ELEMENTS OF BRST THEORY

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I. THE GHOST, YOU'VE COME A LONG WAY BABY

1.1 Introduction

The BRST symmetry shows that ghosts are real. I have chosen to start the lectures like this, because I think that, at least for a theoretical physicist, the statement is correct.

Furthermore, in order to have a least a chance of fully exploiting the richness inherent in the BRST symmetry, one must try to go all the way. That cannot be done if one does not feel at home with ghosts, or if one thinks that they are some kind of embarrassing mathematical artefact that must be included to fix up some details. Therefore, it is important to cross that threshold immediately, so that one can relax and enjoy what follows.

In order to make that step easier, I would first like to make some general comments on the historical evolution of the idea of ghost in physics and its relation with the basic principles of quantum mechanics. These comments will not be logically needed for the subsequent presentation, which does not follow the historical route.

1.2 Quantum mechanics, the art of finding and combining simple elementary processes

In particle physics language, a ghost is a particle that obeys the wrong relation between spin and statistics. The first kind of ghosts were introduced by Feynman [1] in 1963, when studying quantum theory of gravitation. They were vector particles, that is, particles of integer spin (1 and 0) obeying Fermi statistics. Probably the first mention of the ghost in this sense in the physics literature are the following phrases in Feynman's article,

> *...I found it by trial and error...you must substract from the answer...the result you get imagining that...an artificial dopey particle is coupled to it (the graviton). It's a vector particle.*

Well, since 1963 the ghost has come a long way. We do not, or rather we should not think of it anymore as an "artificial" or "dopey" thing, although many of us still have that tendency. Rather, the lesson that we have been taught, sometimes the hard way, is that the distinction between ghosts and "real matter" is not one to be taken too seriously.

To see this, it is best to look at quantum mechanics from the point of view of the sum over histories. In that formulation, the amplitude $K(2,1)$ to propagate from configuration 1 to configuration 2 is given by,

$$K(2,1) = \sum_{\substack{histories \\ joining\ 1\ and\ 2}} e^{\frac{i}{\hbar}S[history]} \tag{1.1}$$

The idea is illustrated in Fig. 1.

Figure 1 The quantum amplitude is obtained by summing over all possible histories joining the configurations 1 and 2.

In Dyson's words [2] on occasion of the Einstein centennial in 1979 , this formula is described as follows,

> *...Thirty-one years ago, Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes,' he said. 'It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function.' I said to him, 'You're crazy.' But he wasn't.*

In the classical limit of actions which are large compared to Planck's constant, due to destructive interference, only histories close to the one which makes the action stationary contribute to the sum. This is why action principles arise in classical physics.

Now, the point here is that although the resulting amplitude $K(2,1)$ may be quite complicated, the amplitude for an elementary process, or history, is simple. It takes the form,

$$\exp \frac{i}{\hbar}S \tag{1.2}$$

But this is not all, another important ingredient must be added. In field theory, (and by field theory here I mean something very general, including things like quantum gravity and strings), the action may be written as,

$$S = S_{kin}(\text{free}) + S_{interaction} \qquad (1.3)$$

The splitting (1,3) is not just a technicality or a calculational tool. It determines our whole physical picture of objects (particles, say) that propagate freely in between interactions. If we did not have this splitting, there would not be that much information contained in (1.2). We would only know that the elementary amplitude must have an absolute value equal to unity, and that is just too flexible.

The sum over histories, together with the splitting of the action into free part and an interaction, lead directly to Feynman diagrams, like the familiar one of quantum electrodynamics shown in Fig. 2, which describes the interaction between two electrons due to the exchange of one photon. To obtain the amplitude for that process one must sum, according to the general rule, over all alternative ways of exchanging a photon.

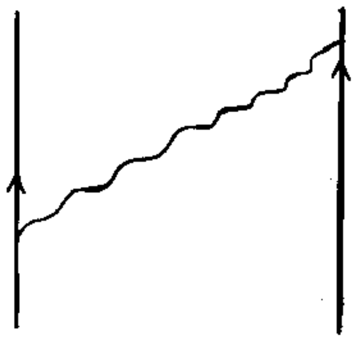

Figure 2 Interaction between two electrons due to exchange of a photon.

There are different ways of doing this. The most convenient and physically transparent one is that which exhibits manifest Lorentz invariance. In that case, one includes in the sum over all possible exchanged photons, not only those which are polarized transverse to the direction of motion, but also those whose polarization vector lies along the momentum (longitudinal photons) and along the time axis (time-like photons). There is also another way of calculating the same amplitude in which only transverse photons are exchanged, but a supplementary instantaneous Coulomb interaction must then be added.

Here we see in a familiar context the first example of a general occurrence which appears to be of importance; namely, if we insist in formulating a theory in terms of as few variables as possible, the elementary machinery becomes less transparent and we lose understanding rather than gaining it.

In fact, look at what happens with the exchanged photons. We may view the instantaneous Coulomb interaction as resulting from performing first the sum over longitudinal and time-like modes, while leaving the sum over transverse polarization yet undone. However, the result of that partial sum is no longer of the form exp $(\frac{i}{\hbar}S)$ with S of the form (1.3).

Thus, this seemingly innocent (and technically correct) step carries with it abandoning the picture that the interaction is completely accounted for by an exchange of particles. This is not just a philosophical loss, since the Lorentz invariance becomes also obscured. Conversely, if one insists in a formulation based on a simple, uniform, elementary process, one gains as a bonus Lorentz invariance.

What I am trying to say is that it appears to be a good principle to insist on an elementary amplitude of the form $exp(\frac{i}{\hbar}S)$. If we start giving up and allowing things like $A\ exp(iB)$ we might as well give up completely and try to write the full amplitude $K(2,1)$ right away, which is not likely to work.

This important example of quantum electrodynamics also teaches us another lesson. We must be willing to pay a certain price in order to have a simple, uniform, elementary process. We must not panic too easily. The time-like photons have negative kinetic energy and their contribution to the probability enters with a minus sign. Yet we must have them; they are a good thing.

It is often said that no catastrophe takes place due to these minus signs because the time-like and (longtitudinal) photons are not "real" but they are just "virtual." By this it is meant that they do not take part in processess like the one illustrated in Fig. 3b but they only appear in diagrams such as the one of Fig. 3a.

This is correct, but the terminology is somewhat unfortunate and misleading. Indeed, the virtual photons are not less real that the other ones in the sense of having observable physical effects. They contribute to energy levels as reflected, for example, in a small shift (the Lamb shift) they produce in the spectrum of light from excited hydrogen atoms. It is just that they cannot be observed directly with something like a photo-cell.

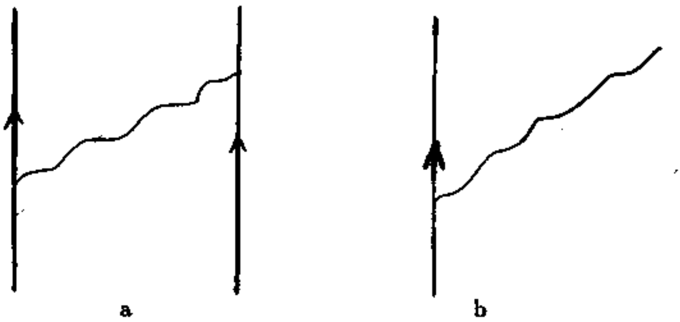

Figure 3 The process shown in (3a) is possible for both real and virtual photons while in (3b) only a real photon can appear.

I would like to indulge here in a slight digression: The more familiar one gets with quantum mechanics, the more blurred in one's mind the distinction between "real" and "virtual" processes becomes. Indeed, relativistic quantum mechanics is somehow engineered so that it is consistent to have a distorted view of the world. For example, a relativistic particle has a non-zero amplitude to cross the light-cone and can even turn back in time. Yet we can, because of that very fact, reinterpret things so that our usual notions of causality are not violated. In the same way, the consistency of a world in which only traverse photons can be observed by a particle detector may be thought of as being possible because of the existence of "virtual" photons of all four polarizations, otherwise the sum of all probabilities for all mutually exclusive "real" processes would not be equal to unity.

It would seem that the more we progress toward systems that are less familiar and reach into smaller scales, the more useful it should be to take quantum mechanics as it naturally comes, without manacling it early in the game with things like the observer or the measuring process.

As an example of how respectable people are willing to gamble along this line, I would like to quote from a paper by Hawking [3] on the path integral approach to quantum gravity. To put the quotation in context I should mention that for the gravitational field, the "initial configuration" is one three-dimensional space and the "final configuration" is another three-dimensional space. The elementary process or history which interpolates between them is a four-dimensional space-time which has the initial and final three-spaces boundaries.

