Part II

QUANTUM FIELD THEORY IN CURVED SPACETIME

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Abstract

These lectures will deal with selected aspects of quantum field theory in curved spacetime. The basic outline of this series of lectures will be as follows:

- Lecture 1. Quantization of fields on a curved background, particle creation by gravitational fields, particle creation in an expanding universe.
- Lecture 2. The Hawking effect - particle creation by black holes.
- Lecture 3. Negative energy in quantum field theory, its gravitational effects, and inequalities which limit fluxes of negative energy.
- Lecture 4. Ultraviolet and infrared divergences, renormalization of the expectation value of the stress tensor.
- Lecture 5. The semiclassical theory of gravity and its limitations, breakdown of this theory due to metric fluctuations, metric fluctuations as an ultraviolet regulator.

Chapter 6

Basic Formalism Particle Creation

1 Second Quantization in Curved Space

There are four basic ingredients in the construction of a quantum field theory. These are

- The Lagrangian, or equivalently, the equation of motion of the classical theory.
- A quantization procedure, such as canonical quantization or the path integral approach.
- The characterization of the quantum states.
- The physical interpretation of the states and of the observables.

In flat spacetime, Lorentz invariance plays an important role in each of these steps. For example, it is a guide which generally allows us to identify a unique vacuum state for the theory. However, in curved spacetime, we do not have Lorentz symmetry. This is not a crucial problem in the first two steps listed above. The formulation of a classical field theory and its formal quantization may be carried through in an arbitrary spacetime. The real differences between flat space and curved space arise in the latter two steps. In general, there does not exist a unique vacuum state in a curved spacetime. As a result, the concept of particles becomes ambiguous, and the problem of the physical interpretation becomes much more difficult.

The best way to discuss these issues in more detail is in the context of a particular model theory. Let us consider a real, massive scalar field for which the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha\varphi\partial^\alpha\varphi - m^2\varphi^2 - \xi R\varphi^2). \tag{1.1}$$

(We adopt the sign conventions of Birrell and Davies [1], which are the $(- - -)$ conventions in the notation of Misner, Thorne, and Wheeler [2]. In particular, the metric signature will be $(+ - --)$. Unless otherwise noted, units in which $G = c = \hbar = 1$ are used.) The corresponding wave equation is

$$\Box\varphi + m^2\varphi + \xi R\varphi = 0. \tag{1.2}$$

Here R is the scalar curvature, and ξ is a new coupling constant. There are two popular choices for ξ: minimal coupling ($\xi = 0$) and conformal coupling ($\xi = \frac{1}{6}$). The former leads to the simplest equation of motion, whereas the latter leads to a theory which is conformally invariant in four dimensions in the massless limit. For our purposes, we need not settle this issue, but rather regard ξ on the same footing as m, as a parameter which specifies our theory. Note that here $\Box$ denotes the generally covariant d'Alembertian operator, $\Box = \nabla_\mu \nabla^\mu$.

A useful concept is that of the *inner product* of a pair of solutions of the generally covariant Klein-Gordon equation, Eq.(1.2). It is defined by

$$(f_1, f_2) = i \int (f_2^* \overleftrightarrow{\partial}_\mu f_1) d\Sigma^\mu, \tag{1.3}$$

where $d\Sigma^\mu = d\Sigma\, n^\mu$, with $d\Sigma$ being the volume element in a given spacelike hypersurface, and n^μ being the timelike unit vector normal to this hypersurface. The crucial property of the inner product is that it is independent of the choice of hypersurface. That is, if Σ_1 and Σ_2 are two different, non-intersecting hypersurfaces, then

$$(f_1, f_2)_{\Sigma_1} = (f_1, f_2)_{\Sigma_2}. \tag{1.4}$$

The proof of this property is straightforward. We assume that f_1 and f_2 are both solutions of Eq. (1.2). Furthermore, if the space is such that the hypersurfaces are non-compact, we assume that these functions vanish at spatial infinity. Let V be the four-volume bounded by Σ_1 and Σ_2, and, if necessary, time-like boundaries on which $f_1 = f_2 = 0$. Then we may write

$$(f_1, f_2)_{\Sigma_2} - (f_1, f_2)_{\Sigma_1} = i \oint_{\partial V} (f_2^* \overleftrightarrow{\partial_\mu} f_1) d\Sigma^\mu = i \int_V \nabla_\mu (f_2^* \overleftrightarrow{\partial_\mu} f_1) dV, \tag{1.5}$$

where the last step follows from the four dimensional version of Gauss' law, and dV is the four dimensional volume element. However, we may write this integrand as

$$\begin{aligned}\nabla_\mu (f_2^* \overleftrightarrow{\partial_\mu} f_1) &= \nabla_\mu (f_2^* \partial_\mu f_1 - f_1 \partial_\mu f_2^*) = f_2^* \Box f_1 - f_1 \Box f_2^* \\ &= -f_2^* (m^2 + \xi R) f_1 + f_1 (m^2 + \xi R) f_2^* = 0. \end{aligned} \tag{1.6}$$

Thus Eq. (1.4) is proven.

The quantization of a scalar field in a curved spacetime may be carried out by canonical methods. Choose a foliation of the spacetime into spacelike hypersurfaces. Let Σ be a particular hypersurface with unit normal vector n^μ labelled by a constant value of the time coordinate t. The derivative of φ in the normal direction is $\dot\varphi = n^\mu \partial_\mu \varphi$, and the canonical momentum is defined by

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot\varphi} \quad . \tag{1.7}$$

We impose the canonical commutation relation

$$[\varphi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta(\mathbf{x}, \mathbf{x}'), \tag{1.8}$$

where $\delta(\mathbf{x}, \mathbf{x}')$ is a delta function in the hypersurface with the property that

$$\int \delta(\mathbf{x}, \mathbf{x}') d\Sigma = 1 . \tag{1.9}$$

Let $\{f_j\}$ be a complete set of positive norm solutions of Eq. (1.2). Then $\{f_j^*\}$ will be a complete set of negative norm solutions, and $\{f_j, f_j^*\}$ form a complete set of solutions of the wave equation in terms of which we may expand an arbitrary solution. Write the field operator φ as a sum of annihilation and creation operators:

$$\varphi = \sum_j (a_j f_j + a_j^\dagger f_j^*), \tag{1.10}$$

where $[a_j, a_{j'}^\dagger] = \delta_{j,j'}$. This expansion defines a vacuum state $|0\rangle$ such that $a_j|0\rangle = 0$. In flat spacetime, we take our positive norm solutions to be positive frequency solutions, $f_j \propto e^{-i\omega t}$. Regardless of the Lorentz frame in which t is the time coordinate, this procedure defines the same, unique Minkowski vacuum state.

In curved spacetime, the situation is quite different. There is, in general, no unique choice of the $\{f_j\}$, and hence no unique notion of the vacuum state. This means that we cannot identify what constitutes a state without particle content, and the notion of "particle" becomes ambiguous. One possible resolution of this difficulty is to choose some quantities other than particle content to label quantum states. Possible choices might include local expectation values, such as $\langle\varphi\rangle$, $\langle\varphi^2\rangle$, etc. In the particular case of an asymptotically flat spacetime, we might use the particle content in an asymptotic region. Even this characterization is not unique. However, this non-uniqueness is an essential feature of the theory with physical consequences, namely the phenomenon of particle creation, which we will now discuss.

2 Particle Creation by Gravitational Fields

Let us consider a spacetime which is asymptotically flat in the past and in the future, but which is non-flat in the intermediate region. Let $\{f_j\}$ be positive frequency solutions in the past (the "in-region"), and let $\{F_j\}$ be positive frequency solutions in the future (the "out-region"). We may choose these sets of solutions to be orthonormal, so that

$$\begin{aligned} (f_j, f_{j'}) &= (F_j, F_{j'}) = \delta_{jj'} \\ (f_j^*, f_{j'}^*) &= (F_j^*, F_{j'}^*) = -\delta_{jj'} \\ (f_j, f_{j'}^*) &= (F_j, F_{j'}^*) = 0. \end{aligned} \tag{2.11}$$

Although these functions are defined by their asymptotic properties in different regions, they are solutions of the wave equation everywhere in the spacetime. We may expand the in-modes in terms of the out-modes:

$$f_j = \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*). \tag{2.12}$$

Inserting this expansion into the orthogonality relations, Eq. (2.11), leads to the conditions

$$\sum_k (\alpha_{jk}\alpha^*_{j'k} - \beta_{jk}\beta^*_{j'k}) = \delta_{jj'}, \tag{2.13}$$

and

$$\sum_k (\alpha_{jk}\alpha_{j'k} - \beta_{jk}\beta_{j'k}) = 0. \tag{2.14}$$

The inverse expansion is

$$F_k = \sum_j (\alpha^*_{jk} f_j - \beta_{jk} f_j^*). \tag{2.15}$$

The field operator, φ, may be expanded in terms of either the $\{f_j\}$ or the $\{F_j\}$:

$$\varphi = \sum_j (a_j f_j + a_j^\dagger f_j^*) = \sum_j (b_j F_j + b_j^\dagger F_j^*). \tag{2.16}$$

The a_j and $a_j^\dagger$ are annihilation and creation operators, respectively, in the in-region, whereas the b_j and $b_j^\dagger$ are the corresponding operators for the out-region. The in-vacuum state is defined by $a_j|0\rangle_{in} = 0$, $\forall j$, and describes the situation when no particles are present initially. The out-vacuum state is defined by $b_j|0\rangle_{out} = 0$, $\forall j$, and describes the situation when no particles are present at late times. Noting that $a_j = (\varphi, f_j)$ and $b_j = (\varphi, F_j)$, we may expand the two sets of creation and annihilation operator in terms of one another as

$$a_j = \sum_k (\alpha^*_{jk} b_k - \beta^*_{jk} b_k^\dagger), \tag{2.17}$$

or

$$b_k = \sum_j (\alpha_{jk} a_j + \beta^*_{jk} a^\dagger_j). \tag{2.18}$$

This is a Bogolubov transformation, and the α_{jk} and β_{jk} are called the Bogolubov coefficients.

Now we are ready to describe the physical phenomenon of particle creation by a time-dependent gravitational field. Let us assume that no particle were present before the gravitational field is turned on. If the Heisenberg picture is adopted to describe the quantum dynamics, then $|0\rangle_{in}$ is the state of the system for all time. However, the physical number operator which counts particles in the out-region is $N_k = b^\dagger_k b_k$. Thus the mean number of particles created into mode k is

$$\langle N_k \rangle = {}_{in}\langle 0|b^\dagger_k b_k|0\rangle_{in} = \sum_j |\beta_{jk}|^2. \tag{2.19}$$

If any of the β_{jk} coefficients are non-zero, i.e. if any mixing of positive and negative frequency solutions occurs, then particles are created by the gravitational field.

