on the strength of the anisotropy. It yields the result that the higher the primordial anisotropy, the bigger the entropy production. Thus, as a subsidiary consequence, the above isotropization mechanism based on the standard Landau-deGennes phase transition can supply the Universe with some additional entropy. The question we would like to answer is this: is this entropy production enough to explain the total actual value S_{total} ?

Chapter 28

The Eternal Universe

1 General Comments

In the Old Standard Cosmological model the geometry of the Universe has a singularity. At time $t = 0$ both the curvature and the matter density diverge. It has been a common practice to identify this as the moment of *creation* of the Universe. The creation of a true initial Cauchy surface.

That the presence of this singularity is not a mere consequence of the high degree of symmetries exhibited by the FRW geometry was proved by [2], [3]. The so-called *singularity theorems of Cosmology* state that under *reasonable* conditions any spacetime geometry controlled by Einstein's General Relativity develops a singularity.

In the last decade a large number of proposals appeared in which acceptable cosmological models that violate the conditions of applicability of these *theorems* and in which the Universe does not have a singular origin were constructed. In the Vth Brazilian School of Cosmology and Gravitation [32] I started to elaborate a list of these proposals. Here we continue this list by adding some new items that appeared in the last years. We next enumerate all these proposals[1]:

- Λ Cosmology
- Non Minimal Coupling of Fields to Gravity
- Quadratic Curvature Terms in Lagrangeans for Gravity
- Torsion Effects
- Quantum Matter Properties
- Quantum Gravity
- Viscous Process
- Non Riemannian Geometry: WIST
- Time Dependent Cosmological Constant
- MicroPhysics Process
- New Short Range Gravitational Forces

We will analyse here some of the proposals that were not discussed in the Vth School.

[1]This list is not exhaustive. We will present a more complete list in a forthcoming publication.

2 Varying Λ

The deSitter cosmological model has many similitudes to the Minkowskii spacetime. Among those, we are interested here in its high degree of symmetries. Both geometries have the maximum number of Killing vectors.

Exercise. Evaluate the Killing vectors of the (flat section) deSitter geometry

$$ds^2 = dt^2 - exp(2Ht)(dx^2 + dy^2 + dz^2)$$

This has led some authors to conjecture that deSitter Universe is nothing but Minkowskii spacetime heat at an effective temperature T which, by simple dimensional considerations, is

$$T \sim \sqrt{\Lambda}.$$

Following this idea, the fact that the temperature depends on the global time implies a possible time-dependence of Λ. Actually, this can be presented in a less arbitrary manner if we note that the cosmological constant may be atributted to a certain particular state of a self-interacting field (see 4). Indeed we saw in a precedent section that in the fundamental state of a scalar field an effective cosmological constant is generated by the identification $\Lambda = V(\phi_0)$. This simple expression was the consequence of a spontaneous symmetry breaking in which the surrounding spacetime is taken to be at zero temperature. However, the Universe is not at zero temperature during all its history.

Consequently the equilibrium state of the ϕ field is not at the minimum of the potential V but instead, at the minimum of the free energy F. From a naif application of Thermodynamics (see [10], [21]) the potential $F = V(\phi, T)$ can be expanded in a series in the temperature:

$$V(\phi, T) = V(\phi) + aT^2\phi^2 + bT^4 + \dots . \tag{1}$$

It then follows that the states of the equilibrium of the system become time-dependent. This is a direct consequence of the assumption of the evolution of the Universe; consequently, the temperature becomes time-dependent too.

In this approach the possibility of a time-dependent cosmological *constant* $\Lambda(t)$ is related to its origin in terms of the fundamental state of a given (scalar) field embedded in a non stationary Universe.

We are thus led to make a second modification in the original Einstein's equation by the introduction of a spacetime dependent vacuum energy:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T^{(m)}_{\mu\nu} + \Lambda(x)g_{\mu\nu} \tag{2}$$

in which $T^{(m)}_{\mu\nu}$ stands for the energy-momentum tensor of matter. We should point out the interdependence of Λ with matter: in the absence of the surrounding matter the Ricci identity implies the true constancy of Λ. The vacuum interchanges energy with matter in such a way that only the overall energy-momentum tensor consisting of the substance (matter) plus the energy of the vacuum becomes a conserved quantity.

