Some notes on the Propagation of Discontinuities in Solutions to the Einstein Equations

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Abstract

After a review of the theory of non-null boundary surfaces and surface layers in general relativity, the evolution of spherical bubbles in vacuum, and voids in cosmology, are studied in detail. The notes conclude with a study of the transition from Minkowski space to Schwarzschild spacetime via null boundary surfaces.

Outline

List of Symbols

Σ	3 space of a spacetime
V	4 manifold (spacetime)
$g_{\alpha\beta}$	metric tensor of spacetime
C^n	n times continuously differentiable
x^α	coordinates for V
$[\Psi] \equiv \Psi \|_p^+ - \Psi \|_p^-$	for all points p on Σ
ξ^i	coordinates intrinsic to Σ
$g_{ij} \equiv g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^i}\frac{\partial x^\beta}{\partial \xi^j}$	metric intrinsic to Σ
n^α	4-normal to $\Sigma (n^\alpha n_\alpha \neq 0)$
$K_{ij} \equiv \frac{\partial x^\alpha}{\partial \xi^i}\frac{\partial x^\beta}{\partial \xi^j}\nabla_\alpha n_\beta$	(∇_α the covariant derivative of V), extrinsic curvature of Σ
$\Gamma^\gamma_{\alpha\beta}$	Christoffel symbols of the 2^{nd} kind for V
$G_{\alpha\beta}$	Einstein tensor in V
u^α	4-tangent to Σ
$u^i \equiv g^{ij}\frac{\partial x^\alpha}{\partial \xi^j}u_\alpha$	intrinsic tangent
ρ	energy density
p	isotropic pressure
ζ	bulk viscosity
$\Theta \equiv \nabla_\alpha u^\alpha$	4-expansion
η	shear viscosity
$\sigma_{\alpha\beta} \equiv \nabla_{(\alpha}u_{\beta)} + \dot{u}_{(\alpha}u_{\beta)} - \frac{\Theta}{3}(g_{\alpha\beta} + u_\alpha u_\beta)$	shear tensor ($\sigma^\alpha_\alpha = \sigma^\alpha_\beta u^\beta = 0$)
$\dot{u}^\alpha \equiv u^\beta \nabla_\beta u^\alpha$	acceleration ($\dot{u}_\alpha u^\alpha = 0$)
q^α	heat flux ($q^\alpha u_\alpha = 0$)
$\Delta \equiv u^\alpha u_\alpha$	= -1, timelike Σ; + 1, spacelike Σ
$m \equiv \frac{1}{2}(g_{\theta\theta})^{\frac{3}{2}} R_{\theta\phi}{}^{\theta\phi}$	effective gravitational mass (spherical symmetry)
Λ	cosmological constant
δ^α_β	Kronecker symbol
$f'(x_o)$	df/dx evaluated at $x = x_o$ (similarly for more independent variables)

$R^\delta{}_{\gamma\alpha\beta}$ Riemann-Christoffel tensor of V (constructed from $g_{\alpha\beta}$,

$$\equiv \frac{\partial}{\partial x^\alpha}\Gamma^\delta_{\gamma\beta} - \frac{\partial}{\partial x^\beta}\Gamma^\delta_{\gamma\alpha} + \Gamma^\epsilon_{\gamma\beta}\Gamma^\delta_{\epsilon\alpha} - \Gamma^\epsilon_{\gamma\alpha}\Gamma^\delta_{\epsilon\beta})$$

$R^i{}_{jkl}$ Riemann-Christoffel tensor constructed as above, but from g_{ij}. ${}^3R$ is the associated Ricci scalar.

$\gamma_{ij} \equiv [K_{ij}]$, $\gamma \equiv g^{ij}\gamma_{ij}$.

$S_{ij} \equiv \Delta(\gamma_{ij} - g_{ij}\gamma)/8\pi$,	surface energy tensor
$\overline{\Psi} \equiv (\Psi \|_p^+ + \Psi \|_p^-)/2$	for all points p on Σ
∇_i	intrinsic covariant derivative

$\sigma \equiv -\Delta S_{ij} u^i u^j$,	surface energy density
$P \equiv (\sigma + S)/2$,	surface pressure
$\Phi \equiv \nabla_i u^i$	3-expansion
$\hat{m} = m - \Lambda r^3/6$	generalized Schwarzschild mass
$\varepsilon \equiv P/\sigma,\ \dot{\varepsilon} = 0$	=-1 for a domain wall
$k \equiv MR^{2\epsilon}$	$\dot{k} = 0$ for shell motion (vacuum)
$S_k(\chi) \equiv \{\sin(\sqrt{k}\chi)\} / \sqrt{k}$	in Robertson-Walker
$\alpha \equiv m_-/m_{+i}$	for vacuum voids
$\beta \equiv \chi_i/\eta_i$	initial void size
$\gamma \equiv (d\chi/d\eta)_i$	initial growth rate
$\delta \equiv \log_{10}(1 + z_i)$	initial epoch

1. Introduction to Boundary Surfaces

Following the pioneering work of Lanczos (1924), Darmois (1927), now some sixty years ago, formulated the junction conditions appropriate to 3-surfaces of discontinuity in the spacetime manifold of general relativity. Since in Newtonian gravity at a mild discontinuity (what we will call a boundary surface) one imposes the continuity of the gravitational potential and its first derivatives, it is natural in a similar situation in general relativity to require the continuity of the metric tensor and its first derivatives. This condition (which is sometimes called the Lichnerowicz condition) is clearly coordinate dependent and so, as Darmois pointed out, of rather limited use.

In general relativity what is needed is a coordinate independent formulation of the above ideas concerning smoothness. This was accomplished by Darmois at non-null 3-surfaces by requiring the continuity of the first and second fundamental forms of the surface. These are the conditions in use today. Curiously, however, it took a long time for these conditions to take hold. It was not until 1966, when the influential paper by Israel appeared, that Darmois' conditions became widely appreciated. (Israel was primarily interested in surface layers where the second fundamental form exhibits discontinuities. This is discussed in these notes starting in section 2.)

1.1 Junction Conditions *

Consider a 3-space Σ which divides spacetime into two distinct four-dimensional manifolds V^+ and V^- with metric tensors $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$ each of class C^3 (except on Σ). Σ is said to satisfy the Lichnerowicz conditions (Lichnerowicz 1955) if there exists a system of coordinates (x^α say, $\alpha = 0, 1, 2, 3$) which cover Σ and for which

$$[g_{\alpha\beta}] = 0, \tag{1.1}$$

and

$$[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}] = 0, \tag{1.2}$$

where for all $\Psi, [\Psi] \equiv \Psi\mid^+_p - \Psi\mid^-_p$ for all points p on Σ. With the conditions (1.1) and (1.2) satisifed, the coordinates x^α are called *admissible*. Σ is to be a *boundary surface* if it can be covered by admissible coordinates. Unfortunately, however, Σ need not be a boundary surface, and even if it is there is no general algorithm for the construction of

* Througout these notes we use standard geometrical units, a metric signature of +2, Latin 3-indices, Greek 4-indices, and the sign convention of Israel for the second fundamental form.

admissible coordinates from those which are not. What is needed, as Darmois pointed out, is an invariant characterization of Σ. As long as Σ is not null (for null hypersurfaces see section 5) this is achieved by considering the first and second fundamental forms associated with it.

Let the coordinates intrinsic to Σ be $\xi^i, i = 1,2,3$ so that the equations for Σ in $V^\pm$ are $x^\alpha_\pm = x^\alpha_\pm(\xi^i)$ which are assumed to be of class C^3. It is not difficult to see that conditions (1.1) and (1.2) are equivalent to

$$[g_{ij}] = 0, \tag{1.3}$$

and

$$[K_{ij}] = 0, \tag{1.4}$$

where

$$g_{ij} \equiv g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j}$$

is the first fundamental form (instinsic metric) of Σ and

$$K_{ij} \equiv \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \nabla_\alpha n_\beta,$$

n^α the 4-normal to Σ *(directed from V^- to V^+)* is the second fundamental form (extrinsic curvature) of Σ (see, e.g., Eisenhart 1926). Conditions (1.3) and (1.4) are called the Darmois conditions for a boundary surface.

It is convenient to record here an alternative form for the extrinsic curvature K_{ij} (e.g. Cocke 1966). Write Σ (assumed non-null) as the identity

$$f(x^\alpha(\xi^i)) = 0, \tag{1.5}$$

so that

$$n_\alpha = \pm \frac{1}{(| g^{\beta\gamma} \frac{\partial f}{\partial x^\beta} \frac{\partial f}{\partial x^\gamma} |)^{1/2}} \frac{\partial f}{\partial x^\alpha}. \tag{1.6}$$

We take $n_\alpha \neq 0$. Differentiating the identify (1.5) then gives

$$n_\alpha \frac{\partial x^\alpha}{\partial \xi^i} = \frac{\partial f}{\partial \xi^i} = 0. \tag{1.7}$$

Differentiating the identify again gives

$$n_\alpha \frac{\partial^2 x^\alpha}{\partial \xi^i \partial \xi^j} + \frac{\partial n_\alpha}{\partial x^\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} = 0. \tag{1.8}$$

With this last expression the extrinsic curvature can be given as

$$K_{ij} = -n_\gamma\left(\frac{\partial^2 x^\gamma}{\partial\xi^i\partial\xi^j} + \frac{\partial x^\alpha}{\partial\xi^i}\frac{\partial x^\beta}{\partial\xi^j}\Gamma^\gamma_{\alpha\beta}\right), \tag{1.9}$$

which is somewhat more useful than the standard definition. The Darmois conditions (1.3) and (1.4), though equivalent to the Lichnerowicz conditions for non-null boundary surfaces, circumvent the often difficult problem of finding admissible coordinates explicitly. There is, however, a complication which arises when applying condition (1.4). We must ensure that the orientation of n^α is preserved through Σ. This can in practise be rather difficult to do, on the basis of conditions (1.3) and (1.4) alone, when the coordinates used in $V^\pm$ are not "admissible". We return to this point below.

In addition to the Lichnerowicz and Darmois conditions, one finds reference to the junction conditions of O'Brien and Synge (1952). Subject to coordinate transformations, these reduce to the present conditions. (See Bonnor and Vickers 1981 for a discussion of the O'Brien-Synge conditions.)

It is worth mentioning here a consequence of the contracted Gauss-Codazzi equations (see section 2) for boundary surfaces. From conditions (1.3) and (1.4) and the contracted Gauss-Codazzi equations (see equations (2.3) and (2.4)) it follows that

$$[G_{\alpha\beta}n^\alpha n^\beta] = 0, \tag{1.10}$$

and that

$$[G_{\alpha\beta}\frac{\partial x^\alpha}{\partial\xi^i}n^\beta] = 0, \tag{1.11}$$

where $G_{\alpha\beta}$ is the Einstein tensor. In particular, it does *not* follow in general that $[G_{\alpha\beta}u^\alpha u^\beta] = 0$ where u^α is the 4-tangent to Σ. Conditions (1.10) and (1.11) follow if Σ is a boundary surface, but they do not alone guarantee that it is. (Note that since $u^\alpha = \frac{\partial x^\alpha}{\partial\xi^i}u^i$, we can also write condition (1.11) as $[G_{\alpha\beta}u^\alpha n^\beta] = 0$.)

1.2 Some Examples

We start with some general, but elementary, observations. Suppose V^- (say) represents a fluid with flow lines $\bar{u}^\alpha$ with respect to which the energy momentum tensor can be decomposed as

$$T^\alpha_\beta = (\rho + p - \zeta\Theta)\bar{u}^\alpha\bar{u}_\beta + (p - \zeta\Theta)\delta^\alpha_\beta - 2\eta\sigma^\alpha_\beta + \bar{u}^\alpha q_\beta + q^\alpha\bar{u}_\beta, \tag{1.12}$$

where, as usual, ρ represents the energy density, p the isotropic pressure, ζ the bulk viscosity, η the shear viscosity, and q^α the heat flux. Further, suppose that $\bar{u}^\alpha = u^\alpha$

where u^α is the 4-tangent to Σ. (Note that in general this will not be the case.) Then, from condition (1.10) and the Einstein equations, it follows that

$$[p - (\zeta - \frac{2}{3}\eta)\Theta - 2\eta n^\alpha n^\beta \nabla_\beta u_\alpha] = 0 \tag{1.13}$$

(timelike Σ). With the identify (1.7) and condition (1.11) we also have

$$[n^\alpha q_\alpha + \eta n^\alpha u^\beta \nabla_\alpha u_\beta] = 0. \tag{1.14}$$

For an ideal fluid ($p = \zeta = \eta = q^\alpha = 0$) conditions (1.13) and (1.14) reduce simply to the continuity of the isotropic pressure. (Note that for a fluid without heat flux and without a shear viscosity $p = \zeta\Theta$ at Σ for junction onto vacuum.)