The most straightforward application of the concepts developed above is to particle creation by an expanding universe. This phenomenon was first hinted at in the work of Schrödinger [3], but was first carefully investigated by Parker [4]. Let us restrict our attention to the case of a spatially flat Robertson-Walker universe, for which the metric may be written as

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\eta)\,(d\eta^2 - d\mathbf{x}^2), \tag{2.20}$$

where a is the scale factor. We may use either the comoving time t or the conformal time η, but the solutions of the wave equation are simpler in terms of the latter. The positive norm solutions of Eq. (1.2) in this metric may be taken to be

$$f_\mathbf{k}(\mathbf{x},\eta) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{a(\eta)\sqrt{(2\pi)^3}}\,\chi_k(\eta), \tag{2.21}$$

where $\chi_k(\eta)$ satisfies

$$\frac{d^2\chi_k}{d\eta^2} + [k^2 - V(\eta)]\chi_k = 0\,, \tag{2.22}$$

with

$$V(\eta) \equiv -a^2(\eta)\Big[m^2 + \Big(\xi - \frac{1}{6}\Big)R(\eta)\Big]. \tag{2.23}$$

The norm of $f_{\mathbf{k}}$ being equal to one is equivalent to the Wronskian condition

$$\chi_k \frac{d\chi_k^*}{d\eta} - \chi_k^* \frac{d\chi_k}{d\eta} = i. \tag{2.24}$$

Let us consider the idealized situation in which the universe is static both in the past and in the future. In this case, we have the necessary asymptotically flat regions needed to define in and out vacua. Let us make the further simplification that the field is massless, $m = 0$. We have chosen modes which are pure positive frequency in the past, the in-modes:

$$\chi_k(\eta) \sim \chi_k^{(in)}(\eta) = \frac{e^{-i\omega\eta}}{\sqrt{2\omega}}, \qquad \eta \to -\infty. \tag{2.25}$$

Their form in the future is

$$\chi_k(\eta) \sim \chi_k^{(out)}(\eta) = \frac{1}{\sqrt{2\omega}}(\alpha_k e^{-i\omega\eta} + \beta_k e^{i\omega\eta}), \qquad \eta \to \infty, \tag{2.26}$$

where the coefficients α_k and β_k are determined by solving Eq. (2.22) for a given $a(\eta)$. They are related to the Bogolubov coefficients by $\alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k \delta_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'} = \beta_k \delta_{\mathbf{k},-\mathbf{k}'}$. Thus the number density of created particles per unit proper volume is given at late times by

$$N = \frac{1}{(2\pi a)^3} \int d^3k \, |\beta_k|^2, \tag{2.27}$$

and their energy density by

$$\rho = \frac{1}{(2\pi a)^3 a} \int d^3k \, \omega |\beta_k|^2. \tag{2.28}$$

These formulas are to be understood to hold in the asymptotic region where the particle creation has effectively stopped, and a is the scale factor in that region. Thus the number of particles per unit proper volume is proportional to a^{-3}, and energy per particle is proportional to a^{-1}. Note that we are here discussing massless particles whose wavelength is sufficiently short that they redshift as would conformally invariant massless particles.

Unfortunately, it is difficult to solve Eq. (2.22) for the mode functions in all but the simplest examples. However, there is a perturbative method, developed by Zeldovich and Starobinsky [5] and by Birrell and Davies [6], which is often useful. The first step is to rewrite the differential equation for χ_k as an integral equation:

$$\chi_k(\eta) = \chi_k^{(in)}(\eta) + \omega^{-1} \int_{-\infty}^{\eta} V(\eta')\, sin\omega(\eta - \eta')\, \chi_k(\eta') d\eta'. \tag{2.29}$$

This integral equation is equivalent to Eq. (2.22) plus the boundary condition Eq. (2.25). We now wish to assume that V is sufficiently small that we may iterate this integral equation to lowest order by replacing $\chi_k(\eta')$ by $\chi_k^{(in)}(\eta')$ in the integrand. If we compare the resulting formula with Eq. (2.26), the Bogolubov coefficients may be read off:

$$\alpha_k \approx 1 + \frac{i}{2\omega} \int_{-\infty}^{\infty} V(\eta)\, d\eta, \tag{2.30}$$

and

$$\beta_k \approx -\frac{i}{2\omega} \int_{-\infty}^{\infty} e^{-2i\omega\eta} V(\eta)\, d\eta. \tag{2.31}$$

Let us restrict our attention to the case where $m = 0$. In this case, the mean number density becomes

$$N = \frac{(\xi - \frac{1}{6})^2}{16\pi a^3} \int_{-\infty}^{\infty} a^4(\eta)\, R^2(\eta)\, d\eta, \tag{2.32}$$

and the energy density becomes

$$\rho = -\frac{(\xi - \frac{1}{6})^2}{32\pi^2 a^4} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \;\; \Big\{ ln(|\eta_1 - \eta_2|\mu) \frac{d}{d\eta_1}\Big[a^2(\eta_1)\, R(\eta_1)\Big] \\ \times \frac{d}{d\eta_1}\Big[a^2(\eta_1)\, R(\eta_1)\Big] \Big\}. \tag{2.33}$$

Here μ is an arbitrary quantity with the dimensions of mass; ρ is independent of μ provided that $a^2(\eta)\,R(\eta) \to \infty$ as $\eta \to \pm\infty$. The approximation which is being used here amounts to perturbation around the conformally invariant theory in powers of $(\xi - \frac{1}{6})$.

As an application of these formulas, let us consider particle creation at the end of an inflationary expansion. A typical inflationary scenario involves the universe making a transition from deSitter space to a radiation dominated Robertson-Walker universe on a relatively short time scale. It is usually assumed that there is a mechanism for creating matter via particle interactions. However, there will also be at least some matter generated by gravitational particle creation. We may use the above results to make some order of magnitude estimates for massless, non-conformal scalar particles [7]. Let Δt be the duration of the transition in co-moving time. The scalar curvature in the deSitter phase is given by $R = 12H^2$, where H^{-1} is the e-folding time of the inflationary expansion. The scalar curvature drops to zero in the radiation dominated phase. If we assume that the transition occurs rapidly so that $\Delta t \ll H^{-1}$, then we have approximately that

$$N \approx \frac{(\xi - \frac{1}{6})^2}{12\pi a^3} H^3, \tag{2.34}$$

and that

$$\rho \approx \frac{(\xi - \frac{1}{6})^2 H^4}{8\pi^2 a^4} ln\Big(\frac{1}{H\Delta t}\Big). \tag{2.35}$$

Comparison of these two results indicates that the mean energy of the created particles is of order $H ln[(H\Delta t)^{-1}]$. Note in the limit $\Delta t \to 0$, that N is finite, but ρ is unbounded. The vacuum energy density, ρ_V, which drives the expansion, is related to H by the Einstein equation:

$$H^2 = \frac{8\pi\rho_V}{3\sqrt{\rho_{Pl}}}, \tag{2.36}$$

where $\rho_{Pl} \approx (10^{19} GeV)^4$ is the Planck density. We can express our estimate for the energy density of the created particles just after the end of inflation as

$$\rho \approx (1 - 6\xi)^2 \frac{\rho_V^2}{\rho_{Pl}}. \tag{2.37}$$

If, for example, we were to take $\rho_V \approx (10^{15}GeV)^4$, which is a typical value for inflation at the GUT (Grand Unified Theory) scale, then we obtain the estimate

$$\rho \approx (1-6\xi)^2(10^{11}GeV)^4. \tag{2.38}$$

This energy density is much less than ρ_V, and would hence be negligible if there is efficient reheating. However, if other reheating mechanisms are not efficient, then particle creation by the gravitational field could play a significant role in cosmological evolution.

Chapter 7

The Hawking Effect

In this lecture, we will apply the notions of particle creation by gravitational fields to black hole spacetimes. This leads to the Hawking effect[8, 9], the process by which black holes emit a thermal spectrum of particles. For the sake of definiteness, we will concentrate on the case of a massless, scalar field in the Schwarzschild spacetime, but the basic ideas may be applied to any quantum field in a general black hole spacetime. For the most part, we will follow the original derivation given by Hawking[9]. We imagine that the black hole was formed at some time in the past by gravitational collapse. The spacetime of a collapsing star is illustrated in Fig. 1. This is not only physically reasonable, but also avoids the issue of boundary conditions on the past horizon which would arise if we were to consider the full Schwarzschild spacetime.

Let us assume that no scalar particles were present before the collapse began. In this case, the quantum state is the in-vacuum: $|\psi\rangle = |0\rangle_{in}$. The in-modes, $f_{\omega\ell m}$, are pure positive frequency on $\mathcal{I}^-$, so $f_{\omega\ell m} \sim e^{-i\omega v}$ as $v \to -\infty$, where $v = t + r^*$ is the advanced time coordinate. Similarly, the out-modes, $F_{\omega\ell m}$, are pure positive frequency on $\mathcal{I}^+$, so $F_{\omega\ell m} \sim e^{-i\omega u}$ as $u \to \infty$, where $u = t - r^*$ is the retarded time coordinate. As before, we need to find the relation between these two sets of modes in order to calculate the Bogolubov coefficients and determine the particle creation. Fortunately, it is not necessary to explicitly solve the wave equation for the modes everywhere in order to determine the Bogolubov coefficients. We are primarily interested in particle emission at late times (long after the collapse occurs). This is dominated by modes which left $\mathcal{I}^-$ with very high frequency, propagated through the collapsing body just before the horizon formed, and then underwent a large redshift on the way out to $\mathcal{I}^+$. Because these modes had an extremely high frequency

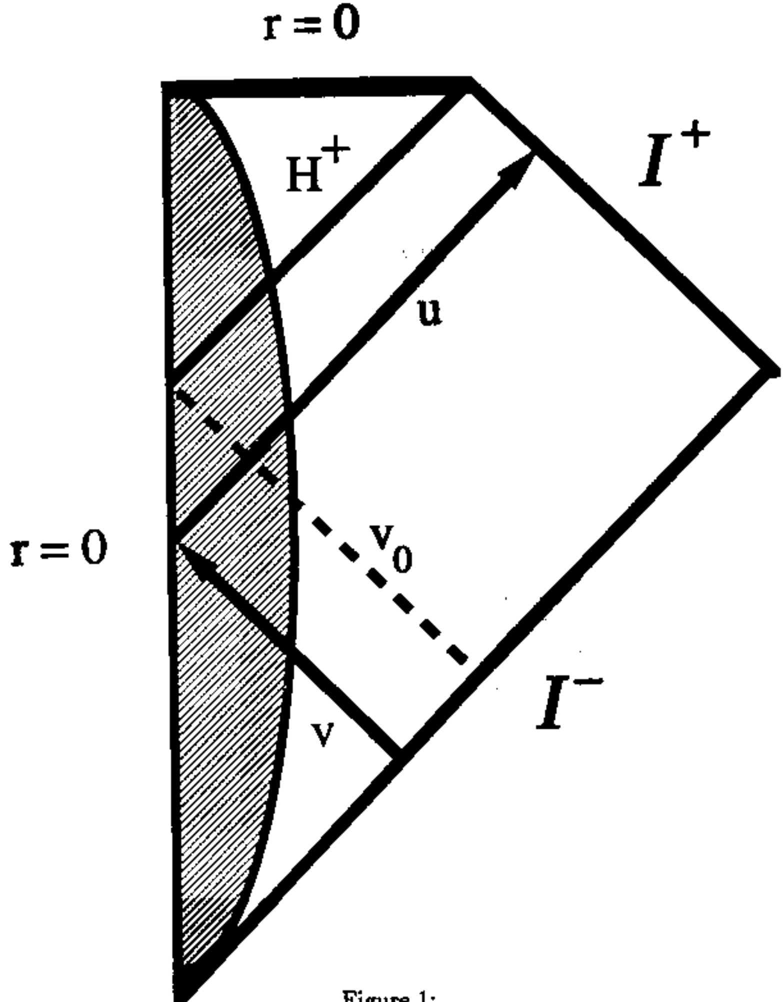

Figure 1:
The Penrose diagram for the spacetime of a black hole formed by gravitational collapse. The shaded region is the interior of the collapsing body, the $r = 0$ line on the left is worldline of the center of this body, the $r = 0$ line at the top of the diagram is the curvature singularity, and H^+ is the future event horizon. An ingoing light ray with $v < v_0$ from $\mathcal{I}^-$ passes through the body and escapes to $\mathcal{I}^+$ as a $u = constant$ light ray. Ingoing rays with $v > v_0$ do not escape and eventually reach the singularity.

during their passage through the body, we may describe their propagation by use of *geometrical optics.*

A $v = constant$ ingoing ray passes through the body and emerges as a $u = constant$ outgoing ray, where $u = g(v)$ or equivalently, $v = g^{-1}(u) \equiv G(u)$. The geometrical optics approximation leads to the following asymptotic forms for the modes:

$$f_{\omega \ell m} \sim \begin{cases} e^{-i\omega v}, & \text{on } \mathcal{I}^- \\ e^{-i\omega G(u)}, & \text{on } \mathcal{I}^+ \end{cases} \tag{0.1}$$

and

$$F_{\omega \ell m} \sim \begin{cases} e^{-i\omega u}, & \text{on } \mathcal{I}^+ \\ e^{-i\omega g(v)}, & \text{on } \mathcal{I}^-. \end{cases} \tag{0.2}$$

Hawking[9] gives a general ray-tracing argument which leads to the result that

$$u = g(v) = -4M\, ln\Big(\frac{v_0 - v}{C}\Big), \tag{0.3}$$

or

$$v = G(u) = v_0 - C e^{-u/4M}, \tag{0.4}$$

where M is the black hole mass, C is a constant, and v_0 is the limiting value of v for rays which pass through the body before the horizon forms.