Let us limit our study here only to the standard spatially homogeneous and isotropic background geometry

$$ds^2 = dt^2 - A^2(t)d\sigma^2 \tag{3}$$

Then the modified Einstein's equations are

$$3\dot{A}^2 - \rho A^2 = -\Lambda A^2 - 3\epsilon. \tag{4}$$

$$2A\ddot{A} + \dot{A}2 = -pA^2 - \Lambda A^2 - \epsilon. \tag{5}$$

The total conservation law (substance plus vacuum) gives

$$\dot{\rho} + \rho\dot{v}_{ac} + (\rho + p)\Theta = 0. \tag{6}$$

Just to see some curious properties of this model let us concentrate in a particular type of the vacuum.

We take the matter to be represented by a perfect fluid with the equation of state $p = \frac{1}{3}\rho$. This seems a reasonable hypothesis once we are considering very high energy effects (remember that we are looking for a non singular Universe). Let us then follow [33] and make the additional hypothesis that the density of matter takes for all times the critical value

$$\rho = \rho_c = 3H^2 \tag{7}$$

in which we set $H \equiv \frac{\dot{A}}{A}$. Then we obtain

$$\Lambda = -\frac{3}{A^2} \tag{8}$$

and

$$\epsilon = 1. \tag{9}$$

Thus, the scale factor takes the form

$$A(t) = \sqrt{t^2 + Q^2}. \tag{10}$$

This particular form for the non singular radius of the Universe yields the same geometry found ten years before by Novello-Salim and, independently, by Melnikov-Orlov ([30], [31]) in another context. The source of the NS solution is the non minimally coupled photons; and the source of MO geometry is a non linear scalar field. We will not describe further these solutions once they were treated in the Vth Brazilian School of Cosmology and Gravitation [32]. Let me only point out here that the fact that all these distinct sources yield the same geometry is not surprising if we analyse more carefully the properties that allow their existence.

Exercise. Show that (10) is a deformation of Milne geometry.

Prove that the source of this geometry can be each one of the following:

- Photons coupled non minimally to Gravity;
- Self-interacting scalar field;
- Time-dependent cosmological constant.

Can you provide another particular kind of source for the same non singular Friedmann-like geometry?

What can we conclude from this overabundance of equivalent geometrical effects generated by distinct material sources? Compare with the old tentatives of the so-called *Already Unified Theory* by Rainich and others in [34], [35].

Exercise. Thermodynamics of the matter-vacuum equilibrium state.

Show that, in order to guarantee that the cosmological and the thermodynamical arrows point in the same direction, we must treat the system matter-vacuum as a single entity.

Hint: Consider the Gibbs law

$$TdS = dE + pdV. \tag{11}$$

If we take into account only the radiation term we have

$$TdS_r = 6dA. \tag{12}$$

Thus, in the collapsing era $TdS_r < 0$. However, taking into account the complete system of matter and vacuum it follows that the system evolves in an adiabatic way.

Exercise. Consider a general varying $\Lambda(x)$. Set

$$\Lambda \sim A^{-n}.$$

Study the thermodynamical properties of the combined system of vacuum plus radiation field.

3 New Short-Range Gravitational Forces

A direct analysis of the properties of the known forces allows us to claim that a theory of Gravity founds a Cosmology. This is the reason which led Einstein to deal with his General Theory of Relativity as a modification of the traditional Newtonian ideas on global properties of the Universe. There have been some proposals in recent years concerning the possible extension of the fundamental forces in Nature in both complementary domains, that is:

- Short-Range Gravitational Forces.
- Long-Range Cosmical Forces.

Although there is not a single evidence that supports these speculations, it seems worth, from a theoretical point of view, to analyse some of these suggestions. We will not consider any proposal of the Long-Range modification of Gravity (see, for instance, Okun [26]) but instead present a rough overview of a model of the existence of new gravitational-like short-range forces (see [27]).

