THE PROGRAM OF AN ETERNAL UNIVERSE

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"Ce monde-ci, le même pour tous,
aucum des dieux ni des hommes ne
l'a créé, mais il a toujours été,
est et sera." *

Heraclite

(*) The proof of this assertion - which is still missing - has been left by Heraclite to the subsequent generations.

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INTRODUCTION

It is said that the most crucial of the unsolved problems in Einstein's Cosmological Program can be put as follows: is the Universe eternal or did it have a beginning?

In other words, has space-time always been there or was there a time in which reality was not reducible to a succession of events represented in a four-dimensional continuum?

Is space-time a useful concept to be invoked in all physical situations in order to describe the flux of our experiences or does one need to appeal to transcendent "complete cosmological models" (Grischuck and Zeldovich, 1984) which aim at generating classical space-time itself from more general structures of quantum character?

The inevitable need to go into such unusual questions (among physicists) appeared more dramatically in the last decade, due mainly to the alternation of success and failure of the so called standard Hot Big Bang cosmological model.

In this Vth School probably most of the lecturers will mention this problem from different aspects - either from a historical point of view (Eisenstaedt), for conventional (Ellis) or unconventional (Narlikar) models.

The simplest and most direct way to present this problem is to consider the situation in a given specific model like for instance, as it appears in Friedmann's Hot Big Bang cosmology. This model represents a homogeneous and spatially isotropic geometry the dynamics of which is characterized by a unique function A(t) of the global time. The fact that A(t) is a monotonic function which can attain the value zero represents an evolution process of the Universe from a singularity - which is to be identified with the vanishing of the radius of the Universe.

A further exam of the consequences of the existence of such singularity in our real Universe divided the physicists into two groups.

In the first group, we have those who believed that the singularity exhibited in Friedmann's model was just a mere artificial consequence of the high degree of symmetry contained in that geometry. They argued that this metric is nothing but an approximation of the real world and that a more realistic, less symmetric model, would reveal the true regular nature of the global geometry throughout the whole space-time. In the other one, we find those scientists who believed that the singularity is an intrinsic property of Einstein's General Relativity which cannot be eliminated by any realistic theory. The history of this controversy is a fascinating subject that should be carefully analysed.

At the end of the sixties and beginning of seventies an important event was the appearance of a series of theorems which pretended to solve unequivocally the question of the inevitability of the presence of a singularity in a general solution of Einstein's equations.

Although the theorems did not succeed in proving that the curvature or any equivalent function of the metric of space-time indeed attained an infinite value (which one naturally should expect in order to characterize a given solution as singular) they led to the belief that General Relativity plus some "reasonable" conditions (like, for instance, the positivity of the energy) induce the presence of particular domains in space-time in which properties like continuity, would no more be reliable. For sake of completeness, let us enunciate here one of those theorems, due to S. Hawking. He says:

"The following requirements on a space-time M are mutually inconsistent:

(i) There exists a compact space-like hypersurface (without boundary) H.

(ii) The divergence θ of the unit normals to H is positive at every point of H.

(iii) $R_{\mu\nu}V^{\mu}V^{\nu} \leq 0$ for each time like vector V^{μ}.

(iv) M is geodesically complete in past time like directions."

A demonstration of this theorem is presented in the book of Hawking and Ellis (1973) or in Penrose's (1967).

We remark that, as it has been emphasized by many authors, condition (iv) should be retained as a sort of equivalence to the presence of singularity, for classically we can hardly conceive the annihilation of the world line of a real particle in an accessible region.

Restriction (iii) is guaranteed to be true in general relativity if we accept the strong condition on the stress-energy tensor:

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq \frac{1}{2}T$$

or

$$\rho + 3p \geq 0$$

which for a classical fluid is almost a dogma for physicists.

These theorems almost completely dominated the scene of cosmology along the seventies. This situation has led some scientists (like C.W. Misner, for instance) to propose a way to "tolerate" the existence of such ugly properties of general relativity by making a mathematical effort to remove the inevitable cosmical singularity to a very remote past.

It was only when some of the difficulties of the standard Hot Big Bang model revealed to be unsurmountable in the classical context that the alternatives to generate a non-singular cosmology gained some real interest. Later on, in the eighties, the work of those scientists which dared to propose non-singular cosmological models, started to be taken seriously.

In the search for such regular cosmos, at least one of the conditions of applicability of the singularity theorems had to be abandoned. The straightforward way to do this is to modify at least one of the three elements which are present in Einstein's equations (that is: the geometry, the gravitational coupling constant and the matter) by adopting any of the following alternatives[1]:

(i) Introduction of a cosmological constant Λ.

(ii) Violation of the condition $R_{\mu\nu}V^{\mu}V^{\nu} \leq 0$.

(iii) Use of more complex expression for the representation of the matter content of the right hand side of Einstein's equations (non-equilibrium configuration) which in turn satisfy the proposal (ii) above.

(iv) Modification of the right hand side of Einstein's equations by stochastic fluctuation or quantum effects.

(v) Introduction of an independent affine connection (e.g. torsion).

(vi) Acceptation of more general metric property like a WIST structure, for example.

(vii) Non-minimal coupling of gravity with long range fields.

(viii) Violation of Lorentz invariance (for very high energies).

(ix) Non-linear Lagrangians.

(x) Generation of a negative effective gravitational constant.

And so on.

It is to be understood that the present status of these theories are not the same and change from time to time. For instance, in the seventies no merit was seen in introducing a cosmological constant. In the eighties, however, the discovery of many different ways to induce an effective Λ modified this situation - and it even turned to make possible the existence of the most fashionable cosmological model of nowadays (inflation).

We will examine in these lectures some features of most of the above suggestions.

COMMENTS AND REFERENCES

(1) We will give the references for the different proposals (i) to (x) in the following section except for (v). In this case we can quote Trautman, A. (1973) Nature 242, 7 in which a non-singular cosmology is presented modifying Einstein's equations due to the influence of the spin on the geometry of space-time (Cartan geometry). This kind of theory has been presented in the IIIrd Brazilian School of Cosmology by Dr. Colber G. de Oliveira.

Although we are presenting these alternatives in the light of the search of a global regular geometry, all of them have an interest by themselves and should not be considered as a by-product of the singularity quarrel.

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2. THE COSMOLOGICAL CONSTANT-I

The modified Einstein's equations take the form

$$R^{\mu}{}_{\nu} - \frac{1}{2} R\delta^{\mu}{}_{\nu} = - \kappa T^{\mu}{}_{\nu} - \Lambda\delta^{\mu}{}_{\nu} \qquad (2.1)$$

Then for an arbitrary time like vector:

$$R_{\mu\nu}V^{\mu}V^{\nu} = - (\rho + 3p) - \Lambda \qquad (2.2)$$

Thus the sign of $R_{\mu\nu}V^{\mu}V^{\nu}$ depends on the value of the constant Λ. This seems to be the easiest way to create conditions for the existence of a non-singular cosmos. Indeed, since the very beginning of relativistic cosmology, Einstein, de Sitter and many others have used this freedom of a non-controlable cosmical constant. Soon, the idea of $\Lambda \neq 0$ was abandoned and revived only in the last years thanks to an artificial generation of an effective Λ by quantum methods. We will come back to this later on.

Let us just retain that although a $\Lambda \neq 0$ could imply a non-singular cosmos there is no experimental support to believe that the curvature of our actual Universe was dominated by such effect.

COMMENTS AND REFERENCES

In the early days of Relativistic Cosmology the cosmological constant Λ was identified to a perfect fluid of density ρ and pressure p with the anomalous equation of state $p + \rho = 0$ which fulfils all space. The origin of this fluid was supposed to be primordial.

In the last decade Λ received a new version: it was associated to the energy of the vacuum. It has been shown (see chapter 5) that in the vacuum state of (say) a scalar field, the total energy-momentum tensor of the field reduces to the product of a constant times $g_{\mu\nu}$ - which yields the conventional form for the presence of the cosmological constant in Einstein's equations of gravity.

Besides these two respectable suggestions a lot of highly speculative proposals have been set forth either as a relic of phantasmagoric extra-dimensions of the space-time (see references in Kolb's lectures) or as a consequence of imposing higher symmetries on unobserved extra-fields (Aurilia, A.; Nicolai, H. and Townsend, P.K. (1980) Nucl. Phys. B 176, 509).

In any case, the origin of the cosmological constant is a fascinating subject which still divides the opinion of physicists. For some, it is nothing but a simplified trick which synthesizes a lot of phenomena that occur in large domains of space-time. Others, however, claim that it contains a germ of the inconsistency of cosmology pointing out the necessity to undertake a Critique of the Cosmological Reason. Just to give one example of how this criticism could produce alternative cosmological theories let me quote the search of new long-range forces by the russian physicist L.B. Okun (1980) Sov. Phys. JETP 52, 351.

3. QUANTUM VACUUM EFFECTS (THE CASE OF THE SCALAR FIELD)

There is not a unique procedure to couple scalar fields to gravity (see Novello and Oliveira, 1987 for a review). Since there is not yet any observational evidence which could be used, physicists have been guided by general principles in order to obtain a valuable prescription for such interaction. The two most employed principles which have been used are:

(i) Minimal Coupling

(ii) Conformal Coupling

In case (i) one adopts the view that the passage from flat space to curved space (which is to be identified with gravity) is made by simple replacement in the Lagrangian which describes the dynamical of ϕ of simple derivatives by covariant derivatives. No functional of the curvature is present (except, of course, for the free part of the gravitational field).

In case (ii) one asks for conformal invariance of the theory (in case the scalar field is massless).

We thus write for L_I and L_{II} : the Lagrangian minimally coupled and conformally coupled, respectivelly:

$$L_I = \sqrt{-g} \;\{\partial_\mu\phi^* \,\partial_\nu\phi \, g^{\mu\nu} + \frac{1}{\kappa} R + V(\phi) + 2\Lambda \} \tag{3.1}$$

in which we have added a cosmological constant just for completeness. In these lectures , we limit the form of the potential V to:

$$V(\phi) = - M^2\phi^*\phi + \sigma(\phi^*\phi)^2 \tag{3.2}$$

For the conformal coupling:

$$L_{II} = L_I - \sqrt{-g} \;\frac{1}{6} R \,\phi^*\phi \;. \tag{3.3}$$

The equation of motion for this L_{II} is given by

$$\Box\phi + \frac{1}{6} R\phi - \frac{\delta V}{\delta\phi^*} = 0 \qquad (3.4)$$

and by varying the metric tensor we obtain the modified Einstein's equation:

$$\left(\frac{1}{\kappa} - \frac{1}{6} \phi^*\phi\right)G_{\mu\nu} = - \frac{1}{2} [\phi^*_{,\mu}\phi_{,\nu} + \phi^*_{,\nu} \phi_{,\mu}] +$$

$$+ \frac{1}{2} g_{\mu\nu}(\phi^*_{,\lambda}\phi^{,\lambda} + V + 2\Lambda) - \frac{1}{6}(\phi^* \Box\phi + \phi\Box\phi^* + 2\phi^*_{,\lambda}\phi^{,\lambda})g_{\mu\nu} +$$

$$+ \frac{1}{6}(\phi^*_{,\mu;\nu}\phi + \phi^*\phi_{,\mu;\nu} + \phi^*_{,\mu}\phi_{,\nu} + \phi^*_{,\nu}\phi_{,\nu}) \qquad (3.5)$$

in which $\Box\phi = \phi_{,\mu;\nu}\, g^{\mu\nu}$.

Taking the trace of (3.5) we obtain (using the form 3.2):

$$\frac{1}{\kappa} R = M^2\phi^*\phi - 4\Lambda \quad . \qquad (3.6)$$

Remark that the trace is independent of the value of the constant σ.

There are many remarkable consequences of assuming conformal coupling. Let us review some of them.

COMMENTS AND REFERENCES

Novello, M. and Oliveira, L.A.R. (1987) Non-minimal Interaction of Gravity with other physical fields: an overview (to appear in Rev. Bras. Física).

These last years the literature envolving scalar fields in cosmology has considerably grown, mainly in two directions: either exploring the consequences of spontaneous symmetry breaking or the properties of conformally invariant coupling. In the fourth Brazilian School of Cosmology we had the presentation of the belgian group (Brout, Englert and Gunzig) work which explored the possibility of the quantum creation of the Universe. They start with the action

$$S = \int \sqrt{-g}\, d_4x \,(\psi^*_{,\mu}\psi^{,\mu} - M^2\psi^*\psi - \frac{1}{6} R\psi^*\psi - \frac{1}{\kappa} R)$$

and restrict the variation of $g_{\mu\nu}$ to the set of conformally flat geometries. We can then substitute the scalar of curvature R by a new scalar field ϕ to obtain a theory of two fields ϕ and ψ in Minkowski space, one of them with the "wrong" sign of contribution to the total energy. They explore then the fact that a state can be created with total energy zero, showing that the Minkowski flat space-time is unstable and yields after perturbation a Friedmann-like Universe.

Exercise: Show that, in general, imposing the condition that the metric is conformally flat in the above action does not commute with the variational calculation. In what conditions do they commute?

In 1974, J. Dreitlein (Phys. Rev. Lett., 33, 1243) gave an interesting example of how a scalar field can produce a cosmological constant using the mechanism of symmetry breaking.

4. THE MELNIKOV-ORLOV SOLUTION

In 1979, Melnikov and Orlov examined the possibility to treat the interaction of a quantum scalar field $\phi(x)$ in a classical geometry (in which the metric tensor $g_{\mu\nu}$ is a c-number). Guided by the features of the mechanism of spontaneous symmetry breaking, they tried to find a solution such that in the fundamental state we have

$$< 0|\phi|0 > = \sqrt{\frac{3}{\lambda}}\ \frac{f(\eta)}{A(\eta)} \tag{4.1}$$

in which η is the conformal time of the open Friedmann geometry given by

$$ds^2 = A^2(\eta)[d\eta^2 - d\chi^2 - \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.2}$$

For a massless field and without cosmological constant the equation of the trace reduces to

$$\frac{A''}{A} = 1 \tag{4.3}$$

in which a dash means derivative with respect to η. The equation of ϕ is

$$\phi'' + 2\phi'\ \frac{A'}{A} + 2\sigma A^2\phi^3 = 0 \tag{4.4}$$

Then, compatibility of these equations with assumption (4.1) yields the relation

$$\sigma = \frac{\lambda}{6} \quad , \tag{4.5}$$

and for the radius of the Universe (as function of the cosmical time t defined by $\eta = \int \frac{dt}{A(t)}$) we obtain

$$A(t) = \sqrt{t^2 + L^2_{(MO)}} \tag{4.6}$$

($L_{(MO)}$ is a constant)
and

$$f'' - f + f^3 = 0 \tag{4.7}$$

Melnikov and Orlov considered the basic solutions of this equation: $f = 0$, $f = +1$ and $f = -1$. The trivial solution ($f = 0$) is unstable and the other two $f^2 = 1$ are stable. A simple direct demonstration of this can be made by changing the equation for f (4.7) into a planar autonomous system. Indeed, set $x = f$ to arrive to the equivalent set

$$\frac{dx}{d\eta} = y$$

$$\frac{dy}{d\eta} = x - x^3 \tag{4.8}$$

The critical points of this system are given by:

$$(x_o, y_o) = (0,0)$$

$$(x_A, y_A) = (-1,0)$$

$$(x_B, y_B) = (1,0)$$

We can arrive at the features seen in graph 1.