Most work on surfaces of discontinuity has been restricted to spherical symmetry. Then, the metric intrinsic to Σ can be given as

$$ds^2_\Sigma = g_{ij} d\xi^i d\xi^j = R^2(\tau)(d\theta^2 + sin^2\theta d\phi^2) + \Delta d\tau^2, \tag{1.15}$$

where $\Delta \equiv u^\alpha u_\alpha$. Without loss of generality we take θ and ϕ continuous through Σ. With the symmetry, it follows from (1.9) that there are but two independent non- vanishing components of K_{ij}, $K_{\tau\tau}$ and $K_{\theta\theta}(= sin^{-2}\theta K_{\phi\phi})$. These components follow from (1.9) as

$$K_{\tau\tau} = -n_\alpha \dot{u}^\alpha, \tag{1.16}$$

where $\dot{u}^\alpha (\equiv u^\beta \nabla_\beta u^\alpha)$ is the 4-acceleration to u^α, and

$$K_{\theta\theta} = -n_\alpha \Gamma^\alpha_{\theta\theta}. \tag{1.17}$$

In terms of the mixed angular components of the Reimann Christoffel tensor we can, as usual, define the effective gravitational mass,

$$m \equiv \frac{1}{2}(g_{\theta\theta})^{3/2} R_{\theta\phi}{}^{\theta\phi}. \tag{1.18}$$

Then, from (1.17), we find

$$K_{\theta\theta} = \pm R(\dot{R}^2 - \Delta(1 - \frac{2m}{R})^{1/2}), \tag{1.19}$$

where $\dot{R} \equiv dR/d\tau$, and the sign of $K_{\theta\theta}$ depends on the orientation of n^α. In any event, since Δ is preserved across Σ, it follows that

$$[m] = 0 \tag{1.20}$$

for spherical boundary surfaces. It is important to note that $[K_{\theta\theta}] = 0 \Rightarrow [m] = 0$ but that $[m] = 0 \not\Rightarrow [K_{\theta\theta}] = 0$. The sign of $K_{\theta\theta}$ in V^+ and in V^- must be considered. Moreover, one must ensure, as mentioned above, that the orientation of n^α (which can alter the sign of $K_{\theta\theta}$) is preserved through Σ. There are examples in the literature where this orientation has been inverted to Σ. This can lead to the fallacious conclusion that the configuration is encased with a negative mass layer. It can also happen that this inversion misrepresents Σ as a boundary surface when it is not. We return to this point once we have explored the junction conditions in a special simple case.

i) The spherical dust/vacuum interface

It is instructive to begin with a rather simple consideration of the junction of dust onto vacuum in the case of spherical symmetry. (We begin here with Λ, the cosmological constant, $= 0$ but we relax this condition later in these notes.) For uniform dust this construction (with a vacuum "interior") gives the familiar "Swiss cheese" model pioneered by Einstein and Straus (1945).

From (1.16) and (1.20) this junction looks deceptively simple, though it has been discussed many times in the literature. Since the flow lines in dust are always geodesic, we have $K_{\tau\tau} = 0$. We need only ensure that the Schwarzschild mass of the vacuum equals the effective gravitational mass of the dust at Σ (assuming that the continuity of the sign of $K_{\theta\theta}$ has been checked and that the orientation of n^α is preserved through Σ). Of course the continuity of the intrinsic metric links the coordinates used to describe V^- and V^+. But how do we know that Σ is generated by the geodesic streamlines? Perhaps the easiest way to see this is to consider the evolution of m along Σ. In vacuum clearly $\dot{m} = 0$. Suppose we choose comoving synchronous coordinates in the dust $(r, \theta\, \phi, t)$. Since, for dust, m = m(r) (see below), we have

$$\dot{m}_\Sigma = \frac{dm}{dr}\frac{dr}{d\tau}\,|_\Sigma = 0 \qquad (1.21)$$

so that unless dm/dr = 0 at Σ we have Σ comoving and hence geodesic as expected.

We now look at the above situation a little more closely. Let us label the dust V^- and write the metric in coordinates (r, θ, ϕ, t) as

$$ds_-^2 = e^\alpha dr^2 + R^2(d\theta^2 + sin^2\theta d\phi^2) - e^\beta dt^2 \qquad (1.22)$$

where α, R and β are functions of r and t. (The formulae recorded here are more general than those required for dust, but will be made use of lat er in these notes.) The components of the Einstein tensor relevant to the present discussion are

$$G^r_r = \frac{-1}{R^2}\,\{1 - e^{-\alpha}(R'^2 + R\beta' R') + e^{-\beta}(\dot{R}^2 + 2R\ddot{R} - R\dot{\beta}\dot{R})\}, \qquad (1.23)$$

$$G_t^t = \frac{-1}{R^2}\,\{1 + e^{-\beta}(\overset{*}{R}^2 + R\overset{*}{\alpha}\overset{*}{R}) - e^{-\alpha}(R'^2 + 2RR'' - R\alpha' R')\}, \tag{1.24}$$

and

$$e^{\alpha} G_t^r = -e^{\beta} G_r^t = \frac{1}{R}\,\{-2\overset{*}{R}' + R'\overset{*}{\alpha} + \overset{*}{R}\beta'\}, \tag{1.25}$$

where $^* \equiv \partial/\partial t$ and $' \equiv \partial/\partial r$. Along *any* radial timelike trajectory of the metric (1.22) the tangent and normal 4-vectors are

$$u^{\alpha} \equiv (\dot{r}, 0, 0, \dot{t}) \textit{ and } n^{\alpha} = \pm(\dot{t}/\delta, 0, 0, \delta\dot{r}), \tag{1.26}$$

where $\delta \equiv e^{(\alpha-\beta)/2}$ and, as before $\;\equiv d/d\tau, \tau$ the proper time along the trajectory. From (1.18) and (1.22) we have

$$m = \frac{R}{2}\,\{1 + e^{-\beta}\overset{*}{R}^2 - e^{-\alpha}R'^2\}. \tag{1.27}$$

Now in the dust we can take comoving synchronous coordinates (e.g. Landau and Lifshitz 1975) so that $\beta = 0$ and along the streamlines we have $u^{\alpha} = \delta_t^{\alpha}$ with $n^{\alpha} = \pm e^{-\alpha/2}\delta_r^{\alpha}$. Then from (1.25) the statement $G_{\beta}^{\alpha} n_{\alpha} u^{\beta} = 0$ gives

$$R' = \Gamma e^{\alpha/2} \tag{1.28}$$

where $\Gamma = \Gamma(r)$, so that (1.27) can be rearranged to give

$$\overset{*}{R}^2 = \Gamma^2 - 1 + \frac{2m}{R}, \tag{1.29}$$

and hence $K_{\theta\theta}^2 = R^2\Gamma^2$. The statement $G_{\beta}^{\alpha} n_{\alpha} n^{\beta} = 0$, (1.23), and (1.28) now give $m = m(r)$ in the dust, as is well known. It is useful to record here some scalars associated with a general timelike trajectory *through* the dust:

$$\left.\begin{aligned} G_{\beta}^{\alpha} n_{\alpha} n^{\beta} &= \frac{2m'R'}{\Gamma^2 R^2}\overset{*}{r}^2, \\ G_{\beta}^{\alpha} n_{\alpha} u^{\beta} &= \pm\frac{2m'}{\Gamma R^2}\overset{*}{r}\overset{*}{t}, \\ \text{and} \qquad G_{\beta}^{\alpha} u_{\alpha} u^{\beta} &= \frac{2m'}{R'R^2}\overset{*}{t}^2. \end{aligned}\right\} \tag{1.30}$$

We observe from conditions (1.10) and (1.11), and the first two scalars, that for junction onto vacuum Σ must be comoving unless $m'(r_{\Sigma}) = 0$. If $m'(r_{\Sigma}) = 0$ and $\overset{*}{r} \neq 0$ it follows that the "dust" is not dust, but rather vacuum. (The circumstance $m' = \overset{*}{r} = \dot{r} = 0$ is allowed, see below.)

Now let us label the vacuum V^+ and use Kruskal-Szekers coordinates (u, θ, ϕ, v) so that

$$ds^2_+ = \frac{-32m^3}{\imath} e^{-\imath/2m} du dv + \imath^2 (d\theta^2 + sin^2\theta d\phi^2) \tag{1.31}$$

where $\imath(u, v)$ is defined, as usual, by $uv = (1 - \imath/2m)e^{\imath/2m}$. From the continuity of the intrinsic metric (condition (1.3)), and the metrics (1.15) ($\Delta = -1$), (1.22), and (1.31) it follows that

$$\imath_\Sigma(u, v) = R(r_\Sigma, t) = R(\tau). \tag{1.32}$$

We call this the "history" of Σ. This is a statement of the continuity of $g_{\theta\theta}$ and $g_{\phi\phi}$. The continuity of the remaining component g_{rr} gives

$$\frac{32m^3}{\imath} e^{-\imath/2m} \dot{u}\dot{v} = \dot{t}^2 - \frac{R'2}{\Gamma^2}\dot{r}^2 = \dot{t}^2 = 1. \tag{1.33}$$

This is simply a statement that Σ is timelike. The radial timelike geodesic equation in V^+ gives

$$\dot{\imath}^2 = \gamma^2 - 1 + \frac{2m}{\imath}, \tag{1.34}$$

where $\gamma \equiv (v/\dot{v} - u/\dot{u})/8m$ so that $\dot{\gamma} = 0$. From equation (1.29) and the history (1.32) then, as expected,

$$\Gamma^2(r_\Sigma) = \gamma^2 \tag{1.35}$$

where, from (1.20), we have set

$$m(r_\Sigma) = m. \tag{1.36}$$

What remains to be given here is a specification of what I will call the "topology" of the situation. The Kurskal-Szekeres diagram is (physically) symmetric about $v = u$ so we need consider only $v \geq u$. With this in mind, it is necessary to specify whether the vacuum is "exterior" to Σ (isolated dust ball) or "interior" to Σ (the vacuum forms a bubble in a background of dust). The easiest way to do this is to pick a point p in the $u - v$ diagram. (The point is to represent an event in the history of a particle on Σ). At p construct n^α. We find

$$n^\alpha = \pm(-\dot{u}, 0, 0, \dot{v}). \tag{1.37}$$

If we take the "+", n^α "points", to increasing values of v. With this choice for n^α, if n^α is directed into vaccum, we have an isolated dust ball. Otherwise we have a bubble. Notice that for the bubble case Σ can, in principle, be viewed from the parallel asymptotically flat "universe". (I return to this point below.)

The topological distinction sketched above is meaningless for Σ given by $u = v$. However, in this case it follows that $\gamma = K_{\theta\theta} = 0$ (i.e. $\mathcal{R}_{max} = 2m$). With $\Gamma(r_\Sigma) = 0$ the metric (1.22) is regular at Σ for $R'(r_\Sigma, t) = 0$, and so from (1.30) we must have $m'(r_\Sigma) = 0$. We return to other cases with $R' = 0$ below. For the remainder of this section we take R' and $\Gamma \neq 0$ at Σ.

I would like now to return to a consideration of the sign of $K_{\theta\theta}$. For dust joined onto vacuum we have $K^2_{\theta\theta}/R^2 = \gamma^2$ so that the sign of $K_{\theta\theta}$ is fixed throughout the non-singular history of Σ. (The sign of K_{ij} is *not* fixed in more general situations.) With the choice "+" in (1.37) one easily calculates

$$K_{\theta\theta} = \frac{4m^2}{e^{\mathcal{R}/2m}}\left\{v\dot{u} - u\dot{v}\right\} = \mathcal{R}\,\gamma. \tag{1.38}$$

Observe that $K_{\theta\theta} > 0$ for $v > u$. (The choice $\gamma \geq 0$ for $v \geq u$ gives the standard future orientation). Label the vacuum as V^+ and the dust as V^-. We then have a dust ball for $K^{\pm}_{\theta\theta} > 0$ (and a bubble for $K^{\pm}_{\theta\theta} < 0$ having taken the "-" in (1.37)). The question to be answered now is that given $n^\alpha = (-\dot{u}, 0, 0, \dot{v})$ in V^+, which is the correct choice of $n^\alpha = \pm e^{-a/2}\delta^\alpha_r$ in V^- which retains the required orientation from V^- to V^+? For the case at hand, a direct calculation gives the following: In V^- for $n^\alpha = \pm\frac{\Gamma}{R'}\delta^\alpha_r$, $K_{\theta\theta} = \pm R\Gamma$ and $\nabla_\alpha n^\alpha = \pm\frac{2\Gamma}{R}$; and in V^+ for $n^\alpha = \pm(-\dot{u}, 0, 0, \dot{v})$, $K_{\theta\theta} = \pm\mathcal{R}\,\gamma$ and $\nabla_\alpha n^\alpha = \pm\frac{2\gamma}{\mathcal{R}}$.