4 The Principle of Unification

On the basis of the model which we will examine here we find the Principle that states that the Unification Program of Physics led to the idea that all long range force has an effective local counterpart of finite range. The reason for this suggestion is the relationship between the Electromagnetic and the Weak forces. Indeed, after the Unified Program of Weinberg and Salam, one can treat weak (Fermi) forces as a local counterpart of long-range electromagnetic processes. This has led to the idea that Gravity should also have an associated local (short-range) interaction. Let us see how this idea could be implemented.

5 The Interaction of Leptons

Leptons interact with photons, the intermediate bosons of the Weak interactions and with Gravity. The electro-weak processes are described by the Salam-Weinberg standard $SU(2)xU(1)$ gauge theory[2]. In this model the description of the interactions of electrons and neutrinos are made by means of an isodoublet L and a singlet R:

$$L = \frac{1}{2}(1-\gamma_5)\begin{bmatrix} \Psi_\nu \\ \Psi_e \end{bmatrix} \tag{13}$$

$$R = \frac{1}{2}(1+\gamma_5)\Psi_e \tag{14}$$

where γ_5 represents the usual Dirac's matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

The gauge group $SU(2)xU(1)$ being a local symmetry group it follows that the usual derivative ∂_μ has to be replaced by the *covariant* derivatives:

$$D_\mu L = \left(\partial_\mu - i\, g\, \frac{\vec{\tau}}{2}.\vec{W}^\mu - \frac{i}{2}\, g'\, B_\mu\right)\, L \tag{15}$$

and $D_\mu R = \left(\partial_\mu - (i/2)\, g'\, B_\mu\right)\, R$ where $\vec{W}_\mu$ and B_μ are the gauge fields (connection of the symmetry group).

The dynamics of these fields is given by the Standard Electroweak theory through the $\mathcal{L}_{GSW}$ Lagrangian:

[2]We note that the Gravitational interaction is not contained in this description.

$$\mathcal{L}_{GSW} = i\,[\overline{L}\,\gamma_\mu\,D_\nu\,L\,\gamma^{\mu\nu} + \overline{R}\,\gamma_\mu\,D_\nu\,R\,\gamma^{\mu\nu}] - \frac{1}{4}\,B_{\mu\nu}\,B^{\mu\nu} - \frac{1}{4}\,\vec{W}_{\mu\nu}.\,\vec{W}^{\mu\nu} \tag{16}$$

where:

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ W_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + g\,\vec{W}_\mu \times \vec{W}_\nu \end{aligned} \tag{17}$$

and $\gamma^{\mu\nu}$ represents, as usual, the flat Minkowski metric written in an arbitrary coordinate system.

We note that all vector fields are taken to be massless. The intermediate vector bosons of the pure Fermi interacting sector are however massive. The most economical way, which does not spoil certain important properties of the theory, like renormalizability, to give mass to vector bosons is by introducing a Higgs doublet

$$\phi = \begin{bmatrix} \phi^\dagger \\ \phi^0 \end{bmatrix} \tag{18}$$

of the complex scalar field, one electrically charged and the other being neutral.

The total Lagrangian will then include a term which couples the gauge bosons (GB) to the scalar field ϕ:

$$\mathcal{L}_{GB\phi} = |D_\mu \phi|^2 - V(\phi^\dagger\,\phi) \tag{19}$$

where D_μ is the covariant derivative operator defined in (15) and $V(\phi^\dagger\,\phi)$ the Higgs potential:

$$V(\phi^\dagger \phi) = \mu^2(\phi^\dagger\,\phi) + h(\phi^\dagger\,\phi)^2, \qquad h > 0,\ \mu^2 < 0 \tag{20}$$

We just have now to add to the above Lagrangian $\mathcal{L}_{GSW}$ and $\mathcal{L}_{GB\phi}$ the interaction of the Higgs field with leptons through:

$$\mathcal{L}_{F-S} = -G_E\,(\overline{L}\,\varphi\,R + \overline{R}\,\varphi^\dagger\,L) \tag{21}$$

and the total Lagrangian of the Electroweak field in interaction with Higgs scalar fields becomes:

$$\mathcal{L} = \mathcal{L}_{GSW} + \mathcal{L}_{GB\phi} + \mathcal{L}_{F-S} \tag{22}$$

The conventional Spontaneous Symmetry Breaking mechanism uses the minimum value of the Higgs scalar field:

$$<\phi> = \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

with

$$\lambda = \frac{v}{\sqrt{2}}$$

and

$$v^2 = -\frac{\mu^2}{h}$$

and by replacing $\phi^\dagger$ by zero and ϕ^0 by $\frac{1}{\sqrt{2}}\,(v + \chi(x))$, some mediating bosons become massive while the photon is inposed to remain massless.