We will derive this result for the explicit case of a thin shell. The spacetime inside the shell is flat and may be described by the metric

$$ds^2 = dT^2 - dr^2 - r^2\, d\Omega^2. \tag{0.5}$$

Thus, in the interior region, $V = T + r$ and $U = T - r$ are null coordinates which are constant on ingoing and on outgoing rays, respectively. The exterior of the shell is a Schwarzschild spacetime with the metric

$$ds^2 = \Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 - r^2\, d\Omega^2. \tag{0.6}$$

As noted above, the null coordinates here are $v = t + r^*$ and $u = t - r^*$, where

$$r^* = r + 2M\, ln\Big(\frac{r - 2M}{2M}\Big) \tag{0.7}$$

is the "tortoise coordinate". Let $r = R(t)$ describe the history of the shell. The metric in this three dimensional hypersurface must be the same as seen from both sides of the shell. (The intrinsic geometry must match.) This leads to the condition

$$1 - \Big(\frac{dR}{dT}\Big)^2 = \Big(\frac{R - 2M}{R}\Big)\Big(\frac{dt}{dT}\Big)^2 - \Big(\frac{R - 2M}{R}\Big)^{-1}\Big(\frac{dR}{dT}\Big)^2. \tag{0.8}$$

There is a second junction condition, that the extrinsic curvatures of each side of this hypersurface match[10]. This leads to the equation which determines $R(t)$ in terms of the stress-energy in the shell. For our purposes, this equation is not needed, and we may assume an arbitrary $R(t)$.

There are now three conditions to be determined: the relation between the values of the null coordinates v and V for the ingoing ray, the relation between V and U at the center of the shell, and finally the relation between U and u for the outgoing ray. This sequence of matchings is illustrated in Fig. 2.

- Let us suppose that our null ray enters the shell at a radius of R_1, which is finitely larger than $2M$. At this point, both $\left(\frac{R-2M}{R}\right)^{-1}$ and $\frac{dR}{dT}$ are finite and approximately constant. Thus $\frac{dt}{dT}$ is approximately constant, so $t \propto T$. Similarly, r^* is a linear function of r in a neighborhood of $r = R_1$. Thus, we conclude that

$$V(v) = av + b \tag{0.9}$$

  in a neighborhood of $v = v_0$, where a and b are constants.

- The matching of the null coordinates at the center of the shell is very simple. Because $V = T + r$ and $U = T - r$, at $r = 0$ we have that

$$U(V) = V. \tag{0.10}$$

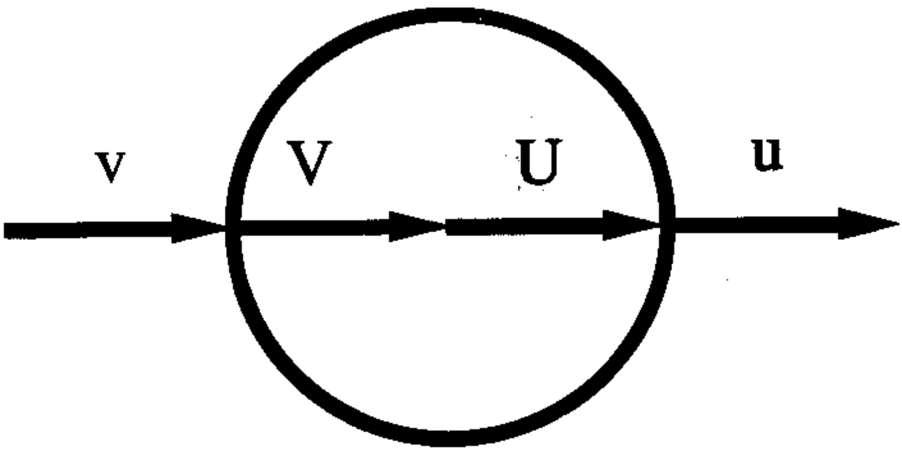

Figure 2:
An ingoing ray enters the collapsing shell, passes through the origin, and exits as an outgoing ray. This is illustrated schematically in this diagram. Note that the rays in question are actually imploding or exploding spherical shells of light.

- We now consider the exit from the shell. We are interested in rays which exit when R is close to $2M$. Let T_0 be the time at which $R = 2M$. (Note that this occurs at a finite time as seen by observers *inside* the shell.) Then near $T = T_0$,

$$R(T) \approx 2M + A(T_0 - T), \tag{0.11}$$

where A is a constant. If we insert this into Eq. (0.8), we have that

$$\left(\frac{dt}{dT}\right)^2 \approx \left(\frac{R-2M}{2M}\right)^{-2}\left(\frac{dR}{dT}\right)^2 \approx \frac{(2M)^2}{(T-T_0)^2}, \tag{0.12}$$

which implies

$$t \sim -2M\, ln\left(\frac{T_0 - T}{B}\right), \qquad T \to T_0. \tag{0.13}$$

Similarly, as $T \to T_0$, we have that

$$r^* \sim 2M\, ln\left(\frac{r-2M}{2M}\right) \sim 2M\, ln\left[\frac{A(T_0 - T)}{2M}\right], \tag{0.14}$$

and hence that

$$u = t - r^* \sim -4M\, ln\left(\frac{T_0 - T}{B'}\right). \tag{0.15}$$

(Again, B and B' are constants.) However, in this limit we have that

$$U = T - r = T - R(T) \sim (1 + A)T - 2M - AT_0. \tag{0.16}$$

Combining these results with Eqs. (0.9) and (0.10) yields our final result, Eq. (0.3). Although we have performed our explicit calculation for the special case of a thin shell, the result is more general, as is revealed by Hawking's derivation. We can understand why this is this case; the crucial logarithmic dependence which governs the asymptotic form of $u(v)$ comes from the last step in the above sequence of matchings. This step reflects the large redshift which the outgoing rays experience after they have passed through the collapsing body, which is essentially independent of the interior geometry.

¿From Eq. (0.2), we see that the out-modes, when traced back to $\mathcal{I}^-$, have the form

$$F_{\omega\ell m} \sim \begin{cases} e^{4Mi\omega \ln[(v_0-v)/C]}, & v < v_0 \\ 0, & v > v_0. \end{cases} \tag{0.17}$$

We can find the Bogolubov coefficients by Fourier transforming this function. Recall that

$$F_{\omega\ell m} = \int_0^\infty d\omega' \Big(\alpha^*_{\omega'\omega\ell m} f_{\omega'\ell m} - \beta_{\omega'\omega\ell m} f^*_{\omega'\ell m} \Big). \tag{0.18}$$

Here we use the notation $\alpha_{\omega'\omega\ell m} = \alpha_{\omega'\ell m,\omega\ell m}$ and $\beta_{\omega'\omega\ell m} = \beta_{\omega'\ell -m,\omega\ell m}$, which is inspired by the fact that the dependence upon the angular coordinates must be the same for each term in the above equation. Thus,

$$\alpha^*_{\omega'\omega\ell m} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv\, e^{i\omega' v}\, e^{4Mi\omega \ln[(v_0-v)/C]}, \tag{0.19}$$

and

$$\beta_{\omega'\omega\ell m} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv\, e^{-i\omega' v}\, e^{4Mi\omega \ln[(v_0-v)/C]}, \tag{0.20}$$

or, equivalently,

$$\alpha^*_{\omega'\omega\ell m} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{-i\omega v_0} \int_0^\infty dv'\, e^{-i\omega' v}\, e^{4Mi\omega \ln(v'/C)}, \tag{0.21}$$

and

$$\beta_{\omega'\omega\ell m} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{-i\omega v_0} \int_0^\infty dv'\, e^{i\omega' v}\, e^{4Mi\omega \ln(v'/C)}, \tag{0.22}$$

where $v' = v_0 - v$.

Both of the above integrands are analytic everywhere except on the negative real axis, where the branch cut of the logarithm function is located. Thus

$$\oint_C dv' \, e^{-i\omega' v} \, e^{4Mi\omega \ln(v'/C)} = 0, \tag{0.23}$$

where the integration is around the closed contour C illustrated in Fig. 3. We may now write

$$\int_0^\infty dv' \, e^{i\omega' v} \, e^{4Mi\omega \ln(v'/C)} = \int_0^\infty dv' \, e^{-i\omega' v} \, e^{4Mi\omega \ln(-v'/C - i\epsilon)}$$
$$= e^{4\pi M\omega} \int_0^\infty dv' \, e^{-i\omega' v} \, e^{4Mi\omega \ln(v'/C)}. \tag{0.24}$$

In the first step, we used Eq. (0.23) and a $v' \to -v'$ change of variables. In the second step, we used the relation $ln(-v'/C - i\epsilon) = -\pi i + ln(v'/C)$. Comparison of this result with Eqs. (0.21) and (0.22) leads to the result

$$|\alpha_{\omega'\omega\ell m}| = e^{4\pi M\omega} |\beta_{\omega'\omega\ell m}|. \tag{0.25}$$

The condition, Eq. (2.13), on the Bogolubov coefficients may be written as

$$\sum_{\omega'} \Big(|\alpha_{\omega'\omega\ell m}|^2 - |\beta_{\omega'\omega\ell m}|^2 \Big) = \sum_{\omega'} \Big(e^{8\pi M\omega} - 1 \Big) |\beta_{\omega'\omega\ell m}|^2 = 1. \tag{0.26}$$

The mean number of particles created into mode $\omega\ell m$ is now given by

$$N_{\omega\ell m} = \sum_{\omega'} |\beta_{\omega'\omega\ell m}|^2 = \frac{1}{e^{8\pi M\omega} - 1}. \tag{0.27}$$

This is a Planck spectrum with a temperature of

$$T_H = \frac{1}{8\pi M}, \tag{0.28}$$

which is the Hawking temperature of the black hole.