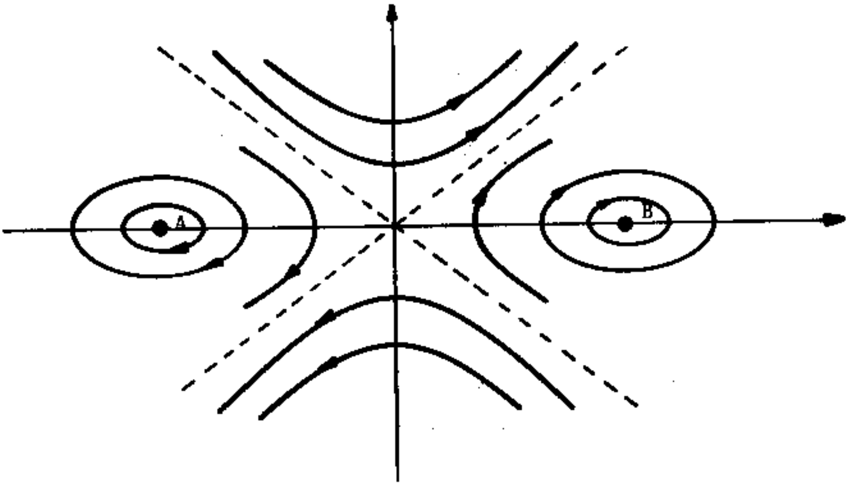

Figure 1 - (see the text)

Points A.B represent the stable solution

$$\phi_{(A,B)} = \pm \frac{1}{\sqrt{2\sigma}} \frac{1}{A} \tag{4.9}$$

From the equation of $g_{\mu\nu}$ (eq. 3.5) and specializing for $\mu = \nu = 0$ we obtain the value of the constant $L_{(MD)}$ which appeared in (4.6):

$$L^2_{(MD)} = \frac{\kappa}{24\sigma} \tag{4.10}$$

for the minimum allowable value of the radius of the Universe.

Is the interest of this solution purely academic? Melnikov and Orlov argue that it is not. Although such long-range scalar field was never observed in any event in Nature, they claim that certainly a ϕ field in those very intense gravity field (near the minimum value L_{MD} for the radius of the Universe) behaves as a massless one - and should be treated in its quantum regime. Remember that the Planck lenght $L_{P\ell}$ is given by

$$L_{P\ell} = \sqrt{\frac{G\hbar}{c^3}} \simeq \sqrt{\kappa \hbar c} \approx 10^{-33}\text{cm} \tag{4.11}$$

In natural units ($\hbar = c = 1$),

$$L_{(MD)} \simeq L_{P\ell} \tag{4.12}$$

for the typical value of σ of the order of unity. It must be remarked that Melnikov-Orlov solution $A(t) = \sqrt{t^2 + L^2_{(MD)}}$ gives values of the maximal red-shift, *decceleration parameter and the age of the* Universe which are in accordance with observations.

4.1. Energy of the fundamental state of Melnikov-Orlov solution

If we adopt the standard Einstein's formula

$$G_{\mu\nu} = -\kappa_{(ren)} T_{\mu\nu}$$

we obtain

$$E_{|0>} = -\frac{3L^2}{A^4} < 0$$

which shows explicitly the expected violation of the positivity condition (in standard notation) and makes understandable the absence of singularity in this model.

Remark that the gravitational constant, in this state, becomes renormalized:

$$\frac{1}{\kappa_{ren}} = \frac{1}{\kappa} - \frac{\phi^2}{6} = \frac{12\sigma t^2 - \kappa/2}{12\sigma\kappa A^2}$$

Thus, for

$$t^2 < \frac{\kappa}{24\sigma} \Rightarrow \kappa_{ren} < 0$$

and for

$$t^2 > \frac{\kappa}{24\sigma} \Rightarrow \kappa_{ren} > 0$$

This simple observation makes explicit a crucial fact: the possibility of a change in the sign of the gravitational constant induced by the non-minimal coupling of a scalar field with gravity. Does a similar phenomenon occur in our actual Universe?

COMMENTS AND REFERENCES

Melnikov, V.N. and Orlov, S.V. (1979) Phys. Lett., 70A, 263.

5. THE COSMOLOGICAL CONSTANT-II

At the end of the seventies a new interpretation to the origin of the cosmological constant Λ appeared: the identification of Λ as a consequence of a spontaneous symmetry breaking mechanism. Let us briefly review its main lines.

In flat space-time the dynamics of a scalar field ϕ is given by

$$L = \phi^*_{,\mu}\, \phi_{,\nu}\, \eta^{\mu\nu} - W(\phi) \tag{5.1}$$

with

$$W(\phi) = M^2\phi^*\phi - \sigma(\phi^*\phi)^2$$

Thus, the equation of motion is given by

$$\Box\phi + \frac{\partial W}{\partial \phi^*} = 0 \tag{5.2}$$

or

$$\Box\phi + M^2\phi - 2\sigma(\phi^*\phi)\phi = 0$$

The energy of this field is

$$t_{\mu\nu}[\phi] = \frac{1}{2}(\phi^*_{,\mu}\,\phi_{,\nu} + \phi^*_{,\nu}\,\phi_{,\mu}) - \frac{1}{2}\,\eta_{\mu\nu}(\phi^*_{,\lambda}\,\phi^{,\lambda} - W) \tag{5.3}$$

There exists a fundamental solution $\phi = \phi_0$ = constant such that by (5.2).

$$\frac{\partial W}{\partial \phi^*} = 0 \tag{5.4}$$

and by (5.3) (with $E = t^o_o$)

$$E = \frac{1}{2} W \tag{5.5}$$

We see that the condition of minimum for the energy coincides with the condition which extremize the potential W. In this case we have

$$|\phi_o|^2 = \frac{M^2}{2\sigma} \tag{5.6}$$

Now, let us examine the field ϕ when minimally coupled to gravity. From (3.1) we obtain

$$\Box\phi + \frac{\partial W}{\partial \phi^*} = 0 \tag{5.7}$$

and for the energy

$$t_{\mu\nu} = \frac{1}{2}(\phi^*_{,\mu}\phi_{,\nu} + \phi^*_{,\nu}\phi_{,\mu}) - \frac{1}{2} g_{\mu\nu}(\phi^*_{,\lambda}\phi^{,\lambda} - W) \tag{5.8}$$

and it is left as an exercise to show that all previous results of free scalar field follow.

Let us now consider the modifications imposed by non-minimal (i.e. conformal) coupling (see equations 3.3, 3.4 and 3.5). There has been some misunderstanding in the literature on the way to define the energy of the field ϕ when coupled (non-minimally) to gravity. (This question is valid for any spin.) However, there has been some agreement between the standard Einstein's definition (see Coleman et al, Tagirov, Zel'dovich, etc.) and we accept that the stress energy tensor $T_{\mu\nu}$ is given by

$$\delta\int \sqrt{-g}\, L d_4x = \int \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}\, d_4x \tag{5.9}$$

Using the conformal Lagrangian (3.1) we obtain

$$T_{\mu\nu} = \frac{1}{2}(\phi^*_{,\mu}\phi_{,\nu} + \phi^*_{,\nu}\phi_{,\mu}) - \frac{1}{2}\, g_{\mu\nu}(\phi^*_{,\lambda}\phi^{,\lambda} - W) - \frac{1}{6}\phi^2 G_{\mu\nu} + \frac{1}{6}(\Box\phi^2 g_{\mu\nu} - \phi^2{}_{,\mu;\nu}) \tag{5.10}$$

or representing by $t_{\mu\nu}$ the energy of the field ϕ when minimally coupled we can write

$$T_{\mu\nu} = t_{\mu\nu} + \frac{1}{6}(\Box\phi^2 g_{\mu\nu} - \phi^2_{,\mu;\nu}) - \frac{\phi^2}{6}\, G_{\mu\nu} \tag{5.11}$$

Using equation (3.5) of the evolution of ϕ:

$$\left(\frac{1}{\kappa} - \frac{1}{6}\phi^2\right)G_{\mu\nu} = -\, t_{\mu\nu} - \frac{1}{6}(\Box\phi^2 g_{\mu\nu} - \phi^2_{,\mu;\nu}) + \Lambda g_{\mu\nu} \tag{5.12}$$

or

$$G\mu\nu = \frac{t_{\mu\nu} + \frac{1}{6}(\Box\phi^2 g_{\mu\nu} - \phi^2_{,\mu;\nu}) - \Lambda g_{\mu\nu}}{-\frac{1}{\kappa} + \frac{1}{6}\phi^2} \tag{5.13}$$

Substitution of (5.13) into equation (5.11) yields:

$$T_{\mu\nu} = \frac{1}{\frac{1}{\kappa} - \frac{\phi^2}{6}}\left[t_{\mu\nu} + \frac{1}{6}\,\Box\phi^2 g_{\mu\nu} - \frac{1}{6}\,\phi^2_{,\mu;\nu} - \frac{\Lambda}{6}\,\phi^2 g_{\mu\nu}\right] \tag{5.14}$$

taking the trace

$$T = T_{\mu\nu}g^{\mu\nu} = \frac{t + \frac{1}{2}\Box\phi^2 - \frac{2}{3}\Lambda\phi^2}{\frac{1}{\kappa} - \frac{1}{6}\phi^2} \quad .$$

We have

$$t = t_{\mu\nu}g^{\mu\nu} = -\phi^*_{,\lambda}{}^{,\lambda} + 2M^2\phi^2 - 2\sigma(\phi^*\phi)^2$$

(Remark that contrary to all other fields with different spin, the trace t of the minimally coupled field does not vanish in the limit $M \to 0$).

Then

$$T = (\frac{1}{\kappa} - \frac{\phi^2}{6})^{-1}(-\phi^*_{,\lambda}\phi^{,\lambda} + 2M^2\phi^2 - 2\sigma(\phi^*\phi)^2 + \frac{1}{2}\Box\phi^2 - \frac{2}{3}\Lambda\phi^2) \qquad (5.15)$$

Now,

$$\Box\phi^* + \frac{1}{6}R\phi^* + \frac{\partial W}{\partial\phi} = 0 \qquad (5.16)$$

and

$$\phi^*\Box\phi + (\Box\phi^*)\phi = -\frac{1}{3}R\,\phi^*\phi - 2M^2\phi^*\phi + 4\sigma(\phi^*\phi)^2$$

which yields for the trace

$$T = \frac{M^2\phi^*\phi - \frac{1}{6}R\phi^*\phi - \frac{2}{3}\Lambda\phi^*\phi}{\frac{1}{\kappa} - \frac{1}{6}\phi^*\phi} \qquad (5.17)$$

or using (3.6), that is

$$\frac{1}{\kappa} R = M^2\phi^*\phi - 4\Lambda$$

we arrive at

$$T = M^2\phi^*\phi \tag{5.18}$$

which has the good limit for $M \to 0$.

Let us come back to the equation of ϕ. For $\phi = \phi_o$ = constant we have

$$\frac{1}{6} R\phi + \frac{\partial W}{\partial \phi^*} = 0$$

or, substituing the value of R we obtain (for the non-trivial solution $\phi_o \neq 0$):

$$\phi_o^2 = \frac{4\kappa\Lambda - 6M^2}{\kappa M^2 - 12\sigma} \tag{5.19}$$

From the definition of $T^{\mu\nu}$ we obtain the energy

$$E = T^o{}_o = \frac{\frac{W}{2} - \frac{\kappa}{6}\Lambda\phi^2}{\frac{1}{\kappa} - \frac{\phi^2}{6}}$$

or

$$E = \frac{3M^2\phi^2 - 3\sigma\phi^4 - \kappa\Lambda\phi^2}{6 - \kappa\phi^2} \tag{5.20}$$

The condition for an extremum $\frac{\delta E}{\delta\phi} = 0$ implies

$$\kappa\sigma\phi_o^4 - 12\sigma\phi_o^2 + 6M^2 - 2\Lambda = 0 \qquad (5.21)$$

which is compatible with the constant solution ϕ_o if and only if

$$\Lambda = \frac{M^4}{8\sigma} \qquad (5.22)$$

and in this case, the fundamental solution (equation 5.19) takes the value

$$\phi_o^2 = \frac{M^2}{2\sigma} \qquad (5.23)$$

which is the same value (5.6) as in flat-space case.

5.3. A Mechanism for Generating an Effective Gravity Repulsion

We will consider now the effects in ordinary matter of the existence of a scalar field ϕ in its fundamental state ϕ_o.

The total Lagrangian is given by

$$L = \sqrt{-g}\,\{\phi^*_{,\mu}\,\phi_{,\nu}\,g^{\mu\nu} - W(\phi) + \frac{1}{\kappa}R - \frac{1}{6}R\phi^2 + 2\Lambda + L_M \qquad (5.24)$$

in which L_M represents all other kind of fields and matter in the world. From (5.24) we obtain

$$(\frac{1}{\kappa} - \frac{1}{6}\phi^2)G_{\mu\nu} = - T_{\mu\nu}[\phi] + \Lambda g_{\mu\nu} + T_{\mu\nu}^{(mat)} \qquad (5.25)$$

in which $T_{\mu\nu}[\phi]$ is given by (5.11) and $T_{\mu\nu}^{(mat)}$ is the energy-momentum tensor obtained from varying L_M.

When ϕ is in its fundamental state $\phi = B = constant$ *we have*

$$\left(\frac{1}{\kappa} - \frac{1}{6} B^2 \right) G_{\mu\nu} = - \frac{W}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu}^{(mat)}$$

or

$$G_{\mu\nu} = - K_{ren} T_{\mu\nu}^{(mat)} - \Lambda_{eff} g_{\mu\nu} \tag{5.26}$$

in which

$$\Lambda_{eff} = \frac{1}{2} \frac{M^2B^2 - \sigma B^4 - 2\Lambda}{\frac{1}{\kappa} - \frac{1}{6} B^2} \tag{5.27}$$

$$\frac{1}{K_{ren}} = \frac{1}{\kappa} - \frac{B^2}{6} \tag{5.28}$$

We remark that:

(i) The sign of K_{ren} depends on the value of B.

(ii) The above result is not a privilege of the scalar field. Any field coupled non-minimally to gravity can induce a similar modification - eventually turning gravity into a repulsive force.

Exercise: Examine the theory of two-scalar fields given by

$$L = \sqrt{-g} \{ \phi^*_{,\mu} \phi_{,\nu} g^{\mu\nu} - V(\phi) + \psi^*_{,\mu} \psi_{,\nu} g^{\mu\nu} - W(\psi) - \frac{1}{6} R(\phi^2 + \psi^2) +$$

$$+ \xi \phi^2 \psi^2 + 2\Lambda + \frac{1}{\kappa} R + L_{mat} \} \tag{5.29}$$

for

$$V(\phi) = m^2\phi^2 - \sigma(\phi^*\phi)^2$$

$$W(\psi) = M^2\psi^2 - \eta(\psi^*\psi)^2$$

Answer: (as σ, η do not appear in the equation for the trace R we will give the answer only in the special case $\sigma = \eta = 0$).