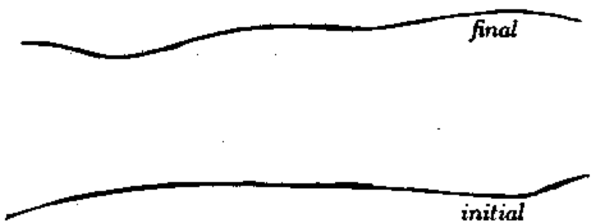

Figure 4 Initial and final configurations for the gravitational field are space-like hypersurfaces.

The quotation is the following:

Ultimately I suspect that one should do away with all boundary surfaces and should deal only with closed space-time manifold.

You see what this means. It means that there would be no analog of what one calls "external lines" in a Feynman diagram. In other words, there would be no "real" space-times. The whole theory would refer only to an enormous maze of "virtual" space-times.

At the present state of development this is just speculation. I have mentioned it only to illustrate the way of thinking.

1.3 Ghosts necessary to keep elementary processes simple

I realize that you are becoming impatient because I never seem to get to the point. However, after all this softening-up you should be ready to believe in ghosts.

We have so far been using quantum electrodynamics as an example. There, one may view the need for the longitudinal and time-like photons as arising from the demand that the sum of all the probabilities for processes involving only "real" particles in the initial and final states should be equal to unity. However this turns out not to be sufficient in more complicated theories. One needs in general, another, more exotic kind of "virtual" particle, and the price to pay in mental flexibility in order to have a simple and uniform elementary process is stiffer. One must accept that the new modes obey a relation between spin and statistics which is opposite to that of usual matter. These new modes are called ghosts. They seem to be more shocking or, I suppose I should say, "ghostlier" than, say, the time-like photons. The reason is that they are either fermions with integer spin or bosons with half-integer spin. However, in my opinion this is mostly psychological since we have had already forty years or so to learn how to live with those other ghosts– the longitudinal and time-like photons.

One may view this "wrong" connection between spin and statistics in the same way as we viewed above the fact that in quantum mechanics particles can propagate faster than light and backward in time. That is, at a basic level there is really no connection between spin and statistics. Anything can happen. One can have bosons with integer spin, fermions with integer spin, bosons with half-integer spin and fermions with half-integer spin. Yet, things are somehow engineered so that in the world that is directly accessible to us, we may consistently imagine that particles do not turn around in time, do not travel faster than light and they do obey the spin- statistics theorem.

To go further, I have to explain what I meant by "more complicated theories" above. I meant a nonabelian gauge theory.

A gauge theory is one which is invariant under a symmetry that acts independently at different points. Since one may define a geometrical object as something which is invariant under a set of transformations, one may say that gauge theories are field theories with geometrical content. This is what makes them theoretically attractive.

The simplest of gauge theories is electrodynamics and the gauge transformation for the photon is the familiar one,

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{1.4}$$

Here, $\Lambda = \Lambda(x)$ is an arbitrary function over space-time. The fact that there is only one function involved, means that the symmetry upon which the theory is built, has only one parameter and hence it is abelian. One then says that electrodynamics is an abelian gauge theory. The extension of this idea to a nonabelian group is due to Yang and Mills. In that case one has a set of functions $\Lambda^a = \Lambda^a(x)$ where the index A runs over the generators of a Lie group. One also has in that kind of theory, not just one "photon" field but a whole collection of them $A^a_\mu(x)$. The gauge transformation (1.4) is modified to read,

$$A^a_\mu \to A^a_\mu + \partial_\mu \Lambda^a + C^a_{bc} A^b_\mu \Lambda^c \tag{1.5}$$

where the real numbers C^a_{bc} are the structure constants of the (nonabelian) gauge group.

The symmetries present in electrodynamics and in its extension, the Yang-Mills theory, are what we call internal gauge symmetries. Technically this appears in the fact that the gauge transformation does not contain derivatives of the fields and hence does not connect different space-time points. There is another kind of gauge symmetry, which is perhaps even more interesting, and which does act on space-time and hence it is not internal. In that case, the gauge parameter Λ is labeled by a space-time index instead of an internal gauge index. For example, in general relativity one has the gauge transformation for the graviton (reparametrization),

$$g_{\mu\nu} \to g_{\mu\nu} + \Lambda_{\mu,\nu} + \Lambda_{\nu,\mu} + \ldots \tag{1.6}$$

and the gauge parameter carries a vector index. Another important instance of a noninternal symmetry is the gauge supersymmetry ("the square root" of a reparametrization) present in supergravity, where the change in the "gravitino" field is given by

$$\psi^A_\mu \to \psi^A_\mu + \partial_\mu \Lambda^A + \ldots \tag{1.7}$$

In this case, the gauge parameter Λ carries a spinor index A and it is anticommutative. For this reason the transformation mixes fermions with bosons and is called a supersymmetry transformation.

The general practical rule for the appearance of ghosts in a gauge theory can now be stated. In each case there will be a pair of new fields, usually called "ghost" and "anti-ghost," for each gauge function present in the gauge transformation. The ghost field will be anticommuting (fermionic) if the corresponding gauge parameter was commuting and vice versa.

Therefore, in electrodynamics there is one pair ($C\bar{C}$) of fermionic scalar (spin 0) ghosts, whereas in the Yang-Mills case we have a whole collection (C^a and $\bar{C}_a$) of fields with the same characteristics.

It is important to emphasize that there are indeed ghosts in quantum electrodynamics. The reason that they were not seen to be necessary before is that in the path integral for a gauge theory one has the freedom of choosing a gauge condition. In electrodynamics there is a simple gauge condition, the Lorentz gauge, which is linear in the fields and does not destroy the simple nature of the elementary process of photon exchange. If that gauge condition is used, the ghosts decouple completely from the photon and can therefore be ignored. However, if one would use other, nonlinear, gauge conditions it would not be possible to ignore the ghosts.

The rise of the ghost has an interesting history with its ups and downs which have reflected the influence of other advances in field theory. As I said at the beginning, the first ghost that was seen to be necessary was the fermionic vector ghost of quantum gravity, coming from the Λ_μ in (1.7). This ghost was found by Feynman as he says, "by trial and error." He was trying to make the sum of all the probabilities for graviton-graviton scattering to be equal to unity and he realized that even including the time-like and longitudinal modes of the graviton, as in electrodynamics, would not do. But of course he, for one, was very fond of Feynman diagrams and tried to overcome the difficulty by bringing in new diagrams which would involve new particles.

There is good reason to be fond of the diagrams because they are not just a calculational tool. They carry with them the key message that the whole complicated theory is built by putting together a few simple elementary processes which are very similar to each other. Thus, with this idea in mind, it was natural to attempt to cure the disease by simply extending what had already been done for electrodynamics, and "introducing" yet another "virtual" degrees of freedom besides the longitudinal and time-like gravitons. I have used here quotation marks for the words "introducing" and "virtual," to emphasize that one is not putting something artificial in by hand, but rather one is discovering something quite real that was there all the time.

This approach, let to a satisfactory formulation which was already written in the desired form in which all fields present, including the time-like and longitudinal modes and the ghost, appeared on the same footing describing particles that propagate freely in between interactions. That is, the total action was written in the form (1.3)

Next, there took place another development which has been hailed with good reason, but from the point of view taken here could be thought of as "reactionary." Faddeev and Popov [4] observed in 1967 that, since the gauge transformations that we wrote before do not change the physical fields (by definition of a physical field), one should include in the sum over histories only classes of equivalence of gauge-related histories, rather than histories themselves. From this nice geometrical point of view, they concluded that the amplitude for an elementary process was not of the form $exp(\frac{i}{\hbar}S)$ but instead of the form

$$\det \mathrm{M}\ (\exp \frac{i}{\hbar} S) \tag{1.8}$$

where the Faddeev-Popov determinant det M, is a certain functional of the fields. They then went on to say [5],

> *the expression...for the S-matrix contains the nonlocal functional det M and therefore does not look like the familiar integral of the Feynman functional exp (i action)... We may, however, use for det M the integral representation...*

Now, this "integral representation" was given, of course, in terms of the ghost fields and brought us back to the expression previously obtained directly from the diagrams. One gained with this a nice and useful connection between the ghosts and the geometry of the gauge field. There was, however, at the same time, a negative effect that came from the psychological impact of the word "representation." We became used to thinking that the understandable thing was the Faddeev-Popov determinant and that the ghosts were merely a technicality to represent it. Perhaps we would have been more flexible if we had kept in mind that this way of thinking was analogous to taking the instantaneous Coulomb interaction in electrodynamics more seriously than the time-like and longitudinal photons.

1.4 BRST symmetry: ghosts and matter become different components of single geometrical object

Perhaps this psychological blockade is one of the reasons that made it necessary for seven more years to pass before the important discovery [6] which put the ghost once and for all at the same level with "real" matter, the BRST symmetry. The other, and more significant reason for this delay would seem to be that the concept of supersymmetry [7] was only formulated in 1973. Indeed, otherwise the development in question could have naturally taken place already in 1967.