To show that these created particles produce a steady flow of energy to $\mathcal{I}^+$, we need to use either an analysis involving wavepackets [9], or else the following argument[11]. Note that the modes are discrete only if we regard the system as

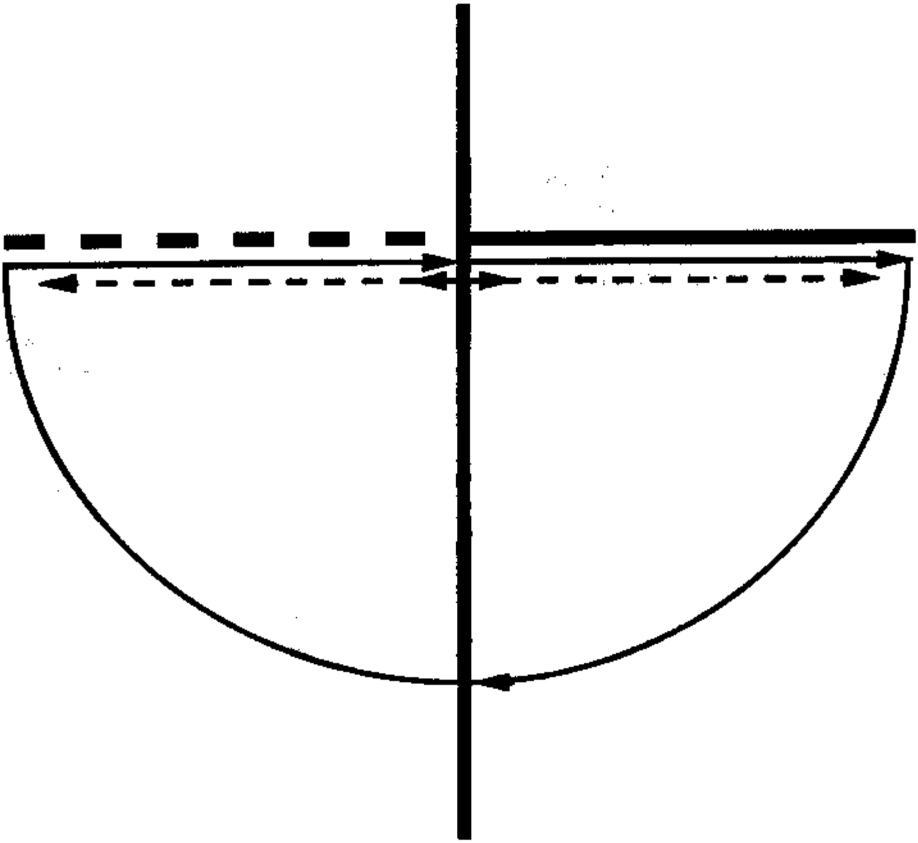

Figure 3:
The closed contour of the integration in Eq. (0.23) is illustrated. The fact that this integral vanishes implies that the integrals along each of the dotted segments are equal, which implies the first equality in Eq. (0.24).

being enclosed in a large box, which we may take to be a sphere of radius $\mathcal{R}$. Then in the limit of large $\mathcal{R}$ we have

$$\sum_{\omega} \to \frac{\mathcal{R}}{2\pi} \int_0^\infty d\omega. \tag{0.29}$$

The total energy of the created particles is

$$E = \sum_{\omega \ell m} \omega N_{\omega \ell m} = \frac{\mathcal{R}}{2\pi} \int_0^\infty d\omega \, \omega N_{\omega \ell m}. \tag{0.30}$$

This quantity would diverge in the limit that $\mathcal{R} \to \infty$. However, this simply reflects a constant rate of emission over an infinitely long time (when backreaction of the radiation on the black hole is ignored). We may find the luminosity by noting that it takes a time $\mathcal{R}$ for an outgoing particle emitted by the black hole to reach the boundary of the spherical cavity. Thus, Eq. (0.30) is the amount of energy emitted in a time $\mathcal{R}$, and the luminosity is

$$L = \frac{E}{\mathcal{R}} = \frac{1}{2\pi} \int_0^\infty d\omega \, \omega N_{\omega \ell m}. \tag{0.31}$$

This result does not include the effect of the backscattering of the particles off of the spacetime curvature surrounding the black hole. Let $\Gamma_{\ell m}$ denote the probability that a particle created in mode ℓm near the horizon escapes to infinity. Now our expression for the luminosity becomes

$$L = \frac{1}{2\pi} \int_0^\infty d\omega \, \omega \frac{\Gamma_{\ell m}}{e^{8\pi M \omega} - 1}. \tag{0.32}$$

The sum on ℓ converges because $\Gamma_{\ell m} \to 0$ as $\ell \to \infty$.

This result shows that black holes emit radiation with a (filtered) Planckian spectrum. However, it may be shown that the radiation is indeed thermal [12, 13, 13]. This may be done, for example, by calculating the higher moments of the distribution, $\langle N^2_{\omega \ell m} \rangle$, etc, and showing that they also take the forms required for thermal radiation. This indicates that there are no correlations among the emitted particles. This thermal character of a black hole may be attributed to the loss of information across the event horizon.

Thus one is lead to the subject of *black hole thermodynamics*, which was anticipated by Bekenstein[15] before Hawking's discovery. With the Schwarzschild black hole temperature as given in Eq. (0.28), the First Law of Thermodynamics now takes the form

$$dS_{BH} = \frac{dM}{T_H}, \tag{0.33}$$

where the black hole's entropy is given by $S_{BH} = 4\pi M^2$ for the Schwarzschild black hole, and more generally by $\frac{1}{4}A_H$, where A_H is the horizon area. The assignment of an entropy to the black hole resolves the apparent paradox discovered by Bekenstein, who realized that otherwise the Second Law of Thermodynamics would be violated when hot matter is thrown into a black hole. The Second Law now takes the form

$$\Delta S = \Delta S_{BH} + \Delta S_{matter} \geq 0. \tag{0.34}$$

Black hole evaporation provides a beautiful unification of aspects of quantum theory, gravitation, and thermodynamics. However, there are still a few unresolved issues. Among these are the questions of the use of ultra-high frequencies, and of the final state of black hole evaporation. The ultra-high frequency issue arises when one calculates the frequency ω' on $\mathcal{I}^-$ that is needed to produce the thermal radiation at some time t well after the collapse has occurred. ¿From Eq. (0.4), this frequency is of the order of

$$\omega' = M^{-1}e^{\frac{t}{4M}}. \tag{0.35}$$

If we take t to be of the order of the expected black hole lifetime, M^3, then $\omega' \approx M^{-1}e^{(M/M_{Pl})^2}$, where M_{Pl} is the Planck mass. For a $1g$ black hole this leads to $\omega' \approx 10^{10^{10}}g$, which is enormously larger than the mass of the observable universe. A cutoff at any reasonable energy scale would appear to quickly kill off the Hawking radiation. (See Ref.[16] for a discussion of attempts to circumvent this conclusion.)

Finally there is the unresolved and much-debated issue of the final state[17]. There seem to be three logical possibilities for the end result:

- **A singularity.** This would mean that quantum theory does not solve the classical singularity problem, and there is a loss of predictability. Such an outcome would seem to imply that the theory is still incomplete.

- **A Stable Remnant.** If this remnant retained all of the information which fell into the black hole during its history, it would appear to have a huge number of internal degrees of freedom, and thus might give an unacceptably large contribution to virtual processes.

- **Total Evaporation.** This possibility seems to be the most natural. It does lead to the conclusion that information is lost during the black hole formation-evaporation process, unless there are subtle correlations in the Hawking radiation that are not predicted by the semiclassical theory presented in this lecture.

Chapter 8

Negative Energy Densities and Fluxes

This lecture will discuss one of the special properties of the local energy density in quantum field theory, namely, that it can be negative. Negative energy is crucial for an understanding of the Hawking effect, in that a negative energy flux across the horizon is needed to implement the backreaction of the spacetime metric to the outgoing radiation. (Note that one cannot think of the backreaction as simply due to the positive energy outgoing radiation, as such radiation would undergo an infinite blueshift when traced back to $r = 2M$.) Rather, one may think of pairs of particles being created in the region outside $r = 2M$, one member of the pair escapes to infinity, and the other falls into the horizon. The latter particle carries negative energy as measured at infinity. This picture is consistent with calculations of the quantum field stress tensor near the black hole horizon in both two [18] and four [19, 20] dimensional models.

However, negative energy densities and fluxes arise even in flat spacetime. A simple example is the Casimir effect [21], where the vacuum state of the quantized electromagnetic field between a pair of conducting plates separated by a distance L is a state of constant negative energy density

$$\rho = \langle T_{tt} \rangle = -\frac{\pi^2}{720L^4}. \tag{0.1}$$

Negative energy density can also arise as the result of quantum coherence effects, which will be the principal concern of this lecture. Although we will restrict our attention to free fields in Minkowski spacetime, the basic considerations are much more general. In fact, it may be shown under very general assumptions that all quantum field theories admit states for which the energy density may be arbitrarily negative at a given point[22].

We can illustrate the basic phenomenon with a very simple example. Let the quantum state of the system be a superposition of the vacuum and a two particle state:

$$|\Psi\rangle = \frac{1}{\sqrt{1+\epsilon^2}}(|0\rangle + \epsilon|2\rangle). \tag{0.2}$$

Here we take the relative amplitude ϵ to be a real number. Let the energy density operator be normal-ordered:

$$\rho =: T_{tt} :, \tag{0.3}$$

so that $\langle 0|\rho|0\rangle = 0$. Then the expectation value of the energy density in the above state is

$$\langle\rho\rangle = \frac{1}{1+\epsilon^2}\left[2\epsilon Re(\langle 0|\rho|2\rangle) + \epsilon^2\langle 2|\rho|2\rangle\right]. \tag{0.4}$$

We may always choose ϵ to be sufficiently small that the first term on the right hand side dominates the second term. However, the former term may be either positive or negative. At any given point, we could choose the sign of ϵ so as to make $\langle\rho\rangle < 0$ at that point.

Note that the integral of ρ over all space is the Hamiltonian, which does have non-negative expectation values:

$$\langle H\rangle = \int d^3x \langle\rho\rangle \geq 0. \tag{0.5}$$

In the above *vacuum + two particle* example, the matrix element $\langle 0|\rho|2\rangle$, which gives rise to the negative energy density, has an integral over all space which vanishes, so only $\langle 2|\rho|2\rangle$ contributes to the Hamiltonian.

This example is a limiting case of a more general class of quantum states which may exhibit negative energy densities, the squeezed states. A general squeezed state for a single mode can be expressed as [23]

$$|z,\zeta\rangle = D(z)\,S(\zeta)\,|0\rangle, \tag{0.6}$$

where $D(z)$ is the displacement operator

$$D(z) \equiv exp(za^\dagger - z^* a) = e^{-|z|^2/2}\, e^{za^\dagger}\, e^{-z^* a} \tag{0.7}$$

and $S(\zeta)$ is the squeeze operator

$$S(\zeta) \equiv exp[\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2]. \tag{0.8}$$

Here

$$z = se^{i\gamma} \tag{0.9}$$

and

$$\zeta = re^{i\delta}. \tag{0.10}$$

are arbitrary complex numbers. The displacement and squeeze operators satisfy the relations

$$D^\dagger(z)\,a\,D(z) = a + z, \tag{0.11}$$

$$D^\dagger(z)\,a^\dagger\,D(z) = a^\dagger + z^*, \tag{0.12}$$

$$S^\dagger(\zeta)\,a\,S(\zeta) = a\,\cosh r - a^\dagger e^{i\delta}\sinh r, \tag{0.13}$$

and

$$S^\dagger(\zeta)\,a^\dagger\,S(\zeta) = a^\dagger\,\cosh r - ae^{-i\delta}\sinh r. \tag{0.14}$$

When $\zeta = 0$ we have the familiar coherent states, $|z\rangle = |z,0\rangle$, which describe classical excitations. In such a state, the expectation value of a quantum field ϕ is

$$\langle\phi\rangle = zf + z^* f^*, \tag{0.15}$$

where f is the mode function for the excited mode. This is a solution of the classical wave equation. Furthermore, the quantum fluctuations in this state are minimized:

$$\langle :\phi^2:\rangle = \langle\phi\rangle^2. \tag{0.16}$$

The opposite limit from a coherent state is a "squeezed vacuum state", $|0,\zeta\rangle$, for which $z = 0$. Sufficiently squeezed states can exhibit negative energy density, and a squeezed vacuum state always has $\langle\rho\rangle < 0$ somewhere. One may think of the effect of the squeezing as decreasing the quantum uncertainty in one variable, but increasing it in the conjugate variable. Squeezed vacuum states are of particular interest to us because they are the states which arise as a result of quantum particle creation. That is, the in-vacuum state is a squeezed vacuum state in the out-Fock space.