Dynamics is given by

$$\Box\phi + \frac{1}{6} R\phi - \xi|\psi|^2\phi + \frac{\delta V}{\delta\phi^*} = 0 \tag{5.30a}$$

$$\Box\psi + \frac{1}{6} R\psi - \xi|\phi|^2\psi + \frac{\delta W}{\delta\psi^*} = 0 \tag{5.30b}$$

$$[\frac{1}{\kappa} - \frac{1}{6}(\phi^2 + \psi^2)]G_{\mu\nu} = - T_{\mu\nu}[\phi] - T_{\mu\nu}[\psi] + \frac{1}{2}\xi\phi^2\psi^2 g_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu}^{(mat)}. \tag{5.30}$$

Remark that we can extend the results of the previous analysis and look for the fundamental states ϕ = constant = A and ψ = constant = B. In this case, a straightforward calculation yields:

$$A^2 = \frac{\kappa M^2(m^2 - M^2) - 2\xi(2\kappa\Lambda - 3M^2)}{\xi[6\xi - \kappa(m^2 + M^2)]}$$

$$B^2 = \frac{\kappa m^2(M^2 - m^2) - 2\xi(2\kappa\Lambda - 3m^2)}{\xi[6\xi - \kappa(m^2 + M^2)]}$$

In the fundamental state $(\phi, \psi, g) = (A,B,g)$ the equation for the geometry reduces to:

$$G_{\mu\nu} = \Lambda_{eff} g_{\mu\nu} - \kappa_{ren} T_{\mu\nu}^{(mat)}$$

with

$$\Lambda_{eff} = \frac{-3m^2M^2}{6\xi - \kappa(m^2 + M^2)} \kappa$$

$$\kappa_{ren} = \frac{\xi [6\xi - \kappa(m^2 + M^2)]}{6 [\xi - \xi_{(-)}][\xi - \xi_{(+)}} \kappa$$

in which

$$\xi_{(\pm)} = \frac{\kappa}{6} (M \pm m)^2 .$$

6. VECTOR FIELDS

Since the early days of General Relativity the interaction of vector fields to gravity has been studied assuming the validity of the minimal coupling principle. Only very recently the exam of non-minimal coupling has attracted the interest of physicists. The only reason for this is the absence of experiments on that coupling, which induces one to choose doubtfully simplicity as a criterion.

The success of theories of scalar fields interacting conformally to gravity opened the way to the exam of more complex theories in a curved space.

Although we do not see in the near horizon any perspective of inequivocally solving that question, physicists have started a program in which different alternatives are being considered in an almost equal footing.

This situation seems very likely the case of weak forces. In the fifties physicists deal with the following question: covariance requirements induce to believe that the weak interaction Lagrangian have to be constructed with objects of a Clifford algebra. Among the different possibilities generated by the basis of that algebra in a four-dimensional space-time which one is better adapted to describe matter desintegration?

This question theoretically remained unsolved until Wu and collaborators gave an experimental answer to the problem. We are still waiting for an analogous decision in the case of gravitational interaction of vector fields. Let us thus examine in general what such an interaction should be in principle.

There are seven possible Lagrangians which can be constructed as linear functions of the curvature tensor. We divide them into two classes:

Class One

$L_1 = R\, W_\mu W^\mu$

$L_2 = R_{\mu\nu}\, W^\mu W^\nu$

Class Two

$L_3 = R\, F_{\mu\nu}\, F^{\mu\nu}$

$L_4 = R\, F_{\mu\nu}\, \overset{*}{F}^{\mu\nu}$

$L_5 = R_{\mu\nu}\, F^\mu{}_\alpha\, F^{\alpha\nu}$

$L_6 = W^{\alpha\beta\mu\nu}\, F_{\alpha\beta}\, F^{\mu\nu}$

$L_7 = \overset{*}{W}^{\alpha\beta\mu\nu}\, F_{\alpha\beta}\, F_{\mu\nu}$

The two first class Lagrangians are gauge dependent but have the right dimension. They do not need the introduction of a new coupling constant. Class two are gauge independent but they all need the introduction of a lenght ℓ_0 in order to have the correct dimension.

One could be tempted to restrict class one Lagrangians to vector mesons and look for the equations of motion of electrodynamics only among class two functionals. However, one should be less radical and leave this decision to an ulterior analysis.

Exercise: Some authors (Goenner, 1984) quote an eight possible form of Lagrangian:

$$L_8 = R^{*}_{\alpha\beta\mu\nu}{}^{*} F^{\alpha\beta} F^{\mu\nu}$$

Show that this L_8 is not an independent one, but that we can write

$$L_8 = - L_6 - 2L_5 - \frac{2}{3} L_3$$

Proof: It is well know that the double dual $R^{*}_{\alpha\beta\mu\nu}{}^{*}$ satisfies the identity

$$R^{*}_{\alpha\beta\mu\nu}{}^{*} = R_{\alpha\beta\mu\nu} - 2W_{\alpha\beta\mu\nu} - \frac{1}{6} R\, g_{\alpha\beta\mu\nu} \tag{6.1}$$

or, equivalently

$$R^{*}_{\alpha\beta\mu\nu}{}^{*} = - W_{\alpha\beta\mu\nu} + \frac{1}{2} (R_{\alpha\mu}\, g_{\beta\nu} + R_{\beta\nu}\, g_{\alpha\mu} - R_{\alpha\nu}\, g_{\beta\mu} - R_{\beta\mu}\, g_{\alpha\nu}) - \\ - \frac{1}{3} R\, g_{\alpha\beta\mu\nu} \tag{6.2}$$

Thus

$$L_8 = - L_6 - \frac{1}{3} R(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})\, F^{\alpha\beta} F^{\mu\nu} + \frac{1}{2}(R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - \\ - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu})\, F^{\alpha\beta} F^{\mu\nu}$$

$$L_8 = - L_6 - \frac{2}{3} R\, F_{\mu\nu} F^{\mu\nu} - 2R_{\mu\nu} F^{\mu}{}_{\alpha} F^{\alpha\nu}$$

or

$$L_8 = - L_6 - \frac{2}{3} L_3 - 2L_5 \qquad \text{qed.} \tag{6.3}$$

Let us examine an example of a class one theory which is given by the Lagrangian

$$L = \sqrt{-g}\{\frac{1}{\kappa} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta R W_\mu W^\mu\} \qquad \textbf{(6.4)}$$

in which $F_{\mu\nu} = W_{\mu,\nu} - W_{\nu,\mu}$.

Variation of $g_{\mu\nu}$ and W_μ yield the equation of motion:

$$(\frac{1}{\kappa} + \beta W^2)G_{\mu\nu} = \beta\Box W^2 g_{\mu\nu} - \beta W^2{}_{,\mu;\nu} - \beta R W_\mu W_\nu - E_{\mu\nu} \qquad (6.5a)$$

$$F^{\mu\nu}{}_{;\nu} = - \beta R W^\mu \qquad (6.5b)$$

in which $E_{\mu\nu}$ is the Maxwell's stress tensor given by

$$E_{\mu\nu} = F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

Remark that as in other non-minimally coupled interactions this set of equations allows a renormalization of gravitational constant. Indeed, consider the special case in which $W_\mu W^\mu = Z =$ constant; then

$$(\frac{1}{\kappa} + \beta Z)G_{\mu\nu} = - \beta R W_\mu W_\nu - E_{\mu\nu} \qquad (6.6)$$

Taking the trace of this equation we obtain

$$\left(\frac{1}{\kappa} + \beta Z \right) R = \beta Z R$$

which implies

$$R = 0$$

Then inserting this result into equation (6.3)

$$R_{\mu\nu} = - \kappa_{ren} E_{\mu\nu}$$

in which

$$\frac{1}{\kappa_{ren}} = \frac{1}{\kappa} + \beta Z$$

Thus in this case equations (6.2) reduce to

$$R_{\mu\nu} = - \kappa_{ren} E_{\mu\nu} \tag{6.7a}$$

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{6.7b}$$

which is nothing but Maxwell's electrodynamics coupled minimally to gravity with a different gravitational coupling (given by (6.7)) plus the (gauge) *condition*

$$W_{\mu} W^{\mu} = \text{constant} = Z.$$

Suppose now that besides this field there are other forms of matter present described by the traceless energy-momentum tensor $T^{\mu\nu}_{(rest)}$.

That is, take the theory given by

$$L = \sqrt{-g} \{ \frac{1}{\kappa} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta R W_\mu W^\mu + L_{(rest)} \} \tag{6.8}$$

in which $L_{(rest)}$ represents the Lagrangian for all other kinds of matter, such that

$$T^{(rest)}_{\mu\nu} g^{\mu\nu} = T^{(rest)} = 0 \tag{6.9}$$

The equation of motion is given by

$$(\frac{1}{\kappa} + \beta W^2) G_{\mu\nu} = \beta \Box W^2 g_{\mu\nu} - \beta W^2{}_{,\mu;\nu} - \beta R W_\mu W_\nu - E_{\mu\nu} - T^{(rest)}_{\mu\nu} \tag{6.10a}$$

$$F^{\mu\nu}{}_{;\nu} = - \beta R W^\mu \tag{6.10b}$$

Let us analyse the same case as above in which $W_\mu W^\mu = Z$ = constant.

Due to (6.9) we obtain, as previously, that

$R = 0$.

Then (6.10a,b) takes the form

$$R_{\mu\nu} = - \kappa_{ren} E_{\mu\nu} - \kappa_{ren} T^{(rest)}_{\mu\nu}$$

$$F^{\mu\nu}{}_{;\nu} = 0$$

Thus, in this particular state of the W^μ vector, the non-minimal coupling represented by the presence of the term $RW_\mu W^\mu$ in the Lagrangian is nothing but a simple mechanism by means of which gravity with a coupling constant distinct from the newtonian value can be generated by ordinary matter whose stress-energy tensor is trace-free, e.g. photons, neutrinos, etc. One can then contemplate the possibility that such process yields a non-positive renormalized constant (κ_{ren}).

Exercise: *Using this interaction (formula 8) can you envisage the* possibility in which the effective gravitational coupling constant changes its sign during the evolution of the Universe?

What can we say about the Cauchy problem of equations (6.10a,b) in this case?

The above embarras du choix of the non-minimal interaction led to the search of alternatives ways of selecting the Lagrangian which describes the coupling of a vector field to gravity.

The most appealing criteria should be one envolving invariance under a given transformation which characterizes a symmetry property. It promptly comes to one's mind to examine dual rotation.

A dual map is a transformation on the set of the bi-tensors $F_{\mu\nu}$ such that

$$F_{\mu\nu} \to F'_{\mu\nu} = \cos\theta\ F_{\mu\nu} + \sin\theta\ F^*_{\mu\nu} \qquad (6.11)$$

Classical Maxwell's electrodynamics is invariant under such transformation for any constant angle θ (but it is not invariant if $\theta=\theta(x)$).

One should then be guided to propose the generalisation of such invariance for the non-minimal Lagrangian L. It is a rather simple exercise to show that this is accomplished by the unique combination.

$$L_{(in)} = L_5 + \frac{1}{4} L_3 = (R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}) F^{\mu}{}_{\alpha} F^{\alpha\nu} \qquad (6.12)$$

The interesting fact is that the invariance of $L_{(in)}$ by the dual map (equation 6.11) is not breaked if we transform the constant angle θ into a function $\theta(x)$. This is a remarkable property which has no counterpart in the flat space limit. This certainly points in favour of $L_{(in)}$ but not too strong to eliminate its currents.

Exercise: The invariance of Maxwell's electrodynamics under a rotation of the dual angle has led to the so called already unified program (Rainich, 1925; Misner and Wheller, 1957). Examine the possibility of generalization of that program in case of non-minimal coupling.

We leave to another ocasion to proceed with such generic study (which, I believe, the reader will not have too much difficulty in going alone through it).

Let us now turn to an example of a class one Lagrangian.

COMMENTS AND REFERENCES

Accioly, A.J. and Pereira da Silva, N.L.P. (1986) Phys. Lett.A, 118,271.

Remark that in this paper, which considers the theory L_3, there is a mistaken in the equations of motion. The correct equations are those given in the text, equation (6.5).

Goenner, H.F.M. (1984) Foundation of Physics, 14, 865.

7. THE NOVELLO-SALIM-HEINTZMANN SOLUTION

In a paper of 1979 Novello and Salim undertook the task of examining the cosmological consequences of class one Lagrangians mainly in respect to the possibility of finding non-singular solutions for the geometry of the Universe.

They started with the theory given by L_1 of Chapter 6 and looked for a spatially homogeneous and isotropic metric such that

$$ds^2 = dt^2 - A^2(t)d\sigma^2 \qquad (7.1)$$

Let us consider the ansatz that the potential W^μ is given by

$$W_\mu = (W(t),\ 0,\ 0,\ 0) \qquad (7.2)$$

then

$$F_{\mu\nu} = 0 \qquad (7.3)$$

(remark that due to non-minimal coupling, from the vanishing of $F_{\mu\nu}$ does not follow the vanishing of the total energy).

Define the function $\Omega(t)$ by

$$\Omega(t) = \frac{1}{\kappa} + \beta W^2 \qquad (7.4)$$

Then equations (7.2) reduces to the set:

$$3\,\frac{\ddot{A}}{A} = -\,\frac{\ddot{\Omega}}{\Omega} \qquad (7.5a)$$

$$\frac{\ddot{A}}{A} + 2\left(\frac{\dot{A}}{A}\right)^2 + \frac{2\varepsilon}{A^2} = - \frac{\dot{A}}{A}\frac{\dot{\Omega}}{\Omega} \qquad (7.5b)$$

$$\frac{\ddot{A}}{A} + 2\left(\frac{\dot{A}}{A}\right)^2 - \frac{1}{A^2}\left(\frac{\sigma''}{\sigma} + \frac{\sigma'^2 - 1}{\sigma^2}\right) = - \frac{\dot{A}}{A}\frac{\dot{\Omega}}{\Omega} \qquad (7.5c)$$

$$\Box\,\Omega = 0 \qquad (7.5d)$$

From (7.5b,c) we obtain the same restriction as in Friedmann models that is, ε can take the values $0, \pm 1$.

From (7.5d) we obtain

$$\dot{\Omega} = (\text{constant})\ A^{-3}$$

A particular solution of this set of equations (7.5) - the Novello-Salim-Heintzmann proposal - is given by

$$W^2(t) = \frac{1}{\kappa}\left[\,1 - \frac{t}{A}\,\right] \qquad (7.6a)$$

$$A(t) = \sqrt{t^2 + Q^2} \qquad (7.6b)$$

in which Q is a constant that measures the minimum value of the radius of the Universe. When Q = 0 the system reduces to empty Minkowski space-time written in Milne coordinates.

Exercise: Show that the so called Milne Universe

$$ds^2 = e^{2\eta}(d\eta^2 - dx^2 - dy^2 - dz^2)$$

is nothing but Minkowskii flat space-time written in a non-inertial coordinate system.

For $Q \neq 0$ this model represents an Eternal Universe without singularity.

Remark that Novello-Salim-Heintzmann solution coincides with Melnikov-Orlov geometry given by (4.6). The difference between them being only the interpretation of the minimum radius Q.