It is evident then that the condition $[\nabla_\alpha n^\alpha] = 0$ picks out for us the correct orientation of n^α. Moreover, since $\nabla_\alpha n^\alpha = K^i_i$ (as is easily verified by reference to e.g. Gaussian coordinates at Σ), condition (1.4) gives

$$[\nabla_\alpha n^\alpha] = 0. \tag{1.39}$$

Though not independent, condition (1.39) is a useful supplement to the Darmois conditions (1.3) and (1.4) as the above simple example demonstrates.

ii) Comoving spherical dust/dust interface

Consider a comoving boundary surface Σ in dust. As long as R' and $\Gamma \neq 0$ at Σ it is clear from the discussion above that the Darmois conditions require the continuity of R and Γ (and, of course, m) at Σ. An interesting situation arises for Σ defined by $r = r_o$ where $R(r_o, t) > 0$ and

$$R'(r_o, t) = 0, \Gamma(r_o) \neq 0. \tag{1.40}$$

It then follows from (1.6) and (1.40) that $n_\alpha = 0$ at Σ. Retreating to the Lichnerowicz conditions, it follows that with R, Γ, and m of class C^1 at Σ the coordinates $(r, \theta\, \phi, t)$ are admissible, as long as $R''(r_o, t)$ is finite. With Γ and m of class C^1 and R of class C^{2-} then Σ, defined by (1.40), is a boundary surface. This conclusion is at variance with that of Bonnor (1985) and Hellaby and Lake (1985). Moreover, there is, evidently, no formal requirement that $m'(r_o) = 0$. If $m'(r_o) \neq 0$ then it follows that Σ is a scalar polynomial singularity in the sense that $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges along Σ ($R_{\alpha\beta\gamma\delta}$ the Riemann Christoffel tensor). I am unaware of any other boundary surface which is also an s.p. singularity in the above sense.

We return to the study of boundary surfaces in Chapter 5.

References

This is not meant to be a historical document. The theory of junction conditions in general relativity has a long history. The references chosen here are, as I understand the subject, the most important for the points covered in the text. Numerous further references can be found by way of the Science Citation Index. For example, there are over 90 references to Israel's paper in the last ten years.

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2. Introduction to Surface Layers

In this section we examine non-null 3-surfaces of discontinuity Σ at which condition (1.4) (that is, $[K_{ij}] = 0$) fails to hold. The "Gauss-Codazzi formalism" of Israel (1966) is used. (The approach by Papapetrou and Hamoui (1968) is not.) This "thin shell" formulation fits into a more general analysis by Taub (1980).

2.1 Review of Basic Equations

We start with the equations of Gauss

$$R_{\alpha\beta\gamma\delta}\frac{\partial x^\alpha}{\partial \xi^i}\frac{\partial x^\beta}{\partial \xi^j}\frac{\partial x^\gamma}{\partial \xi^k}\frac{\partial x^\delta}{\partial \xi^\ell} = R_{ijk\ell} - \Delta(K_{i\ell}K_{jk} - K_{ik}K_{j\ell}), \tag{2.1}$$

and of Codazzi

$$R_{\alpha\beta\gamma\delta}n^\alpha\frac{\partial x^\beta}{\partial \xi^j}\frac{\partial x^\gamma}{\partial \xi^k}\frac{\partial x^\delta}{\partial \xi^\ell} = \nabla_\ell K_{jk} - \nabla_k K_{j\ell}, \tag{2.2}$$

(e.g. Eisenhart 1926). Contracting these we have

$$G_{\alpha\beta}n^\alpha n^\beta = (\Delta(^3R) + K^2 - K_{ij}K^{ij})/2, \tag{2.3}$$

and

$$G_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^i}n^\beta = \nabla_j K^j_i - \nabla_i K, \tag{2.4}$$

(see, e.g., Misner, Thorne and Wheeler (1973) for details) where $G_{\alpha\beta}$ is the Einstein tensor, $K \equiv g^{ij}K_{ij}$, and $^3R \equiv$ Ricci scalar constructed from g_{ij}.

Consider the intrinsic 3-tensor S_{ij} defined by

$$8\pi S_{ij} \equiv \Delta(\gamma_{ij} - g_{ij}\gamma), \tag{2.5}$$

where $\gamma_{ij} \equiv [K_{ij}]$, and $\gamma \equiv g^{ij}\gamma_{ij}$. We call S_{ij} the surface energy tensor and (2.5) the Lanczos equation. From equations (2.4) and (2.5) we have

$$[G_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^i}n^\beta] = 8\pi\Delta\nabla_j S^j_i. \tag{2.6}$$

With $\overline{\psi} \equiv (\psi\,|^+_p + \psi\,|^-_p)/2$ for all points p on Σ, from equation (2.4) we also have

$$\overline{G_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^i}n^\beta} = \overline{\nabla_j K^j_i} - \overline{\nabla_i K}. \tag{2.7}$$

From equations (2.3) and (2.5) we find

$$[G_{\alpha\beta}n^\alpha n^\beta] = -8\pi\Delta S^{ij}\overline{K_{ij}}. \tag{2.8}$$

Finally, from equations (2.3) and (2.5) we also find

$$\overline{2G_{\alpha\beta}n^\alpha n^\beta} = -16\pi^2(S_{ij}S^{ij} - S^2/2) + \Delta(^3R) + (\overline{K})^2 - (\overline{K_{ij}})(\overline{K^{ij}}), \tag{2.9}$$

where, from equation (2.5), we have used the trace

$$16\pi^2 S^2 = \gamma^2. \tag{2.10}$$

Note that the right hand side of equations (2.6) through (2.9) involve only quantities intrinsic to Σ.

2.2 Thin Shells

We presume that V^+ and V^- are given solutions to the Einstein equations. Then equations (2.7) and (2.9) may be viewed as redundant since, given a surface equation fo state for Σ, equation (2.6) gives the intrinsic conservation laws for Σ, and (2.8) gives the equation of motion for Σ (see below). By way of Einstein's equations, and equation (1.7), we can rewrite equation (2.6) as

$$[T_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^i}n^\beta] = \Delta\nabla_j S^j_i, \tag{2.11}$$

and equation (2.8) as

$$[\Delta T_{\alpha\beta}n^\alpha n^\beta + \frac{\Lambda}{8\pi}] = -S^{ij}\overline{K_{ij}}. \tag{2.12}$$

To offer an interpretation of equations (2.11) and (2.12), consider the (discontinuous) 4-tensor $S^{\alpha\beta}$ defined by

$$S^{\alpha\beta} = \begin{cases} \frac{\partial x^\alpha}{\partial \xi^i}\frac{\partial x^\beta}{\partial \xi^j}S^{ij} & \text{on } \Sigma; \\ 0 & \text{off } \Sigma. \end{cases} \tag{2.13}$$

In terms of the 3-tangent $u_i = u_\beta \partial x^\beta/\partial \xi^i$ it follows from equations (2.11) and (2.13) that

$$u_\alpha \nabla_\beta S^{\alpha\beta} = u_i \nabla_j S^{ij} = \Delta[T_{\alpha\beta}u^\alpha n^\beta]. \tag{2.14}$$

That is, the tangential "stress" on Σ is continuous across Σ and equal to the jump across Σ of the "flux". From the definition K_{ij} and equations (1.7) and (2.13) we have

$$n_\alpha \nabla_\beta S^{\alpha\beta} = -K_{ij} S^{ij}, \tag{2.15}$$

so that from equation (2.12)

$$\overline{n_\alpha \nabla_\beta S^{\alpha\beta}} = [\Delta T_{\alpha\beta} n^\alpha n^\beta + \frac{\Lambda}{8\pi}]. \tag{2.16}$$

That is, the average normal "stress" on Σ is equal to the jump in "pressure" across Σ. Finally, from the Lanczos equation (2.5) and equation (2.15) we find

$$[n_\alpha \nabla_\beta S^{\alpha\beta}] = -8\pi\Delta(S^{ij}S_{ij} - S^2/2). \tag{2.17}$$

That is, the jump in the normal "stress" across Σ is due to the "self-interaction" of the layer.

In analogy to a fluid, we define the surface energy density σ by

$$\sigma \equiv -\Delta S_{ij} u^i u^j,$$

so that from the Lanczos equation we have

$$8\pi\sigma = \Delta\gamma - \gamma_{ij} u^i u^j. \tag{2.18}$$

Similarly, we define the surface pressure P by

$$2P \equiv -\Delta S_{ij} u^i u^j + S$$

so that, again from the Lanczos equation, we have

$$16\pi P = -\Delta\gamma - \gamma_{ij} u^i u^j. \tag{2.19}$$

Equations (2.18) and (2.19) give

$$[n_\alpha \dot{u}^\alpha] = -\gamma_{ij} u^i u^j = 8\pi(P + \sigma/2). \tag{2.20}$$

As a result, if Σ is geodesic ($\dot{u}^\alpha = 0$) it follows that $P = -\sigma/2$. In general, however, Σ is *not* geodesic.

We now summarize the fundamental equations for Σ. From equation (2.11) (or, equivalently (2.14)), the Lanczos equation (2.5), and relations (2.18) and (2.19) we have

$$\dot{\sigma} + (\sigma + P)\Phi = -\Delta[T_{\alpha\beta} u^\alpha n^\beta], \tag{2.21}$$

where, as usual, $\dot{\sigma} \equiv u^i \nabla_i \sigma$, and $\Phi \equiv \nabla_i u^i$ gives the 3-expansion associated with Σ. Similarly, from equation (2.12) (or, equivalently (2.16)), the Lanczos equation (2.5) and relations (2.18) and (2.19) we find

$$\Delta(\sigma + P)\overline{n_\alpha \dot{u}^\alpha} + P\overline{K} = -\Delta[T_{\alpha\beta}n^\alpha n^\beta + \frac{\Delta\Lambda}{8\pi}]. \tag{2.22}$$

Much of what I have given here has been known for close to twenty years (see Chase 1970). Despite this, parts of the above can be found repeated numerous times in the current literature.

Equations (2.18) and (2.19) relate the intrinsic "thermodynamic" quantities σ and P to the intrinsic geometric structure of Σ. Equations (2.21) and (2.22) relate this intrinsic structure to the enveloping 4-dimensional spacetimes. In particular, equation (2.21) governs the conservation laws which must hold on Σ and, as will become clear below, equation (2.22) gives the history of Σ in the background spacetimes.

In what follows we presume that the background spacetimes are known. Then, given the history of Σ, we can solve for the evolution of σ and P. Alternatively, given the surface equation of state (say $\mathrm{P} = \mathrm{P}(\sigma)$) we can solve for the evolution of Σ. In either case the foregoing equations indicate that the situation requires a fair amount of calculation. For example, it has been known for about ten years (Cunningham (unpublished)) that even the motion of a dust shell ($\mathrm{P} = 0$), with $V^\pm$ given by Reissner-Nordström metrics ($\Lambda = 0$), can be classified into 80 types! Except for some limiting calculations in the Kerr metric (see De La Cruz and Israel 1968) most calculations involving thin shells have assumed spherical symmetry, to which we now turn.

2.3 Spherical Shells

From section (1.2) recall that the intrinsic metric can be given as in (1.15) and that without loss of generality θ and ϕ can be taken continuous through Σ. Moreover, there are but two independent non-vanishing components of K_{ij} and these are given by expressions (1.16) and (1.17). Writing

$$u^i \equiv (\dot{\theta}, \dot{\phi}, \dot{\tau}) = (0, 0, 1)$$

it follows from the intrinsic metric (1.15) and equation (2.18) that

$$4\pi R^2 \sigma = \Delta\gamma_{\theta\theta} \equiv M(\tau) \tag{2.23}$$

where we call M the mass intrinsic to Σ.