We may illustrate this by considering a Bogolubov transformation involving a single mode. Let

$$a = \alpha^* b - \beta^* b^\dagger, \tag{0.17}$$

or, equivalently,

$$b = \alpha a + \beta^* a^\dagger, \tag{0.18}$$

where $|\alpha|^2 - |\beta|^2 = 1$. These are single mode versions of Eqs. (2.17) and (2.18). As before, the in-vacuum satisfies $a|0\rangle_{in} = 0$, and the out-vacuum satisfies $b|0\rangle_{out} = 0$. We wish to express $|0\rangle_{in}$ as a state in the out-Fock space. This may achieved by the action of some operator Σ upon $|0\rangle_{out}$:

$$|0\rangle_{in} = \Sigma|0\rangle_{out}. \tag{0.19}$$

Act with the operator $\Sigma^\dagger a$ on both sides of this equation to obtain

$$\Sigma^\dagger a \Sigma\,|0\rangle_{out} = 0. \tag{0.20}$$

Hence we may identify $\Sigma^\dagger a \Sigma = b = \alpha a + \beta^* a^\dagger$. However, this is basically of the same form as Eq. (0.13). We may choose the phase of our mode so that α is real. Then, if we let r and δ be such that $\alpha = \cosh r$, and $\beta = -e^{-i\delta}\sinh r$, we see that $\Sigma = S$, the squeeze operator. Hence $|0\rangle_{in}$ is a squeezed vacuum state in the out-Fock space.

Squeezed states of light have recently been created in the laboratory by use of nonlinear optics[24]. The essential idea is that a nonlinear medium acts like a material with a time-dependent dielectric function when a strong, time-varying classical electromagnetic field is applied. Photon modes propagating through a time-dependent dielectric will undergo a mixing of positive and negative frequencies, just as in a time-dependent gravitational field, and photons will be quantum mechanically created into a squeezed vacuum state. Thus, the observation of squeezed states of photons can be regarded as an experimental confirmation of the formalism developed in Lecture I.

In the present context, squeezed states are of interest as examples of negative energy density states. Although it has not been possible to directly detect negative energy densities in the laboratory, the reduction of quantum noise due to squeezing has been observed[24]. We can think of negative energy density as a related reduction. When the quantum fluctuations are momentarily suppressed so that the energy density falls below the vacuum level, we have negative energy density. This suppression of quantum fluctuations could in principle be detected by a spin system, and would manifest itself in the net magnetic moment of the system increasing *above* the vacuum value[25]. We can visualize this as occurring because the vacuum fluctuations tend to depolarize the spins, and negative energy tends to reduce this depolarizing effect, allowing the spins to become more perfectly aligned.

A closely related phenomenon to a negative energy density is a negative energy flux. This arises when we have a quantum state in which all of the particles are moving in one direction, but the instantaneous flow of energy is in the opposite direction. Such a state would also have a locally negative energy density, but not all states with negative energy density carry a negative energy flux. The examples given above for negative energy density also exhibit a negative energy flux if the mode in question is a travelling wave mode (as opposed to a standing wave). Another example of a negative energy flux arises in moving mirror models in two-dimensional spacetime[26]. Here if the acceleration of the mirror is increasing in the direction of the observer, then the mirror emits a negative energy flux.

If one could have arbitrary fluxes of negative energy, it would seem that one could shine the negative energy on a hot object and cause a net decrease in entropy, and hence violate the second law of thermodynamics[27]. For example, the object could be a black hole. The absorption of negative energy would seem to decrease the black hole's entropy without a compensating increase in the entropy of radiation.

However, there are some constraints upon negative energy fluxes. The first is that the net energy must be non-negative. Let $F(t)$ be the instantaneous flux. Then

$$\int_{-\infty}^{\infty} F(t)\, dt \geq 0. \tag{0.21}$$

This inequality alone is not sufficient to prevent an arbitrarily large violation of the second law from occurring before the compensating positive energy arrives. There are stronger restrictions on negative energy fluxes which constrain the magnitude and duration of a pulse of negative energy. In flat two-dimensional spacetime such an inequality is of the form

$$|F| < t^{-2}, \tag{0.22}$$

where $|F|$ is the magnitude of the negative flux and t is its duration. This inequality implies that $|F|t$, the amount of negative energy which passes by a fixed location in time t is less than the quantum energy uncertainty on that timescale, t^{-1}. A more precise version of this inequality is obtained by multiplying F by a peaked function of time whose time integral is unity and whose characteristic width is t_0. A suitable choice of such a function is $t_0/[\pi(t^2 + {t_0}^2)]$. The inequality is [28]

$$\hat{F} \equiv \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{F(t)dt}{t^2 + {t_0}^2} \geq -\frac{1}{16\pi {t_0}^2}. \tag{0.23}$$

which holds for any quantum state for which only modes with $k > 0$ are excited. (This restriction is needed in order to distinguish a flow of negative energy to the right from a flow of positive energy to the left.) An illustration of the application of this inequality is afforded by the case of a negative energy pulse followed at a later time by a compensating positive energy pulse. The most efficient separation of positive and negative energy is obtained by delta-function pulses. Consider the following flux:

$$F(t) = |\Delta E|[-\delta(t) + \delta(t - T)]. \tag{0.24}$$

This represents a pulse of negative energy followed a time T later by an exactly compensating pulse of positive energy. The inequality, Eq. (0.23), yields

$$|\Delta E| \leq \frac{T^2 + t_0{}^2}{16 t_0 T^2}. \tag{0.25}$$

This relation is true for all t_0, but the best constraint on $|\Delta E|$ is obtained by setting $t_0 = T$. Then we find

$$|\Delta E| \leq \frac{1}{8T}. \tag{0.26}$$

This inequality tells us that there is a maximum separation in time between the two pulses, which is within the limits allowed by the uncertainty principle.

A similar inequality applies to the massless scalar field in four-dimensional flat spacetime:

$$\hat{F}_x \equiv \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{F_x(t)dt}{t^2 + t_0{}^2} \geq -\frac{3}{32\pi^2 t_0{}^4}. \tag{0.27}$$

Here $F_x(t) = \langle T^{xt} \rangle$ is the flux in the x-direction, and the expectation value is taken in any quantum state in which only modes with $k_x \geq 0$ are excited. Again we can apply it to the case of separated negative and positive energy pulses. Let

$$F_x(t) = \frac{|\Delta E|}{A}[-\delta(t) + \delta(t - T)]. \tag{0.28}$$

This represents a plane delta function pulse of negative energy which has a magnitude $|\Delta E|$ over a collecting area A, and which is followed a time T later by compensating positive energy. Here we may regard T as being the timescale for the duration of the negative energy. In order that the all parts of the collecting system be causally connected on a time T, this time should be larger than the linear dimensions of the collector. If we insert Eq. (0.27) into Eq. (0.28), set $t_0 = T$ and require that $A \leq T^2$, then we find that

$$|\Delta E| \leq \frac{3}{16\pi T}. \tag{0.29}$$

Again there is a constraint which requires the positive energy to arrive within a time $1/|\Delta E|$.

Similar inequalities may be proven in black hole spacetimes[29]. Let us consider an attempt to create a naked singularity using negative energy. We could start with an extreme, $Q = M$, charged black hole. We then shine some negative energy on it so as to decrease M with no change in the charge Q. This should result in the naked singularity Reissner-Nordstrom spacetime. Even if subsequent positive energy converts it back into a black hole, there might be a finite interval when signals from the singularity can escape to $\mathcal{I}^+$ ("cosmic flashing"). However, the analog of Eq. (0.29) for the four-dimensional Reissner-Nordstrom spacetime leads to the constraint

$$|\Delta M| < \frac{1}{t}, \tag{0.30}$$

where ΔM is the change in the black hole's mass due to absorption of negative energy, and t is the duration of the naked singularity. This implies[29] that the change in the background geometry is less than the expected quantum metric fluctuations on the timescale t. Thus it is doubtful that the naked singularity is observable.

We may also use the quantum inequalities of the form of Eq. (0.29) to limit any violations of the second law of thermodynamics due to negative energy. Let us consider the use of negative energy to decrease the mass of a black hole by ΔM. This decreases the entropy of the black hole by an amount of order

$$\Delta S \approx M \Delta M. \tag{0.31}$$

This entropy decrease can only be sustained for a period of time $T \leq (\Delta M)^{-1}$. If we wish to be able to measure the area of the horizon and verify that it has decreased, we should require that this time be larger than the light travel time across the black hole, so $T > M$. These conditions together imply that

$$\Delta S < 1. \tag{0.32}$$

This small entropy corresponds to less than one bit of information. Thus, it is clear that negative energy constrained by an inequality of the form of Eq. (0.29) cannot produce a macroscopic violation of the second law.

It should be noted that there are examples of negative energy fluxes that do not obey an inequality of this form. An example is the case of an observer at a fixed value of r just outside of the horizon of an evaporating black hole. Such an observer sees a constant negative flux going into the hole on a timescale of the order of the black hole's lifetime. However, this observer is non-inertial. If one wants to describe measurements made by a detector carried by such an observer, one needs to take into account the Unruh radiation effects, by which an accelerated detector responds as though it were immersed in a thermal bath. In fact, Unruh[17] has shown that for a detector near the black hole horizon, this effect dominates any effect due to the ingoing negative energy. There are other apparent counterexamples to the inequality, Eq. (0.29), which involve inertial motion through a negative energy background. These include observers moving through the Casimir energy in a cylinder universe, and observers orbiting an evaporating black hole[31]. However, in all of these cases, the quantum field is in its natural ground state, and it is not clear that it is possible to absorb any of this negative energy.

In addition to the above inequalities on fluxes of negative energy, there may also be restrictions on negative energy density[28]. However, this is a topic which has not yet been carefully investigated. Such restrictions, if they exist, could limit "traversable wormholes" [32]. Such wormholes would require negative energy densities in order that their metric be a solution of the semiclassical Einstein equations. Wormhole solutions have the possibility of containing closed time-like curves and apparent causality violation. Thus it is of interest to discover whether there are restrictions imposed by quantum field theory which prevent either the existence of wormholes, or their use to violate causality.

Chapter 9

Green's Functions and $\langle T_{\mu\nu} \rangle$ in Curved Spacetime

In this lecture, we will discuss the removal of the ultraviolet divergences and the calculation of $\langle T_{\mu\nu} \rangle$ on a curved background spacetime. In addition, we will discuss some issues related to the infrared behavior of Green's functions in curved spacetime.

1 Ultraviolet Behavior

The ultraviolet divergences of a quantum field theory are related to the short distance behavior of the vacuum expectation values of products of field operators. We will be considering only free fields, so our primary interest is in the two-point function. If one calculates a two-point function for some choice of the vacuum state in a curved spacetime, the typical short distance behavior which one finds is

$$G(x,x') \sim \frac{1}{2\pi^2}\Big(-\frac{1}{\sigma} + \ln\sigma + finite\ parts\Big), \qquad x' \to x, \tag{1.1}$$

where σ is one-half of the square of the geodesic distance between x and x'. Thus, in flat spacetime or in the $x' \to x$ limit in curved spacetime, $\sigma = \frac{1}{2}(x-x')^2$. To be more precise, let us discuss the *Hadamard function* for the scalar field ϕ:

$$G_1(x,x') \equiv \langle 0|\phi(x)\phi(x')|0\rangle, \tag{1.2}$$

where $|0\rangle$ is a chosen vacuum state. This function is said to have the *Hadamard form* if it can be expressed as

$$G_1(x,x') = \frac{U(x,x')}{\sigma} + V(x,x')\, ln\, \sigma + W(x,x'), \tag{1.3}$$

where U, V, and W are regular functions for all choices of x and x'. The functions U and V are geometrical quantities independent of the quantum state, and only W carries information about the state. For most situations of interest to us, this form will hold for all quantum states, and hence the singular part of $G_1(x,x')$ in the coincidence limit will be state-independent. (The significance of the Hadamard function *not* having the Hadamard form will be discussed below when we deal with infrared divergences.)