We can examine the system (7.5) of equations in a more transparent way if we define new variables x and y by setting:

$$x = 3\,\frac{\dot{A}}{A} \qquad (7.7a)$$

$$y = \frac{\dot{\Omega}}{\Omega} \qquad (7.7b)$$

Then, equations (7.5) reduces to a planar autonomous system:

$$\dot{x} = -\frac{1}{3}x^2 + xy \qquad (7.8a)$$

$$\dot{y} = -y^2 - xy \qquad (7.8b)$$

C. Romero has studied this set of equations and drawn the integral curves of this system on the Poincaré sphere (compactifying the whole phase space (x,y)).

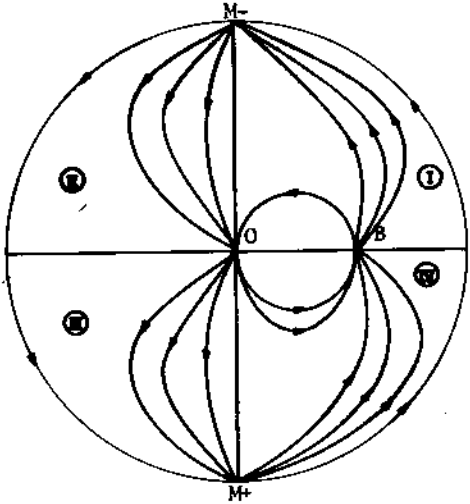

Figure 1 - Phase portrait of dynamical system (7.8a,b) with compactification of the infinity. Points $M_{(\pm)}$ represents Minkowskii space-time.

Let us follow naïvely the (time) parameter t throughout the figure. From $M_{(-)}$ (which corresponds to the infinite past $t = -\infty$) to the origin 0 ($t = o$) we have a contracting Novello-Salim-Heintzmann phase. From the minimum value for the radius of the Universe at 0 until the future infinite the model represents an expanding Novello-Salim-Heintzmann ending at Minkowskii $M_{(+)}$. If we insist and follow the graph from $M_{(+)}$ to point B we find models which starts in Minkowskian Nothing at past infinity and ends at finite time $t = -|Q|$ in a singularity at B.

Then we find a new set of solutions: those which starts at the singularity B (for $t = -|Q|$), pass through 0 ($t = 0$) and ends at the singularity in B. Here we find a class of closed solutions (all others geometries are open). Finally, there are the solutions which start at the singularity B ($t = + |Q|$) and ends in the infinite future in a Minkowskian regime.

Although the identification of $M_{(-)}$ to $M_{(+)}$ is not free of ambiguities, one should be tempted to interpret the above figure as being the representation of eternal cycles of Universes of infinite duration beginning at Minkowskian Nothing, as fluctuation, and ending at Minkowskian Nothing, and so on.

Exercise: a) Interpret Novello-Salim-Heintzmann solution as being originated by a fluid in Einstein's standard theory

$$G_{\mu\nu} = - \kappa T_{\mu\nu} \quad .$$

What condition of the singularity theorem is violated in such way as to allows this fluid to generate a non-singular cosmos?

b) Do equations (2a,b) have a well-defined Cauchy problem? Examine this problem for Novello-Salim-Heintzmann Universe. (S.Jorda, 1987) .

c) Is Novello-Salim-Heintzmann solution linearly stable?

d) Consider the equation of $F_{\mu\nu}$ in the presence of an external current:

$$F^{\mu\nu}{}_{;\nu} = - \beta R W^{\mu} + J^{\mu}_{ext}$$

Show that although in general $J^{\mu}{}_{(ext);\mu} \neq 0$ in a Novello-Salim-Heintzmann type of solution the conservation law $J^{\mu}{}_{;\mu} = 0$ is obeyed.

COMMENTS AND REFERENCES

Novello, M. and Heitzmann, H. (1984) GRG Journal 16 (535).
Novello, M. and Romero, C. (1987) (to appear in GRG Journal).
Novello, M. and Salim, J.M. (1979) Phys. Rev. D20 (377).

8. WIST (WEYL INTEGRABLE SPACE TIME)

1. Some years ago, during the IIIrd Brazilian School of Cosmology, Vittorio Canuto gave us some very beautiful lessons on the state of the so-called Dirac's scale-covariant theory of gravitation.

In this Vth School we have the opportunity to follow Narlikar's lessons on Hoyle-Narlikar theory of gravity. Both theories have its roots on a deep analysis of the role of conformal transformation in Physics. I will not extend here in Hoyle-Narlikar's theory, but invite the reader to study Narlikar's lectures in this Vth School and other references quoted therein.

In this section, I intend to emphasize a common feature of both theories by presenting some curious properties which appear when one considers the most general conformally invariant structure of space-time, that is, Weyl's generalization of Riemann space.

2. In the 1984 Chicago Meeting "Inner Space/Outer Space", some scientists speculated on the possibility to elaborate a complete cosmological theory, i.e., a theory which "pretends to describe the Universe from the very beginning (including the quantum gravitational stage) and up to the present time". This includes the mechanism of creation of space-time itself, as a classical counterpart of some (yet-to-be-discovered) quantum process.

Such line of research is the *aggiornamento* of the ambitious Einstein's cosmological program, which intend to describe completely the metric properties of our Universe as a unique solution of the equations of gravity from a given distribution of matter.

From time to time scientists criticize this program and elaborate, by many different ways, less ambitious analysis pointing out the limitations of the naïf applicability of globalisation which is nothing but the extension of local physical rules to the whole Universe.

In this lecture I will present an instigating model in which the global structure of space-time becomes dependent on an entirely uncontrollable function.

This solution represents the opposite of the above scheme and leads to the complete abandon of the laws which could be attached to the creation of the Universe, or, less dramatically, of a complete classical picture of the Cosmos.

3. Let us consider once more the non-minimal coupling of a vector field Z^{μ} with gravity given by the Lagrangian

$$L = \sqrt{-g}\,\{ \frac{1}{\kappa} R - \frac{1}{4} Z_{\mu\nu}Z^{\mu\nu} + \beta R Z_{\mu} Z^{\mu} \} \tag{8.1}$$

in which $Z_{\mu\nu} = Z_{\mu,\nu} - Z_{\nu,\mu}$.

In the standard procedure one varies $g_{\mu\nu}$ and Z_{μ} independently assuming a priori the Riemannian nature of space-time. However, one could abandon this restriction and let the symmetric affine connection $\Gamma^{\alpha}_{\mu\nu}$ to vary independently. This technique, which we owe to Palatini, yields in the standard Einstein (linear) Lagrangian, $L_E = \sqrt{-g}\,R$, the beautiful result that space-time is riemannian. However, such result is not maintened if matter is coupled non-minimally to gravity. Let us prove this for the case of the vector field Z^{μ} in Lagrangian (8.1). Let us vary $g_{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu}$ and Z^{μ} independently.

$$\delta L = -\frac{1}{2} L\, g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{\kappa}\sqrt{-g}\, R_{\mu\nu}\delta g^{\mu\nu} + E_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g} +$$

$$+ \beta\sqrt{-g}\, R\, Z_{\mu}Z_{\nu}\,\delta g^{\mu\nu}$$

$$L = [(\frac{1}{\kappa} + \beta Z^2)G_{\mu\nu} + E_{\mu\nu} + \beta R Z_{\mu}Z_{\nu}]\delta g^{\mu\nu} + \sqrt{-g}\,(\frac{1}{\kappa} + \beta Z^2)g^{\mu\nu}\delta R_{\mu\nu} + L_{\mu}\delta Z^{\mu} \tag{8.2}$$

Then we obtain

$$(\frac{1}{\kappa} + \beta Z^2)G_{\mu\nu} = - E_{\mu\nu} - \beta R Z_\mu Z_\nu \tag{8.3a}$$

$$g_{\mu\nu;\alpha} = K_\alpha g_{\mu\nu} \tag{8.3b}$$

$$Z^{\mu\nu}\|_\nu = - \beta R Z^\mu \tag{8.3c}$$

in which

$$K_\alpha = - \nabla_\alpha \ln(\frac{1}{\kappa} + \beta Z^2) \tag{8.4}$$

The symbol ; represents covariant derivative evaluated with the affine connection $\Gamma^\alpha_{\mu\nu}$. The symbol double bar $\|$ represents covariant derivative evaluated with the Christoffel symbol $\{ {\alpha \atop \mu\nu} \}$.

Remark that (8.3b) implies that space time is not riemannian but instead Weyl-like. Besides, the vector K_α is a gradient and this space is a very special one called WIST (Weyl Integrable Space Time).

The defining property of Weyl space-time

$$g_{\mu\nu;\alpha} = K_\alpha g_{\mu\nu} \tag{8.5}$$

implies that there is a variation of the lenght ℓ_σ of any vector which is parallel-transported, given by

$$\Delta\ell = \ell \,_{o}K_{\mu}\, \Delta x^{\mu} \tag{8.6}$$

Such property has the undesirable consequence that the measure of lenghts depends on the previous history of the measurement aparatus. In the early twenties, this fact led to the abandon of such geometry.

However, using the fact that in a WIST the vector K_{μ} is irrotational, in a closed trajectory we have

$$\oint \Delta\ell = 0 \tag{8.7}$$

which eliminates the major part of the criticism against the application of Weyl geometry in the actual Universe.

From the definition (8.5) of Weyl geometry we obtain the connection:

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} - \frac{1}{2} (K_{\mu}\, \delta^{\alpha}_{\nu} + K_{\nu}\delta^{\alpha}_{\mu} - K^{\alpha}g_{\mu\nu}) \tag{8.8}$$

We leave it to the reader to show that a WIST is conformally related to a Riemann space.

Let us now come back to the fundamental equations (8.3) and look for a solution which represents an homogeneous and isotropic (Friedmann-like) geometry, that is,

$$ds^2 = dt^2 - A^2(t)[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{8.9}$$

Now, using 8.9 we can write the contracted curvature tensor $R_{\mu\nu}$ in terms of the tensor $\overset{(o)}{R}_{\mu\nu}$ of the associated Riemann space (that is, constructed with Christoffel symbol) and we obtain (exercise)

$$R_{\mu\nu} = \overset{(o)}{R}_{\mu\nu} - \frac{3}{2} K_{\mu}\|_{\nu} + \frac{1}{2} K_{\nu}\|_{\mu} - \frac{1}{2} K^{\alpha}\|_{\alpha} g_{\mu\nu} + \frac{1}{2} K^2 g_{\mu\nu} - \frac{1}{2} K_{\mu}K_{\nu} \tag{8.10}$$

in which we have used the fact that

$$K_{\mu;\nu} = K_{\mu}\|_{\nu} + K_{\mu}K_{\nu} - \frac{1}{2} K^2 g_{\mu\nu} \tag{8.11}$$

Taking the trace of equation (8.10):

$$R = \overset{(o)}{R} + \frac{3}{2} K^2 - 3K^{\mu}\|_{\mu} \tag{8.12}$$

in which $\overset{(o)}{R}$ is the scalar of curvature of the Riemann (associated) space.

We make the ansatz that the vector field Z^{μ} takes the form

$$Z^{\mu} = (Z(t), 0, 0, 0) \tag{8.13}$$

Then it follows that $Z_{\mu\nu} = 0$ and $E_{\mu\nu} = 0$.

From (8.3c) the trace vanishes:

$$R = 0 \tag{8.14}$$

Using these results into equation (8.3c)

$$\left(\frac{1}{K} + \beta Z^2 \right) R_{\mu\nu} = 0$$

or

$$R_{\mu\nu} = 0 \tag{8.15}$$

Using (8.10) and the form (8.9) for the metric the whole system reduces to the set

$$2\,\frac{\ddot{A}}{A} = \ddot{f} + \dot{f}\,\frac{\dot{A}}{A} \tag{8.16a}$$

$$\frac{\ddot{A}}{A} + 2\left(\frac{\dot{A}}{A}\right)^2 - \frac{2}{A^2}\,\frac{\sigma''}{\sigma} = \frac{1}{2}\,\ddot{f} - \frac{1}{2}(\dot{f})^2 + \frac{5}{2}\,\frac{\dot{A}}{A}\,\dot{f} \tag{8.16b}$$

in which we have defined the function f by the expression:

$$f(t) = -\,\ell n\left(\frac{1}{\kappa} + \beta Z^2\right)$$

Now comes the curious fact that, as a consequence of our ansatz, equations (8.16a,b) reduce to just one single equation

$$\dot{f} = 2\,\frac{\dot{A}}{A} \pm \frac{2\sqrt{-\varepsilon}}{A} \tag{8.17}$$

in which $\varepsilon = -\frac{\sigma''}{\sigma}$. Remark that the model excludes the possibility of a closed ($\varepsilon = 1$) section.

Once the whole system of equations rests only on the unique condition (8.17) to be satisfied by the two functions A(t) and Z(t) we conclude that the function Z(t) - which is the real cause of the curvature of the space-time - is completely indeterminate. This property led us to call it a Marionette Universe.

Exercise: Using Einstein's standard form for the equation of gravity evaluate the energy of the Z-fields.

Answer: $\rho_{(Z)} = 3\left(\frac{\dot{A}^2 - 1}{A^2}\right)$

Exercise: Consider a more general non-minimal Lagrangian e.g.,

$$L_a = \sqrt{-g}\, R_{\mu\nu} W^{\mu} W^{\nu}$$

or

$$L_b = \sqrt{-g}\, R_{\mu\nu} F^{\mu}{}_{\alpha} F^{\alpha\nu}$$

Use Palatini's variational process (independent variables: $g_{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu}$ and W_{μ}). What is the nature os the space-time which these Lagrangians describe?

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9. PHASE TRANSITION IN COSMOLOGY

1. The analogy between superconductivity theory and the phenomenon of spontaneous symmetry breaking in field theory, largely explored these recent years, led to a new program of research in cosmology. A new scenario for the behavior of the Universe in a very condensated regime is therefore being developed. Phase transition in cosmology was discussed in the IIIrd School of Cosmology and more recently many articles of review on the subject have appeared (Linde, 1979).

The purpose of the present lecture is to examine a particular kind of theory which has not been previously analysed and whose main characteristic is to have phase transition induced by gravity.

2. In the present school we have seen the difficulty in conciliating standard cosmological HBB model with the observed isotropy of the cosmic fluid. In principle, one should expect that in an Eternal Universe (for instance, with a previous collapsed era) this problem disappeared, for, in general, the horizons are not present anymore and consequently any region had enough time to be exposed to the all-embracing cosmic interaction. However, this question is not so easy and depends crucially on the very specific properties of the matter which dominates the gravitational behavior of the Universe.

Let us present a model associated to the phenomenon of gravitationally induced phase transition in which anisotropy may not be an evanescent process.

We start by considering a non-perfect fluid which is described by the stress-energy tensor

$$T_{\mu\nu} = \rho V_\mu V_\nu - p\, h_{\mu\nu} + \pi_{\mu\nu} \qquad (9.1)$$

in which $h_{\mu\nu}$ is the projector operator (in the 3-dimensional rest space of V^μ) defined by

$$h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - V^{\mu}V_{\nu} \tag{9.2}$$

and

$$P = p_{th} + \pi$$

$$\pi_{\mu\nu}V^{\mu} = 0$$

$$\pi_{\mu\nu} = \pi_{\nu\mu}$$

$$\pi_{\mu\nu}g^{\mu\nu} = 0$$

The extra-term (π, $\pi_{\mu}{}^{\nu}$) in expression (9.1) represents the (isotropic, anisotropic) contribution of viscous processes.