In general it is argued that, since the coupling of gravity with matter is so weak, one can work in a flat spacetime background neglecting completely the action of gravity. However, from the fundamental theoretical point of view, the strictly correct theory should include gravity. The simplest way to achieve such a purpose is by just using the Equivalence Principle[3].

It leads to a non-equivocal way for such a coupling, which is usually taken to be given by an interaction Lagrangian of electrons and neutrinos with gravity in the form:

$$\begin{aligned} \mathcal{L}_{grav.EW} &= \sqrt{k_E}\,[T_{\mu\nu}(e) + T_{\mu\nu}(\nu_e)]\,\phi^{\mu\nu} \\ &= \sqrt{k_E}\,[e\,\gamma_\mu\,\partial_\nu e + \overline{\nu}\,\gamma_\mu\,\partial_\nu \nu]\,\phi^{\mu\nu} \\ &= \sqrt{k_E}\,[\overline{L}\,\gamma_\mu\,D_\nu\,L + \overline{R}\,\gamma_\mu\,D_\nu\,R]\,\phi^{\mu\nu} \end{aligned} \tag{24}$$

Let us emphasize that this is not an approximation procedure but it gives the exact correct interaction scheme. Indeed one can reobtain Einstein 's standard geometrical vision by employing identification of the metric in terms of the $\phi^{\mu\nu}$ field (see Deser, GPP, [28])

$$\sqrt{-g}\,g^{\mu\nu} = \sqrt{-\gamma}\,(\gamma^{\mu\nu} + \sqrt{k_E}\,\phi^{\mu\nu})$$

and using the action

$$\begin{aligned} S = -\frac{1}{2\kappa}\int \sqrt{-\gamma}\,[h^{\mu\nu}{}_{;\alpha}\,(-K_{\mu\nu}{}^{\alpha} + \frac{1}{2}\,\delta_\mu{}^{\alpha}\,K_\nu + \frac{1}{2}\,\delta_\nu{}^{\alpha}\,K_\mu) \\ + (h^{\mu\nu} + \gamma^{\mu\nu})\,(KK)_{\mu\nu}] \end{aligned} \tag{25}$$

[3]We have also examined a possible violation of the Equivalence Principle by considering that the coupling constants of electron and neutrino with gravity were not the same, thus setting:

$$\mathcal{L}_{int} = \sqrt{k_E}\,T_{\mu\nu}(e) + \xi\,T_{\mu\nu}(\nu_e)\,\phi^{\mu\nu}$$

After some algebraic manipulation it came out that:

$$T_{\alpha\beta}(e) + \xi\,T_{\alpha\beta}(\nu_e) = \frac{(\xi+1)}{2}\,\overline{L}\,\gamma_{(\alpha}\,D_{\beta)}\,L + \overline{R}\,\gamma_{(\alpha}\,D_{\beta)}\,R + \frac{(\xi-1)}{2}\,\overline{L}\,\gamma_{(\alpha}\,D_{\beta)}\,\tau_3\,L + h.c.$$

The important fact was a τ_3 operator ocurrence. To complete the algebra we had to introduce the charged tensor currents $\overline{L}\,\gamma_\mu\,D_\nu\,\tau^{\pm}\,L$. Due to charge conservation these new currents did not couple directly to gravity but to new spin-2 charged bosons.

where

$$(KK)_{\mu\nu} \equiv K_{\mu\nu}{}^{\alpha} K_{\alpha} - K_{\mu\alpha}{}^{\beta} K_{\nu\beta}{}^{\alpha}$$

However it is evident that this procedure does not take into account the possible existence of a short range counterpart of gravity.