The BRST symmetry, where the initials stand for Becchi-Rouet-Stora and Tyutin, may be taken to be the basic invariance of the quantum mechanics of a geometrical system. It contains and extends the concept of gauge invariance. In it, the "original fields" get mixed with the ghosts. For example, the BRST transformation of the Yang-Mills field is given by

$$\delta A^a_\mu = \epsilon \partial_\mu C^a + ... \tag{1.9}$$

where ϵ is an anticommuting parameter. It is necessary that ϵ should be anticommuting, because the field A^a_μ is bosonic, whereas the ghost C^a is fermionic. In this sense, the BRST transformation is a supersymmetry transformation.

The fact that the ghosts and the other fields take part in an invariance which mixes them with each other, means that they are all to be thought of as components of a single geometrical object. This is much in the same way as we consider the electric and magnetic fields as components of one single field, the electromagnetic field, because they get mixed under the Lorentz transformation. Thus, from now on, we have to get used to thinking in terms of "BRST multiplets" instead of "original fields" and ghosts. The task of finding geometrical quantum theories is then the task of developing a BRST "tensor calculus."

This would be already attractive by itself but it is made even more so by a spectacular property of the BRST transformation: If we apply it twice we get zero. That is, if we call Ω the BRST generator, we have the equation

$$\Omega^2 = 0 \tag{1.10}$$

and we say that Ω is nilpotent.
The construction of a nilpotent BRST charge for any gauge system, including non-internal symmetries, was completed with the work of Fradkin, Batalin, Vilkovisky and Henneaux [8].

Equation (1.10), if shown to a mathematician, makes him tremble, because it expresses as John Wheeler likes to put it, that "the boundary of the boundary is zero." Thus, one expects that there should be a deep connection between BRST invariance and topology. This relation has not yet been fully unraveled.

The BRST invariance has already had a number of useful practical applications, both in Yang-Mills and string theory. I will just mention some of them here.

The first one is connected with the business of the Faddeev-Popov determinant. If one takes the original Faddeev-Popov expression for the amplitude and "represents" the determinant as an integral over "additional variables" called ghosts, the resulting ghost action is always quadratic in the ghost field. This means that the ghosts interact with the other fields of the theory but they do no interact directly among themselves. However, if one looks in detail, in terms of diagrams, at the balance of probability in a complicated theory like supergravity, one finds that it is necessary to include terms which are quartic in the ghosts, that is, self-interactions of the ghosts. Now, there is no way in which this can come out of a direct application of the Faddeev-Popov presectiption which only gives a quadratic ghost action. This came as a bit of a shock because it clearly invalidates the view that the ghosts are simply a way to represent a determinant. However, if one looks at the situation from the point of view of BRST invariance this is not an exceptional case at all; the interactions of the ghosts among themselves are necessary in such a case just to ensure BRST invariance.

Another important application of the BRST invariance is the one which actually directly motivated its discovery. It is the fact that it is responsible for the Ward-Slavnov identities which are crucial in proving the renormalizability of the Yang-Mills theory. As a result of this understanding, a greatly simplified proof of the renormalizability became available.

The importance of the BRST invariance becomes even greater for generally covariant theories like gravity, supergravity and strings. The reason is that in that kind of theory, gauge invariance is everything. There is no distinction between dynamics and a gauge transformation, because the latter is just a localized displacement in space-time.

The condition for the BRST invariance of permissible states,

$$\Omega|\psi>=0 \tag{1.11}$$

is then *the* equation of the theory.

In string theory, the nilpotency condition provides a particularly transparent way to understand the need for the critical dimension. Indeed, if one computes Ω^2 for, say, the free bosonic string immersed in a D-dimensional spacetime, one finds that it has two contributions. One of the form D times a factor coming from the "original fields," and another equal to -26 times the same factor, coming from the ghosts. The condition $D = 26$ results thus from precisely balancing the "matter" and the "ghosts." What else could one ask to be convinced that the ghosts are on the same footing with matter!

The applications of the BRST symmetry, and hence, of the concept of ghost which I have just described are quite significant. But one has the feeling of having seen only the tip of the iceberg. Probably the deepest developments along these lines should lie in developing theories where elementary extended objects (strings in string theory, three dimensional spaces in quantum gravity) can undergo quantum mechanical decay. A field theory of strings along these lines has been proposed by Witten [9], not long ago. Another interesting line of development, perhaps more reachable, lies along the understanding of anomalies in gauge theories. Of course all these developments, being based on BRST invariance, could not be even thought of without allowing ghosts to play a prominent role.

The ghost has indeed come a long way since it was called in 1963 an "artificial dopey particle" by its own discoverer.

II. BRST SYMMETRY IN CLASSICAL MECHANICS

2.1 Ghosts have role in classical mechanics

As it was reviewed in the first lecture, the need for ghosts and the symmetry that reveals their profound importance were first established in quantum mechanics. It was only afterwards realized that they have a natural and necessary place within classical mechanics as well. Indeed, the BRST symmetry could have been discovered in the last century within a strictly classical context by mathematicians dealing with the geometry of phase space had they only been willing to extend their analysis to Grassmann variables.

Having said this, it should be immediately clarified that I am not advocating a direct physical meaning for ghosts within classical mechanics. Their physical implications can only be infered by explicit use of quantum mechanical laws. It is nevertheless extremely useful to be able to discuss them and the BRST symmetry classically. One can then bring in these concepts as a powerful tool in the actual construction of the quantum theory.

The role of ghosts in classical mechanics emerges most clearly through the Hamiltonian formulation of the classical dynamics of gauge systems, which provides the most general ground for systematically discussing the BRST symmetry. This will be the starting point for our whole discussion of BRST invariance. It will appear that through Hamiltonian methods one obtains a formulation of great generality and power. In particular one frees oneself from the assumption that the gauge transformations obey a group composition law. The results cover therefore, the general case of an "open algebra." Furthermore, they are valid "off-shell."

2.2 Gauge invariance and constraints

One says that a dynamical system is a gauge system if the general solution of the equations of motion contains arbitrary functions of time which are not fixed by the initial conditions.

In practice, a gauge system is most often given by specifying the action integral in lagrangian form. The procedure for passing from the lagrangian to the hamiltonian was worked out by Dirac long ago [10]. It will be assumed here that the system is already given in hamiltonian form.

By definition, one says that all the histories which spring from the same initial condition are physically indistinguishable and are related to each other by a gauge transformation. The gauge transformation turns out to be a canonical transformation whose generators will be denoted by $G_a(q,p)$.

2.3 Classical mechanics over Grassmann algebra necessary

The notation (q,p) is being used to represent a set of canonically conjugate coordinates in phase space. Some of these coordinates might be commuting and other anticommuting. More precisely, they will be assumed to be elements of a Grassmann algebra with definite Grassmann parity. One must allow for anticommuting (q,p) in order to have a classical description of fermions, but even in a theory which has no fermions to start with, they will be brought in when the ghosts are included. Therefore, we have to be prepared and have a classical mechanics capable of dealing with anticommuting numbers, so we allow for that possibility from the very beginning. Also, it is not necessary for what follows to have canonically conjugate pairs. Indeed, there are cases of interest such as the classical description of spin, in which the dimension of phase space is odd and conjugate pairs are not available. Those cases are included in the present discussion which just needs the existence of a Poisson bracket. I have chosen to use the notation (q,p) anyway, because it has a phase space ring.

2.4 Higher order structure functions

In order to avoid unnecessary cluttering of the equations with sign factors in this introductory account, I will only deal with the case in which there are no fermionic coordinates among the (q,p). Actually it suffices to assume that the G_a are commuting. All the important results carry over unchanged to the general Bose-Fermi case.

It will also be assumed for simplicity that the constraints G_a are independent or "irreducible," as one says. That is, the Jacobian matrix $(\partial G_a/\partial q,\ \partial G_a/\partial p)$ is of maximal rank everywhere on the constraint surface. This means that one can locally take the G_a's as the first m (non-canonical) coordinates in phase space. The reducible case will not be dealt with. Again, the main results still hold (provided one adds even more ghosts!– usually called "ghosts of ghosts").

To begin with, the constraints G_a will be also called "zeroth order structure functions" and will be indistinctly denoted by $\overset{(0)}{U}_a$. From the point of view of the action principle these constraints are not uniquely determined. They can always be replaced by an equivalent set,

$$\overset{(0)}{U}_a(q,p) = M_a^b(q,p)\,\overset{(0)}{U}_b(q,p), \;\; det\;\; M \neq 0 \tag{2.1}$$

because the equations $\overset{(0)}{\bar{U}} = 0$ are equivalent to $\overset{(0)}{U} = 0$. This change in the generators implies of a change in the description of the gauge transformations. If, for example, the structure functions C^c_{ab} associated to the first set are independent of (q,p) (structure constants of a group), that property will not hold for the new set. One expects, of course, that the two descriptions will be equivalent, but that equivalence is not transparent. It will be one of the virtures of bringing in the ghosts to make the equivalence manifest.