Two observations follow from equation (2.23). First, suppose that $[T_{\alpha\beta}u^\alpha n^\beta] = 0$ and that P = 0. Then, from equations (2.21) and (2.23) we have

$$\dot{M} = 0. \tag{2.24}$$

That is, "dust" shells through which the "flux" is constant have fixed surface mass. In general, however, there is no a priori reason to take P = 0, though this is frequently done in the literature. Moreover, in part of what follows we will be interested in cases for which $[T_{\alpha\beta}u^\alpha n^\beta] \neq 0$. In terms of M, equation (2.21) reads as

$$\dot{M} + 8\pi R\dot{R}P = -4\pi R^2\Delta[T_{\alpha\beta}u^\alpha n^\beta]. \tag{2.25}$$

Next, from equation (2.23) note that we have the identity

$$(K^+_{\theta\theta})^2 = \frac{1}{4M^2}([K^2_{\theta\theta}] + M^2)^2. \tag{2.26}$$

For any spherically symmetric spacetime it follows that

$$K^2_{\theta\theta} = R^2(\dot{R}^2 - \Delta(1 - \frac{2m}{R})) \tag{2.27}$$

where, as before,

$$m \equiv \frac{R^3}{2}R_{\theta\phi}{}^{\theta\phi}. \tag{2.28}$$

Equations (2.26) and (2.27) give us the useful relation (Lake 1979)

$$\dot{R}^2 = \Delta + (\frac{[m]}{M})^2 - \frac{2\Delta\overline{m}}{R} + (\frac{M}{2R})^2. \tag{2.29}$$

For example, in vacuum (with $\Lambda = 0$ so m is the Schwarzschild mass) equation (2.29) clearly gives the history of Σ. The evolution of M (which is required to solve (2.29)) follows from (2.25) given the surface equation of state P = P(M). What equation (2.29) does not do is to distinguish m_+ from m_- as regards the dynamics of the shell. This is complicated by the fact that, as yet, we have no physical distinction of V^+ from V^-. To do this suppose $K_{\theta\theta} > 0$ on one side of Σ, label this as V^+, and call it the "outside". Next, consider timelike Σ and impose the intrinsic weak energy condition

$$M > 0 \tag{2.30}$$

so that from the definition (2.23) we have

$$K^-_{\theta\theta} > K^+_{\theta\theta} > 0. \tag{2.31}$$

Equations (2.27) and (2.31) then give the condition

$$m_+ > m_- \tag{2.32}$$

which is exactly what one might guess. The problem is, however, that there exist situations (e.g. Σ located beyond an "Einstein-Rosen bridge") where the above construction doesn't work. We return to this situation in section 3. Note that condition (1.39) is not applicable here since, for surface layers, $\gamma \neq 0$ except for $P = \sigma/2$.

Both equations (2.22) and (2.29) give the dynamics of the shell. Writing equation (2.22) out explicitly we obtain the proper time derivative of equation (2.29). We can, therefore, view (2.29) as the general first integral of the motion of spherical shells.

Now whereas σ has a natural interpretation leading to the definition (2.23), it is worthwhile here to take another look at P. From (2.22), with spherical symmetry, we find

$$(\sigma + \frac{2P}{R^2})\,\overline{K_{\tau\tau}} = -\frac{2a}{R}$$

where a, motivated by the standard definition of the surface tension ($[p] \equiv -2a/R$), is defined by

$$a \equiv \frac{R}{2}\Delta[T_{\alpha\beta}n^\alpha n^\beta + \frac{\Delta\Lambda}{8\pi}].$$

In the weak-field ($m_\pm << R$) slow-motion ($\dot{R} << 1$) limit it then follows that $P \rightarrow -a$, as expected.

References

The thin wall approximation has seen widespread use in the recent literature. Most of this work has been involved with either the evolution of "bubbles" in vacuum, or the evolutions of "voids" in cosmology. These subjects are considered in the following two sections.

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3. Evolution of Bubbles in Vacuum

On the basis of guage field theories with spontaneous symmetry breaking it has become popular to argue that the Universe may have undergone vacuum phase transitions. Because of these transitions the Universe may also have undergone an exponential expansion phase with important consequences for the horizon and flatness problems of the standard model (Guth 1981). "Old" inflation concerns a Higgs effective potential $V(\phi)$ with local minima at ϕ_{false} and ϕ_{true} such that the difference between the real and false vacua is much smaller than the potential barrier between them. In this situation one can argue that the bubbles of a true vacuum in a background of a false vacuum can be described by the thin-wall approximation (see Coleman 1977). Through "New" inflation (see Linde 1982 and Albrecht and Steinhardt 1982) is less amenable to a thin- walled treatment, one still needs some idea about the role of gravitational effects and so the thin-wall approximation is a useful first step.

In the last few years there have been many discussions in the literature concerning vacuum bubbles in the thin wall approximation. In this section a systematic treatment is attempted along the lines of Lake and Wevrick (1986). Pseudoeffective potentials are developed for the motion of a bubble, and, as a result, the qualitative history of a bubble is reduced to an algebraic problem. It is assumed here that the intrinsic surface tension and density are proportional. This restriction includes the important domain wall case ($P = -\sigma$).

3.1 Formulation of pseudopotentials

The spherical vacuum in "curvature" coordinates (r, θ, ϕ, t) is given by

$$ds^2_\pm = \left(\frac{dr^2}{f(r)} + r^2 d\Omega^2 - f(r)dt^2\right)_\pm, \tag{3.1}$$

with

$$f(r) = 1 - \frac{2\hat{m}}{r} - \frac{\Lambda r^2}{3}, \tag{3.2}$$

and $\hat{m}$ constant. Note that

$$m = \hat{m} + \frac{\Lambda r^3}{6}. \tag{3.3}$$

The curvature coordinates are not only defective near horizons, they do not, in general, map one to one onto events of the spacetime. (The elliptic interpretation is not used in these notes.) Double-null coordinates for the metrics (3.1) are constructed in Appendix A. It is convenient to work in curvature coordinates, but to interpret the history of Σ in the generalized Kruskal-Szekeres diagrams.

For metrics of the form (3.1) it is convenient to rewrite the equation of motion (2.29) in the form

$$\dot{R}^2 = (\frac{R}{2M})^2\ [f]^2 - \overline{f} + (\frac{M}{2R})^2 \tag{3.4}$$

for timelike Σ. (Whereas it is possible to consider a "simultaneous" transition of the vacuum (Hawking and Moss 1982), which would require spacelike Σ, we do not consider this possibility here.) In this section we take the ansatz

$$P = \epsilon\sigma,\ \dot{\epsilon} = 0 \tag{3.5}$$

so that equation (2.25) reduces to

$$MR^{2\epsilon} = k,\ \dot{k} = 0. \tag{3.6}$$

(With $\epsilon = -1$ we have the familiar domain wall case for which $\dot{\sigma} = 0$.)

From equations (3.4) and (3.6) it follows that

$$4k^2\ R^{4\epsilon+2}\dot{R}^2 = (k^2 - k_+^2)(k^2 - k_-^2) \tag{3.7}$$

where

$$k_\pm^2 \equiv R^{4\epsilon+2}(f_+ + f_- \pm 2\sqrt{f_+ f_-}). \tag{3.8}$$

Note that $\dot{R}^2 > 0$ for $k^2 > k_+^2$ and for $k^2 < k_-^2$ and that $\dot{R} = 0$ for $k^2 = k_\pm^2$. Even though equation (3.7) cannot, in general, be integrated analytically, the evolution of Σ can be understood qualitatively from the function $k_\pm^2(R)$ given by (3.8). The construction of a dimensionless equivalent to this function depends on the explicit forms of $f_\pm$.

3.2 History of Σ

Since R does not, in general, map uniquely onto events in the embedding space-times, it is necessary to distinguish the history of Σ in the neighbourhood of a Killing horizon (assumed here to be non degenerate). This is done in figure 1 about a non-cosmological horizon (By "cosmological horizon" I mean the largest root to $f = 0$ for $\Lambda > 0$.) For a cosmological horizon the roles of A and C are interchanged. Further, it is useful to exhaust the possibilities leading to the equation of motion (3.4). This is done in Appendix B. We conclude that

$$\frac{M}{R} = \begin{cases} (\sqrt{\dot{R}^2 + f_-} - \sqrt{\dot{R}^2 + f_+}) < \sqrt{f_- - f_+} & (3.9) \\ \sqrt{f_- - f_+} & (3.10) \\ (\sqrt{\dot{R}^2 + f_-} + \sqrt{\dot{R}^2 + f_+}) > \sqrt{f_- - f_+} & (3.11) \end{cases}$$

for the trajectories A, B, and C, respectively, as distinguished in figure 1. Moreover, from the intrinsic weak energy condition (2.30) ($M > 0$) we conclude that

$$f_- > f_+ \tag{3.12}$$

for trajectories of type A.

From equations (3.7) through (3.11) it follows that

$$k^2 \begin{cases} \leq k_-^2 \neq k_+^2 \Leftrightarrow A \\ = k_-^2 = k_+^2 \Leftrightarrow B \\ \geq k_+^2 \neq k_-^2 \Leftrightarrow C \end{cases} \tag{3.13}$$

in the regions for which $k_\pm^2$ are defined. Further, it follows that if f_+ has an inner root at R_0 (that is, a non cosmological horizon for which $f_+(R_0) = 0$ and $f_+'(R_0) > 0$) then

$$k_+^2 = k_-^2 = R_0^{4\epsilon+2} f_-(R_0) > 0 \tag{3.14}$$

and $k_\pm^2$ is undefined for $R < R_0$. If f_+ has an outer root at R_1 (that is, a cosmological horizon for which $f_+(R_1) = 0$ and $f_+'(R_1) < 0$) then

$$k_+^2 = k_-^2 = R_1^{4\epsilon+2} f_-(R_1) > 0 \tag{3.15}$$

and $k_\pm^2$ is undefined for $R > R_1$. Further details regarding $k_\pm^2$ depend on the explicit forms of f_+ and f_-.

Before we move to some examples, it is necessary to address a somewhat thorny issue. That is, can we take all the trajectories in figure 1 to be part of physical reality? Contrary to some recent work (see, for example, Blau, Guendelman and Guth 1987) I do not believe so. The reason is that non-degenerate Killing horizons, to the past of their bifuraction, are unstable. This is known from perturbations with test electromagnetic fields (see Appendix C), from global arguments (Tipler 1977), from perturbation methods (e.g. Gürsel *et al* 1979), and from semi-classical calculatios (e.g. Wald and Ramaswamy 1980). This means that a bubble boundary located beyond an "Einstein-Rosen bridge" (e.g. trajectories of type C in figure 1 when "our" vacuum has $v > 0$ and $u < 0$) is a mathematical artifact. The complete pseudopotential $k_\pm^2$ is, however, included in what follows.

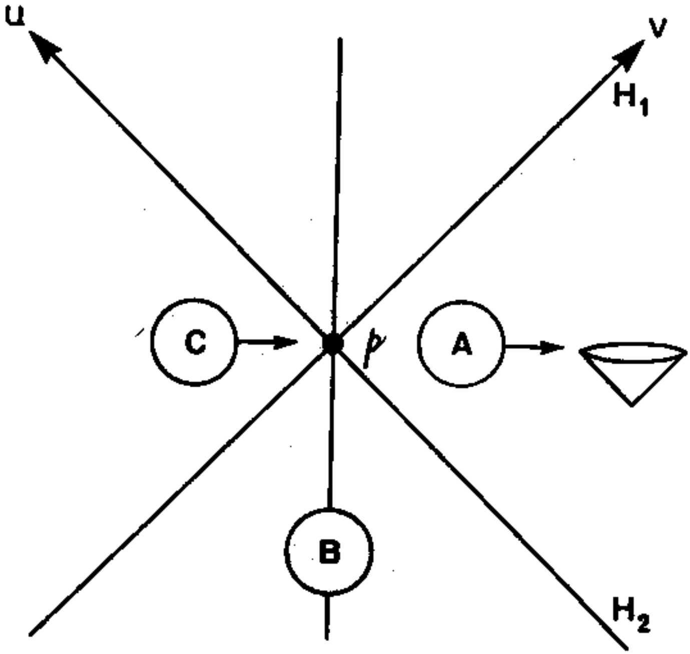

Fig. 1 Generalized Kruskal-Szekeres diagram for the totally geodesic 2-surface θ and ϕ constant for, say, the embedding "+" in the neighbourhood of a nondegenerate Killing horizon H. The branches of H are designated H_1 for $u = 0$ and H_2 for $v = 0$. Null geodesics on the 2-surface are represented by constant u or v lines. The chosen (global) fugure orientation is shown. $\not{p}$ represents the bifurcation of H and is the image of spacelike 2-surface of the full space-time. Surfaces of constant R are space- like for $uv > 0$. Three distinct types of timelike trajectories are labelled A, B, and C and are distinguished as follows: trajectories A reach $v > 0, u < 0$; trajectories B have $uv > 0$; and trajectories C reach $u > 0, v < 0$. The sense of the normal to Σ is shown, and we use the convention that $K^+_{\theta\theta} > 0$ for $A, = 0$ for B, and < 0 for C. (It can, of course, happen that trajectories $A(C)$ reach $uv > 0$ and that $K^+_{\theta\theta} = 0$ at, say, $R = R_*$ where $k^2 = R_*^{4\epsilon+2}(f_-(R_*) - f_+(R_*)$. This is in no way affects the above labelling.) Because of the reflection symmetry allowed about $u = v$, there is no loss of generality by the choice of labelling used here.