In order to preserve the $SU(2)_L \times U(1)$ gauge symmetry even in presence of tensorial coupling, let us introduce a triplet of tensors $\varphi^{(i)}_{\mu\nu}$ and a singlet $\psi_{\mu\nu}$. Now leptons L and R couple to the *intermediate tensor bosons* $\varphi^{(i)}_{\mu\nu}$ and $\psi_{\mu\nu}$. These tensor bosons are true vectors in the $SU(2)$ algebra and not connections. In this sense they are not gauge fields, so that we can note that gravity in our approach is not treated as a gauge field. By the way, there is no need to treat garvity as a gauge field.

The Lagrangian (24) thus becomes:

$$\mathcal{L}_{grav.EW} = \sqrt{k_a}\, \left(\overline{R}\, \gamma_\mu\, D_\nu\, R + \frac{2^{1/4}}{\sqrt{G_W}}\, \overline{L}\, \varphi\, \gamma_\mu\, D_\nu\, S\right)\, \psi^{\mu\nu} + \sqrt{k_b}\, \overline{L}\, \gamma_\mu\, D_\nu\, \overline{\tau}\, L.\overline{\varphi}^{\mu\nu} + h.c. \tag{26}$$

in which k_a and k_b will be related to the k_E Einstein's constant later on (cf. equation (30)).

The ocurrence of Fermi's constant is the residual consequence of two parts: the unification of Electroweak process and the SSB mechanism of the Higgs bosons. Besides the right-handed electron singlet we have introduced for convenience its left-hand part $S = \frac{(1-\gamma_5)}{2}\, e$. It does not change anything in the EW interaction as we can be easily convinced just by checking the following properties:

$$\begin{array}{lll} \overline{L}\, \varphi\, S = 0 & and & \overline{R}\, \gamma_\mu\, S = 0 \\ \overline{L}\, \varphi\, R = \lambda\, \overline{e}\, \frac{(1+\gamma_5)}{2}\, e & and & \overline{R}\, \varphi^\dagger\, L = \lambda\, \overline{e}\, \frac{(1-\gamma_5)}{2}\, e \end{array} \tag{27}$$

in fundamental state of the Higgs field in which:

$$\varphi = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{2^{-1/4}}{\sqrt{2G_W}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let us emphasize the fact that in order to make a contact of this Lagrangian (26) with the above one (24) and reproduce the universality coupling of electrons and neutrinos through the same constant coupling k_E, we have to choose the gravitational field to become a certain combination of $\psi_{\mu\nu}$ and $\varphi^{(3)}_{\mu\nu}$ in an analogous way as it occurs for the long-range (massless) electromagnetic field. We shall see that this procedure will be compatible with the fact that only one of these tensor fields will remain massless, which is of course the one to be identified with gravity. Let us define the linear combination of $\varphi^{(3)}_{\mu\nu}$ and $\psi_{\mu\nu}$ given by a rotation linked to an η mixing angle:

$$\begin{bmatrix} \varphi^{\mu\nu(3)} \\ \psi^{\mu\nu} \end{bmatrix} = \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} Z^{\mu\nu} \\ \phi^{\mu\nu} \end{bmatrix} \tag{28}$$

We thus obtain for the gravity field $\phi^{\mu\nu}$ the expression:

$$\phi^{\mu\nu} = -sin\eta\, \varphi^{\mu\nu(3)} + \cos\eta\, \psi^{\mu\nu} \tag{29}$$

The same rotation applied to the Lagrangian (24) leads to the following relation:

$$\sqrt{k_a} = \frac{\sqrt{k_E}}{\cos\eta} \qquad and \qquad \sqrt{k_b} = -\frac{\sqrt{k_E}}{\sin\eta} \tag{30}$$

we can the extract the k_E Einstein's coefficient in terms of the coupling constants k_a and k_b:

$$\frac{1}{k_E} = \frac{1}{k_a} + \frac{1}{k_b} \tag{31}$$

All these tensor fields, in analogy with vector fields are created massless [4] Reasons for this are correspondingly:

1. a term like $m^2\,\vec{\varphi}_{\mu\nu}.\vec{\varphi}^{\mu\nu}$ can be constructed with the bosons since $\vec{\varphi}_{\mu\nu}$ is not a gauge field (it is not an affinity but a true $SU(2)$ vector). Such a term is a true scalar not only with respect to the coordinate indices but also with respect to the $SU(2)$ indices.