A new notation and a new name will be also introduced for the structure functions $C^c_{ab}\ (q,p)$ appearing in

$$[G_a, G_b] = C^c_{ab} G_c \tag{2.2}$$

They will be denoted by $-2\,\overset{(1)}{U}{}^{c}_{ab}$ and the $\overset{(1)}{U}$'s will be called "first order structure functions." The first class property of the constraints reads then,

$$\left[\overset{(0)}{U}_a, \overset{(0)}{U}_b\right] = -2\,\overset{(1)}{U}{}^{d}_{ab}\,\overset{(0)}{U}_d \tag{2.3}$$

The first order structure functions carry with them an ambiguity over and above that implied by the ambiguity (2.1) in $\overset{(0)}{U}$. Indeed, once $\overset{(0)}{U}$ is fixed, equation (2.3) determines $\overset{(1)}{U}$ only up to

$$\overset{(1)}{U}{}^{c}_{ab} \rightarrow \overset{(1)}{U'}{}^{c}_{ab} = \overset{(1)}{U}{}^{c}_{ab} + M^{cd}_{ab}\,\overset{(0)}{U}_d \tag{2.4a}$$

with

$$M^{cd}_{ab} = -M^{dc}_{ab} = -M^{cd}_{ba} \tag{2.4b}$$

Having introduced an appropriate notation for $\overset{(0)}{U}$ and $\overset{(1)}{U}$, the analysis proceeds as follows. One takes the Poisson bracket of equation (2.3) with $\overset{(0)}{U}_d$ and fully antisymmetrizes in a, b and c. The left-hand side then vanishes by virtue of the Jacobi identity, while the right side becomes, with the help of (2.3), a linear combination of the $\overset{(0)}{U}$'s. This yields,

$$\overset{(0)}{U}_d \left(\left[\overset{(1)}{U}{}^d_{[ab}, \overset{(0)}{U}_{c]} \right] + 2\, \overset{(1)}{U}{}^e_{[ab} \overset{(1)}{U}{}^d_{c]e} \right) = 0 \qquad (2.5)$$

Now, (2.5) does not imply that the coefficient of $\overset{(0)}{U}$ vanishes, because that equation is clearly identically satisfied if one sets

$$\left[\overset{(1)}{U}{}^d_{[ab}, \overset{(0)}{U}_{c]} \right] + 2\, \overset{(1)}{U}{}^e_{[ab} \overset{(1)}{U}{}^d_{c]e} = 2\, \overset{(2)}{U}{}^{de}_{abc} \overset{(0)}{U}_e \qquad (2.6)$$

where the thereby-defined second order structure functions $\overset{(2)}{U}{}^{de}_{abc}(q,p)$ are antisymmetric in (d,e) and (a,b,c). Again, once $\overset{(0)}{U}$ and $\overset{(1)}{U}$ are fixed, the $\overset{(2)}{U}$'s bring in their own ambiguity. They are determined up to

$$\overset{(2)}{U}{}^{de}_{abc} \rightarrow \overset{(2)}{U}{}^{de}_{abc} + M^{def}_{abc} \overset{(0)}{U}_f \qquad (2.7)$$

where M^{def}_{abc} is completely antisymmetric in both the upper and lower indices. It may be shown [8] that, under the assumption that the constraints are irreducible, equation (2.6) with the ambiguity (2.7) is the most general solution of (2.5) and it always exists.

The construction leading to the appearance of the second order structure functions may be systematically continued. Thus, third order structure functions will appear by taking the Poisson bracket of (2.6) with $\overset{(0)}{U}_f$ and fully antisymmetrizing in a, b, c and f. Using the Jacobi identity and the defining equations (2.3) and (2.6) of the structure functions of lower order, yields the identity

$$\overset{(2)}{D}{}^{[a_1 a_2]}_{[b_1 b_2 b_3 b_4]} \; \overset{(0)}{U}{}_{a_2} = 0$$

where,

$$\overset{(2)}{D}{}^{a_1 a_2}_{b_1 b_2 b_3 b_4} = -\tfrac{1}{2}\sum_{q=0}^{2}\left[\overset{(q)}{U}{}^{a_1 \ldots a_q}_{b_1 \ldots\ldots b_{q+1}} \; , \; \overset{(2-q)}{U}{}^{a_{q+1} \ldots a_2}_{b_{q+2} \ldots\ldots b_4}\right]$$

$$-\sum_{q=0}^{1}(q+1)(3-q) \overset{(q+1)}{U}{}^{a_1 \ldots a_q c}_{b_1 \ldots\ldots b_{q+2}} \overset{(2-q)}{U}{}^{a_{q+1} \ldots a_2}_{b_{q+3} \ldots\ldots b_4 c} \tag{2.9}$$

From (2.9), it follows that

$$\overset{(2)}{D}{}^{[a_1 a_2]}_{[b_1 b_2 b_3 b_4]} = 3 \overset{(3)}{U}{}^{a_1 a_2 a_3}_{b_1 b_2 b_3 b_4} \overset{(0)}{U}{}_{a_3} \tag{2.10}$$

where the "third order structure functions" are completely antisymmetric in both the a and b indices.

For the higher order structure functions, one finds, in an analogous manner, the identities

$$\overset{(n)}{D}{}^{[a_1 \ldots a_n]}_{[b_1 \ldots\ldots b_{n+2}]} \overset{(0)}{U}{}_{a_n} = 0 \tag{2.11}$$

which imply in turn the existence of the structure functions of order $n+1$

$$\overset{(n+1)}{U}{}^{a_1 \ldots a_{n+1}}_{b_1 \ldots\ldots\ldots b_{n+2}} \equiv \overset{(n+1)}{U}{}^{[a_1 \ldots a_{n+1}]}_{[b_1 \ldots\ldots\ldots b_{n+2}]}$$

obeying

$$\overset{(n)}{D}{}^{[a_1 \ldots a_n]}_{[b_1 \ldots\ldots b_{n+2}]} = (n+1) \overset{(n+1)}{U}{}^{a_1 \ldots a_{n+1}}_{b_1 \ldots\ldots\ldots b_{n+2}} \overset{(0)}{U}{}_{a_{n+1}} \tag{2.12}$$

In (2.11) and (2.12), $\overset{(n)}{D}{}^{a_1 \ldots a_n}_{b_1 \ldots\ldots b_{n+2}}$ stands for

$$\overset{(n)}{D}{}^{a_1 \ldots a_n}_{b_1 \ldots\ldots b_{n+2}} = \tfrac{1}{2}\sum_{q=0}^{n}\left[\overset{(q)}{U}{}^{a_1 \ldots a_q}_{b_1 \ldots\ldots b_{q+1}} \; , \; \overset{(n-q)}{U}{}^{a_{q+1} \ldots a_n}_{b_{q+2} \ldots\ldots b_{n+2}}\right](-)^{nq+1}$$

$$-\sum_{q=0}^{n-1}(q+1)(n-q+1)\overset{(q+1)}{U}{}^{a_1\ldots a_q c}_{b_1\ldots\ldots\ldots b_{q+2}}\overset{(n-q)}{U}{}^{a_{q+1}\ldots a_n}_{b_{q+3}\ldots b_{n+2}c}(-)^{n(q+1)} \tag{2.13}$$

and only involves structure functions of order $\leq\ n$.

Finally, one easily checks that (2.12) determines the structure functions of order $n+1$ only up to

$$\overset{(n+1)}{U}{}^{a_1\ldots a_{n+1}}_{b_1\ldots\ldots b_{n+2}} \to \overset{(n+1)}{U}{}^{a_1\ldots a_{n+1}}_{b_1\ldots\ldots b_{n+2}} + M^{a_1\ldots a_{n+2}}_{b_1\ldots b_{n+2}}\overset{(0)}{U}{}_{a_{n+2}} \tag{2.14}$$

(for given structure functions of order $\leq\ n$), where $M^{a_1\ldots a_{n+2}}_{b_1\ldots b_{n+2}}$ possesses the appropriate antisymmetry properties.

The proof of the existence of the structure functions of order two and higher may be found in [8]. It will not be dealt with here. I will turn instead to the relation between the structure functions and to the BRST generator Ω. Afterwards, an independent proof of the existence of Ω will be given. That proof will, therefore, provide in turn a proof of the existence of the structure functions. However, this indirect method does not make the direct construction superfluous for two reasons. The first is that the direct construction provides an explicit way to write down the BRST generator for a given set of G_a's. This is done by systematically following the steps indicated in the previous paragraph. Second, the indirect proof as given below holds only locally in phase space, while no such restriction applies to the other procedure. (The indirect proof can also be extended to hold globally. See [11].)