The fluid is assumed to be stokesian, that is, the dynamical objects π and $\pi_{\mu}{}^{\nu}$ are given as functionals of the associated kinematical quantities, e.g., the expansion factor $\theta = V^{\alpha}{}_{;\alpha}$ and the shear

$$\sigma_{\mu\nu} = \frac{1}{2} h_{(\mu}^{\alpha} h_{\nu)}^{\beta} V_{\alpha;_{\beta}} - \frac{\theta}{3} h_{\mu\nu}$$

From the theory of matrices the most general expression for $\pi_{\mu}{}^{\nu}$ is given by

$$\pi^{\mu}{}_{\nu} = f_1 h^{\mu}{}_{\nu} + f_2 \sigma^{\mu}{}_{\nu} + f_3 \sigma^{\mu}{}_{\varepsilon} \sigma^{\varepsilon}{}_{\nu} \tag{9.3}$$

in which the scalar coeficients f_k are defined by a series expansion, typically the following

$$f_K = f_{Ko} + f_{K1} I_1 + [f_{K2}(I_1)^2 + f'_{K2}I_2] + $$

$$+ [f_{K3}(I_1)^3 + f'_{K3} I_1 I_2 + f''_{K3} I_3] + \dots \tag{9.4}$$

for K = 1. 2. 3. In this expression (9.4) the I_k are the canonical invariants of matrix $\tilde{\theta}$ (with components $\theta^{\mu}{}_{\nu} = \sigma^{\mu}{}_{\nu} + \frac{1}{3}\theta h^{\mu}{}_{\nu}$), given by

$$I_1 = \mathrm{Tr}\,\tilde{\theta} = \theta$$

$$I_2 = \mathrm{Tr}\,\tilde{\theta}^2 = \sigma^2 + \frac{1}{3}\theta^2$$

$$I_3 = \mathrm{Tr}\,\tilde{\theta}^3 = \sigma_{\mu}{}^{\alpha}\,\sigma_{\alpha}{}^{\lambda}\,\sigma^{\mu}{}_{\lambda} - \theta\sigma^2 - \frac{1}{9}\theta^3 \qquad (9.5)$$

In order to simpliry our demonstration let us limit the expansion (9.4) up to third order and write

$$\pi_{\mu}{}^{\nu} = [\alpha_0 + \alpha_1\theta + \beta\theta^2]\,\sigma_{\mu}{}^{\nu} + [A_0 + A_1\theta + \ldots]h_{\mu}{}^{\nu} + \delta\sigma_{\mu}{}^{\alpha}\,\sigma_{\alpha}{}^{\nu} \qquad (9.6)$$

or redefining the constants α_0 and α_1 as: $\alpha_0 = -a^2\theta^*$ and $\alpha_1 = a^2$ for later convenience, we set

$$\pi_{\mu}{}^{\nu} = [a^2(\theta - \theta^*) + \beta\sigma^2]\,\sigma_{\mu}{}^{\nu} + [A_0 + A_1 + \ldots]h_{\mu}{}^{\nu} + \delta\sigma_{\mu}{}^{\alpha}\,\sigma_{\alpha}{}^{\nu}$$

in which a, θ^*, β and δ are constants with the additional hypothesis that $\beta > 0$ and $\delta < 0$.

The next step is to search for the fundamental equilibrium configurations of such fluid taking into account the gravitational field it generates. This is achieved by looking for the extremum (minimum) of a unique function which generalizes for the non equilibrium case the free energy F = E - TS.

Let us suppose that the cosmic fluid in its non-equilibrium era is represented by (9.1). As we are contemplating the possibility to describe the evanescence of anisotropy into an isotropic era, let us proceed as in Landau's exam of the behavior of irregularities in the neighborhood of a phase transition.

The first step is to choose a convenient macroscopic parameter which netly caracterizes the state of order/disorder of the system. As we are considering the existence or absence of privileged directions in space it seems rather natural to select the shear ($\sigma^{\mu}{}_{\nu}$) or any functional of it to represent such parameter. We can thus follow Gramsbergen et al (1986) and identify the fluid either as a liquid (less ordered, more symmetric) or isotropic phase from a crystal structure (less symmetric, more ordered) or anisotropic phase.

The free energy F is to be described, following Landau's treatment, as a polynomial expansion in the invariants I_1, I_2 and I_3. We call such basic expression (in the absence of gravitational interaction) F_0. Landau's phase transition mechanism depends crucially on this expansion: for instance, the lowest order should be at least quadratic in the order parameter to allow the isotropic state to be accessible. Besides, the coeficient of the lowest term should change sign with the increasing of temperature - the real parameter which controls the transition. We will not follow this procedure and show that in the case of Einstein's theory of gravity the control parameter is not the temperature.

The first question we are faced to is: what is the influence of gravity in the free energy? This is a problem which we cannot, for the time being, solve by first principles nor by direct experimentation. Thus, we can only proceed by analogy.

In case of an external magnetic field B_{μ} the contribution of this field to F is given by

$$(\Delta F)_{mag} = \xi B_\mu B^\nu \sigma^\mu{}_\nu \tag{9.7}$$

We are, then induced to admit that the gravitational effect is given by

$$(\Delta F)_{grav} = \gamma R_\mu{}^\nu \sigma^\mu{}_\nu \tag{9.8}$$

This expression is indeed unique if we are limited to linear terms in the order parameter (remember that $\sigma^\mu{}_\nu$ is trace-free). (The extension of this theory to non-linear terms in $\sigma^\mu{}_\nu$ is an interesting non-trivial exercise.)

Using Einstein's equations

$$(\Delta F)_{grav} = -\gamma[T_\mu{}^\nu - \frac{1}{2} T\delta_\mu{}^\nu + \Lambda\delta_\mu{}^\nu]\sigma^\mu{}_\nu$$

$$(\Delta F)_{grav} = -\gamma T_\mu{}^\nu \sigma^\mu{}_\nu \tag{9.9}$$

From the expression (9.1) for the stress-energy tensor:

$$(\Delta F)_{grav} = -\gamma \pi_\mu{}^\nu \sigma^\mu{}_\nu \tag{9.10}$$

Let us simplify our exposition and restrict ourselves to the case in which there is a plane of isotropy and write

$$\sigma^\mu{}_\nu = \begin{pmatrix} -\frac{1}{2}\Sigma & & \\ & -\frac{1}{2}\Sigma & \\ & & \Sigma \end{pmatrix} \tag{9.11}$$

which implies

$$T_r\,\hat{\sigma}^2 = \frac{3}{2}\,\Sigma^2$$

$$T_r\,\hat{\sigma}^3 = \frac{3}{4}\,\Sigma^3$$

[Exercise: Abandon this restriction and re-do the following calculations in the general case.]

Thus we obtain the expression

$$(\Delta F)_{grav} = -\,\gamma[a^2(\theta-\theta^*) + \beta\sigma^2]\,\sigma^2 \quad -$$

$$-\,\gamma\,\delta\,\sigma^{\mu}{}_{\alpha}\,\sigma^{\alpha}{}_{\lambda}\,\sigma^{\lambda}{}_{\mu}$$

or using (9.11)

$$(\Delta F)_{grav} = -\,\frac{3}{2}\,\gamma[a^2(\theta-\theta^*)\Sigma^2 + \frac{1}{2}\,\delta\Sigma^3 + \frac{3}{2}\,\beta\Sigma^4] \qquad (9.12)$$

The most probable state of equilibrium is given by the minimum of the free-energy $F = F_o + (\Delta F)_{grav}$, that is

$$3a^2(\theta-\theta^*)\Sigma + \frac{9}{4}\,\delta\Sigma^2 + 9\beta\Sigma^3 = 0 \qquad (9.13)$$

There are two possible solutions:

$$\Sigma = 0 \tag{9.14a}$$

corresponding to the isotropic phase, and

$$3\beta\Sigma_o^2 + \frac{3}{4}\,\delta\Sigma_o + a^2(\theta - \theta^*) = 0 \tag{9.14b}$$

corresponding to an anisotropic state Σ_o. The possibility of the existence of the solution $\Sigma_o \neq 0$ is given by the inequality

$$\Delta_{(1)} = \frac{9}{16}\,\delta^2 - 12\,\beta a^2(\theta - \theta^*) \geq 0 \tag{9.15}$$

or

$$\theta < \theta^* + \frac{3}{64}\,\frac{\delta^2}{a^2\beta} \equiv \theta_c \tag{9.16}$$

This is a remarkable result: it states that the very possibility of *existence of an equilibrium configuration which is not isotropic depends* on the intensity of the expansion factor θ: for values of θ bigger than θ_c we can have at least one extremum $\Sigma_o \neq 0$. It remains to examine if it is a minimum. In this case, we must have

$$-3\gamma[a^2(\theta - \theta^*) + \frac{3}{2}\,\delta\Sigma + 9\beta\Sigma^2] > 0 \tag{9.17}$$

Let us restrict here our exam to the case $\gamma = -m^2 < 0$ (we left as an exercise to consider the case of a positive γ).

Then, we must have

$$9\beta\Sigma^2 + \frac{3}{2}\,\delta\Sigma + a^2(\theta - \theta^*) > 0 \tag{9.18}$$

The associated equation

$$9\beta\Sigma^2 + \frac{3}{2}\,\delta\Sigma + a^2(\theta - \theta^*) = 0 \tag{9.19}$$

will have real roots if

$$\Delta_{(2)} = \frac{9}{4}\,\delta^2 - 36\,\beta a^2(\theta - \theta^*) > 0$$

or

$$\delta^2 > 16\,\beta a^2(\theta - \theta^*) \tag{9.20}$$

On the other hand, if $\Delta_{(2)} < 0$ then inequality (9.18) will be satisfied for any value of Σ. We note, however, that condition (9.20) is a consequence of the existence of an extremum. Thus we are obliged to compare the relative position of the roots $\Sigma_{(\pm)}$ of (9.19) with the extremum points $\Sigma_{(1,2)}$.

We have

$$\Sigma_{(\pm)} = -\frac{\delta}{2\beta}\left[1 \mp \sqrt{1 - \frac{16a^2(\theta - \theta^*)}{\delta 2}}\right] \tag{9.21}$$

$$\Sigma_{(1,2)} = -\frac{\delta}{8\beta}\left[1 \mp \sqrt{1 - \frac{64a^2\beta(\theta - \theta^*)}{3\delta^2}}\right] \tag{9.22}$$

It is easy to show that, for instance, $\Sigma_{(2)} > \Sigma_{(+)}$. Thus, if there is an extremum $\Sigma_{(2)} \neq 0$ it is a minimum. We are now faced to the following question: if there exists both extrema $\Sigma = 0$ and $\Sigma_o \neq 0$ which one will be the absolute minimum? Let us examine the conditions in which $\Sigma = 0$ is a global minimum. We must have

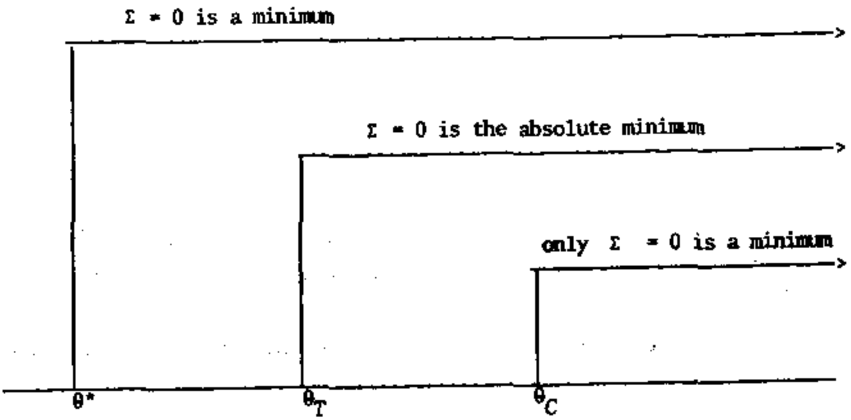
$$F(0) < F(\Sigma_o) \qquad\qquad , \text{ or}$$

$$3\beta\Sigma_o^2 + \delta\Sigma_o + 2a^2(\theta - \theta^*) > 0$$

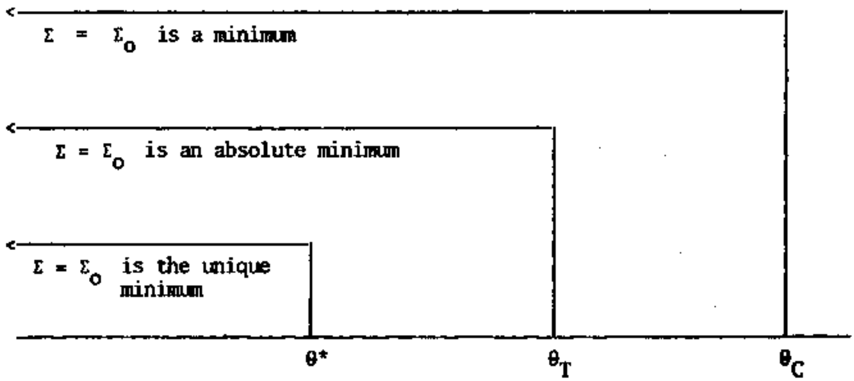
which is satisfied if

$$\theta > \theta^* + \frac{\delta^2}{24\beta a^2} \equiv \theta_T \tag{9.23}$$

Let us represent graphically these results in graphs 1, 2:



Graph 1. The conditions for isotropic phase



Graph 2. The conditions for anisotropic phase

Note:

$$\theta_C = \theta^* + \frac{3}{64} \quad \frac{\delta^2}{\beta a^2}$$

$$\theta_T = \theta^* + \frac{1}{24} \quad \frac{\delta^2}{\beta a^2}$$

We can then summarize these results:

$\theta > \theta_C$: The most favorable state (MFS) is an isotropic phase.

$\theta_T < \theta < \theta_C$: The most favorable state is the isotropic phase, but there is a local minimum corresponding to a small anisotropy.

$\theta^* < \theta < \theta_T$: The most favorable state is the anisotropic one, but there is a local minimum corresponding to an isotropic phase.

$\theta < \theta^*$: Anisotropic phase.

The above mechanism of phase transition is controlled by the expansion factor (θ) which mimics the role of temperature in standard Landau's model. The shear ($\sigma^{\mu}{}_{\nu}$) is the macroscopic order parameter in terms of which the whole theory is based. The present model depends on three basic conditions:

(i) Einstein's theory of gravity.

(ii) The expression (9.6) for the (anisotropic) pressure.

(iii) Formula (9.8) for the contribution of gravity to the free energy.

If all these three conditions are fulfilled then the consequences (9.24) indubitably follow.

Although one can very safely believe in (i) and accepts (iii) without much pain, there is not any evidence neither pro nor against the hypothesis that the cosmic fluid of our actual Universe satisfies (ii). Thus the application of the present mechanism to our Universe rests a matter to be decided in the future.

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10. NON LINEAR LAGRANGIANS

Since the early days of General Relativity many scientists (Eddington, Weyl, Lanczos, etc) have examined an alternative, more complex, form for the Lagrangian which describes the dynamical evolution of gravitational fields, using non-linear functionals of the curvature. The leitmotiv to such a study has changed from time to time. In particular, it has been avocated that (as it occurs in electrodynamics) quadratic (or higher orders) terms are an almost direct consequence of quantum phenomena.