2. because of the well-known fact of the non-renormalizability of massless spin-2 interactions.

and one can use the same Higgs SSB mechanism to provide mass for three of these bosons leaving one massless (precisely the $\phi_{\mu\nu}$ defined by (29)) which is to be identified to the long-range gravity.

Using these generalized gauge-invariant procedure plus Deser-GPP action (25) for the general case in which there are four spin-two fields one obtains the necessary equation of motion for the complete set of fields.

[4]It should be noted that in the case of EW force one is strongly compelled to start with massless spin-1 gauge fields. There are two reasons for this. The first one is that we could not consider a term like$m^2\,\vec{A}_\mu.\vec{A}^\mu$ in the theory since this term is not gauge-invariant, and secondly because of the well-known difficulties related to the non-renormalizability of massive spin-1 fields interacting with matter. These difficulties however do not have the same consequence of the spin-2 mediating tensor bosons.

Chapter 29

Traditional Cosmological Models

In the early days of Relativistic Cosmology some simple solutions of Einstein's equations were found. Although the interest on these traditional geometries has diminished they still constitute a basic reference. Here we will consider just a few ones, namely:

- Einstein's model: a static homogeneous geometry;
- Friedmann's model: a dynamic spatially homogeneous and isotropic geometry;
- Empty Kasner's model: a dynamic spatially homogeneous and anisotropic geometry;
- Gödel's model: a rotating universe.

We can characterize these models by the kinematical properties of a given set of fundamental free-falling matter (geodesics of the background geometry) and by choosing a coordinate system in which their four-vector velocity take the simple form $V^\mu = \delta^\mu_0$. Thus,

- Einstein Universe: it has no expansion ($\Theta = 0$), no shear ($\sigma_{\mu\nu} = 0$), no vorticity ($\omega_\mu = 0$).
- Friedmann Universe: it has no shear ($\sigma_{\mu\nu} = 0$), no vorticity ($\omega_\mu = 0$) but has a non null expansion ($\Theta \neq 0$).
- Kasner Universe: empty configuration with expansion and shear; it has no vorticity.
- Gödel Universe: it has no expansion ($\Theta = 0$), no shear ($\sigma_{\mu\nu} = 0$), but has a non null vorticity ($\omega_\mu \neq 0$).

1 Einstein's Universe

The framework: Einstein's General Relativity with cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -T_{\mu\nu} \tag{1}$$

The source is a perfect fluid with no pressure

$$T_{\mu\nu} = \rho\delta^0_\mu\delta^0_\nu \tag{2}$$

The geometry is static. It can be described as the topological product $R \otimes S_3$. We set

$$ds^2 = dt^2 - \frac{1}{\Lambda}[d\xi^2 + (sin\xi)^2 d\Omega^2)]. \tag{3}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$.

Exercise. Show that $\rho = 2\Lambda$.

Exercise. Show that the Einstein Universe can be written in the conformal form

$$ds^2 = \frac{\alpha^2}{[1+(\eta+r)^2][1+(\eta-r)^2]}[d\eta^2 - dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2)] \tag{4}$$

in which

$$\eta = \frac{1}{2}(V+W)$$

$$r = \frac{1}{2}(V-W)$$

and

$$t = \frac{\alpha}{2}(\arctan V + \arctan W)$$

$$\xi = \frac{\alpha}{2}(\arctan V - \arctan W)$$

where $\alpha^2 = \frac{1}{\Lambda}$.

