# The Role of Quantum Mechanics in the Specification of the Structure of Space-Time *

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## 1. INTRODUCTION

General-relativity theory as a metric theory formulated in Riemann or Riemann-Cartan space is now accepted as the most satisfactory theory of gravitation as far as quantum effects of gravitation may be neglected. During the last centuries there have been many attempts to deduce the Riemannian structure of general-relativity theory from a few axioms.

Up to 1970 the common axiomatic approach to space-time structure was the one of Synge,[1] which is based on the behaviour of standard clocks. The main objection against this chronometric approach [2] is that the real clocks of physicists and astronomers (e.g., atomic clocks) are highly complicated systems which work on the basis of quantum mechanics. Because one can construct ideal clocks showing gravitational time in a more geometric way by means of light rays and freely falling particles, the chronometric axiom reduces to the claim that gravitational and atomic time agree. This, on the other hand, should better be deduced from theory [3] and measured experimentally. Accordingly there have been several efforts after 1970 to describe an alternative constructive approach to general relativity based on more primitive concepts.[5] All these different approaches end up with assigning to space-time a Weyl geometry instead of the further restricted Riemann geometry of general

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relativity.

A typical axiomatic scheme which results in a Weyl geometry is the one described in Ref. 2. It can be very briefly summarized as follows: primitive concepts are event, light ray, and freely falling particle. The light propagation determines the null cones and therefore a conformal structure, i.e. an equivalence class $e^{\Lambda(x)}g_{\alpha\beta}$ of metrics. The freely falling particles determine the affine geodesics and therefore a projective structure, i.e., an equivalence class of symmetric affine connections. The compatibility requirement that the null geodesics of the conformal structure belong to the geodesics defined by the projective structure then finally results in a Weyl structure.

There seems to be no way to close the remaining gap between Weyl and Riemann space if one is restricted to the use of these primitive concepts only. The axiomatic scheme remains incomplete. *The aim of this paper is to show that the gap can in fact be closed if quantum mechanics as the theory of matter is made part of the total scheme. Quantum mechanics proves that space-time must be a Riemann (or Riemann-Cartan) space.*

This proof is based on two demands a gravitation theory has to fulfill [5]:

(i) Completeness: The theory must mesh with and incorporate all nongravitational laws, in particular the quantum mechanical.

(ii) Self-consistency: If one calculates the prediction for the outcome of an experiment by different methods, one always gets the same result.

The demand (i) forces us to include quantum mechanically described matter into the scheme of general relativity. Quantum mechanics must contain classical mechanics as a limiting case. The demand (ii) then requires that this classical limit on one hand and the axiomatically introduced classical mechanics on the other agree. It is this demand which will finally lead to the conclusion that gravity as a space-time theory must be described by a Riemannian instead of the more general Weylian structure.

## 2. WEYL SPACE

The Weyl space is characterized by the fact, that all scalar fields and tensor fields $\underline{A}(x)$ are subject to the following position-dependent real one-parameter gauge transformation

$$\underline{A} \to \underline{A}' = e^{\frac{1}{2} w(A)\Lambda(x)} \underline{A} \tag{1}$$

(accordingly for f(x)) which are called *Weyl transformations*. $\Lambda(x)$ is thereby a real function and w(A) is a real number, called the *Weyl type* of $\underline{A}(x)$.

The gauge covariant *Weyl derivative*

$$D_\mu A^\alpha = \partial_\mu A^\alpha + \Gamma^\alpha_{\mu\varepsilon} A^\varepsilon + \frac{w(A)}{2} a_\mu A^\alpha \tag{2}$$

conserves the Weyl type. The real valued *Weyl potential* $a_\mu$ transforms thereby according to

$$a_\mu \to a'_\mu = a_\mu - \partial_\mu \Lambda(x) . \tag{3}$$

The axiomatic approach [2] is not able to provide the manifold with torsion:

$$\Gamma^\alpha_{\beta\mu} = \Gamma^\alpha_{\mu\beta} . \tag{4}$$

A corresponding Weyl derivative for spinors can be introduced.

There is a metric $g_{\mu\nu}$ of Weyl type $w(g_{\mu\nu}) = 2$

$$g_{\alpha\beta} \to g_{\alpha\beta}' = e^{\Lambda(x)} g_{\alpha\beta} \tag{5}$$

with vanishing Weyl derivative

$$D_\mu g_{\alpha\beta} = 0 \tag{6}$$

$$\partial_\mu g_{\alpha\beta} - \Gamma^\varepsilon_{\alpha\mu} g_{\alpha\varepsilon} - \Gamma^\varepsilon_{\mu\beta} g_{\alpha\varepsilon} = -a_\mu g_{\alpha\beta} . \tag{7}$$

This implies

$$\Gamma^{\rho}_{\mu\nu} = \{^{\rho}_{\mu\nu}\} + 1/2\,(\delta^{\rho}_{\mu}\, a_{\nu} + \delta^{\rho}_{\nu}\, a_{\mu} - g_{\mu\nu}\, g^{\rho\varepsilon}\, a_{\varepsilon}) \quad . \tag{8}$$

The curvature tensor of the Weyl connection

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_{\mu}\,\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\,\Gamma^{\sigma}_{\rho\mu} + \Gamma^{\varepsilon}_{\rho\nu}\,\Gamma^{\sigma}_{\varepsilon\mu} - \Gamma^{\epsilon}_{\rho\mu}\,\Gamma^{\sigma}_{\varepsilon\nu} \tag{9}$$

contains as a trace the *field strength of the Weyl potential*

$$f_{\mu\nu} = \partial_{\mu}\, a_{\nu} - \partial_{\nu}\, a_{\mu} = R^{\varepsilon}{}_{\varepsilon\mu\nu} \quad . \tag{10}$$