3.3 Some examples

i) $\Lambda_\pm = 0, \hat{m}_\pm \neq 0$

Set

$$\alpha \equiv \hat{m}_-/\hat{m}_+ \tag{3.16}$$

and write $x \equiv R/\hat{m}_+$ with $\ell^2 \equiv k^2/\hat{m}_+^{4\epsilon+2}$. The equation of motion (3.7) takes the form

$$4\hat{m}_+^2 \ell^2 x^{4\epsilon+2} \dot{x}^2 = (\ell^2 - \ell_+^2)(\ell^2 - \ell_-^2) \tag{3.17}$$

and the pseudopotential (3.8) takes the dimensionless form

$$\ell_\pm^2 = 2x^{4\epsilon+1}\{x - (1+\alpha) \pm \sqrt{(x-2)(x-2\alpha)}\} \ . \tag{3.18}$$

Figure 2 shows the function $\ell_\pm(x)$ for the case $\alpha = 1/2$. It is a straightforward matter to sketch the qualitative history of Σ in the Kruskal-Szekeres diagrams (for V^+ and for V^-) given the general behaviour of the pseudopotential. (The case $\alpha = \epsilon = 0$ has been considered in detail previously by Kodama *et al.* 1981.)

From the form (3.18) it follows that $\ell_\pm \neq 0$ for finite x and that as $x \to \infty, \ell_+ \sim 2x^{2\epsilon+1}$ and $\ell_- \sim (1-\alpha)x^{2\epsilon}$. For vacuum phase transitions we are interested primarily in transitions with $\epsilon < 0$ and which can be carried to completion $(x \to \infty, \Sigma \to i^+$ in $V^+)$. The asymptotic form for ℓ_- gives us the important result that these vacuum phase transitions cannot be completed with $\Lambda_\pm = 0$.

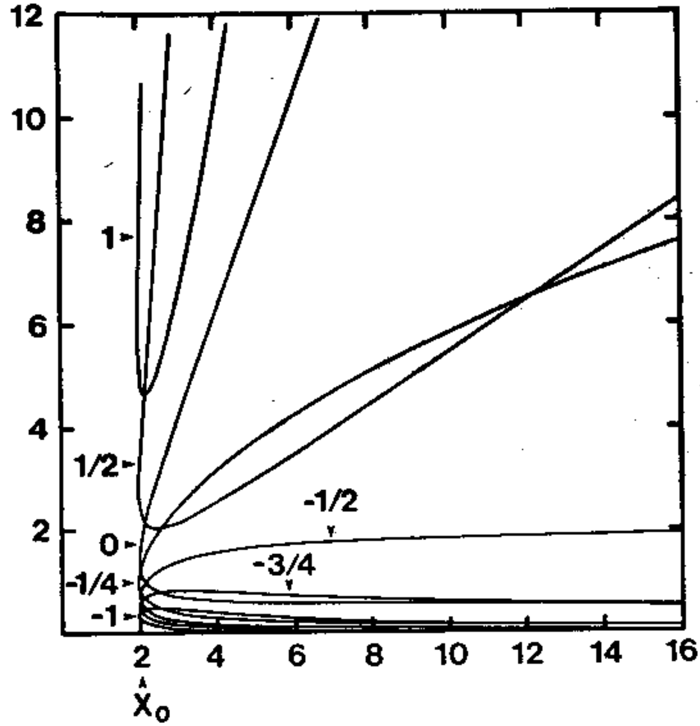

Fig. 2: Pseudopotential for $\Lambda_\pm = 0.\ell(x)$, given by (3.18), is shown for $\alpha = 1/2$. The curves are labelled by ϵ.

ii) $\hat{m}_\pm = 0, \Lambda_\pm \neq 0$

Set

$$\Lambda \equiv \begin{cases} \Lambda_+, & \Lambda_+ \neq 0 \\ \Lambda_-, & \Lambda_+ = 0 \end{cases} \tag{3.19}$$

$$\beta \equiv \begin{cases} \Lambda_-/\Lambda_+, & \Lambda_+ \neq 0 \\ 0, & \Lambda_+ = 0, \end{cases} \tag{3.20}$$

$$y \equiv \mid \Lambda \mid R^2, \tag{3.21}$$

$$\gamma \equiv k \mid \Lambda \mid^{\epsilon+1/2}, \tag{3.22}$$

and

$$L \equiv \Lambda / \mid \Lambda \mid . \tag{3.23}$$

The equation of motion (3.7) now takes the form

$$\gamma^2 y^{2\epsilon} \dot{y}^2 = \mid \Lambda \mid (\gamma^2 - \gamma_+^2)(\gamma^2 - \gamma_-^2), \tag{3.24}$$

with the dimensionless pseudopotentials

$$\gamma_\pm^2 = \frac{2}{3} y^{2\epsilon+1} \{3 - \frac{L(1+\beta)}{2} y \pm \sqrt{(y-3L)(\beta y - 3L)}\}. \tag{3.25}$$

Note that as $y \to 0, \gamma_+^2 \sim 4y^{2\epsilon+1}$ and $\gamma_-^2 \sim (1-\beta)^2 y^{2\epsilon+3}/36$.

First consider $L = +1$. Some examples are shown in figure 3. (The de Sitter-Minkowski case has been examined recently in detail by Laguna-Castillo and Matzner 1986). Cases with $\beta \geq -1$ are much like those in figure 3, but for $\beta < -1$ the character changes. For $-1/2 > \epsilon > (\sqrt{-\beta} + \beta - 1)/(1-\beta) \equiv \beta^*$, two stationary points arise along γ_+, the inner one representing stable equilibrium. For $-(2+\epsilon^*) > \epsilon > -2/3$, two stationary points arise along γ_-, again the inner one representing stable equilibrium.

Next, for $L = -1$, the pseudopotentials (3.25) are defined on $0 \leq y < \infty$ for $\beta \geq 0$ and on $0 \leq y \leq -3/\beta$ for $\beta < 0$. If $\beta \geq 0$ and $\epsilon > -1/2$, then γ_+ and γ_- intersect only at $y = 0$. If $\epsilon \leq -1/2$ then γ_+ and γ_- do not intersect. For $\beta < 0$ the pseudopotential is closed for $\epsilon > -1/2$, and γ_+ and γ_- intersect only at $y = -3/\beta$ for $\epsilon \leq -1/2$. As $y \to \infty (\beta \geq 0)$

$$\gamma_\pm^2 \sim 2y^{2\epsilon+2}[(1+\beta)/2 \pm \sqrt{\beta}]/3, \tag{3.26}$$

so phase transitions cannot be completed (in the sense that $y_\Sigma \to \infty$) for $\epsilon < -1$.

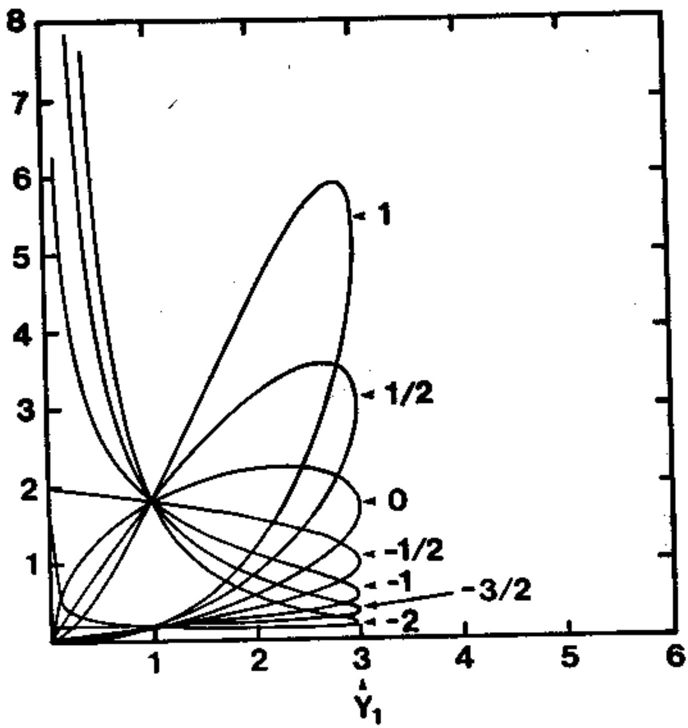

Fig. 3: The Pseudopotential for $\hat{m}_\pm = 0.\gamma_\pm(y)$ is given by (2.35) and is shown for $L = +1$ and $\beta = 0$. The curves are labelled by ϵ.

iii) $\hat{m}_\pm \neq 0, \Lambda_\pm \neq 0$

Set

$$\delta \equiv \begin{cases} \hat{m}_+\sqrt{|\Lambda|}, & \hat{m}_+ \neq 0 \\ \hat{m}_-\sqrt{|\Lambda|}, & \hat{m}_+ = 0, \end{cases} \tag{3.27}$$

$$\mathscr{M} \equiv \begin{cases} +1, & \hat{m}_+ \neq 0 \\ -1, & \hat{m}_+ = 0, \end{cases} \tag{3.28}$$

$$\mathscr{L} \equiv \begin{cases} +1, & \Lambda_+ \neq 0 \\ -1, & \Lambda_+ = 0, \end{cases} \tag{3.29}$$

and use α as defined in (3.16) for $\hat{m}_+ \neq 0$, but set $\alpha = 0$ for $\hat{m}_+ = 0$. The equation of motion (3.7) again reduces to the form (3.24), but the pseudopotentials are now given by

$$\gamma_\pm^2 = \frac{2}{3}y^{2\varepsilon+1}\ \{3 - \frac{3(1+\alpha)\delta}{\sqrt{y}} - \frac{L(1+\beta)}{2}y \pm \sqrt{\Psi},\} \tag{3.30}$$

where

$$\begin{aligned} \Psi \equiv 9(1 + \frac{4\alpha\delta^2}{y} - \frac{2\delta(1+\alpha)}{\sqrt{y}} + \frac{L\delta}{3}[(1+\mathscr{M}\mathscr{L})(\alpha+\beta) \\ + (1-\mathscr{M}\mathscr{L})(1+\alpha\beta)]\sqrt{y} - \frac{L(1+\beta)}{3}y + \frac{\beta y^2}{9}). \end{aligned} \tag{3.31}$$

(The case $\alpha = \beta = 0, \varepsilon = \mathscr{L} = -1$, and $L = \mathscr{M} = +1$ has been examined recently by Blau, Guendelman and Guth 1987 , and the more general case $\alpha = \mathscr{M} = 1$, and $\varepsilon = -1$ by Hiscock 1987.) Note that condition (3.12) holds for

$$6\delta\Big(\alpha\big(1+\mathscr{M}\big) - 2\mathscr{M}\Big) < Ly^{3/2}\Big(2\mathscr{L} - \beta\big(1+\mathscr{L}\big)\Big), \tag{3.32}$$

and that $\gamma_- = 0$ at y_*, where y_* is the solution to (3.32).

With $L = +1$, the cases allowed by condition (3.32) are summarized in Table 1 for $\mathscr{M} = \mathscr{L} = 1$. Some examples are shown in figure 4. With $L = -1$, as $y \to \infty$, it follows from (3.30) that $\gamma_\pm^2$ has the same asymptotic form as for the case $\hat{m}_\pm = 0$ discussed above except that

$$\gamma_-^2 \sim 3(1-\alpha)^2\delta^2\ y^{2\varepsilon-1} \tag{3.33}$$

for $\beta = 1$ (which rules out the completion of a phase transition in this single case without $\varepsilon \geq 1/2$). The cases allowed by condition (3.32) are summarized in Table 2.