2.5 Rank defined. Open algebras

One knows that for a Lie group, all the (local) geometric structure is contained in the structure constants C^c_{ab}. In that case one may take the $\overset{(2)}{U}$ and all the higher order structure functions equal to zero. However, in the general case, this will not be possible and structure functions up to some order n will appear. One then says that the theory is of rank n. It should be emphasized here that the notion of rank is not intrinsic to a given theory, but it depends on the choice of the constraints G_a (structure functions of order zero) and also on how the ambiguities in the choice of the structure functions of order one and higher are resolved. Indeed, as we shall see below, one may always choose, at least locally in phase space, a set of G_a such that any theory is of rank zero ("abelianization"). However, in general, that choice is cumbersome. For a field theory it typically leads to generators which are non local in space. In practice, there is always a choice, or perhaps a few choices, of the G_a which are privileged because of locality properties, covariance, etc. Thus, when I indulge, from now on, in speaking of "rank of a theory" I will have in mind the lowest possible rank associated with those natural choices. In this sense, the Maxwell field has rank zero, the Yang-Mills field, the relativistic string and Einstein theory of gravitation have rank 1, $N = 1$ supergravity in four space-time dimensions has rank 2, and the n-dimensional relativistic membrane has rank n. For a theory of rank n, the idea is that the local geometrical structure is contained not only in the first order structure functions, but also in those up to order n.

When the first order structure functions are not constant, one often says that "the gauge algebra only closes on-shell". This means that the commutator of two gauge transformations is a new gauge transformation only on the constraint surface. Note that for this to happen it is not necessary that the rank be higher than one. The gravitational field is an example of a theory of rank one whose Hamiltonian gauge algebra only closes on-shell.

The only on-shell closure property arises as follows. The commutator of two gauge transformations with parameters ϵ_1^a and ϵ_2^a, is a new transformation obtained by acting with $C_{ab}^c \epsilon_1^a \epsilon_2^b G_c$. This is a consequence of the Jacobi identity. However, the action δF of that transformation on a generic function F is

$$\begin{aligned} \delta F = [F, C_{ab}^c \epsilon_1^a \epsilon_2^b G_c] = C_{ab}^c \epsilon_1^a \epsilon_2^b [F, G_c] \\ + \epsilon_1^a \epsilon_2^b [F, C_{ab}^c] G_c \end{aligned} \tag{2.14}$$

and this is generated by G_c only on the surface in which G_c is equal to zero. Otherwise the second term of the right side of (2.14) makes this no longer true for every F unless C_{ab}^c is independent of q and p.

[Actually, this last assertion is a bit too strong. Indeed, if the C_{ab}^c depended on the q's and p's only through the generators G_a themselves the second term on the right of (2.14) would also have an overall factor $[F, G_c]$. However, in such a case there is no quarantee that the rank would be equal to one].

2.6 Ghosts. Ghost number. BRST generator as generating function for structure functions

The original phase space of the q's and the p's will e enlarged by introducing an additional canonical pair $(\eta^a, \mathcal{P}_a)$ for each first class constraint G_a present. The canonical pair will be taken to be of Grassmann parity opposite to that of the corresponding G_a. Thus, if the G's are all commuting, as we have been assuming for simplicity, the η's and the $\mathcal{P}$'s will be anticommuting. These extra variables will be called ghosts. More specifically, η will be called ghost and $\mathcal{P}$ anti-ghost.

The ghosts may initially be thought of as a useful bookkeeping device for concisely taking into account the properties of the structure functions. However, this narrow view lends itself to rapid change once one realizes that those properties imply the existence of a symmetry which mixes the ghosts with the original variables. This means that the appropriate space for describing the dynamics of the gauge system is not the original space of the q's and the p's, but rather, the extended one which includes the ghosts.

The basic properties of the ghosts are, therefore,

$$[\mathcal{P}_a, \eta^b] = [\eta^b, \mathcal{P}_a] = -\delta_a^b \tag{2.15}$$

$$(\eta^a)^* = \eta^a \quad , \quad (\mathcal{P}_a)^* = -\mathcal{P}_a \tag{2.16}$$

and

$$[\mathcal{P}_a, q^i] = [\mathcal{P}_a, p_i] = [\eta^a, q^i] = [\eta^a, p_i] = 0 \tag{2.17}$$

Note that η is taken to be real which implies that the conjugate $\mathcal{P}$ is imaginary. This property, and also the symmetry of the bracket (2.15) are due to the anticommuting character of the ghosts for bosonic G_a.

It is also convenient to define an additional structure on the extended phase space, that of ghost number. This is done by attributing the following ghost number to the canonical variables: the q^i, p_i's have ghost number zero, the ghosts η^a have ghost number one, and the antighosts $\mathcal{P}_a$ have ghost number minus one. Moreover, one requires that the ghost number of a product of variables is equal to the sum of their ghost numbers.

Consider now the following function on the extended phase space

$$\Omega = \sum_{n\geq 0} \eta^{b_{n+1}} \ldots \eta^{b_1} \overset{(n)}{U}{}^{a_1 \ldots a_n}_{b_1 \ldots b_{n+1}} \mathcal{P}_{a_n} \ldots \mathcal{P}_{a_1} \tag{2.18}$$

Note that due to the antisymmetry properties of U, there is no loss of information when contracting it with the anticommuting η's and $\mathcal{P}$'s. Indeed, one may recover the U's by repeatedly differentiating Ω with respect to the ghosts and setting them equal to zero afterwards.

The function Ω will be called the BRST generator. It has the following fundamental properties

$$\Omega \text{ is real }, \Omega = \Omega^* \tag{2.19}$$

$$\Omega \text{ has ghost number } +1, \; g(\Omega) = 1 \tag{2.20}$$

$$\Omega \text{ is anticommuting }, \epsilon(\Omega) = 1 \tag{2.21}$$

here ϵ denotes Grassmann parity. More importantly, it is nilpotent

$$[\Omega, \Omega] = 0 \tag{2.22}$$

Properties (2.19) through (2.21) follow because the n-th term in Ω contains n products $\eta\mathcal{P}$ and one "loose" η. Each product $\eta\mathcal{P}$ is real, has ghost number zero and even Grassmann parity while the loose η is also real, has ghost number +1 and is anticommuting.

The crucial property of Ω, its nilpotency, involves the detailed properties of the structure functions. Indeed, one may check directly that (2.22) is equivalent to the identities (2.12), (2.13) which define the U's. This shows that the generator Ω captures in a nutshell the complete gauge structure of the system. For this reason, it is natural to consider Ω as the central geometrical object in a gauge theory.

A remarkable feature of Ω, in the present Hamiltonian formulation, is that the nilpotency holds "off-shell," namely at all points in the extended phase space. This is so even for systems in which the gauge algebra in the original phase space of the q's and the p's only closes on the constraint surface, as discussed in Sec. 2.5.

2.7 Abelianization of constraints. Existence of Ω

As was mentioned in Sec. 2.5, the structure functions can be changed by replacing the constraints G_a by linear combinations of themselves. Since one admits in the linear combination coefficients which depend on q and p, the flexibility in the structure functions is enormously greater than the one available when changing the basis of generators of a Lie algebra. Indeed, one can even achieve in principle that the constraints become locally abelian, that is that all the structure functions vanish in a region of phase space. This possibility is useful for proving general properties such as the one we are interested in now, the existence of Ω, but it is not one which is of much use in practice. The reason is that simplicity requirements on the functional form of the G_a, which are specially important in the passage to quantum mechanics, such as polinomial structure or locality in field theory, are absent in the abelianized constraints.

The problem one faces is the following. Given a set of constraints G obeying

$$[G_a, G_b] = C^c_{ab} G_c \tag{2.23}$$

one wants to find an invertible matrix $M^b_a(p,q)$ such that

$$F_a(p,q) = M^b_a G_b \tag{2.24}$$

obeys

$$[F_a, F_b] = 0 \tag{2.25}$$

One may find a general proof of the existence of M in [8]. I would just like to mention here that the passage from G to F amounts to solving the constraint equation $G_a = 0$ to express some of the momenta in terms of the remaining variables. For example in the Yang-Mills case, one solves Gauss's law, for the longitudinal component of the "electric field." This may only be done in a perturbation series in the coupling strength, which illustrates that F exits only locally in that case. It follows from the work of Gribov [12] that the solution does not exist globally.

For the constraint F, the existence of Ω is immediate. One simply writes

$$\Omega^F = \eta^a F_a \tag{2.26}$$

which obeys all the properties listed in (2.19-2.22).