It has been suggested that the true Lagrangian should be thought as a polynomial series in the curvature, thus searching the identification of gravity to some sort of elasticity of space-time. The same effect seems to occur as a consequence of stochastic fluctuations of the geometry. Others (see the review presented by Folomeshikin, 1971) have argued that any zero mass field should be conformal invariant and consequentely the Lagrangian which describes gravity should be at least of quadratic orders (e.g., $W^{\alpha\beta\mu\nu}W_{\alpha\beta\mu\nu}$).

Finally, it has been claimed by some scientists that non-linear terms are indispensable in order to inhibit singular solutions to appear.

Exercise: Explain this last sentence in the context of the singularity theorems. (Remark that as we have seen in these lectures one should weaken this sentence and substitute the word "indispensable" by "possible".)

Let me emphasize that I do not pretend to give here a complete historical review of the abundant literature on this subject, but only to pick up some examples here and there of the typical situation throughout these years.

In order to capturate the spirit of the controversy which was in the air at the end of the sixties it seems worth to quote the american physicist C.W. Misner who claimed that... "quantum effects do not significantly modify the nature of initial singularity in relativistic cosmology and that there is no suggestion that would allow a contracting Universe to pass through a quantum phase and *emerge as an expanding* Universe".

An opposite point of view is supported by Nariai (1971), Novello and Rodrigues (1975) and many others. In a subsequent paper Nariai gives a specific example of the consequence of quantum effects in the modified Lagrangian, which he takes to be

$$L = \sqrt{-g}\ \{ \frac{1}{\kappa} R + \eta(R^2 + \alpha R_{\mu\nu}R^{\mu\nu}) + L_M \} \tag{10.1}$$

Using numerical integration Nariai and Tomita (1971) have arrived at the regular solution presented in figure 1.

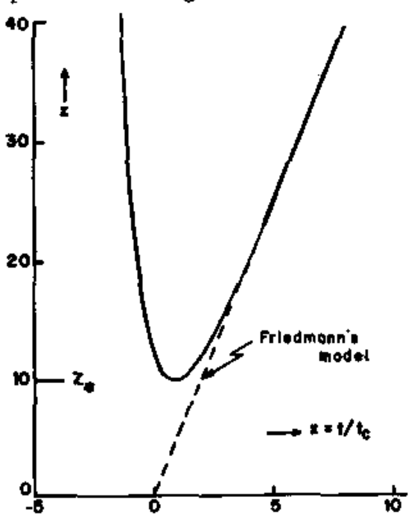

Figure 1 - The 1971 solution by Nariai and Tomita. The dashed line corresponds to the Friedmann model.

Ruzmaiknima and Ruzmaiknin (1970) start a program based on Sakharov's proposal from which the Lagrangian of gravity should be given by

$$L(R) = L(0) + \alpha R + \beta R^2 + \gamma R_{\mu\nu} R^{\mu\nu} + \ldots \qquad (10.2)$$

In the realm of a homogeneous and isotropic Universe, they found that although it is possible to find regular transition from contraction to expansion (at t=0), divergences appear as $t \to +\infty$ and/or $t \to -\infty$.

The year after Gurovich showed that more complex forms for the Lagrangian can induce regularity throughout the whole history of the Universe. He has shown this starting from a Lagrangian

$$L_g = \sqrt{-g}\ \left[R + \frac{1}{\ell^2} (\ell^2 R)^{4/3}\right] \qquad (10.3)$$

in which the constant ℓ has the dimension of lenght. From this Lagrangian (exercise) we obtain

$$G_{\mu\nu} + \ell^{2/3} R^{1/3} \left[\frac{4}{3} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right] + \frac{4}{3} \ell^{2/3} \left[\Box R^{1/3} g_{\mu\nu} - R^{1/3}{}_{,\mu;\nu}\right] =$$

$$= -\kappa T_{\mu\nu} \qquad (10.4)$$

Exercise: Show that equation (10.4) implies conservation of matter, that is, $T^{\mu\nu}{}_{;\nu} = 0$.

In a Friedmann-like Universe, Gurovich found the solution shown in figure 2.

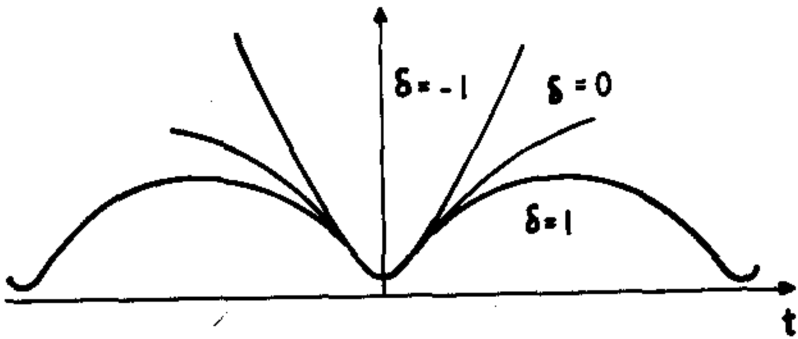

Figure 2 - Evolution of the radius of the Universe for the three possible topologies *in Gurovich's proposal.*

Let us point out that the factor 4/3 is not the only value to allow for such regularity. More *general expressions allow for* a *similar* behavior. For instance, Buchdal (1970) proposed a systematic exam of generic Lagrangians

$$L = F(R) + L_M \tag{10.5}$$

and let for an ulterior decision to fix the form of the function F. Recently, Kenner (1982) has re-examined this suggestion and obtained well behaved solutions for choosing

$$F(R) = \frac{R}{1 - \ell_\rho^2 R} \tag{10.6}$$

Exercise: Calculate the equations of motion from (10.5)

Solution:

$$[R_{,\mu} R_{,\nu} - R_{,\lambda} R^{,\lambda} g_{\mu\nu}] \frac{d^3F}{dR^3} + [R_{,\mu;\nu} - \Box R\, g_{\mu\nu}] \frac{d^2F}{dR^2} +$$

$$+ R_{\mu\nu} \frac{dF}{dR} - \frac{1}{2} F g_{\mu\nu} = - \kappa T_{\mu\nu} \tag{10.7}$$

Show that in a Friedmann-like geometry with $T_{\mu\nu} = \rho V_\mu V_\nu$ and

$$ds^2 = dt^2 - A^2(t) \left[\frac{dr^2}{1 + \varepsilon r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

there remains from (10.7) only one equation

$$\ddot{R} \frac{dF}{dR} - \frac{3\dot{A}}{A} \dot{R} \frac{d^2F}{dR^2} + \frac{1}{2} F = - \kappa\rho \tag{10.8}$$

in which

$$R = 6\left[\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A} \right)^2 + \frac{\varepsilon}{A^2} \right] \tag{10.9}$$

One is then free to choose the function F in order to obtain for A(t) the desirable property of regularity.

The difficulty with these a posteriori models is that all of them lack a more profound principle in order to be reliable.

A different road was undertaken by Ginzburg, Kirzhnits and Lyubushin (1971). They considered Einstein's equations as being a theory for microscopic fields. Macroscopic fields, which yield the observed metric, contain fluctuations in the mean metric. This situation can be represented by the formula

$$g_{\mu\nu} = \langle g_{\mu\nu} \rangle + \Delta g_{\mu\nu} \qquad (10.10)$$

Due to the non-linearity of the equation for $g_{\mu\nu}$ the equation for $\langle g_{\mu\nu} \rangle$ is given by

$$G_{\mu\nu} = -\kappa T_{\mu\nu} + \Phi_{\mu\nu} \qquad (10.11)$$

in which $\Phi_{\mu\nu}$ is a complicated functional of $\Delta g_{\mu\nu}$ and $G_{\mu\nu}$ is the Einstein tensor constructed with the mean metric.

At this point Ginzburg et al made a crucial hypothesis by taking the fluctuations represented by $\Phi_{\mu\nu}$ to admit a polynomial expansion on the mean metric by setting

$$\Phi_{\mu}{}^{\nu} = \Phi_{\mu(o)}{}^{\nu} + \Phi_{\mu(1)}{}^{\nu} + \Phi_{\mu(2)}{}^{\nu} + \cdots \qquad (10.12)$$

The number in parenthesis *(κ) represents the order of the mean* curvature which appears in this expansion.

For instance,

$$\Phi_{\mu(o)}{}^{\nu} = \tilde{\Lambda}\, \delta_{\mu}{}^{\nu} \qquad (10.13)$$

induces the renormalization of the (bare) cosmological constant.

The first order term is given by

$$\Phi_{\mu(1)}^{\ \nu} = AR_{\mu}^{\ \nu} + BR\delta_{\mu}^{\ \nu} = \tilde{A}G_{\mu}^{\ \nu} + \bar{B}R\delta_{\mu}^{\ \nu} \tag{10.14}$$

The first term renormalizes the gravitational constant and the second term must vanish because it violates (in this order of approximation) the conservation law.

Thus we can set

$$\Phi_{\mu(o)}^{\ \nu} = \Phi_{\mu(1)}^{\ \nu} = 0$$

as κ and Λ are already taken by their renormalized values.

Thus, the first non-trivial term which gives a real contribution is given by

$$\Phi_{\mu(2)}^{\ \nu} = \alpha\, RR_{\mu}^{\ \nu} + \beta R_{\mu\alpha}^{\ \ \ \nu\beta} R^{\alpha}_{\ \beta} + \gamma\, R_{\mu}^{\ \alpha} R_{\alpha}^{\ \nu} + \varepsilon\, R_{\mu\alpha\beta\lambda} R^{\nu\alpha\beta\lambda} +$$

$$+ \delta\, R^{2} \delta_{\mu}^{\ \nu} + \sigma\, R_{\alpha\beta} R^{\alpha\beta}\, \delta_{\mu}^{\ \nu} + \rho\, R_{,\mu}^{\ \ ;\nu} + \xi\, \Box R_{\mu}^{\ \nu} + \eta \Box R\delta_{\mu}^{\ \nu} \tag{10.15}$$

But the conditions

$$\Phi_{\mu(2);\nu}^{\ \nu} = 0 \tag{10.16}$$

imposes four relations among the nine constants that appear in (10.14).

It is a beautiful result (Ginzburg et al) that there remains only two independent combination, e.g.

$$I_{\mu\nu(2)} = - 2RR_{\mu\nu} + \frac{1}{2} R^2 g_{\mu\nu} + 2R_{,\mu;\nu} - 2 \Box R \, g_{\mu\nu} \qquad (10.17a)$$

$$J_{\mu\nu(2)} = - 2R^{\alpha\beta} R_{\mu\alpha\nu\beta} + \frac{1}{2} R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + R_{,\mu;\nu} - \Box R_{\mu\nu} - \frac{1}{2} \Box R \, g_{\mu\nu} \qquad (10.17b)$$

Thus,

$$G_{\mu\nu} = - \kappa T_{\mu\nu} + \bar{\alpha} J^{(2)}_{\mu\nu} + \bar{\beta} I^{(2)}_{\mu\nu} + \Lambda g_{\mu\nu} \qquad (10.18)$$

If the fluctuations have quantum character we can set

$$\bar{\alpha} \sim \bar{\beta} \sim L^2_{P\ell}$$

Applying this expression (10.17) into an homogeneous and isotropic Universe one obtains (exercise)

$$G^o{}_o = - \kappa\rho - \Lambda + L_{P\ell} P_2(\rho, p, \Lambda) \qquad (10.19)$$

The term $P_{(2)}$ is obtained after applying Einstein's equations for the microscopic metric into $J^{(2)}_{\mu\nu}$ and $I^{(2)}_{\mu\nu}$.

We can then see that the net effect of this method (in cosmology) is just to provide a correction of the perfect fluid, introducing new viscous terms.

The crucial question of this method is to use quantum gravity theory to evaluate $\bar{\alpha}$ and $\bar{\beta}$, a problem which is still open.

Another approach to deal with such fluctuations has been developed

(Novello, 1978) where an equivalent formulation of Einstein's theory to deal with stochastic or quantum fluctuation is used. The idea is to use Jordan's approach to Einstein's theory.

In 1949 Jordan and his co-workers proposed to use Bianchi identities as dynamical equations of gravity, written in the form:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha}R^{,\beta]} \tag{10.20}$$

and

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu}$$

in which

$$J^{\alpha\beta\mu} = -\frac{\kappa}{2} T^{\mu[\alpha;\beta]} + \frac{\kappa}{6} g^{\mu[\alpha}T^{,\beta]} \tag{10.21}$$

Lichnerowicz (1960) has shown that if Einstein's equations ($G_{\mu\nu} + \kappa T_{\mu\nu} = 0$) are valid on a space like hypersurface Σ , then (10.19) guarantees the validity of Einstein's equations throughout the whole space-time in the future of Σ .

Within Jordan's approach the fluctuation presented in equation (10.15) takes the form

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu} + Q^{\alpha\beta\mu} \tag{10.22}$$

in which $Q^{\alpha\beta\mu}$ depends on the perturbation of $W^{\alpha\beta\mu\nu}$. The use of this form of perturbation in case of a conformally flat back-ground is rather easier to treat then the standard procedure (equation 10.15). See Novello and Salim (1983) in which a complete review of this theory was presented, with emphasis on the modification of the behavior of Friedmann's model near the singularity.

Recently Novello and Neto (1987) have considered a somewhat different program. They started by realizing that cosmology seems to give arguments in favour of the two following statements:

(i) We live in a world represented by a (riemannian) metric which describes a non-stationary spatially homogeneous and isotropic (Friedmann-like) Universe of very slow expansion.

(ii) The Universe has not always been in a conformally flat configuration.

Although (i) does not require any further comment and seems to consist in a well stablished thruth accepted by the great majority of cosmologists, the assertion (ii) needs some explanation concerning its actual meaning.

Recently, many scientist have been very critical with respect to the so called standard cosmology (identified with the Hot Big Bang solution found by Friedmann and developed by many others) mainly due to the well-known difficulties which are inevitably present in this model (see Novello, 1987). Among these we can quote, for instance, the question of the initial singularity (problem 1) and the explanation of the origin of the high degree of isotropy which is present in the $2.7^{0}K$ background radiation (problem 2) associated to the presence, in such geometry, of particle horizons (see chapter 9).

Although it is, in principle, possible to find solutions to these difficulties without abandoning the condition of a conformally flat metric, some scientists have proposed the examination of models in which the Weyl tensor $W_{\alpha\beta\mu\nu}$ is non-null at some prior era. One of the main works which gave a real contribution to the clarification of this question and produced a severe critic to the standard model in the neighborhood of the assumed singularity was undertaken by the russian cosmologists Lifshitz, Belinsky and Khalatnikov (1982). These scientists have shown (at first, in a limited scheme in which all matter in the world is identified to a perfect fluid in equilibrium) that a deep analysis of the coupled behaviour of geometry and matter in the very dense stage of the Universe yields the result that the Weyl tensor $W_{\alpha\beta\mu\nu}$ should vanish only in a later stage of the cosmic evolution, characterizing then the moment in which the Universe enters its present friedmannian era. More than this, from the analysis of Lifshitz et al, one concludes that Weyl curvature effects are, at these "early" times, much more important than the corresponding Ricci terms show this.