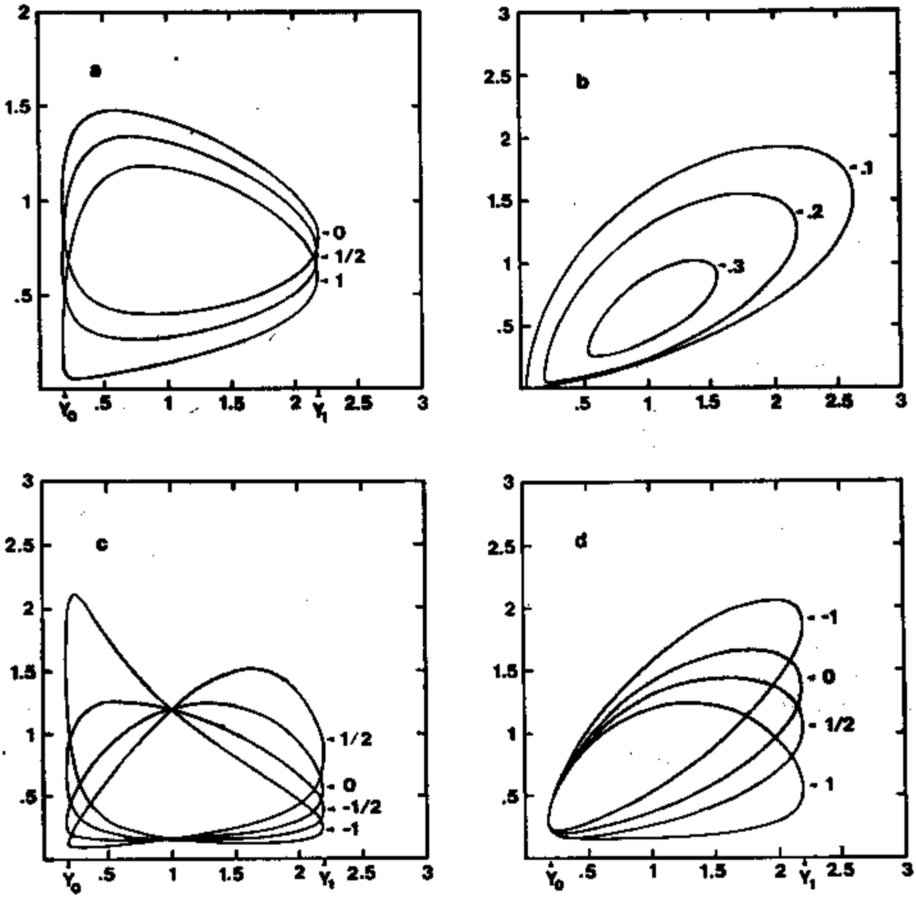

Fig. 4 The pseudopotential for $\widehat{m}_\pm \neq 0$ and $\Lambda_\pm \neq 0.\gamma_\pm(y)$, given by (3.30) with (3.31), is shown for $\mathscr{M}=\mathscr{L}=L=+1, \alpha \leq 1$, and $\beta \leq 1$. *(a)* The curves are labelled by α and have $\varepsilon = -1/2, \beta = \ 1/2$, and $\delta = 0.2$. *(b)* The curves are labelled by δ and have $\varepsilon = 0, \beta = 0$, and $\alpha = 1$. Note that the roots y_0 and y_1 are functions of δ. *(c)* The curves are labelled by ε and have $\beta = 1, \delta = 0.2$, and $\alpha = 1/2$. *(d)* The curves are labelled by β and have $\varepsilon = 0, \delta = 0.2$, and $\alpha = 1/2$.

TABLE 1. Range in α and β given that $\mathcal{M} = \mathcal{L} = L = +1$ and $M > 0$. * indicates no $\gamma_\pm^2$; # indicates no restriction on $y > 0$; and $y_* \equiv (6\delta(\alpha - 1)/(1 - \beta))^{2/3}$

	α		
β	< 1	= 1	> 1
< 1	#	#	$y > y_*$
= 1	#	*	*
> 1	$y < y_*$	*	*

TABLE 2. As in Table 1 but for $L = -1$. Here $y_* = (6\delta(1 - \alpha)/(1 - \beta))^{2/3}$

	α		
β	< 1	= 1	> 1
< 1	$y < y_*$	*	*
= 1	#	*	*
> 1	#	#	$y > y_*$

Appendix A: Construction of Double-Null Coordinates

Consider the totally geodesic 2-surfaces Γ of the metrics (3.1) defined by $d\Omega = 0$. The radial null geodesics on Γ satisfy

$$\frac{dt}{dr} = \pm \frac{1}{f(r)} \tag{A.1}$$

so there exist coordinates (u, v) on Γ such that

$$\left.\begin{aligned} 2CA(u)du &= f^{-1}(r)dr - dt, \\ &\text{and} \\ 2CB(v)dv &= f^{-1}(r)dr + dt \end{aligned}\right\} \tag{A.2}$$

where $C = const$. That is, $u = const.$ and $v = const.$ label null geodesics on Γ. Suppose $A(x) = B(x) = 1/x$ so that from (A.2)

$$\ln | uv | + D = \int \frac{dr}{Cf(r)}, \tag{A.3}$$

where $D = const.$. Suppose f has a simple root at $r = a$. Then

$$f(r) = (r-a)h(r), h(a) \neq 0$$

so that we have the unique decomposition

$$\frac{1}{f(r)} = \frac{1}{(r-a)h(a)} + \frac{g(r)}{h(r)} \tag{A.4}$$

where $g(a)/h(a) \neq 0$ and finite, and $h(a) = f'(a)$. From (A.3) and (A.4) then

$$| uv | = | r - a |^{1/Ch(a)} \exp \{ \int \frac{g(r)}{Ch(r)} dr + E \} \tag{A.5}$$

where $E = const.$. The removal of || depends on how we choose to orientate the uv axes for $r \lessgtr a$. In any event,

$$uv = \pm (r-a)^{1/Ch(a)} \exp \{ \int \frac{g(r)}{Ch(r)} dr + E \}. \tag{A.6}$$

From (A.2)

$$ds_\Gamma^2 = \frac{dr^2}{f(r)} - f(r)dt^2 = \frac{4C^2 f(r)}{uv} dudv, \tag{A.7}$$

so that from (A.6)

$$ds_\Gamma^2 = \pm \frac{4C^2(r-a)h(r)}{(r-a)^{1/Ch(a)}} \exp\left\{-\int \frac{g(r)}{Ch(r)}\, dr - E\right\} dudv. \tag{A.8}$$

The metric (A.8) is regular at $r = a$ only for $C = 1/h(a) = 1/f'(a)$. Let

$$\kappa \equiv \frac{1}{2} f'(a), \tag{A.9}$$

the usual "surface gravity", so finally

$$ds_\Gamma^2 = \pm \frac{ah(r)}{\kappa^2} \exp\left\{-2\kappa \int \frac{g(r)}{h(r)}\, dudv\right\}, \tag{A.10}$$

where, by choice of scale for u and v, we have set $e^{-E} = a$. Once again, the choice of sign in (A.10) follows from the chosen orientation of the $u - v$ axes from (A.6).

The procedure given above (Lake 1979a) yields a regular metric about a simple root $r = a$. About another simple root $b \neq a$ the procedure is repeated (with a replaced by b) to yeild a new chart, say $(\hat{u}, \hat{v})$. If a is not a simple root (that is, the horizon $r = a$ is "degenerate") a different procedure must be used (see Lake 1979b). For general relativity (without a scalar field), the most general form for $f(r)$ is

$$f(r) = 1 - \frac{2\hat{m}}{r} - \frac{\Lambda r^2}{3} + \frac{e^2}{r^2},$$

which gives the Reissner-Nordström-de Sitter metric. The case $e = 0$ was, to my knowledge, first considered by Gibbons and Hawking (1977), and independently by Lake and Roeder (1977).

Appendix B: Possible Histories for Σ

The purpose of this Appendix is to exhaust the possibilities leading to the equation of motion (3.4) for trajectories of type A or C in figure 1. We use curvature coordinates for convenience, though the coordinate t does not enter the argument. Futher, we assume that neither V^+ nor V^- are degenerate, and we refer to some event p in the history of Σ such that $f(r_p) > 0$. We have

$$K_{\theta\theta} = sign(n^r)R\sqrt{\dot{R}^2 + f} \tag{B.1}$$

so that from the definition of the surface mass (2.23)

$$M = \begin{cases} R(\sqrt{\dot{R}^2 + f_-} - \sqrt{\dot{R}^2 + f_+}) & (B.2) \\ -R(\sqrt{\dot{R}^2 + f_-} + \sqrt{\dot{R}^2 + f_+}) & (B.3) \\ R(\sqrt{\dot{R}^2 + f_-} + \sqrt{\dot{R}^2 + f_+}) & (B.4) \\ R(-\sqrt{\dot{R}^2 + f_-} + \sqrt{\dot{R}^2 + f_+}). & (B.5) \end{cases}$$

The intrinsic weak energy condition $M > 0$ rules out (B.3). As a result, if $K^+_{\theta\theta} > 0$ equation (B.2) must hold and it follows that

$$f_- > f_+. \tag{B.6}$$

This case corresponds to r increasing away from Σ into V^+ and Σ (as viewed from V^+) is a trajectory of type A for a non cosmological horizon, and of type C for a cosmological horizon. Now suppose $K^+_{\theta\theta} < 0$ so that r decreases away from Σ into V^+. Then Σ (again as viewed from V^+) is a trajectory of type C for a non cosmological horizon, and of type A for a cosmological horizon. Equation (B.4) gives $M > 0$ without condition (B.6), whereas with equation (B.5) the intrinsic weak energy condition gives

$$f_+ > f_-. \tag{B.7}$$

This latter case is, however, identical to the case (B.2) above upon interchange of the labels + and −.

Appendix C: Stability of Killing Horizons

Consider timelike trajectories r_e and r_o which cross the same branch of a non degenerate Killing horizon H_1 (less p) designated by H_1 for $u = 0$ and H_2 for $v = 0$ as in figure 1. From the general frequency shift relation

$$1 + z = \frac{\nu_e}{\nu_o} = \frac{(u^\alpha k_\alpha)_e}{(u^\alpha k_\alpha)_o} \tag{C.1}$$

and equation (A.6) it follows that

$$(1+z)_H = const. \begin{cases} v_o/v_e & \text{along } \mathrm{H}_1 \\ u_o/u_e & \text{along } \mathrm{H}_2. \end{cases} \tag{C.2}$$

The metric (A.10) is invariant under the transformation

$$\left.\begin{aligned} \hat{v} &= v \exp(\kappa D) \\ \hat{u} &= u \exp(-\kappa D) \end{aligned}\right\} \tag{C.3}$$

for finite constant D. (Transformation (C.3) can be visualized by noting that $\hat{t} = t + D$.) Whereas (C.3) applied simultaneously to e and o leaves $(1+z)$ unchanged, this is clearly not the case when it is applied to e or o separately. Along H_1 fix e and choose $\kappa D > 0$ to shift o to $\hat{o}$ by (C.3). Then

$$\hat{\nu}_o = \nu_o \exp(-\kappa D). \tag{C.4}$$

This means that "eikonal" perturbations along H_1 die out. Next, along H_2 fix o and choose $\kappa D < 0$ to shift e to $\hat{e}$ by (C.3). We again obtain (C.4), but now with $\kappa D < 0$, the perturbations along H_2 grow. (This fundamental asymmetry was first pointed out (for the Schwarzschild metric) by Eardley 1974.) One can view the timescale as the e-folding time $|\kappa|^{-1}$. For the Schwarzschild case this is of the order $20(m/m_\odot)\mu sec$. In contrast, for de Sitter space $|\kappa|^{-1} = \sqrt{3/\Lambda} \sim$ age of the universe.

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4. Development of Voids in Cosmology

It has been known for some time now that the distribution of galaxies may contain voids (e.g. Kirshner et al. 1981, Davis et al. 1982, de Lapparent, Geller, and Huchra 1986). These "voids" are known to contain a few galaxies (Moody et al. 1987) and, perhaps, low-density metal-enriched gas (Brosch and Gondhalekar 1984). In the literature there are a number of Newtonian calculations which indicate that negative perturbations can give rise to "voids" bounded by sharp "walls" (e.g. Hoffman, Salpeter and Wasserman 1983, Hausman, Olson, and Roth 1983, Fillmore and Goldreich 1984). Such voids can appear in n-body simulations (e.g. Aarseth, Gott, and Turner 1979, Efstathiou and Eastwood 1981, Klypin and Shandarin 1983, Centrella and Melott 1983), though the n-body voids and the voids in the distribution of galaxies appear to be different (Ryden and Turner 1984). Recently a Monte Carlo study of two-dimensional voids has been carried out by Icke and van den Weygaert (1986).

In this section we study the development of spherical voids in a Robertson-Walker background within the context of the thin-wall approximation. I summarize and update here the analysis by Lake and Pim (1985) and Pim and Lake (1986). This work follows upon the work of Maeda and Sato (1983a,b) but extends it so as to include a background pressure, an interior pressure and mass, and a surface pressure (or tension). (In comparable situations, however, some of our integrations differ significantly from those presented previously.)

Certainly the thin-wall approximation is applicable to the observed distribution of galaxies at best over a limited part of the evolution of a void. The work reviewed here explores the initial conditions associated with the thin-wall equations, but not the development of the applicability of this approximation in the early universe. Though the analysis is restricted to spherical symmetry, there is some evidence to suggest that this is an entirely reasonable approximation for voids (e.g. Fujimoto 1983, Icke 1984).

In contrast to section 3, where Λ played an essential role, we set $\Lambda = 0$ in this section.