The question now is how to infer from Ω_F the BRST generator Ω_G corresponding to the constraint G_a. Clearly, if one could show that Ω_G is related to Ω_F by a canonical transformation, the problem would be solved. The reason is that the key nilpotency property $[\Omega, \Omega] = 0$, being written in terms of Poisson brackets, is invariant under canonical transformations. The reality, ghost number 1 and Grassmann parity 1, properties of Ω_G would be assured, if the generator of the canonical transformation is real, has ghost number 0 and Grassmann parity 0.

The canonical transformation that I am talking about here, should be a canonical transformation in the extended phase space and should unavoidably mix the ghost with the original p's and q's. This is so, because there is no way that F and G could be related by a canonical transformation in the p's and the q's only since the Poisson brackets are different.

The solution may be easily found in the case when the constraints G_a and F_a differ infinitesimally,

$$G_a = (\delta_a^b + \epsilon_a^b)F_b \tag{2.27}$$

with $\epsilon_a^b(p,q)$ small. The general case can then be obtained by exponentiation.

The question reduces then to that of finding the generator C of the canonical transformation such that

$$\Omega^{(G)} - \Omega^{(F)} = [\Omega^{(F)}, C] \tag{2.28}$$

where

$$\Omega^{(G)} - \Omega^{(F)} = \eta^b \epsilon_b^a F_a + \eta^b \eta^a \overset{(1)}{U}{}_{ab}^{\ c} \mathcal{P}_c + 0(\epsilon^2) \tag{2.29}$$

(the higher structure functions of the set G_a may be taken to be of order ϵ^2 or higher).

This generator C is obviously given by

$$C = \eta^b \epsilon_b^a \mathcal{P}_a \tag{2.30}$$

and has all the desired properties.

The above argument only covers the case in which the invertible linear transformation $M(p,q)$ is in the connected component of the identity, i.e., has positive determinant. The general case with both positive and negative determinant is easily included by observing that the particular matrix $M_b^a = diag(-1, 1, ...1)$, which possesses negative determinant, can be generated by the ghost canonical transformation

$$\eta^1 \to -\eta^1, \mathcal{P}_1 \to -\mathcal{P}_1, \eta^a \to \eta^a, \mathcal{P}_c \to \mathcal{P}_a (a \geq 2).$$

2.8 Uniqueness of Ω

The central idea of the discussion that I have been giving is to take the BRST approach as the basic description of the idea of gauge invariance. The reasoning of the preceding paragraph shows that this view has an important pay-off. It makes evident the equivalence between the descriptions based on different choices of the constraints G_a, an equivalence which is by no means transparent in the original phase space of the q's and the p's. Indeed, it emerges that the BRST generator is unique. For a given system, the Ω obtained from one choice of the whole tower of structure functions (including, in particular, the choice of the G_a) is related to that obtained from another choice by a canonical transformation in the extended phase space.

One may say that this important result shows that the "canonical covariance" of the theory becomes manifest only when one enlarges the original phase space to include the ghosts. Once more, simplicity and understanding are gained by adding variables and not by eliminating them.

When going over to quantum mechanics, it is impossible to realize all canonical transformations as unitary transformations in Hilbert space. Therefore, in the quantum case, different choices of the constraints may lead to BRST generators which are not unitary equivalent. This is not a problem of BRST theory, but rather a general problem of the passage from classical mechanics to quantum mechanics. In practice, one is happy if one can find *a* choice of the constraints which will lead to an Ω simultaneously satisfying the key requirements of nilpotency and hermiticity.

2.9 Classical BRST cohomology

The off-shell nilpotency of the BRST transformation enables one to introduce the notion of classical (non quantum) BRST cohomology for any gauge theory.* This concept permits one to implement the ideas of gauge transformation and gauge invariance in BRST terms.

Indeed, it follows from the nilpotency (2.22) of Ω and the Jacobi identity for the graded Poisson bracket that

$$[[A,\Omega],\Omega] = 0 \tag{2.31}$$

for any $A(q,p,\eta,\mathcal{P})$ on the extended phase space. Hence, one can define BRST-closed functions as functions which are BRST invariant,

$$[A,\Omega] = 0 \tag{2.32}$$

while BRST-exact functions are given by

$$A = [K,\Omega] \tag{2.33}$$

for some K, and are clearly BRST-closed.

The following idea proposes itself for exploration. To what extent can one identify two BRST-closed functions wich differ by a BRST-exact function? Or, what is the same, to what extent is the addition of a function of the form (2.33) the BRST analog of a gauge transformation? This is the question which is addressed by BRST cohomology.

As a result of the uniqueness of Ω, the classical BRST cohomology only depends on the first-class constraint surface defining the dynamical system under consideration, and not on how one represents this surface by the equations $G_a(q,p) = 0$, or on how one removes the ambiguity in the structure functions entering in the construction of Ω.

Because the BRST charge possesses definite ghost number one can study cohomology classes with given ghost number. Two equivalent functions will then differ by a sum of terms, each belonging to an equivalence class with definite ghost number.

One thus defines

$$\left(\frac{Ker\Omega}{Im\Omega}\right)^g_{classical} \tag{2.34}$$

as the set of equivalence classes of BRST-closed functions modulo BRST exact functions, with definite ghost number g.

The following result is then known,

$$\left(\frac{Ker\Omega}{Im\Omega}\right)^g_{classical} = \begin{cases} 0 & g < 0 \\ \left(\frac{Ker\ d}{Im\ d}\right)^g & g \geq 0 \end{cases} \tag{2.35}$$

The proof may be found in [13].

The operator d appearing in (2.35) generalizes the exterior derivative operator acting along the gauge orbits generated by the constraint functions G_a on the constraint surface. It acts on forms which are defined throughout the original phase space of the p's and the q's (without the ghosts), but differs from the phase space exterior derivative operator in that it only takes derivatives along the gauge orbits, and not along directions orthogonal to them. So, this operator d measures the variations of the forms under gauge transformations only, and remains insensitive to how the forms vary in transverse directions.

Closed zero forms are constant along the gauge orbits and thus, are just gauge invariant functions. The statement (2.35) says then that BRST invariants with ghost number zero, may be identified with gauge invariant functions in the original phase space. To see more precisely how the identification goes, one recalls that a gauge invariant function $A(q,p)$ is one that commutes weakly with the constraints. That is, one has

$$[A_o, G_a] = 0 \text{ on the surface } G_a = 0 \tag{2.36}$$

Furthermore, two gauge invariant functions A_o and A'_o are considered as equivalent if they differ by a term which vanishes in the constraint surface or, what is the same, if

$$A'_o = A_o + k^a G_a \tag{2.37}$$

The result that we are describing then says that, starting from any gauge invariant $A_o(q,p)$, one can construct its "BRST invariant extension" by adding to it terms which vanish when the ghosts are set equal to zero, in such a way that the resulting function in the extended phase space has zero Poisson bracket with Ω. The extension is not unique, one can always add to it a BRST exact function. This addition yields, when setting the ghosts equal to zero, the, ambiguity (2.37) in A_o.

The situation is therefore quite clear when the ghost number is zero. However, for g greater than zero the understanding is far from complete. Indeed, although the above theorem provides a phase space geometrical interpretation of the BRST cohomology in the case of non-vanishing ghost number, the physical meaning and the use of the cohomological classes with $g > o$ is still to be uncovered. It is of interest to point out that one may get non trivial gauge invariant functions from non trivial closed p-forms by integrating them along non trivial closed p-surfaces immersed the gauge orbits.

III. QUANTUM BRST THEORY

3.1 States and operators

It should be apparent at this point that the most natural and powerful description of the dynamics of a gauge system is that which treats the ghosts in the same footing with the "original" dynamical variables. To implement this same view point in quantum mechanics one must realize the q's, the p's, the ghosts η and their conjugate momenta $\mathcal{P}$ as linear operators in a Hilbert space.

In particular, the BRST charge becomes a linear operator. Since the Poisson bracket of two anticommuting functions becomes upon quantization an anticommutator, the nilpotency property of Ω reads,

$$[\Omega, \Omega] = 2\Omega^2 = 0 \tag{3.1}$$

Furthermore, since Ω was real in the classical theory one now demands that it should be a self-adjoint operator

$$\Omega^\dagger = \Omega \tag{3.2}$$

As a consequence of (3.1) and (3.2) the Hilbert space inner product must contain states with zero norm. Moreover, it follows from the ghost anticommutation rules that there must actually be negative norm states, as well as others with positive norm.

One also defines, by the same arguments used in the classical theory, a BRST observable as a linear operator A which commutes with the BRST charge

$$[A, \Omega] = 0 \tag{3.3}$$

Here the word 'commutes' is used in a generalized sense, the bracket in (3.3) is to be understood as an anticommutator when A is anticommuting.