A Weyl space reduces to a Riemann space if and only if the field strength $f_{\mu\nu}$ vanishes. The necessary and sufficient condition for this is that $a_{\mu}$ is a gradient of a scalar

$$f_{\mu\nu} = 0 \iff a_{\mu} = \partial_{\mu}\,(\text{scalar}) \; . \tag{11}$$

This implies: If in a Weyl space the covariant derivative of a scalar $\phi$ of non-trivial Weyl weight $w(\phi) \neq 0$ vanishes,

$$\begin{aligned} D_{\mu}\,\phi = 0 \;&\Longrightarrow\; \partial_{\mu}\,\phi + \frac{w}{2}\, a_{\mu}\,\phi = 0 \\ &\Longrightarrow\; a_{\mu} = -\frac{2}{w}\,\partial_{\mu}(\ln\phi) \Rightarrow f_{\mu\nu} = 0 \;, \end{aligned} \tag{12}$$

then the Weyl space reduces to a Riemann space.

## 3. DIRAC THEORY IN WEYL SPACE

The Dirac field is described by two 2-spinors $\chi^{\dot{A}}$ and $\varphi_{A}$. The corresponding Lagrangian scalar density is:

$$\begin{aligned} \mathcal{L} = \frac{i\hbar}{\sqrt{2}}\sqrt{-g}\,\{\, & \chi^{\dot{x}}\,\sigma^{\alpha}{}_{A\dot{x}}\, D_{\alpha}\,\chi^{A} - \chi^{A}\,\sigma^{\alpha}{}_{A\dot{x}}\, D_{\alpha}\,\chi^{\dot{x}} \\ & - \varphi_{\dot{x}}\,\sigma^{\alpha A\dot{x}}\, D_{\alpha}\,\varphi_{A} + \varphi_{A}\,\sigma^{\alpha A\dot{x}}\, D_{\alpha}\,\varphi_{\dot{x}}\} - m\,(\varphi_{A}\,\chi^{A} + \varphi_{\dot{x}}\,\chi^{\dot{x}}) \; . \end{aligned} \tag{13}$$

This implies for the corresponding Weyl types:

$$w(\chi^{\dot{A}}) = w(\varphi_A) = -\,{}^{3}/_{2}\ . \tag{14}$$

As expected, the mass has to be of non-trivial Weyl type.

The corresponding *Dirac equations* are

$$\begin{aligned} i\,\sigma^{\alpha}{}_{A\dot{X}}\, D_\alpha \chi^{A} - \frac{m}{\sqrt{2}\hbar}\ \varphi_{\dot{X}} &= 0 \\ i\,\sigma^{\alpha A\dot{X}}\, D_\alpha \varphi_{A} + \frac{m}{\sqrt{2}\hbar}\ \chi^{\dot{X}} &= 0 \end{aligned} \tag{15}$$

The *Dirac 4-current*

$$j^{\alpha} = \sqrt{2}\ \sigma^{\alpha A\dot{X}}(\varphi_A\, \varphi_{\dot{X}} + \chi_A\, \chi_{\dot{X}}) \tag{16}$$

is conserved

$$D_\alpha\, j^{\alpha} = 0\ . \tag{17}$$

## 4. CLASSICAL LIMIT

We obtain the limit of the classical particle paths by means of the first step of a *WKB approximation*

$$\begin{aligned} \chi^{\dot{A}} &= e^{i\,S(x)/\hbar} \sum_{n=o}^{\infty} (-i\hbar)^n\, \chi_n^{\dot{A}} \\ \varphi_A &= e^{i\,S(x)/\hbar} \sum_{n=o}^{\infty} (-i\hbar)^n\, \varphi_{nA} \end{aligned} \tag{18}$$

which leads in lowest order to the Hamilton Jacobi equation

$$(\partial_\alpha S)\ (\partial_\beta S)\ g^{\alpha\beta} = m^2 \tag{19}$$

and therefore to

$$2(\partial^{\kappa} S) D_{\kappa} \partial_{\alpha} S = D_{\alpha} m^2 \quad , \tag{20}$$

where we have used

$$D_{\varepsilon} \partial_{\alpha} S = D_{\alpha} \partial_{\varepsilon} S \tag{21}$$

which is a consequence of $w(S) = 0$ .

In the WKB limit the 4-current takes the form:

$$j_{\mu} = -\frac{B_o}{m} \partial_{\mu} S + O(\hbar) =: j_{o\mu} + O(\hbar) \quad . \tag{22}$$

For its directional Weyl derivative we obtain:

$$(D_{\kappa} j_{o\alpha}) j_o^{\kappa} = D_{\kappa}\left[\ln \frac{\sqrt{jo_{\mu}\, jo^{\mu}}}{m}\right] j_o^{\kappa} j_{o\alpha} + j_{o\mu} j_o^{\mu} D_{\alpha}(\ln m) \quad . \tag{23}$$

The corresponding streamlines of $j_{o\mu}$ are not autoparallels. *Accordingly, in a Weyl space, the quantum mechanically defined particle trajectories on one hand, and the free-fall trajectories of classically defined structureless test particles on the other, do not in general agree.* The *requirement* that they should agree results in

$$D_{\alpha} m = 0 \tag{24}$$

so that with (11)

$$f_{\mu\nu} = 0 \quad . \tag{25}$$

The consequence of the requirement is therefore that the Weyl space reduces to a Riemann space and the gap described in section 1 is closed. *Quantum mechanics will contain classical mechanics as a limiting case if and only if the Weyl space is reduced to a Riemann space.*

Comments on the possibility to enlarge the axiomatix scheme [2] in order to introduce torsion, i.e. to construct a Weyl-Cartan space, are given in the lectures.

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