2 Friedmann's Universe

In the Gaussian coordinate system $(t, \xi, \theta, \varphi)$ the Friedmann geometry takes the form

$$ds^2 = dt^2 - A^2(t)[d\xi^2 + \sigma^2(\xi)d\Omega^2)]. \tag{5}$$

in which the 3-section curvature is fixed by the form of the σ-function. We distinguish three cases

- Closed section: $\sigma = \sin\xi$.
- Open section: $\sigma = \sinh\xi$.
- Euclidean section: $\sigma = \xi$.

corresponding to the parameter ϵ equal, respectively to $+1, -1$ and 0 (see above in the text). Einstein's equations reduce to the set

$$\epsilon = \frac{1}{3}A^2(\rho - \rho_c), \tag{6}$$

and

$$2\frac{\ddot{A}}{A} + (\frac{\dot{A}}{A})^2 + \frac{\epsilon}{A^2} = -p - \Lambda, \tag{7}$$

where we have defined

$$\rho_c \equiv 3(\frac{\dot{A}}{A})^2.$$

Note that this geometry is conformally flat, that is, Weyl conformal tensor (the traceless part of the Riemann curvature) vanishes.

3 Lemaitre-De Sitter Universe

The source is a perfect fluid endowed with the equation of state $p + \rho = 0$. This is interpreted as the cosmic vacuum or the cosmological constant Λ.

The fundamental lenght has the same form as in the previous case (5).

Show that the expansion factor Θ is constant.

4 Kasner's Universe

See the previous chapter.

5 Gödel's Universe

Exercise.

Show that a perfect fluid without pressure yields the following solution of Einstein's equations

$$ds^2 = a^2[dt^2 - dr^2 - dz^2 + 2h(r)dtd\varphi + g(r)d\varphi^2] \qquad (8)$$

in which the functions g and h are

$$g(r) = \sinh^2 r(\sinh^2 r - 1).$$

and

$$h(r) = \sqrt{2}\sinh^2 r.$$

Show that the congruence of the fluid that generates this geometry has no expansion ($\Theta = 0$), no shear ($\sigma_{\mu\nu} = 0$) but has a non null vorticity. The vorticity vector has components $\omega_\alpha = (0, 0, 0, \Omega)$ such that

$$\rho = 2\Omega^2 = \frac{4}{a^2} = -2\Lambda.$$

Exercise.

Find a local Gaussian system for this geometry. Prove that any such system cannot cover the entire manifold. This yields the so called causal cosmological problem of Gödel's model.

Answer.

$$ds^2 = d\tilde{t}^2 - a^2(\mu^2 - 1)d\tilde{\xi}^2 + a^2 g(\tilde{t}, \tilde{\xi})d\tilde{\eta}^2 + 2a^2 h(\tilde{t}, \tilde{\xi})d\tilde{\xi}d\tilde{\eta} - a^2 d\tilde{z}^2 \tag{9}$$

in which the functions g and H are the same as above, with the substitution of the radial coordinate r in terms of the new ones.

Hint: evaluate the time-like geodesics of Gödel's geometry which are not orthogonal to a global space-like hypersurface.

The complete analysis of this case can be found in [29].

The Gaussian system of coordinates $(\tilde{t}, \tilde{\eta}, \tilde{\xi}, \tilde{z})$ is related to the older variables by the expressions

$$\tilde{t} = \mu a t + \frac{1}{2} a \sqrt{\mu^2 + 1} \arcsin \Psi + \frac{1}{\sqrt{2}} \mu a \arcsin \Delta.$$

$$\tilde{\xi} = t + \frac{\mu}{2\sqrt{\mu^2 + 1}} \arcsin \Psi + \frac{1}{\sqrt{2}} \arcsin \Delta.$$

$$\tilde{\eta} = \varphi - \frac{\pi}{4} + \frac{1}{2} \arcsin \Delta.$$

$$\tilde{z} = z.$$

in which

$$\Psi = 1 - 2\frac{\mu^2 + 1}{\mu^2 - 1} \sinh^2 r$$

$$\Delta = \frac{3\mu^2 + 1}{\mu^2 - 1} \frac{\sinh^2 r}{\sinh^2 r + 1} - \frac{1}{\sinh^2 r + 1}.$$

Exercise. Consider a trajectory which has the following velocity four-vector: $(0, 0, \frac{1}{a \sinh r \sqrt{\sinh^2 r - 1}}, 0)$. Show that this trajectory is a closed one in the four-dimensional space-time.

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