A similar result is obtained when the basic properties of the standard cosmological model are carefully examined, at those regions of very high curvature, using a more general model for the matter content of the Universe. It has been shown that indeed, at those extreme regions ordinary matter has an unimportant role on the evolution of the geometry. This property has been used by Starobinsky (1983) and others to create a model of an assymptotic de Sitter regime in which a cosmological constant becomes the main agent for the isotropization of space-time. Such "vacum domain" made its appearance also in some recent models of cosmology - e.g., the so-called inflationary scenario. However, what seems to restricts strongly all these models is the difficulty of obtaining a mechanism by which the de Sitter era should be replaced by a friedmannian one.

In another context, quantum fluctuations and the properties of non-equilibrium thermodynamics in an intense gravitational field, with its intrinsic generation of entropy, has led some authours (Penrose, 1978) although in a more speculative way, to put in relief the role of Weyl conformal tensor in the evolution of the cosmological metric.

However, this recognition does not solves our problem, since we still have an infinite set of possibilities for the dynamics of gravity. At this point we shall turn our attention to an old proposal of M. Born (1933, 1935) developed later by him and his collaborator L. Infeld, of an extension of Maxwell's Electrodynamics, in order to guide us in the search of our present aimes.

In the early thirties these authors proposed to develop a (non-linear) electrodynamics based on the hypothesis that the basic fields of physics must be described by a dynamics comprising within itself some sort of limitation on the possible extreme values of the strenghths of the fields. In a subsequent stage, quantum physics has shown that non-linear effects of electrodynamics do appear in Nature and may be described by successive approximations which are in accordance with the expansion in series of the polynomial Lagrangian of Born-Infeld.

The above considerations led us to propose the following Lagrangian to describe gravity:

$$L = \sqrt{-g}\ \left(\frac{1}{\kappa} R + \frac{\beta}{\kappa} \sqrt{1 - \left(\frac{I}{\beta^2} \right)^2} - \frac{\beta}{\kappa} + L_M \right) \tag{10.23}$$

in which constant β measures the maximum intensity admissible for the value of the topological invariant $I = R^{*}_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$. Let us make some comments on L. First we note that the choice of the radical term to be quadratic on I is, of course, not unique. However, had we chosen a linear term, e.g. $\sqrt{1-I}$, expanding this expression for small values of I the first constribution of I is the quadratic one. This is due precisely to the fact that I is a topological invariant and it contributes only with a surface integral to the action principle.

The term $\sqrt{-g}\ \beta$ is introduced in order to eliminate the contribution of the new part of the Lagrangian to the cosmological constant, when I = 0.

The equations of motion then are giving by:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} \frac{\beta^3}{\sqrt{\beta^4 - I^2}}\ g_{\mu\nu} - \frac{4}{\beta} \left[\left(\frac{I}{\sqrt{\beta^4 - I^2}} \right)_{,\lambda} R^{\beta}{}_{\mu\nu}{}^{*\lambda} \right]_{;\beta} =$$

$$= - K T_{\mu\nu} - \frac{1}{2} \beta g_{\mu\nu} \tag{10.24}$$

which reduces to Einstein's theory for $\frac{I}{\beta^2} \ll 1$.

Exercise: Use Bianchi identities to show that equation (10.24) guarantees the conservation of energy-momentum tensor.

10.1. COSMICAL SCENARIO

There are many well-known solutions of Einstein's theory which are solutions of the new set of equations (10.24).

Exercise: Show that the following geometries are solution of equation (10.24).

(i) Schwarzchild,

(ii) Friedmann,

(iii) Kasner,

(iv) Gödel,

(v) Reisner-Nordstron.

Thus, one should ask what the assumption of this new Lagrangian is good for? From a pratical point of view, we can answer this question by examining cosmology: we can perform a new cosmological scenario which could solve difficulties not only of the ancient standard program, but also of some new models like e.g. inflation.

Our new scenario is based on the assumption that during its history the Universe has experienced the whole spectra of permissible values for the topological invariant I. In order to fix our ideas, let us concentrate in a specific configuration and assume that the evolution of I can be represented as in Figure 3.

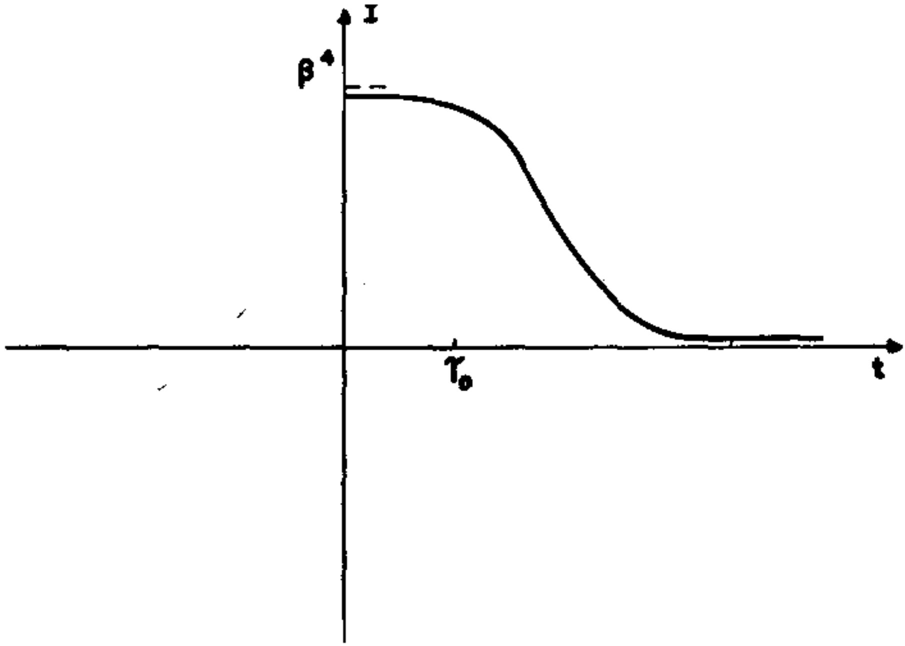

Figure 3 - Representation of the evolution of the topological invariant $I = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ during the cosmic history for an expanding Universe.

In this model, during a period $\Delta\tau = \tau_0$ the Universe experienced a value I_0^2 very near its admissible maximum $I_0^2 = \beta^4$. At this stage the net effect of the I-term in L is to induce a very large cosmological constant (remark that

$$\lim_{I\to\beta^2} \Lambda_{eff} = \lim_{I\to\beta^2} \frac{\beta^3}{\sqrt{\beta^4 - I}} \approx \infty)$$

which could modify drastically the primordial behaviour of the Cosmos, even avoiding the singularity. The Universe at this regime $I^2 = cte \approx \beta^4$ is in a de Sitter like state that is, a region in which the main responsable for the curvature of space-time is the effective cosmological constant. After τ_0 (the time in which I leaves its constant value and starts to diminish) the Universe leaves smoothly this phase, pass through a region in which matter becomes more and more important and finally enters the actual Friedmann era in which I vanishes and, besides, geometry becomes conformally flat.

This simplified scenario, which is in principle allowed by our set of equations, exhibits some features very similar to other sophisticated mechanisms which have been examined in the last years to produce some alternative to the standard cosmological scenario (e.g. cosmic spontaneous breakdown of symmetry of a given scalar field, phase transitions, etc). Furthermore, the equations of motion generated by Lagrangian (10.23) reproduces all observable effects of classical gravity.

Exercise: Examine the possibility of new spherically symmetric (static) solution of equation (10.24).

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11. NON-LORENTZIAN THEORIES

In the middle of the sixties, in a course given in CBPF, Colber G. de Oliveira examined the consequences of strong violations of local Lorentz invariance. Although extremely speculative, the idea that one can deal with a riemannian geometry without accepting the validity of local Lorentz group, comes from time to time as a fashionable model to which deviations of standard predictions of the special theory of relativity are assigned.

In recent years this idea has been revived by S. Weimberg in the realm of higher dimensions theories. It is Weimberg who asks: "Why should one adhere to Lorentz invariance in 4 + M dimensions?". Why should physics in 4 + M dimensions have the generalized Lorentz group SO(3+M,1) as a fundamental symmetry?

We left to other lecturers in this School the exam of theories which propose the existence of hidden dimensions beyond the usual four which constitute the classical arena of physics. Here we limit ourselves to present a model of non-Lorentzian theory which changes the behavior of gravitational interaction in cases of strong field. The main consequence in cosmology is precisely to avoid singularities. One should suspect anyhow that if there is any violation of Lorentz symmetry it certainly should appear at high energies (cf. Nielsen and Picek, 1982). This is precisely what Gasperini (1986) explored.

Let us assume (Gasperini, 1986) that the space-time is a four-dimensional (quasi) Riemannian structure with $G_T = SO(3)$ as the local group of invariance. In order to simplify our calculation we will follow the standard procedure and refer the set of dynamical objects to the Lorentzian ones.

Let $e_\mu^{(i)}$ and $e_\mu^{(o)}$ be, an SO(3) vector and scalar, respectively (i=1,2,3).

Construct the 1 - forms

$$\theta^{(i)} = e^{(i)}{}_{\mu} dx^{\mu}$$

$$\theta^{(o)} = e^{(o)}{}_{\mu} dx^{\mu} \tag{11.1}$$

The original Lorentz connection $\omega^{A}{}_{B}$ is defined by

$$d\theta^{A} = - \omega^{A}{}_{B} \wedge \theta^{B} \ , \tag{11.2}$$

A, B... = (0, 1, 2, 3) being Lorentzian indices, and is decomposed into an SO(3) connection

$$\omega^{i}{}_{j}$$

and $\bar{w}^{i}{}_{o}$, a 1-form which is covariantly transformed under SO(3).

The Lorentz curvature $\Omega^{A}{}_{B}$, defined by

$$\Omega^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B} \tag{11.3}$$

decomposes into

$$\Omega^{ij} = d\omega^{ij} + \omega^{i}{}_{k} \wedge \omega^{kj} + \bar{\omega}^{i}{}_{o} \wedge \bar{\omega}^{oj}$$

$$\Omega^{io} = d\bar{\omega}^{io} + \omega^{i}{}_{j} \wedge \bar{\omega}^{jo} \tag{11.4}$$

Remark that although the Lorentz connection is defined as torsion-free

$$d\theta^{A} + \omega^{A}{}_{B} \wedge \theta^{B} = 0 \tag{11.5}$$

the SO(3) connection acquires a torsion ($\tau^{i} = - \bar{\omega}^{i}{}_{o} \quad \theta^{o}$).

In order to set up a theory which preserves SO(3) invariance one has to construct a Lagrangian as a functional of the independent objects ω^{ij} and $\bar{\omega}^{i}{}_{o}$. The most general theory of this sort has been constructed by de Alwis and Randjbar-Daemi (1985) and later on examined by Gasperini. To our proposal here, we will limit to a simplified version which contains typical new features. Before doing this, let us briefly review how one deals with differential forms to obtain tensorial equations. Let us consider, as an example, the case of Einstein's theory, with SO(3.1) local invariance.

Consider the curvature 2-form

$$\Omega^A{}_B = \frac{1}{2} R^A{}_{BCD}\, \theta^C \wedge \theta^D \quad . \tag{11.6}$$

We set the variational principle for free gravity as

$$\delta \int \Omega^{AB} \wedge t^*{}_{AB} = 0 \tag{11.7}$$

in which, as in chapter 6 , the star * means the dual operation and

$t_{AB} = \theta_A \wedge \theta_B$.

Let us vary (Palatini) independently the basis θ^A and the connections $\omega^A{}_B$:

$$\delta \int \Omega^{AB} \wedge t^*{}_{AB} = \int (\chi_A \delta\theta^A + \chi_{AB} \delta\omega^{AB}) = 0 \ ; \tag{11.8}$$

this yields

$$\chi_{AC} = -\frac{1}{2} d(\epsilon_{AC}{}^{MN} \theta_M \wedge \theta_N) - \omega_{CB} \wedge \epsilon_A{}^{BMN} \theta_M \wedge \theta_N \tag{11.9}$$

which by straightforward manipulation is reduced to

$$d\theta^A + \omega^A{}_B \wedge \theta^B = 0 \tag{11.10}$$

(11.10) is nothing but Palatini's result that space-time is riemannian (torsion-free).

For the vector χ_A we obtain

$$\int \Omega^{\overset{*}{A}B} \wedge \theta_A \wedge \delta\theta_B = 0$$

or

$$\Omega^{\overset{*}{A}B} \wedge \theta_A = 0 \quad , \tag{11.11}$$

which can then be reduced to

$$R^{\overset{*}{A}B}{}_{\overset{*}{A}M} = 0 \ . \tag{11.12}$$

From formula (6.1) that is,

$$R_{\overset{*}{A}B\overset{*}{C}D} = - W_{ABCD} + \frac{1}{2} [R_{AC}\, g_{BD} + R_{BD}\, g_{AC} - R_{AD}\, g_{BC} - R_{BC}\, g_{AD}] -$$

$$- \frac{1}{3} R \, g_{ABCD}.$$

Taking the trace of this formula we obtain that (11.12) reduces to

$$R^{\overset{*}{A}B}{}_{\overset{*}{A}M} = R^B{}_M - \frac{1}{2} R \, {}^B{}_M = 0 \tag{11.13}$$

which is precisely Einstein's equations.

Let us now turn to the SO(3) theory. With the fundamental objects of this theory (θ^i, $\bar{\theta}^o$, ω^{ij} , $\bar{\omega}^{io}$ and corresponding curvatures) we can construct five independent Lagrangians which are

$$I_E = \int \Omega^{AB} \wedge t^{*}_{AB}$$

$$I_1 = \int \Omega^{oB} \wedge t^{*}_{oB}$$

$$I_2 = \int \Omega^{oB} \wedge \Omega^*_{oB}$$

$$I_3 = \int \Omega^{oB} \wedge \theta^o \wedge (\Omega_{oB} \wedge \theta^o)^*$$

$$I_4 = \int \Omega^{Ao} \wedge \theta_A \wedge (\Omega_{Co} \wedge \theta^C)^* \tag{11.14}$$

Thus, the most general action is given by

$$I = I_E + k_1 I_1 + k_2 I_2 + k_3 I_3 + k_4 I_4 \tag{11.15}$$

for arbitrary constants k_1, k_2, k_3 and k_4.

Exercise: Use the systematics employed above to obtain from I_E Einstein's equations to compute from 11.15 the equation of motion and write it in a covariant way under the form:

$$G_{\mu\nu} + \chi_{\mu\nu} = - \kappa T_{\mu\nu} \tag{11.16}$$

Remark that an extra SO(3) invariant matter term was introduced. Although constants k_1, k_2, k_3 and k_4 are completely arbitrary, Gasperini choses a way to fix a relation between them by imposing that $\chi^{\mu\nu}$ is divergence free (which guarantees matter conservation, $T^{\mu\nu}{}_{;\nu} = 0$) and that the motion of the particles with are responsable for the curvature of space-time is geodetic (a property which does not follow automatically from equation of motion 11.16).

If such break of Lorentz invariance occurs for very high energies but below Planck's mass, the classical gravity theory (11.16) generates a cosmological system coherent with Einstein's Cosmological Program.