4.1 Vacuum voids

We start with a vacuum (i.e. Schwarzschild) bubble joined onto a Robertson-Walker background.

i) Basic Equations

The metric exterior to Σ is taken to be the Robertson- Walker metric

$$ds_+^2 = a^2(\eta)\{d\chi^2 + S_k^2(\chi)d\Omega^2 - d\eta^2\} \tag{4.1}$$

where, as usual, $S_k(\chi) \equiv \{ \sin(\sqrt{k}\ \chi)\}/\sqrt{k}$ $(k = \pm 1, 0)$. From the metric (4.1), and Einstein's equations, it follows that

$$T_{\alpha\beta}u^\alpha n^\beta \mid_\Sigma^+ = a^2\dot{\eta}\dot{\chi}(\rho + p) \mid_\Sigma, \tag{4.2}$$

where ρ is the total comoving energy density

$$\rho = \frac{3}{8\pi a^2}\ \{\frac{1}{a^2}(\frac{da}{d\eta})^2 + k\}, \tag{4.3}$$

and p is the comoving isotropic pressure

$$p = \frac{-2}{8\pi a^2}\ \{\frac{2}{a}\frac{d^2a}{d\eta^2} - \frac{1}{a^2}(\frac{da}{d\eta})^2 + k\ \}. \tag{4.4}$$

Note that Σ is, in general, *not* comoving. Equations (2.28) and (4.1) give

$$m_+ = \frac{4}{3}\pi a^3 S_k^3(\chi)\rho \mid_\Sigma . \tag{4.5}$$

The interior Schwarzschild mass (m_-) is, of course, constant. With (4.5) we define

$$\alpha \equiv m_-/m_{+i} \tag{4.6}$$

where m_{+i} is the initial value of m_+. Rearrangement of the equation of motion for Σ (2.29) gives

$$\frac{d\chi}{d\eta} \mid_\Sigma = \frac{(\Psi\ \{S_k'^2 + \Psi - (\frac{S_k}{a}\frac{da}{d\eta})^2)\}\ ^{1/2} - \frac{S_k}{a}\frac{da}{d\eta}S_k'}{S_k'^2 + \Psi} \mid_\Sigma \tag{4.7}$$

where

$$\Psi \equiv\ \{(\frac{m_+ - \alpha m_{+i}}{M})^2 + (\frac{m_+ + \alpha m_{+i}}{aS_k}) + (\frac{M}{2aS_k})^2 - 1\ \} \mid_\Sigma , \tag{4.8}$$

with M defined as in (2.23). As in section 3 we again take $P = \epsilon\sigma$, $\dot{\epsilon} = 0$, so that since

$$T_{\alpha\beta}u^\alpha n^\beta \mid_\Sigma^- = 0, \tag{4.9}$$

it follows from the conservation equation (2.25), and equation (4.2) that

$$\frac{dM}{d\eta} = \left(\frac{abS_k^2 d\chi/d\eta}{\sqrt{1-(d\chi/d\eta)^2}} - 2\varepsilon M \left\{\frac{S_k'}{S_k}\frac{d\chi}{d\eta} + \frac{1}{a}\frac{da}{d\eta}\right\}\right)\Big|_\Sigma , \tag{4.10}$$

where

$$b \equiv -\frac{1}{a}\frac{d^2a}{d\eta^2} + \frac{2}{a^2}\left(\frac{da}{d\eta}\right)^2 + k. \tag{4.11}$$

Note that for $\varepsilon = \alpha = p = 0$, the system of equations (4.7) and (4.10) reduces to the equations integrated by Maeda and Sato (1983b).

ii) Background Model

As a model of the background we use a noninteracting mixture of dust and black-body radiation. Then

$$a(\eta) = a_0^2 = \begin{cases} \sqrt{\nu}\eta + \frac{1}{4}\mu a_o \eta^2, & \mathrm{k} = 0 \\ \sqrt{\nu}\sin\eta + \frac{1}{2}\mu a_o(1-\cos\eta), & \mathrm{k} = +1 \\ \sqrt{\nu}\sinh\eta + \frac{1}{2}\mu a_0(\cosh\eta - 1), & \mathrm{k} = -1 \end{cases} \tag{4.12}$$

where (in units of time^{-2})

$$\nu \equiv \frac{64\pi^6 G k^4 T_0^4}{45 h^3 c^5}$$

and

$$\mu \equiv \frac{128\pi^2 G k^3 (m_e + m_p)\zeta(3) T_0^3}{3h^3 c^3 f_0}.$$

Here f_o gives the photon to baryon number density ratio and T_0 the present background temperature ($\sim 2.7K$). The scale factor today, a_0, is removable for $k = 0$. For $k \neq 0$ it is obtained from the relation

$$a_0^2 = \frac{k}{\mu + \nu - H_0^2} \tag{4.13}$$

where H_o is the present Hubble parameter (taken here to be 100 $km\ s^{-1}\ Mpc^{-1}$). In summary, a flat background has one input parameter (in addition to T_0), H_o (say). If $k \neq 0$ in the background, two parameters must be given, H_0 and f_0 (or, equivalently, $q_0 = (\nu + \mu/2)/H_o^2$).

With the background specified, and with the aid of equations (4.12), the system of equations (4.7) and (4.10) (with the definitions (4.8) and (4.11)) can be reduced to a dimensionless form suitable for numerical integration. These formulae are given explicitly in Appendix A.

iii) Initial Conditions

At an initial epoch η_i (specified by $(1+z)_i$) χ_i and M_i must be given. We specify the first by the dimensionless ratio

$$\beta \equiv (\frac{\chi}{\eta})_i \ , \tag{4.14}$$

the void/horizon (coordinate) size. For the remaining condition either M_i or $(d\chi/d\eta)_i$ can be specified. In the latter procedure M_i is obtained from the definition

$$M_i = R_i \ \{(\dot{R}^2 + 1 - \frac{2am}{R})^{1/2} - (\dot{R}^2 + 1 - \frac{2m}{R})^{1/2} \ \}, \tag{4.15}$$

where m is given by (4.5). Note that of the two values of M_i which follow from (4.15), the larger one is spurious.*

iv) Summary of Integrations

In what follows I use the notation

$$\gamma \equiv (d\chi/d\eta)_i, \tag{4.16}$$

and

$$\delta \equiv \log_{10}(1 + z_i). \tag{4.17}$$

a) $k = 0$

i) $\varepsilon = 0$

There is a minimum γ (γ_{min}) such that for $\gamma < \gamma_{min}$ Σ collapses (to $R = 0$), and for $\gamma > \gamma_{min}$,Σ has the asymptotic form (a.f.) $d \ \ln \ \chi/d \ \ln\eta \sim 0.39$. (This asymptotic form is discussed by Maeda and Sato 1983a,b.)

There is a minimum β (β_{min}) such that for $\beta < \beta_{min}$ Σ collapses, and for $\beta > \beta_{min}$,$\Sigma \rightarrow$ *a.f.* This β_{min} is epoch dependent such that β_{min} increases with δ. This is shown in figure 1.

* We find that the initial conditions stated by Sato (1984) ($m_- = 0, M_i = m_{+i} \neq 0, (d\chi/d\eta)_i = 0$) are not consistent with the equation of motion (2.29).

There is a maximum α (α_{max}) such that for $\alpha > \alpha_{max}$ Σ collapses, and for $\alpha < \alpha_{max}, \Sigma \rightarrow a.f.$ This α_{max} is epoch dependent such that as α_{max} decreases as δ increases. This is shown in figure 2.

ii) $\varepsilon \neq 0$

If $\varepsilon > 0$ we find the asymptotic form $d \ln \chi / d \ln \eta \rightarrow 1$. If $\varepsilon < 0$ we find that Σ collapses. Some examples are shown in figure 3.

b) $k \neq 0$

i) $\alpha = \varepsilon = 0$

For $k = -1$ we find that Σ becomes comoving (to, say, χ_Σ) and that χ_Σ decreases as f_o increases. For $k = +1$, Σ grows rapidly and continues to grow as the background recollapses. Some examples are shown in figure 4.

ii) $\varepsilon = 0, \ \alpha \neq 0$

For $k = -1$, as α is increases, χ_Σ decreases. There is an α_{max} (which is epoch dependent) beyond which Σ collapses. This is shown in figure 2. For $k = +1$, the growth of Σ decreases as α is increased. Again there is an α_{max} beyond which Σ collapses. This is also shown in figure 2.

As for $k = 0$, with $k = \pm 1$ there is a minimum β, dependent on epoch, below which Σ collapses.

iii) $\varepsilon \neq 0$

For $k = -1$ and $\varepsilon > 0$ we find the asymptotic form $d \ln \chi / d \ln \eta \rightarrow 1$, just as with $k = 0$. Moreover, again as with $k = 0$, we find that with $\varepsilon < 0, \Sigma$ collapses. Some examples are shown in figure 5. For $k = +1$ we find that $\varepsilon > 0$ gives rise to a more rapid growth of Σ. For $\varepsilon < 0$ we find $\varepsilon_o < 0$ such that with $\varepsilon < \varepsilon_o < 0, \Sigma$ collapses. Some examples are shown in figure 6.

The role of α is as one might guess. An effective gravitational mass slows the growth of a void. Moreover, as can be seen from figure 1, voids must have a minimum initial "size" if they are to survive. Perhaps most interesting, however, are our results with $\varepsilon \neq 0$. For $\varepsilon > 0$ we found that Σ grows asymptotically like the particle horizon in an open background. In contrast, for $\varepsilon < 0$ we found that Σ eventually collapses.

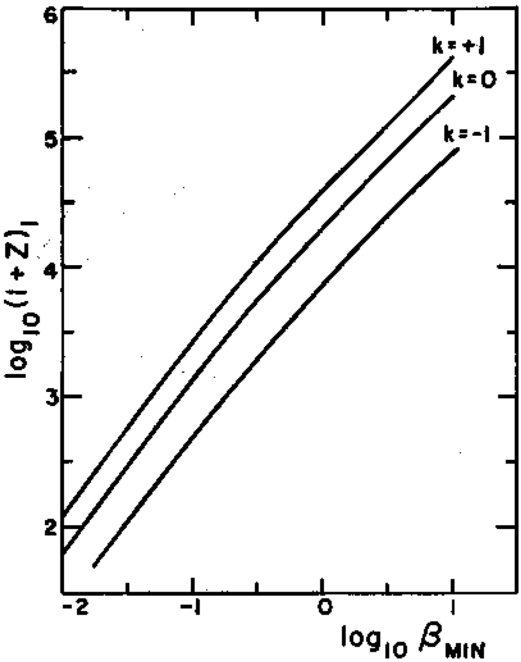

Fig.1: Minimum β for a given epoch. Curves have $\alpha = \epsilon = 0$, and $(d\chi/d\eta)_i = 1 - 10^{-6}$. For $k = +1$, $f_o = 1.8 \times 10^7$, and for $k = -1$, $f_o = 10^8$. Abscissa gives $\log_{10} \beta_{min}$, where $\beta < \beta_{min}$ guarantees collapse of the void. Curves are labelled by k.

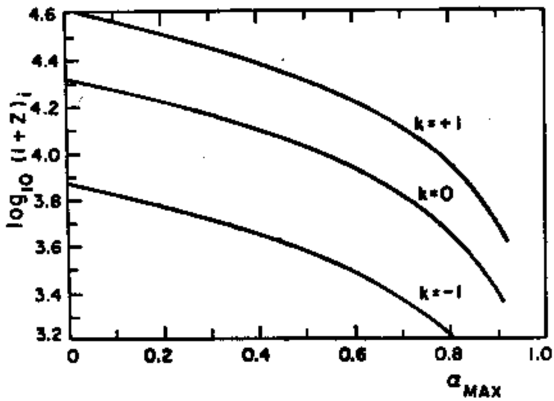

Fig.2: Maximum α for a given epoch. Curves have $\varepsilon = 0$, $\beta = 1$, and $(d\chi/d\eta)_i = 1 - 10^{-6}$. For $k = +1$, $f_o = 1.8 \times 10^7$, and for $k = -1$, $f_o = 10^8$. Abscissa gives α_{max}, where $\alpha > \alpha_{max}$ guarantees eventual collapse of the void.