We can formally construct the expectation value of the stress tensor, $\langle T_{\mu\nu}\rangle$, as a limit of derivatives of $G^{(1)}$. For example, consider the massless, minimally coupled scalar field, for which

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}. \tag{1.4}$$

The formal expectation value of $T_{\mu\nu}$ is

$$\langle T_{\mu\nu}\rangle = \frac{1}{2}\lim_{x'\to x}\Big\{[\partial_\mu\partial_{\nu'} - \frac{1}{2}g_{\mu\nu}\partial_\alpha\partial^{\alpha'}]G^{(1)}(x,x')\Big\}, \tag{1.5}$$

where ∂_μ denotes a derivative with respect to x^μ and $\partial_{\nu'}$ denotes one with respect to x'^ν. So far, this expression is only formal because it diverges in the coincidence limit. However, for $x' \neq x$, it is a regularized form of $\langle T_{\mu\nu}\rangle$. Here we are utilizing point separation regularization[33]. There are several other methods for formally removing the ultraviolet divergences, including dimensional regularization and the zeta function method[34, 33]. Point separation has the advantage of being more generally applicable than do these other methods. Here we give a brief summary of the basic ideas. For more details see, for example, the books by Birrell and Davies[1] and by Fulling[36]. References which employ the properties of the Hadamard form include Adler, et al[37], Wald[38], Castagnino and Harari[39], and Brown and Ottewill[40].

The right-hand side of Eq. (1.5) as it stands depends upon the direction of the separation vector of the points x and x'. This is undesirable, and can be removed by averaging over these directions[37]. If we do this, the asymptotic form for our regularized expression becomes

$$\langle T_{\mu\nu}\rangle \sim A\,\frac{g_{\mu\nu}}{\sigma^2} + B\,\frac{G_{\mu\nu}}{\sigma} + (C_1 H^{(1)}_{\mu\nu} + C_2 H^{(2)}_{\mu\nu})\,\ln\sigma. \tag{1.6}$$

Here A, B, C_1, and C_2 are constants, $G_{\mu\nu}$ is the Einstein tensor, and the $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$ tensors are covariantly conserved tensors which are quadratic in the Riemann tensor. Specifically, they are the functional derivatives with respect to the metric tensor of the square of the scalar curvature and of the Ricci tensor, respectively:

$$\begin{aligned} H^{(1)}_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}[\sqrt{-g}R^2] \\ &= 2\nabla_\nu\nabla_\mu R - 2g_{\mu\nu}\nabla_\rho\nabla^\rho R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} H^{(2)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}[\sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}] &= 2\nabla_\alpha\nabla_\nu R^\alpha_\mu + \nabla_\rho\nabla^\rho R_{\mu\nu} \\ &-\frac{1}{2}g_{\mu\nu}\nabla_\rho\nabla^\rho R - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2R^\rho_\mu R_{\rho\nu}. \end{aligned} \tag{1.8}$$

The divergent parts of $\langle T_{\mu\nu}\rangle$ may be absorbed by renormalization of counterterms in the gravitational action. Write this action as

$$S_G = \frac{1}{16\pi G_0}\int d^4x\,\sqrt{-g}\,\left(R - 2\Lambda_0 + \alpha_0 R^2 + \beta_0 R_{\alpha\beta}R^{\alpha\beta}\right). \tag{1.9}$$

We now include a matter action, S_M, and vary the total action, $S = S_G + S_M$, with respect to the metric. If we replace the classical stress tensor in the resulting equation by the quantum expectation value, $\langle T_{\mu\nu}\rangle$, we obtain the semiclassical Einstein equation including the quadratic counterterms:

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \alpha_0 H^{(1)}_{\mu\nu} + \beta_0 H^{(2)}_{\mu\nu} = -8\pi G_0\langle T_{\mu\nu}\rangle. \tag{1.10}$$

We may remove the divergent parts of $\langle T_{\mu\nu}\rangle$ in redefinitions of the coupling constants G_0, Λ_0, α_0, and β_0. The renormalized values of these constants are then the physical parameters in the gravitational theory. After renormalization, G_0 is replaced by G, the renormalized Newton's constant, which is the value actually measured by the

Cavendish experiment. Similarly, Λ_0 becomes the cosmological constant Λ, which might be taken to be zero if we do not wish to have a cosmological term in the Einstein equations.

In any case, the renormalized value of $\langle T_{\mu\nu}\rangle$ is obtained by subtracting the terms which are divergent in the coincidence limit. However, we are free to perform additional finite renormalizations of the same form. Thus, $\langle T_{\mu\nu}\rangle_{ren}$ is defined only up to the addition of multiples of the four covariantly conserved, geometrical tensors $g_{\mu\nu}$, $G_{\mu\nu}$, $H^{(1)}_{\mu\nu}$, and $H^{(2)}_{\mu\nu}$. Apart from this ambiguity, Wald[41] has shown under very general assumptions that $\langle T_{\mu\nu}\rangle_{ren}$ is unique. Hence, at the end of the calculation, the answer is independent of the details of the regularization and renormalization procedures employed.

An interesting feature of the renormalization of a quantum stress tensor is that it breaks conformal invariance. This leads to the conformal anomaly[42, 43]. A conformally invariant classical theory, such as electromagnetism or the conformally coupled massless scalar field has the property that the trace of the stress tensor vanishes: $T^{\mu}_{\mu} = 0$. However, this classical property is lost in the renormalized quantum theory, and the expectation value of $T^{\mu}_{\mu} = 0$ acquires a nonzero trace. This anomalous trace is independent of the choice of quantum state and is a local geometrical quantity. Furthermore, it is not of a form which could be removed by a finite renormalization of the form discussed above. For the case of the conformal ($\xi = 1/6$) scalar field, it is

$$\langle T^{\mu}_{\mu}\rangle_{ren} = -\frac{1}{2880\pi^2}\Big(R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - R_{\alpha\beta}R^{\alpha\beta} - \nabla_{\rho}\nabla^{\rho}R\Big). \tag{1.11}$$

Although the conformal anomaly is a state independent object, in general $\langle T_{\mu\nu}\rangle_{ren}$ is a state-dependent quantity. This is, of course, necessary so that it carry information about the matter content of particular quantum states. Because of this state-dependence, it is not possible to make any general statements concerning its order of magnitude. However, one typically finds for states which have the appearance of a "vacuum state" (i.e. in some sense, states of minimum excitation) that the order of magnitude (in a local orthonormal frame) is

$$\langle T_{\mu\nu}\rangle_{ren} \approx C\ell^{-4}, \tag{1.12}$$

where ℓ is the characteristic local radius of curvature of the spacetime, and C is a dimensionless constant which tends to be of the order of 10^{-3} to 10^{-4}. For example, in the Einstein universe the energy density for a massless conformal scalar field is[44]

$$\langle T_{tt}\rangle_{ren} = \frac{1}{480\pi^2 a^4} \tag{1.13}$$

in the vacuum state, which in this case is the unique state of lowest energy due to the presence of a global time-like Killing vector. This case is also of interest because the conformal anomaly, Eq. (1.11), vanishes here.

2 Infrared Behavior

In our discussion of the Hadamard form, we noted that it is a common, although not universal property of quantum states. In a state in which the two-point function does not have the Hadamard form, the renormalization procedure outlined above will not remove all of the infinities from the stress tensor. In flat spacetime, a state which does not have the Hadamard form would have to be considered to be unphysical if the normal-ordered energy density were infinite. Fulling, Sweeny and Wald[45] have shown that a two point function which has the Hadamard form at one time will have it at all times. In particular, in any spacetime which is asymptotically flat in the past or in the future, the Hadamard form will hold if it holds in the flat region. Thus, it seems reasonable to require that the two point function having the Hadamard form be a criterion for a physically acceptable state.

Examples of states which do not have the Hadamard form may be constructed even in flat spacetime[46]. Let us first consider a massless scalar field in flat four-dimensional spacetime, which has the mode expansion

$$\varphi = \sum_{\mathbf{k}} (a_{\mathbf{k}} f_{\mathbf{k}} + a^\dagger_{\mathbf{k}} f^*_{\mathbf{k}}), \tag{2.14}$$

where we now take our modes to have the form (box normalization in a volume V)

$$f_{\mathbf{k}} = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\omega V}}\,[\alpha(\omega)e^{-i\omega t} + \beta(\omega)e^{i\omega t}]. \tag{2.15}$$

In order that $f_{\mathbf{k}}$ have unit norm, we must require that

$$|\alpha(\omega)|^2 - |\beta(\omega)|^2 = 1. \tag{2.16}$$

This expansion defines a state $|\psi\rangle$ such that $a_{\mathbf{k}}|\psi\rangle = 0$. This state is the vacuum state only if $\beta = 0$. Nonetheless, we may still define a two-point function by

$$\langle\psi|\phi(x)\phi(x')|\psi\rangle = \frac{1}{2(2\pi)^3}\int d^3k\,\omega^{-1}\left\{[\alpha(\omega)e^{-i\omega t} + \beta(\omega)e^{i\omega t}]\right.$$
$$\left.\times[\alpha^*(\omega)e^{i\omega t'} + \beta^*(\omega)e^{-i\omega t'}]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right\}. \tag{2.17}$$

Let us suppose that the integral is dominated by low frequency modes. Then

$$\langle\psi|\phi(x)\phi(x')|\psi\rangle \sim \frac{1}{(2\pi)^2}\int d\omega\,\omega|\alpha(\omega) + \beta(\omega)|^2. \tag{2.18}$$

There exist choices of the functions $\alpha(\omega)$ and $\beta(\omega)$ which satisfy Eq. (2.16), but for which this integrand diverges as $\omega \to 0$. For example, let

$$\beta(\omega) = \omega^{-c}, \qquad \alpha(\omega) = (1 + \omega^{-2c})^{\frac{1}{2}}. \tag{2.19}$$

In this case

$$|\alpha(\omega) + \beta(\omega)| \sim \omega^{-c}, \qquad \omega \to 0, \tag{2.20}$$

and the two-point function is infinite for all x and x' if $c > 1$. This is an example of an infrared divergence. The result that the Hadamard form is preserved shows that infrared divergences will not arise during the course of time-evolution from a state which is free of them. Thus we are justified in excluding such states as unphysical.