Let us examine what are the consequences of the new theory in the global structure of space-time and assumes for the cosmological metric the Friedmann's form:

$$ds^2 = dt^2 - A^2(t)\, d\sigma^2 \tag{11.17}$$

Take the velocity of the perfect fluid to be $V^\mu = e^\mu_{(o)} = \delta^\mu{}_o$, we obtain from (11.16)

$$\dot\rho + (\rho + p)\theta = 0 \tag{11.18a}$$

$$\dot A^2 = \frac{1}{1 + \frac{k_1}{2}} \left[\frac{1}{3} \kappa\rho A^2 - \epsilon \right] \tag{11.18b}$$

This set (11.18) is obtained under condition $k_1 = k_2 = -k_3 = -k_4$ which is a consequence of imposing that the cosmic fluid follows a geodesic motion.

Exercise: Obtain the set (11.18) under this hypothesis.
For completenesse we consider here the case in which the fluid is radiation $p = \frac{1}{3}\rho$. The expansion factor θ is given by $\theta = 3 \frac{\dot A}{A}$. From (11.18) it follows that

$$\rho = \rho_o A^{-4}$$

The solution of (11.18) for the closed world ($\epsilon = +1$) is given by

$$A = A_0 \sqrt{t^2 + Q^2} \tag{11.19}$$

in which constants A_o and Q are given by

$$A_o = \sqrt{-\left(1 + \frac{k_1}{2}\right)}$$

$$Q^2 = -\frac{\kappa\rho_o}{3} \cdot \frac{1}{1 + \frac{k_1}{2}}$$

with the condition $1 + \frac{k_1}{2} < 0$.

Remark that (11.19) gives the same time-dependence for the radius of the Universe as those suggested by the solutions presented by Novello--Salim and Melnikov-Orlov in the case of gravitational non-minimal coupling of vector and scalar fields, respectively.

Note however that although Novello-Salim and Melnikov-Orlov solutions are exclusive for open Universes, the proposal of Gasperini avoids the singularity only in the case of closed space section.

Exercise: Compare equations (11.18) with the corresponding Einstein's equations.

Solution: In the case of Einstein's theory we have:

$$\dot{\rho} + (\rho + p) = 0$$

$$\dot{A}^2 = \frac{\kappa}{3}\,\rho\, A^2 - \epsilon$$

We see that the Gasperini's model yields a change in the gravitational constant to the renormalized value $\kappa_{ren} = \frac{\kappa}{1 + \frac{k_1}{2}} < 0$.

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12. NON-EQUILIBRIUM THERMODYNAMICS

In standard cosmology (Ellis, 1987), the myriad of events associated to the matter existing in the Universe receives a simplified description in the form of a perfect fluid:

$$T_{\mu\nu} = \rho V_\mu V_\nu - p\, h_{\mu\nu} \tag{12.1}$$

The reader must be aware that form (12.1) does not define a perfect fluid in a unique way. This is related to the fact that we can choose another four-velocity field $\tilde{V}_\mu$ and decompose $T_{\mu\nu}$ in the basis provided by $\tilde{V}_\mu$.

In general, this would imply

$$T_{\mu\nu} = \tilde{\rho}\tilde{V}_\mu \tilde{V}_\nu - \tilde{p}\, \tilde{h}_{\mu\nu} + \tilde{q}_{(\mu}\tilde{V}_{\nu)} + \tilde{\pi}_{\mu\nu} \tag{12.2}$$

in which besides the modified expressions $\tilde{\rho}$ and $\tilde{p}$ a heat flux $\tilde{q}_\mu$ and anisotropic pressures $\tilde{\pi}_{\mu\nu}$ appear. It is a simple, but very elucidative exercise to evaluate the quantities $\tilde{\rho}$, $\tilde{p}$, $\tilde{q}^\mu$ and $\tilde{\pi}_{\mu\nu}$ in terms of the corresponding quantities defined by observer V^μ.

Exercise: Show that for normalized vectors V^μ and $\tilde{V}^\mu$ we obtain, by comparision of (12.1) and (12.2) the following relations:

$$\tilde{\rho} = \beta^2\rho + (\beta^2 - 1)p$$

$$\tilde{p} = \frac{1}{3}(\beta^2 - 1)\rho + \frac{2+\beta^2}{3}\, p$$

$$\tilde{q}_\mu = \beta(\rho + p)(V_\mu - \beta\tilde{V}_\mu)$$

$$\tilde{\pi}_{\mu\nu} = (1 - \beta^2)\rho V_\mu V_\nu - p\, h_{\mu\nu} + p(1 - \beta)\tilde{V}_\mu \tilde{V}_\nu \tag{12.3}$$

in which β is the angle between the two vectors fields:

$$\beta = V_\mu \tilde{V}^\nu .$$

Exercise: In the old interpretation of the cosmological constant, Λ is treated as perfect fluid satisfying the condition $\rho + p = 0$. Show that the "vacuum gas" is the only fluid which has the same equation of state for all observers.

Although the hypothesis (12.1) may be a good description of our equilibrium era, many scientists have started a tentative program to incorporate dissipative terms in the energy-momentum tensor of the galactic fluid.

In 1968 Misner suggested that neutrino viscosity could be an efficient mechanism by means of which any arbitrary initial anisotropy wears out as the Universe expands. A simple phenomenological description of such process is given by the linear expression relating the anisotropic pressure $\pi_{\mu\nu}$ to the shear deformation $\sigma_{\mu\nu}$. Although this mechanism did not succeeded to explain why our Universe is isotropic, it had the merit to call attention upon viscous processes in cosmology.

Since then, many cosmologists have examined such more general non-perfect fluids in different contexts. The second viscosity coefficient has been used by Klimek (1973) and later by Murphy (1973), in order to create a homogeneous and isotropic cosmology without singularity. The effect of the first and second viscosity coefficients on the cosmological singularity has been investigated by Belinski and Khalatnikov (1977) and generalized for non-linear fluids by Novello and Araujo (1980).

Let me stress that non-perfect fluids may appear as a fluid description of some specific configurations of fields interacting with gravity.

Exercise: This last remark must be interpreted as a statement about the non-uniqueness of the representation of the source of a given geometry. For instance, we leave to the readers to show the following equivalences:

i) Quadratic Stokesian Fluid.

Show that the geometry (Novello-Soares, 1976)

$$ds^2 = dt^2 - e^{2Mt}\, dxdt - e^{2Jt}(dy^2 + dz^2) \qquad (12.4)$$

in which M and J are arbitrary parameters constrained to satisfy the requirement $J(J-M) < 0$, has as its source either a neutrino field ψ with a current $j^\mu = \bar{\psi}\gamma^\mu\psi$ direct in the x-direction or a fluid such that

$$T_{\mu\nu} = \rho V_\mu V_\nu - p\, h_{\mu\nu} + q_{(\mu}V_{\nu)} + \pi_{\mu\nu} \qquad (12.5)$$

with

$$q_\mu = \frac{2J(J-M)}{M}\ \dot{V}_\mu$$

$$\pi_{\mu\nu} = \frac{-2J}{J+M}\left(\theta_\mu{}^\lambda\ \theta_{\lambda\nu} - \frac{1}{3}\,\theta_{\rho\sigma}\ \theta^{\rho\sigma}h_{\mu\nu}\right)$$

$$p = \frac{1}{3}\rho$$

ii) Accelerating non-Stokesian Fluid.

Consider the conformally flat Bertotti-Robinson solution

$$ds^2 = \eta_{AB}\ \theta^A\theta^B \qquad (12.6)$$

with $\eta_{AB} = \text{diag}\,(+1, -1, -1, -1)$

$$\theta^0 = \frac{1}{Qr}\, dt$$

$$\theta^1 = \frac{1}{Qr}\, dr$$

$$\theta^2 = \frac{1}{Q}\, d\theta$$

$$\theta^3 = \frac{1}{Q}\, \sin\theta\, d\phi$$

The source of Bertotti-Robinson geometry is usually stated to be a (static) eletromagnetic field. Show that it can be interpreted equivalently as a fluid with

$$T_{AB} = \rho V_A V_B - p\, h_{AB} + \pi_{AB} \qquad (12.8)$$

(the tetrad indices A, B, ... range 1, 2, 3, 4) with $p = \frac{1}{3} Q^2$, for Q^2 = constant.

$$\pi_M{}^N = -2A_M{}^N$$

$$A_M{}^N = \dot{V}_M \dot{V}^N - \frac{1}{3} \dot{V}_C \dot{V}^C h_M{}^N$$

$$V^A = \delta^A{}_0$$

$$\dot{V} = (0, Q, o, o).$$

iii) Rotating Non-Stokesian Fluid

Consider the generalized Gödel-like fluid

$$ds^2 = \eta_{AB}\, \theta^A \theta^B$$

$$\theta^o = dt + h(r)d\phi$$

$$\theta^1 = dr$$

$$\theta^2 = \Delta(r)d\phi = \sqrt{h^2 - g}\ d\phi$$

$$\theta^3 = dz$$

Show that in the co-moving frame $V_A = \delta_A{}^o$ the vorticity tensor is given by $\omega^A = (o, 0, 0, -\frac{1}{2}\frac{h'}{\Delta})$ in which $h' = \frac{dh}{dr}$.

Show that for $\frac{h'}{\Delta}$ = constant = $2\,\Omega$ there is a solution of Eisnteins's equation given by

$$T_{AB} = \rho V_A V_B - p\, h_{AB} + \pi_{AB}$$

with

$$\pi_{AB} = -\gamma^2 \Omega_{AB}$$

$$\Omega_A{}^B = \omega_A \omega^B - \frac{1}{3}\omega^2 h_A{}^B$$

$$\rho = -\Lambda + (1 + \gamma^2)\Omega^2$$

$$p = \Lambda + (1 - \frac{\gamma^2}{3})\Omega^2$$

with $\Delta = \sin\sqrt{\gamma^2 - 2}\ \Omega r$

and $2 < \gamma^2 < 3$.

Show that the source of this solution can be interpreted as a perfect fluid plus an electromagnetic field (Novello, 1980).

From the theoretical point of view all these models suffer of a drastic illness: they do not satisfy causality principle, since in the viscous regime there is no limit for the velocity of propagation of signals. This difficulty has been examined and partially solved by Belinski-Nikomarov and Khalatnikov (1979), who presented a model of a viscoelastic fluid in which the causality requirement is explicitly fulfilled.

This approach has been recently re-examined by Salim and Oliveira (1987) in the context of far-from-equilibrium thermodynamics. They have obtained a non-singular cosmology, which I intend to present as an example. We will limit our analysis here to a Friedmann-like geometry $ds^2 = dt^2 - A^2(t)\, d\sigma^2$ and let the cosmic fluid be described by the expression

$$T_{\mu\nu} = \rho V_\mu V_\nu - (p_{th} + \pi) \qquad (12.9)$$

with

$$p_{th} = \lambda\rho \ .$$

In this expression π represents the effect of the viscosity. It has been of general use (before Belinski et al (1979) paper) to assume that in the Stokesian regime we should have

$$\pi = \pi(\theta)$$

for $\theta = \frac{\dot{V}}{V}$, without any further considerations of the Thermodynamical properties of such fluid and without any restriction on the behavior of the function $\pi(\theta)$ in virtue of causality requirements, for instance.

Let me emphasize that once the fluid is to be considered as being far from the equilibrium, we have to modify the standard Gibbs equations to describe the production of entropy provoked by the presence of viscosity.

The standard procedure (Israel, 1976, Pavon,Jou and Casas-Vāsquez, 1982) is to start by making the hypothesis that the entropy per particle $s = \frac{S}{N}$ depends on the available variables: the density of energy ρ , the specific volume $v = \frac{1}{n} = \frac{V}{N}$ and the viscosity π. In these formulas N is the total number of particles (asumed to be conserved throughout the whole history), V is the total volume. Then, we can write the generalized Gibbs relation

$$\dot{s} = \frac{\partial s}{\partial \varepsilon}\dot{\varepsilon} + \frac{\partial s}{\partial v}\dot{v} + \frac{\partial s}{\partial \pi}\dot{\pi} \quad . \tag{12.10}$$

In the standard procedure we can also define a generalized temperature of non-equilibrium $T = \left(\frac{\partial s}{\partial \varepsilon}\right)^{-1}$, a pressure $p = \left(\frac{\partial s}{\partial v}\right)\left(\frac{\partial s}{\partial \varepsilon}\right)^{-1}$ and a viscous pressure

$$\pi = \frac{n}{\alpha_0}\left(\frac{\partial s}{\partial \pi}\right)\left(\frac{\partial s}{\partial \varepsilon}\right)^{-1} \quad .$$

Then using (12.10) we obtain

$$n\dot{s} = \frac{1}{T}\left(n\dot{\varepsilon} + np\dot{v} + \alpha_0 \pi\dot{\pi}\right) \tag{12.11}$$

From the conservation of $T^{\mu\nu}$ we obtain

$$n\dot{\varepsilon} + \theta(p + \pi) = 0 \tag{12.12}$$

which implies

$$n\dot{s} = \frac{1}{T}\left(\alpha_0 \pi\dot{\pi} - \theta\pi\right) \tag{12.13}$$

The entropy four-current s^{α} is given by

$$s^{\alpha} = ns\, v^{\alpha} + i^{\alpha} \quad , (i^{\alpha} \text{ represents the entropy flux}) . \tag{12.14}$$

Then the entropy production σ is:

$$\sigma = s^{\alpha}{}_{;\alpha} = (\dot{n} + n\theta)s + n\dot{s} + i^{\alpha}{}_{;\alpha} = n\dot{s} + i^{\alpha}{}_{;\alpha} \geq 0 \tag{12.15}$$

in which we used that the total number of particles is a constant of motion.

Following the standard procedure we set:

$$\sigma = \frac{1}{T} \left(\frac{1}{\xi} \pi^2 + 0(\pi^4) \right) \qquad (12.16)$$

to obtain (neglecting higher order terms):

$$\frac{1}{\xi} \pi = \alpha_o \dot{\pi} - \theta \qquad \mathbf{(12.17)}$$

Now, going to the Newtonian limit we obtain

$$\alpha_o \xi = - \tau_o$$

in which τ_o is the relaxation time characteristic of the fluid. Then, we can finally write

$$\tau_o \dot{\pi} + \pi = - \xi\theta \qquad (12.18)$$

Thus, if there is no entropy current $i^\mu = 0$ we have for the entropy production

$$n\dot{s} = \frac{1}{\varepsilon T} \pi^2 \qquad (12.19)$$

Putting (12.9, 18, 19) together in the homogeneous geometry one can obtain a (causal) solution in terms of ξ and τ_o.

In general ξ may still be dependent of the parameters of the equilibrium. If we set $\xi = \beta\rho$, Salim and Oliveira have found a regular solution which gives for the radius of the Universe the value $A(t) = A_o\, e^{mt}\, (\operatorname{sech} \gamma t)^\mu$ in which m, μ and γ are constants defined in terms of β, τ_o, λ and Λ .

Although we have restricted here our analysis to the scalar viscosity π, it is possible to extend, in a straightforward manner this treatment to more complex situations.

However, the main difficulty which one faces in all these treatments is related to the gravitational distortion of the entropy production. In other words, if the entropy flux $I^{\mu}_{(s)}$ depends on gravity, then a correct treatment of this question cannot ignore the fact that space-time is curved and consequently curvature effects must appear as an independent variable in $I^{\mu}_{(s)}$ (Salim, 1987).

COMMENTS AND REFERENCES

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