I will assume that one can find a charge Ω satisfying the nilpotency and hermiticity conditions (3.1) and (3.2). Unlike the situation in the classical case, there is no a priori guarantee that this can always be done starting from a classical theory, since the question of ordering of the factors comes in crucially. For example in string theory, (3.1) and (3.2) only hold quantum mechanically for the critical value of the space-time dimension, whereas no such restriction appears in the classical problem. Indeed, the experience with string theory supports the view that when (3.1) and (3.2) do not hold the quantum theory is ultimately inconsistent.

Thus, we will regard (3.1) and (3.2) as the statements of the gauge invariance of the quantum theory and, for that reason, they will be taken as fundamental. If they cannot be satisfied, it would appear that the theory at hand is to be discarded.

3.2 Ghost number

The ghost number of an operator is defined as in the classical theory. It can be represented by the operator

$$\mathcal{G} = i\eta^a \mathcal{P}_a + \text{ constant} \tag{3.4}$$

which obeys

$$[\mathcal{G}, p_i] = [\mathcal{G}, q^i] = 0 \tag{3.5a}$$

$$[\mathcal{G}, \eta^a] = \eta^a \quad , \quad [\mathcal{G}, \mathcal{P}_a] = -\mathcal{P}_a \tag{3.5b}$$

If desired, one can adjust the constant in (3.4) so that $\mathcal{G}$ is anti-hermitian,

$$\mathcal{G} = \frac{i}{2}(\eta^a \mathcal{P}_a - \mathcal{P}_a \eta^a), \quad \mathcal{G}\dagger = -\mathcal{G} \tag{3.6}$$

It follows from (3.5) that if a state $|f>$ has definite ghost number g, $\mathcal{G}|f>= g|f>$, then, $q^i|f>$ and $p_i|f>$ have also ghost number g, whereas $\eta^a|f>$ and $\mathcal{P}_a|f>$ have ghost number $g+1$ and $g-1$, respectively.

Although anti-hermitian, $\mathcal{G}$ possesses real eigenvalues. This can be seen by expanding the states in the η^a-representation as

$$|\psi> = |\psi^{(o)}> + |\psi_a> \eta^a + \tfrac{1}{2}|\psi_{ab}> \eta^a \eta^b + \ldots \tag{3.7}$$

Here, $|\psi^{(o)}>, |\psi_a>, \ldots$, are states which live in the Hilbert space of the original variables q^i, p_i and do not involve the ghosts. With the choice (3.6) $|\psi^{(o)}>$ has ghost number $-m/2$, $|\psi_a> \eta^a$ has ghost number $-m/2+1$... and the last term in the expansion (3.7) has ghost number $m/2$. Note that when m is odd, the ghost number is half-integer, an occurrence sometimes called "fractionalization of the ghost number." It should also be noticed that the anti-hermiticity of $\mathcal{G}$ and the reality of its eigenvalues imply that its eigenstates have zero norm, except perhaps for those associated with the eigenvalue zero, which appears when m is even.

3.3 BRST invariant states

The description of the system in terms of the extended phase space has redundant variables. This redundancy was already present in the formulation of the gauge theory in terms of the p's and q's and it becomes even larger when one has the ghosts. The whole point above is that by enlarging the redundancy the description becomes ultimately more transparent and, in a sense, the two redundancies cancel each other.

To make the redundancies cancel each other one must bring in a condition that will select the physical subspace. This condition must be the BRST analog of the demand that the gauge generators G_a annihilate the physical states in the formalism without ghosts.

There are not many options for such a condition in the BRST scene. The equation that suggests itself [14] is clearly

$$\Omega|\psi>=0 \tag{3.8}$$

This condition possesses the following good properties:

i . It is linear and hence it selects a subspace.

ii . BRST observables, obeying $[\Omega, A]=0$, map the physical subspace onto itself.

iii. Trivial observables of the form $[K,\Omega]$ have vanishing matrix elements between physical states,

$$<\psi_1|[K,\Omega]|\psi_2>=0 \tag{3.9}$$

if $|\psi_1>$, $|\psi_2>$ obey (3.8). Note that (3.9) needs the hermiticity of Ω to hold.

I will return shortly to the relation between $\Omega|\psi>=0$ and the conditions $G_a|\psi>$ which are implemented in the more conventional formalism. For the moment, I would like to emphasize that while the latter are many equations (typically several per space point in a field theory), the former is just one condition. The reason is that the state vector depends on more variables in the BRST case.

3.4 Quantum BRST cohomology

If one assumes $\Omega|\psi>=0$, one identifies two BRST observables which differ by a "BRST total derivative,"

$$A \rightarrow A+[K,\Omega] \tag{3.10}$$

Two physical states which differ by a BRST total derivative

$$|\psi>\rightarrow|\psi>+\Omega|\chi> \tag{3.11}$$

are also identified. A physical state is therefore an equivalence class and the space of physical states may be characterized as

$$Ker\Omega/Im\Omega, \tag{3.12}$$

just as in the classical case, but with the understanding that now Ω is a linear operator acting on a Hilbert space. The study of the equivalence classes belonging to $(Ker\Omega/Im\Omega)$ constitutes the subject of quantum BRST cohomology.

The key test that a satisfactory quantum theory must pass is that the metric induced on the space $(Ker\Omega/Im\Omega)$ must be positive definite. This means that the norm of any state $|\psi>$ obeying $\Omega|\psi>=0$ must be positive or zero, with the value zero only happening when $|\psi>$ is of the form $\Omega|\psi>$. If the positivity of the induced inner product holds and if the Hamiltonian $H_o(q,p)$ admits an Hermitian BRST invariant extension, the theory is unitary. This happens in the usual cases, otherwise additional conditions which restrict the physical subspace over and above $\Omega|\psi>=0$ must be taken. (See [15,16] for more on this.)

3.5 Equivalence of the BRST physical subspace with the conventional gauge invariant subspace

I would like to show in a simple case how the single condition $\Omega|\psi>=0$ turns out to be equivalent to the many conditions $G_a|\psi>=0$ of the Dirac approach. This simple case will also show that the equivalence may not hold strictly when, for example, topological complications come in.

In view of the local abelianizability of the constraints, a natural case to look at in order to understand the condition $\Omega|\psi>=0$ is that of constraints which are pure momenta,

$$G_a = p_a \quad , \quad [G_a, G_b] = 0 \tag{3.13}$$

Then the coordinates (q^i, p^i) split into two groups (q^α, p_α) and $(q^a, p_\alpha)(\alpha = 1, ..., n-m)$. The variables (q^α, p_α) are true, gauge invariant degrees of freedom, whereas (q^a, p_a) are pure gauge.

The BRST generator reads

$$\Omega = \eta^a \mathcal{P}_a \tag{3.14}$$

and a general BRST state is given by

$$|\psi>= \psi^{(0)} + \psi_a\eta^a + \tfrac{1}{2}\psi_{ab}\eta^b\eta^a + ... \tag{3.15}$$

with $\psi^o, \psi_a, ...$ being functions of q_a and q^α.

One finds that $\Omega|\psi>$ is given by

$$\Omega|\psi> = \frac{1}{i}(\partial_a\psi^{(0)}\eta^a + \tfrac{1}{2}(\partial_b\psi_a - \partial_a\psi_b)\eta^b\eta^a + ...) \tag{3.16}$$

This shows that the BRST operator is the exterior derivative operator d in the space of the q^a. BRST states can be viewed as forms. The ghost number of a state, is equal to the rank of the form plus the overall additive constant $-m/2$ where m is the number of ghosts.

From (3.15), one sees that a physical state, rewritten as

$$|\psi> = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + ... \tag{3.16}$$

($\psi^{(0)}$ = first term in (3.15), 0–form; $\psi^{(1)} = \psi_a dq^a$, 1-form, etc.), must be a closed form

$$d\psi^{(0)} = d\psi^{(1)} = d\psi^{(2)} = ... = 0 \tag{3.17}$$

Furthermore, adding to $|\psi>$ the state $\Omega|\chi>$ amounts to modifying $\psi^{(0)}, \psi^{(1)}, \psi^{(2)}...$ as

$$\psi^{(0)} \to \psi^{(0)}; \psi^{(1)} \to \psi^{(1)} + d\chi^{(0)}, \psi^{(2)} \to \psi^{(2)} + d\chi^{(1)}, ... \tag{3.18}$$

Accordingly, the physical subspace is just given by $Ker\, d/Im\, d$ in the q^a-space.

If the topology of the q^a-space is trivial and no boundary condition is imposed [15], one can set $\psi^{(1)}$ and all the higher order terms equal to zero by an appropriate choice of $|\chi>$. This means that one can take a representative with ghost number $-m/2$ in each equivalence class of physical states. So the requirement of definite ghost number $-m/2$ is not a further assumption but, rather, is a gauge condition on the quantum gauge invariance.