In the above example, it may seem that we had to go to some lengths to find pathological states. However, in other spacetimes, the infrared divergences appear in apparently natural choices of quantum state, and the cure is remarkably similar to the prescription which caused the problems in the above example. Let us now consider a massless scalar field in two-dimensional spacetime. If we follow a construction parallel to that given above in four dimensions, we find that the analog of Eq. (2.18) is now

$$\langle\psi|\phi(x)\phi(x')|\psi\rangle \sim \frac{1}{4\pi}\int d\omega\,\omega^{-1}|\alpha(\omega) + \beta(\omega)|^2. \tag{2.21}$$

If we were to choose the Minkowski vacuum state, for which $\alpha = 1$ and $\beta = 0$, we have an infrared divergence. This is a well-known property of massless fields in two dimensions. However, there exist states which are free of infrared divergences. For example, let

$$\beta(\omega) = -\omega^{-c}, \qquad \alpha(\omega) = (1+\omega^{-2c})^{\frac{1}{2}}. \tag{2.22}$$

Now $|\alpha(\omega) + \beta(\omega)| \sim \frac{1}{4}\omega^{2c}$ as $\omega \to 0$, and the two-point function is finite if $c > 0$. Thus the infrared divergences in two dimensions are the consequence of a poor choice of quantum state and are remedied when a better choice is made. Note that the physically allowable quantum states are all ones which break Lorentz invariance. One may show that in any state which is free of infrared divergences, $\langle\phi^2\rangle$ must be a growing function of time[47]. In particular, in the quantum state defined by Eq. (2.22), one finds

$$\langle\phi^2\rangle \sim t^{2c}, \qquad t \to \infty. \tag{2.23}$$

Another example in which similar behavior occurs is deSitter spacetime. In the representation as a spatially flat Robertson-Walker universe, its metric is

$$ds^2 = \frac{1}{(H\eta)^2}(d\eta^2 - d\mathbf{x}^2) = dt^2 - e^{2Ht}d\mathbf{x}^2. \tag{2.24}$$

These coordinates cover one-half of the full deSitter space, but this is not a serious problem for our purposes. In the context of inflationary models, one is interested in spacetimes which involve only a piece of deSitter space. The massless, minimally coupled scalar field, which satisfies the wave equation

$$\Box\phi = 0, \tag{2.25}$$

has solutions in terms of Hankel functions[48]:

$$f_{\mathbf{k}} \propto e^{i\mathbf{k}\cdot\mathbf{x}}\Big[c_2 H^{(2)}_{\frac{3}{2}}(k\eta) + c_1 H^{(1)}_{\frac{3}{2}}(k\eta)\Big]. \tag{2.26}$$

where $|c_2|^2 - |c_1|^2 = 1$. The vacuum state which is invariant under the action of the deSitter symmetry group is given by

$$c_2 = 1, \qquad c_1 = 0. \tag{2.27}$$

However, because $H^{(2)}_{\frac{3}{2}}(k\eta) \sim k^{-\frac{3}{2}}$ as $k \to 0$, this state is infrared divergent. As before, we may find states which are free of such divergences; here what is required is a choice of $c_1(k)$ and $c_2(k)$ such that $|c_1(k) + c_2(k)| \to 0$ as $k \to 0$. Such states necessarily break deSitter invariance and lead to growth[49, 8, 51] of $\langle\phi^2\rangle$:

$$\langle\phi^2\rangle \sim \frac{H^3 t}{4\pi^2}, \qquad t \to \infty. \tag{2.28}$$

This is similar to the result in two-dimensional flat spacetime, although now the asymptotic rate of growth is independent of the choice of state, so long as the state is well defined.

This growth of $\langle\phi^2\rangle$ in deSitter space has consequences for inflationary models. For example, in the "new inflation" model, one postulates a self-coupled scalar field ϕ with a potential $V(\phi)$ which is very flat near the origin. It is the long period of slow rolling away from the origin which allows sufficient inflation to solve the horizon problem. During this period, ϕ approximately satisfies Eq. (2.28), so the root-mean-square value of ϕ must grow as $\sqrt{t}$. This tends to limit the period of inflation.

Another application of these results is to models of global symmetry breaking. Let us consider the Goldstone model of $U(1)$ symmetry breaking, where a complex scalar field Φ has the Lagrangian density

$$\mathcal{L} = \partial_\alpha \Phi^* \partial^\alpha \Phi - V(\Phi), \tag{2.29}$$

where

$$V(\Phi) = -\frac{1}{2}m^2 \Phi^* \Phi + \frac{1}{4}\lambda(\Phi^*\Phi)^2. \tag{2.30}$$

This potential has an unstable maximum at $\Phi = 0$, but minima at

$$\Phi = \sigma e^{i\phi}, \qquad \sigma = m\lambda^{-1/2}. \tag{2.31}$$

If σ is constant, then the equation of motion for Φ implies that $\Box\phi = 0$. Thus the Goldstone boson ϕ is a massless scalar field.

We now wish to treat ϕ as a quantized field and calculate the expectation value of Φ. This requires that we find the expectation value of the exponential of an operator. Decompose ϕ into its positive and negative frequency parts: $\phi = \phi^+ + \phi^-$, where $\phi^+|0\rangle = 0$ and $\langle 0|\phi^- = 0$. In terms of annihilation and creation operators, $\phi^+ = \sum_j a_j f_j$ and $\phi^- = \sum_j a_j^\dagger f_j^*$. We now write

$$e^{i\phi} = e^{i(\phi^+ + \phi^-)} = e^{i\phi^-} e^{-\frac{1}{2}[\phi^+,\phi^-]} e^{i\phi^+}, \tag{2.32}$$

where in the second step we use the Campbell-Baker-Hausdorff formula. We now take the vacuum expectation value of this expression and use the facts that $e^{i\phi^+}|0\rangle = |0\rangle$ and $\langle 0|e^{i\phi^-} = \langle 0|$, which follow immediately if the exponentials are expanded in a power series. Finally, we use $[\phi^+, \phi^-] = \sum_j f_j f_j^* = \langle \phi^2 \rangle$ to write

$$\left\langle \Phi \right\rangle = \sigma \left\langle e^{i\phi} \right\rangle = \sigma e^{-\frac{1}{2}\langle \phi^2 \rangle}. \tag{2.33}$$

The ultraviolet divergence in $\langle \phi^2 \rangle$ is understood to be absorbed in a rescaling of Φ (a wavefunction renormalization). In spacetimes, such as four dimensional flat space, where one can have $\langle \phi^2 \rangle$ constant in a physically acceptable state, then there are stable broken symmetry states in which $\langle \Phi \rangle \neq 0$. However, in two dimensional flat spacetime or in four dimensional deSitter spacetime, the growth of $\langle \phi^2 \rangle$ forces $\langle \Phi \rangle$ to decay in time: $\langle \Phi \rangle \to 0$ as $t \to \infty$. In these cases, the infrared behavior of the massless scalar field prevents the existence of a stable state of broken symmetry.

Chapter 10

Semiclassical Gravity Theory and Metric Fluctuations

1 Limits of the Semiclassical Theory

The semiclassical theory of gravity is that in which a classical gravitational field is coupled to a quantized matter field through the semiclassical Einstein equations:

$$G_{\mu\nu} = -8\pi\langle T_{\mu\nu}\rangle. \tag{1.1}$$

This theory provides the necessary transition to the classical theory of gravity. It also seems to give a convincing picture of the backreaction to the Hawking radiation. Calculations of $\langle T_{\mu\nu}\rangle$ in the Unruh vacuum state reveal a steady negative energy flux into the horizon which accounts for the decrease in mass of the black hole as evaporation proceeds[19, 20].

However, this theory also suffers from some serious problems. One of these is that the semiclassical equations are typically fourth order equations. The tensors $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$, defined in Eqs. (1.7) and (1.8), involve fourth derivatives of the metric and will generally appear as part of $\langle T_{\mu\nu}\rangle$. This can lead to runaway solutions[52, 53], similar to those in classical electron theory when radiation reaction is taken into account.

Another difficulty of the theory based upon Eq. (1.1) is that it fails when there are large fluctuations in the stress tensor. This may be illustrated with a simple example: Suppose our system is in a superposition state in which the two possibilities are a $1000kg$ mass located on either side of our laboratory, with equal amplitudes. If we measure the resulting gravitational field with a gravimeter, we expect to find either the gravitational field of $1000kg$ on one side of the laboratory or that of $1000kg$ on the other side, and that each will occur with 50% probability. However, Eq. (1.1) predicts that we will always find the gravitational field produced by having $500kg$ on both sides of the laboratory. This difficulty is avoided if we only use Eq. (1.1) when the quantum state of the system is one in which the stress tensor fluctuations are small[54], that is, one in which

$$\langle T_{\alpha\beta}(x)\, T_{\mu\nu}(y)\rangle \approx \langle T_{\alpha\beta}(x)\rangle\langle T_{\mu\nu}(y)\rangle. \tag{1.2}$$

Of course, the expectation values on both sides of the above equation are formally divergent and need to be defined. Let us restrict our attention to free fields in Minkowski spacetime. Then all operators will have finite expectation values if we define them as being normal ordered with respect to the Minkowski vacuum state. Let

$$\Delta(x) \equiv \left|\frac{\langle :{T_{00}}^2(x):\rangle - \langle :T_{00}(x):\rangle^2}{\langle :{T_{00}}^2(x):\rangle}\right|. \tag{1.3}$$

Note that $\langle :T_{00}(x):\rangle$ is the mean energy density at x and $\langle :{T_{00}}^2(x):\rangle$ is the mean squared energy density. Thus Δ is a measure of the energy density fluctuations at point x. We could define similar quantities which measure the fluctuations in other stress tensor components. We should require that $\Delta \ll 1$ in order that the energy density fluctuations be small and that the semiclassical theory of gravity be valid. The fluctuations in the other components of $T_{\mu\nu}$ should also be small; however, we will restrict our attention to the energy density.

Let us consider a massless, scalar field for which the energy density is

$$T_{00} = \frac{1}{2}(\dot{\phi}^2 + |\nabla\phi|^2). \tag{1.4}$$

In a coherent state, one may show that

$$\langle T_{\alpha\beta}(x)\, T_{\mu\nu}(y)\rangle = \langle T_{\alpha\beta}(x)\rangle\langle T_{\mu\nu}(y)\rangle, \tag{1.5}$$

and hence $\Delta = 0$. Thus coherent states are states of minimum stress-energy fluctuations, and are hence states for which the semiclassical gravity theory holds. This is to be expected as coherent states are the quantum states which describe classical field excitations. It is of interest to now consider squeezed states. As we saw in Lecture 3, this two parameter family of states includes the coherent states as one limit, but also includes the squeezed vacuum states with negative energy density as another limit. Recall for the general squeezed state $|z,\zeta\rangle$, that z is the coherent state parameter and ζ is the squeezing parameter. If $|z| \gg |\zeta|$, then the state is close to a coherent state, whereas non-classical behavior such as negative energy densities arise in the opposite limit where $|\zeta| \leq |z|$. In Ref.[55], Δ was calculated numerically for various ranges of these parameters for a single plane wave mode. It was found that $\Delta \ll 1$ holds only in the former limit, $|z| \gg |\zeta|$. In particular, by the point that $|\zeta|$ has increased so that negative energy density appears somewhere, one always seems to have that Δ is of order unity. This result implies that the semiclassical theory of gravity fails for quantum states in which the energy density is negative.

It is also of interest to compute the fluctuations in the Casimir energy density. As noted previously, this can provide an example of negative energy density. In general, the calculation of the Casimir energy for any but the simplest geometries is a very difficult task. Nonetheless, it is possible to establish a lower bound on Δ which is independent of the boundary conditions. For the case of a massless scalar field, this lower bound is[55]

$$\Delta \geq \frac{1}{3}. \tag{1.6}$$

In the particular case of such a field which is periodic in one spatial direction with periodicity length L, the Casimir energy density is

$$\langle :T_{00}(x): \rangle = -\frac{\pi^2}{90\, L^4}, \tag{1.7}$$

and $\Delta = 6/7$. In all cases, Δ is at least of order unity, so there are large energy density fluctuations.

Thus our criterion, Eq. (1.2), for the validity of the semiclassical gravity theory is not fulfilled for the Casimir energy. This brings us to the question of how *do* we describe the gravitational field of the Casimir vacuum. The answer is presumably that we must introduce a fluctuating metric, rather than a fixed classical metric. The concept of a fluctuating metric is perhaps best approached in an operational manner. We can think of a classical metric as encoding information about the trajectories of classical test particles. Similarly, a fluctuating metric may be described in terms of the statistical properties of an ensemble of test particles. In this way we are led to treat the fluctuating gravitational field in terms of the Brownian motion which it produces in test particles.

Brownian motion may be described by means of a Langevin equation. In the case of nonrelativistic motion on a nearly flat background, this equation is

$$m\frac{d\mathbf{v}(x)}{dt} = \mathbf{F}_c(x) + \mathbf{F}(x), \tag{1.8}$$

where m is the test particle mass, $\mathbf{F}_c$ is a classical force, and $\mathbf{F}$ is a fluctuating force. In our case, the latter will be the force produced by the fluctuating gravitational field. The solution of this equation is

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \frac{1}{m}\int_{t_0}^{t} [\mathbf{F}_c(t') + \mathbf{F}(t')]\, dt' = \mathbf{v}_c(t) + \frac{1}{m}\int_{t_0}^{t} \mathbf{F}(t')\, dt' , \tag{1.9}$$

where $\mathbf{v}_c(t)$ is the velocity along a classical trajectory. We assume that the fluctuating force averages to zero, $\langle \mathbf{F} \rangle = 0$, so $\langle \mathbf{v} \rangle = \mathbf{v}_c(t)$, but that quantities quadratic in $\mathbf{F}$ do not average to zero. Thus the mean squared velocity, averaged over an ensemble of test particles is,

$$\langle \mathbf{v}^2 \rangle = \mathbf{v}^2(t_c) + \frac{1}{m^2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, \langle \mathbf{F}(t_1)\, \mathbf{F}(t_2) \rangle \, . \tag{1.10}$$

Typically, the correlation function for a fluctuating force vanishes for times separated by much more than some correlation time, t_c, and is approximately constant for shorter time separations:

$$\langle \mathbf{F}(t_1)\, \mathbf{F}(t_2) \rangle \approx \begin{cases} \langle F^2 \rangle, & |t_1 - t_2| < t_c, \\ 0, & |t_1 - t_2| > t_c. \end{cases} \tag{1.11}$$

In this case, the contribution of the fluctuating force to $\langle \mathbf{v}^2 \rangle$ grows linearly in time:

$$\langle \mathbf{v}^2 \rangle \sim \mathbf{v}^2(t_c) + \frac{1}{m^2}\langle F^2 \rangle t_c t \, , \qquad t \gg t_c \, . \tag{1.12}$$

We can apply these notions to the case of the gravitational field of the Casimir vacuum by considering test particles which interact only gravitationally (that is, have no coupling to the quantized field itself). In the absence of fluctuations, such a classical particle shot down parallel to and midway in between a pair of conducting plates would follow a trajectory half way between the plates indefinitely. However, the fluctuations of the gravitational field will cause it to eventually drift toward one plate or the other. The characteristic time scale for the fluctuations, t_c, will in this case be of order L, the plate separation. More generally, when the semiclassical theory of Eq. (1.1) breaks down because of large fluctuations in the stress tensor, we are forced to replace the notion of a classical gravitational field by a statistical description.

2 Metric Fluctuations and the Ultraviolet Divergence Problem

The metric fluctuations which we have been discussing arise strictly from fluctuations in the source of the gravitational field, and hence might be dubbed "passive" fluctuations. There can also be fluctuations of the gravitational degrees of freedom themselves, which we might call "active" fluctuations. The latter will arise from the quantum nature of gravity and should become important at the Planck scale. In the absence of a full quantum theory of gravity which is capable of treating the Planck scale, we may still discuss active fluctuations in the weak field limit. Consider quantized metric perturbations (gravitons) propagating on a fixed background spacetime. These gravitons could be in a quantum state, such as a squeezed vacuum state, in which there are significant fluctuations. For example, gravitons created in the early universe are expected to be in a squeezed vacuum state[56]. On a Robertson-Walker background, they will produce Weyl curvature fluctuations in the sense that $\langle C_{\alpha\beta\sigma\rho}\rangle = 0$, but $\langle C_{\alpha\beta\sigma\rho}C^{\alpha\beta\sigma\rho}\rangle \neq 0$.

We wish to consider the effect of metric fluctuations upon the ultraviolet behavior of quantum field theory. It was noted long ago by Pauli[57] that metric fluctuations might smear out the light cone and thus remove the singularities of Green's functions on the light cone. This idea was further discussed by Deser[58], but so far as I am aware, has never been implemented in a concrete manner. Let us consider a flat background spacetime with a linearized perturbation $h_{\mu\nu}$ propagating upon it. In the unperturbed spacetime, the square of the geodesic separation of points x and x' is $2\sigma_0 = (x - x')^2$. In the presence of the perturbation, let this squared separation be 2σ, and write

$$\sigma = \sigma_0 + \sigma_1 + O(h_{\mu\nu}^2), \tag{2.13}$$

so σ_1 is the first order shift in σ.

Let us consider the retarded Green's function for a massless scalar field. In flat spacetime, this function is

$$G_{ret}^{(0)}(x - x') = \frac{\theta(t - t')}{4\pi}\delta(\sigma_0)\,, \tag{2.14}$$

which has a delta-function singularity on the future lightcone and is zero elsewhere. In the presence of a classical metric perturbation, the retarded Green's function has its delta-function singularity on the perturbed lightcone, where $\sigma = 0$. In general, it may also become nonzero on the interior of the lightcone due to backscattering off of the curvature. However, we are primarily interested in the behavior near the new lightcone, and so let us replace $G_{ret}^{(0)}(x - x')$ by

$$G_{ret}(x, x') = \frac{\theta(t - t')}{4\pi}\delta(\sigma)\,. \tag{2.15}$$

This may be expressed as

$$G_{ret}(x, x') = \frac{\theta(t - t')}{8\pi^2}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\sigma_0}\, e^{i\alpha\sigma_1}\,. \tag{2.16}$$

We now replace the classical metric perturbations by gravitons in a squeezed vacuum state $|\psi\rangle$. Then σ_1 becomes a quantum operator which is linear in the graviton field operator, $h_{\mu\nu}$. Because a squeezed vacuum state is a state such that σ_1 may be decomposed into positive and negative frequency parts, i.e., we may find σ_1^+ and σ_1^- so that $\sigma_1^+|\psi\rangle = 0$, $\langle\psi|\sigma_1^- = 0$, and $\sigma_1 = \sigma_1^+ + \sigma_1^-$. Thus, the derivation of Eq. (2.33) holds here as well and enables us to write

$$\left\langle e^{i\alpha\sigma_1}\right\rangle = e^{-\frac{1}{2}\alpha^2\langle\sigma_1^2\rangle}\,. \tag{2.17}$$

Thus when we average over the metric fluctuations, the retarded Green's function is replaced by its quantum expectation value:

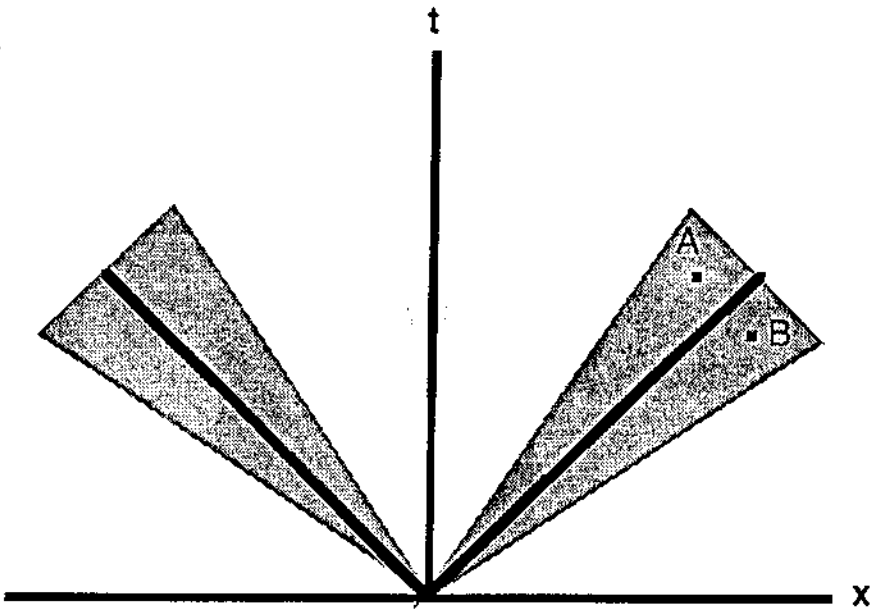

Figure 4 :
The smearing of the lightcone due to metric fluctuations. A photon which arrives at Point A from the origin has been slowed by the effect of metric fluctuations. A photon which arrives at Point B has been boosted by metric fluctuations, and appears to travel at a superluminal velocity in the background metric.

$$\left\langle G_{ret}(x,x')\right\rangle = \frac{\theta(t-t')}{8\pi^2}\int_{-\infty}^{\infty} d\alpha e^{i\alpha\sigma_0} e^{-\frac{1}{2}\alpha^2\langle\sigma_1^2\rangle}\,. \tag{2.18}$$

This integral converges only if $\langle\sigma_1^2\rangle > 0$, in which case it may be evaluated to yield

$$\left\langle G_{ret}(x,x')\right\rangle = \frac{\theta(t-t')}{8\pi^2}\sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}}\, exp\left(-\frac{\sigma_0^2}{2\langle\sigma_1^2\rangle}\right). \tag{2.19}$$

Note that this averaged Green's function is indeed finite at $\sigma_0 = 0$ provided that $\langle\sigma_1^2\rangle \neq 0$. Thus the lightcone singularity has been smeared out. Note that the smearing occurs in *both* the timelike and spacelike directions. This is illustrated in Fig. 4.

This smearing may be interpreted as due to the fact that photons may be either slowed down or boosted by the metric fluctuations. Photon propagation now becomes a statistical phenomenon; some photons travel slower than light on the classical spacetime, whereas others travel faster. We have now the possibility of "faster than light" signals. This need not cause any causal paradoxes, however, because the theory is no longer Lorentz invariant. The graviton state defines a preferred frame of reference.

In a similar way, we may average the other singular functions over metric fluctuations. For example, for the case that $\langle\sigma_1^2\rangle > 0$, the Hadamard function becomes

$$\Big\langle G_1(x,x')\Big\rangle = -\frac{1}{2\pi^2}\Big\langle\frac{1}{\sigma}\Big\rangle = -\frac{1}{2\pi^2}\int_0^\infty d\alpha \, sin\, \alpha\sigma_0 \, e^{-\frac{1}{2}\alpha^2\langle\sigma_1^2\rangle}. \tag{2.20}$$

In the limit that $\sigma_0^2 \gg \langle\sigma_1^2\rangle$, we recover the usual form of G_1 :

$$\Big\langle G_1(x,x')\Big\rangle \sim -\frac{1}{2\pi^2}\frac{1}{\sigma_0}. \tag{2.21}$$

On the other hand, near the lightcone, $\Big\langle G_1(x,x')\Big\rangle$ is finite:

$$\Big\langle G_1(x,x')\Big\rangle \sim -\frac{\sigma_0}{2\pi^2\langle\sigma_1^2\rangle}, \qquad \sigma_0^2 \ll \langle\sigma_1^2\rangle. \tag{2.22}$$

The average of the Feynman propagator over metric fluctuations may be obtained from Eqs. (2.19) and (2.22) and the relation

$$G_F(x,x') = \frac{1}{2}[G_{ret}(x,x') + G_{ret}(x',x)] - iG_1(x,x'), \tag{2.23}$$

and is also finite on the lightcone.

These averaged functions are, however, not finite in the limit of coincident points, that is in the limit that both σ_0^2 and $\langle\sigma_1^2\rangle$ vanish with their ratio finite. This can be understood on the grounds that the effect of metric fluctuations is to cause the propagation time for a photon to fluctuate. This causes an effect which grows with increasing spatial separation, but is small for points which are spatially close to one another.

Thus the metric fluctuations as treated here do not remove all of the ultraviolet divergences of quantum field theory. They do, however, lead to a modification, whose consequences need to be better understood.

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