For ghost number $-m/2$ representatives, the BRST condition reads,

$$\frac{1}{i}\frac{\partial\psi^{(0)}}{\partial q^a} = 0 \tag{3.19}$$

thus, $\psi^{(0)}$ must be independent of q^a. These are exactly the physical state conditions of the Dirac approach.

3.6 Action principle

The propagation amplitude in quantum mechanics is given by a sum over histories. In the BRST formulation of the gauge theory the concept of history includes giving the ghosts as functions of time. This is because the ghosts enter the formalism in the same footing with the original variables. One therefore needs an action appropriate to the extended phase space.

According to the discussion given in Sec. (2.9), the propagation amplitude that we are interest in should be just the matrix element of the evolution operator associated with the BRST invariant extension H of the Hamiltonian H_o. However, that extension is not unique. Two permissible Hamiltonians H differ by what we have called above a "BRST total derivative." Therefore one fixes once and for all one choice of H and allows for the ambiguity in the extension by writing

$$S_K = \int_{t_1}^{t_2} (\dot{q}p + \dot{\eta}^a \mathcal{P}_a - H + [K, \Omega])dt \tag{3.19}$$

The function K may depend on the q's, the p's, the η's and the $\mathcal{P}$'s. Different choices of K correspond to different BRST invariant extensions of the Hamiltonian.

The path integral should be independent of K. This property is the statement in BRST terms of the gauge invariance of the amplitude. Its validity is called the Batalin-Fradkin-Vilkovisky theorem.

The B-F-V theorem is a central result in BRST theory. It gives the most flexible and powerful formulation known to man of the sum over histories for gauge systems. It contains as a particular case the Faddeev-Popov prescription, but it also applies to the situations not covered by the latter. Phenomena such as ghost self-interactions, which are inescapable in theories of rank higher than one, are treated here in equal footing with the more traditional case of rank one or zero. However, even in the simpler cases additional flexibility is gained since the Faddeev-Popov gauge condition may be now taken to depend even on the ghosts themselves. That possibility would be hard to conceive if one takes the view that the ghosts are introduced to represent a pre-existing determinant associated with the gauge condition.

3.7 Path integral. B-F-V theorem

One defines the path integral as [17]

$$Z_K = \int exp[iS_K] DpDqD\mathcal{P}D\eta \tag{3.20}$$

where the measure is the ordinary one, given by the product of the differentials for all times in the interval $[t_1, t_2]$ over which the action is evaluated. The integration over the anticommuting ghosts is understood to be that of Berezin. The boundary conditions which the histories admitted in the integral must obey will be dealt with below.

To prove the independence of Z_K from K, one proceeds as follows. Take K' and K which differ from each other infinitesimally. Call ϵ the time integral of their difference multiplied by the imaginary unit

$$\epsilon = i \int_{t_1}^{t_2} dt(K' - K) \tag{3.21}$$

The integral (3.21) depends on the history in the complete interval $[t_1, t_2]$.

Next, perform a BRST transformation with parameter ϵ. This transformation is not canonical because the parameter depends on the history. Therefore the Liouville measure in the path integral is changed by it. The effect of that change is precisely to replace K' by K in the action.

To see this in detail, one defines new variables of integration by

$$F'(t) = F(t) + [F, \Omega]\big|_t \epsilon \tag{3.22}$$

where F stands generically for q, p, η and $\mathcal{P}$. The transformation (3.22) changes the action by a boundary term. This is because the fact that ϵ depends on the history plays no role when analyzing the action and the Hamiltonian appearing in it is BRST invariant.

On the other hand, the Jacobian of (3.22) may be evaluated directly. A direct calculation yields

$$DF' = DF \;\; exp[-i \int_{t_1}^{t_2} [\Omega, K' - K]dt] \tag{3.23}$$

where DF is an abbreviation for the product of the Liouville measures appearing in (3.20).

If one inserts (3.23) into the definition (3.20) one finds the desired result

$$Z_{K'} = Z_K \tag{3.24}$$

provided the transformation (3.22) does not change the boundary conditions on the histories, and the boundary term by which the action changes vanishes.

Those two issues are analyzed in Sec. 3.9 below.

3.8 Contact with lagrangian notation

I would like to digress briefly and relate the present notation with the one usually employed in the lagrangian form of the path integral. It is appropriate to do so here because the relation will be needed below to write down one of the permissible boundary conditions.

It was stated in Chapter I, just after (1.7), that "the general practical rule for the appearance of ghosts in a gauge theory can now be stated. In each case there will be a *pair* of new fields, usually called 'ghosts' and 'antighosts' for each gauge function present in the gauge transformation."

Now, this seemingly contradicts what we have been doing in Chapters II and III where we introduced just *one* ghost η for each constraint. There is, however, no contradiction. What happens is that, usually, the lagrangian form of a gauge theory is given so that the Lagrange multiplier associated with a first class constraint is included as a dynamical variable. This is the case with the time components A^a_o, $g_{o\mu}$ and ψ^A_o in Yang-Mills, gravity and supergravity, respectively. It also happens in string theory where one introduces the conformal metric components $(-g)^{\frac{1}{2}}g^{ab}$ on the world-sheet, as variables in the lagrangian.

If we generically denote the Lagrange multipliers by λ^α, one finds that their conjugate momenta vanish,

$$\pi_\alpha = 0 \tag{3.25}$$

These equations are called by Dirac "primary constraints." By demanding that (3.25) be maintained in time one then obtains the secondary constraints,

$$\phi_\alpha(q,p) = 0 \tag{3.26}$$

which involve neither the λ^α nor the π_α.

The secondary constraints ϕ_α are those which really contain information about the gauge invariance of the theory, whereas the primary ones (3.25) are somewhat trivial. However, it is useful to include both in the path integral since, for example, one can thus maintain manifest Lorent invariance in the Yang-Mills case.

The situation is then the following. The phase space includes both the (q,p) appearing in ϕ_α and the $(\lambda^\alpha, \pi_\alpha)$. The constraints G_a include both the ϕ_α and the π_α. Their number is thus twice the number of ϕ_α's and that is why the number of ghosts is doubled. The precise correspondence is as follows,

$$G_a = (\phi_a, \pi_\alpha), \quad \eta^a = (C^\alpha, -i\mathcal{P}^\alpha), \quad \mathcal{P}_\alpha = (\bar{\mathcal{P}}_\alpha, i\bar{C}_\alpha) \tag{3.27}$$

3.8 BRST invariant boundary conditions

In order for (3.24) to hold it is sufficient to find conditions at the end points t_1, t_2 which,

(i) are themselves BRST invariant

(ii) make the action BRST invariant, that is are such that

$$\left[f^k \frac{\partial\Omega}{\partial f^k} - \Omega\right]\Big|_{t_1}^{t_2} = 0 \tag{3.28}$$

where f^k stands for all variables fixed at the end points. With the form (3.19) of the action the f^k are the q's and the η's. If one wants to fix a momentum at both end points then $\dot{q}p$ must be replaced by $-q\dot{p}$ in the action.

One may say that selecting BRST invariant boundary conditions amounts to implement in the path integral the demand that the initial and final states be anihilated by the BRST generator Ω. That is one has the equality

$$Z_K = <\psi_2\left|e^{-i(H-[K,\Omega])(t_2-t_1)}\right|\psi_1> \tag{3.29}$$

with

$$\Omega\big|\psi_1> = \Omega\big|\psi_2> = 0 \tag{3.30}$$

No attempt will be made here to give an exhaustive treatment of this important issue. Indeed, it appears that a general procedure which would allow one to exhibit an appropriate boundary condition for each given BRST invariant state $|\psi>$, has not yet been devised. In particular, there seems to be no available criterion which would allow one to relate boundary conditions corresponding to two states which differ by a BRST total derivative.

I will just indicate three different sets of boundary conditions [8], which arise in practice and do satisfy the above mentioned requirements. They are the following requirements at t_1, t_2

$$\eta^a = 0, \qquad q \text{ (or } p\text{) fixed but arbitrary} \tag{3.31}$$

$$C^\alpha = \bar{C}_\alpha = \pi_\alpha = 0, \qquad q \text{ (or } p\text{) fixed but arbitrary} \tag{3.32}$$

This choice uses the "mixed representation" introduced in Sec. 3.7.

If the constraints ϕ_α involve only momenta (this happens in electrodynamics for all times and it also holds for large times in Yang-Mills and gravity if the couplings can then be neglected) one can take

$$\mathcal{P}_\alpha = G_a = 0, \qquad \text{the other } q\text{'s (or } p\text{'s) fixed but arbitrary} \qquad (3.33)$$

It is left to the reader to verify that these three sets of boundary conditions are permissible in the sense explained above.

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