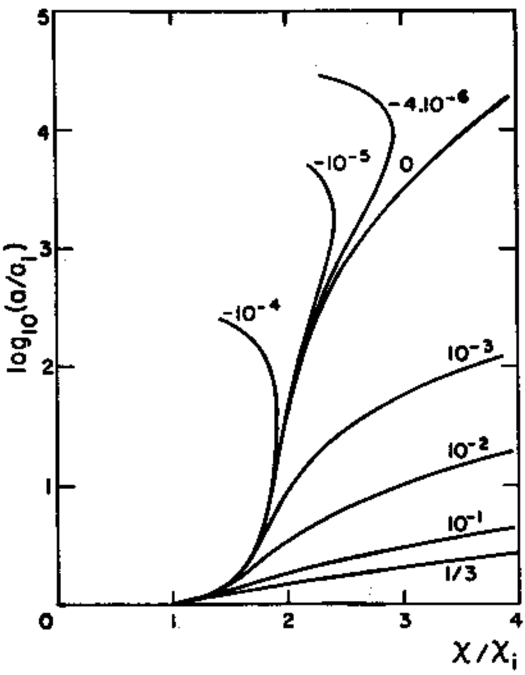

Fig.3: Effect of ε. History of Σ for $(1 + z)_i = 10^3$ and $(d\chi/d\eta)_i = 1 - 10^{-6}$ is shown for $\alpha = k = 0$ and $\beta = 0.1$. Curves are labelled by ε.

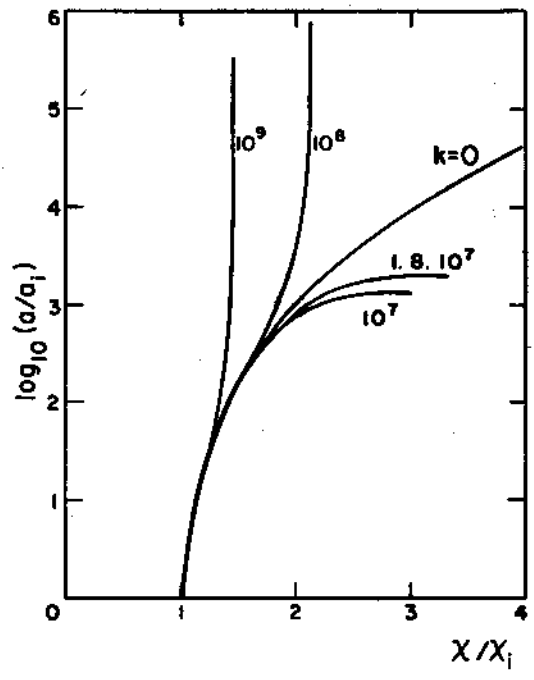

Fig.4: Effect of f_o. History of Σ for $(1+z)_i = 10^3$ and $(d\chi/d\eta)_i = 1 - 10^{-6}$ is shown for $\alpha = \varepsilon = 0$ and $\beta = 5$. Curves are labelled by f_o.

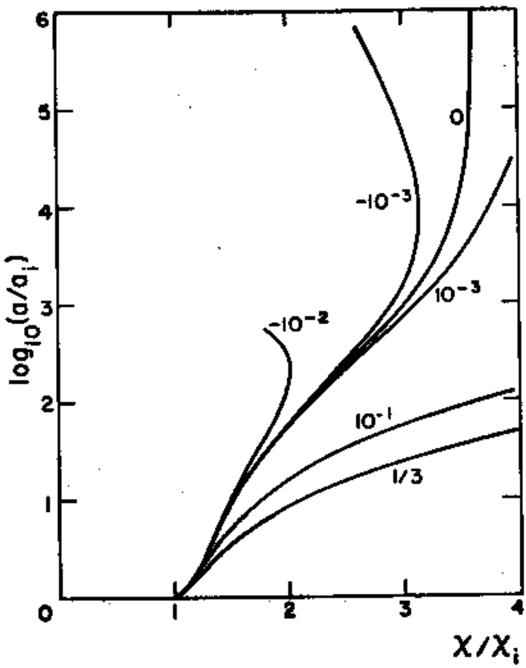

Fig.5: Effect of ε, $k = -1$. History of Σ for $(1+z)_i = 10^3$ and $(d\chi/d\eta)_i = 1 - 10^{-6}$ is shown for $\alpha = 0$, $\beta = 1$, and $f_o = 10^8$. Curves are labelled by ε.

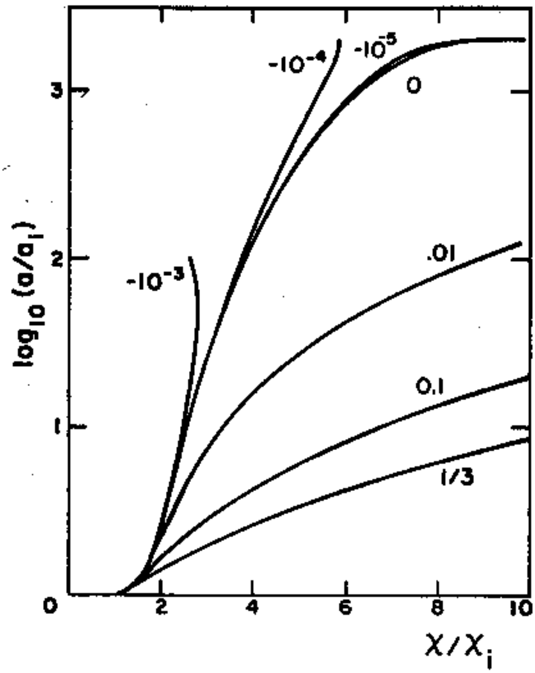

Fig.6: Effect of ε, $k = +1$. History of Σ for $(1+z)_i = 10^3$ and $(d\chi/d\eta)_i = 1 - 10^{-6}$ is shown for $\alpha = 0$, $\beta = 0.1$, and $f_o = 1.8 \times 10^7$. Curves are labelled by ε.

4.2 Radiation Voids

The use of vacuum in the void as described above is certainly idealized. One would like to know, for example, the effect of a radiation field filling the void. One might guess that the effective gravitational mass of the radiation field would slow the growth. However, one might also guess that the pressure associated with the radiation field would drive the expansion. To obtain a definitive answer the model of a void must be refined so as to include the interior radiation field, and the equation of motion must be reintegrated. In the model described below we use an isotropic pure (blackbody) radiation field interior to Σ, and, as above, a noninteracting mixture of dust and blackbody radiation exterior to Σ. In what follows I will only sketch the appropriate refinements to the vacuum model for $k_\pm = 0$ (see Pim and Lake 1986 for further details) and merely summarize our results for $k_\pm \neq 0$.

i) Basic equations for $k_\pm = 0$

In addition to the metric (4.1), the interior metric is given by

$$ds_-^2 = a_-^2(\psi)\{d\zeta^2 + \zeta^2 d\Omega^2 - d\psi^2\}, \tag{4.18}$$

so that with θ and ϕ continuous, $a_-(\psi)\zeta = a_+(\eta)\chi$. We continue with the ansatz $P = \epsilon\sigma$ with constant ϵ. In addition to equation (4.2) we have

$$T_{\alpha\beta}u^\alpha n^\beta \,|_\Sigma^- = a_-^2\dot{\psi}\dot{\zeta}(p_- + \rho_-) \tag{4.19}$$

where ρ_- and p_- are as given in equations (4.3) and (4.4) but replacing a_+ with a_-, and η with ψ. As a result, the conservation equation (2.25) now takes the form

$$\frac{dM}{d\eta} = \frac{R^2}{a_+\sqrt{1-(d\chi/d\eta)^2}}\left(b_+\frac{d\chi}{d\eta} - b_-\frac{d\psi}{d\eta}\frac{d\zeta}{d\eta}\right) - 2\epsilon M\left(\frac{1}{a_+}\frac{da_+}{d\eta} + \frac{1}{\chi}\frac{d\chi}{d\eta}\right), \tag{4.20}$$

where b is given by (4.11) (η replaced by ψ in V^-).

We are interested in the history of Σ in V^+. As a result, expressions are required for $d\psi/d\eta$ and $d\zeta/d\eta$. The latter is obtained from the relation $a_+\chi = a_-\zeta$ by differentiation. We find

$$\frac{d\zeta}{d\eta} = \left\{a_-\left(\frac{da_+}{d\eta}\chi + a_+\frac{d\chi}{d\eta}\right) - a_+\chi\left(\frac{da_-}{d\psi}\frac{d\psi}{d\eta}\right)\right\}/a_-^2\ . \tag{4.21}$$

As a result, if $d\psi/d\eta$ is known, we calculate $dM/d\eta$. Note that with an interior radiation field both m_+ and m_- are functions of the shell's history. In addition to relation (4.5) we have

$$m_- = \frac{4}{3}\pi R^3\rho_- \,|_\Sigma \tag{4.22}$$

Finally, to obtain $d\psi/d\eta$, it follows from the timelike condition

$$a_+^2\{(\frac{d\chi}{d\eta})^2 - 1\} = a_-^2\{(\frac{d\zeta}{d\eta})^2 - (\frac{d\psi}{d\eta})^2\}, \tag{4.23}$$

and equation (4.21), that

$$A(\frac{d\psi}{d\eta})^2 + B(\frac{d\psi}{d\eta}) + C = 0, \tag{4.24}$$

where

$$A = \frac{1}{a_-^2}\{-a_-^4 + a_+^2\chi^2(\frac{da_-}{d\psi})^2\}, \tag{4.25}$$

$$B = \frac{1}{a_-^2}(-2a_+^2 a_-\chi\frac{d\chi}{d\eta}\frac{da_-}{d\psi} - 2a_+a_-\chi^2\frac{da_+}{d\eta}\frac{da_-}{d\psi}), \tag{4.26}$$

and

$$C = \frac{1}{a_-^2}\{(a_+a_-\frac{d\chi}{d\eta})^2 + 2a_+a_-^2\chi\frac{d\chi}{d\eta}\frac{da_+}{d\eta} + a_-^2\chi^2(\frac{da_+}{d\eta})^2\} - a_+^2\{(\frac{d\chi}{d\eta})^2 - 1\}. \tag{4.27}$$

(It can be shown that in solving (4.24) the positive root gives a spurious history of Σ.)

The background V^+ is described by equation (4.12), and V^- by

$$a_-(\psi) = a_{0-}^2\sqrt{\nu_-}\psi \tag{4.28}$$

(a pure radiation field). Note that a_{0-} is, as before, removable for $k_- = 0$. For initial conditions we again use the ratios (4.6) (with m_- given by (4.22)) and (4.14). M_i again follows from (4.15). Note that given α, ψ_i follows from

$$\nu_-\psi_i^4 = (\frac{a_+^4}{\alpha(da_+/d\eta)^2})_i. \tag{4.29}$$

Whereas the value of ν_- depends, say, on the present temperature within the void, its actual value does not alter the exterior history of the void. (In particular, if ν_- is decreased by a constant multiple then ψ_i and $d\psi/d\eta$ decrease by the fourth root of that multiple. However, because of the form of a_-, the final equations describing the external history ($dM/d\eta, d\chi/d\eta$) are *unchanged*. If one is interested only in the external properties (i.e., χ, M) any value for ν_- can be used. Formally this corresponds to the transformation $\sqrt{\nu_-}\psi \to \hat{\psi}$.)

ii) Summary of integrations

a) Given an initial epoch $(1+z)_i$, void size β, and interior photon density α, there is a minimum initial $d\chi/d\eta$ below which the voids collapse. This minimum is lower than for vacuum voids. That is, radiation-filled voids grow much more readily. As before, however, there is a minimum $\beta(\beta_{min})$ below which Σ collapses. This is shown in figure 7.

b) Voids expand more readily by increasing the interior photon density (increasing α). The presence of radiation allows, in some circumstances, Σ to bounce. This is in contrast to the vacuum case where once χ begins to decrease, the void collapses. In contrast to figure 2, figure 8 shows the *minimum* α below which the voids collapse. (The limit $\alpha \to 0$ corresponds to a Minkowski vacuum void.)

c) For *any* surface pressure or tension, if the void does not collapse, the growth is always like the particle horizon at late times ($d\ln\chi/d\ln\eta \to 1$). It is possible for radiation voids to grow for $\varepsilon < 0$, but below $\varepsilon_{crit} < 0$ (which depends on the detailed initial conditions) the voids collapse. This is an interesting contrast to the vacuum void case.

iii) General Results for $k_\pm \neq 0$

With $k_\pm = -1$ and $\varepsilon = 0$ voids become comoving at late times. However, it can take many lifetimes of the Universe for this limiting behaviour to become apparent. Similarly if $\varepsilon > 0$, though the voids eventually expand like the particle horizon, if ε is small the growth can be virtually indistinguishable from the case $\varepsilon = 0$ for several ages of the Universe. With $\varepsilon < 0$ we find that the voids collapse, in contrast to the case $k_\pm = 0$, though this can again take a very long time. Also, our integrations indicate that $T_0^-/T_0^+ \lesssim 1$.

With $k_\pm = +1$, our present integrations indicate that voids do not grow unless $T_0^-/T_0^+ \geq 7$.

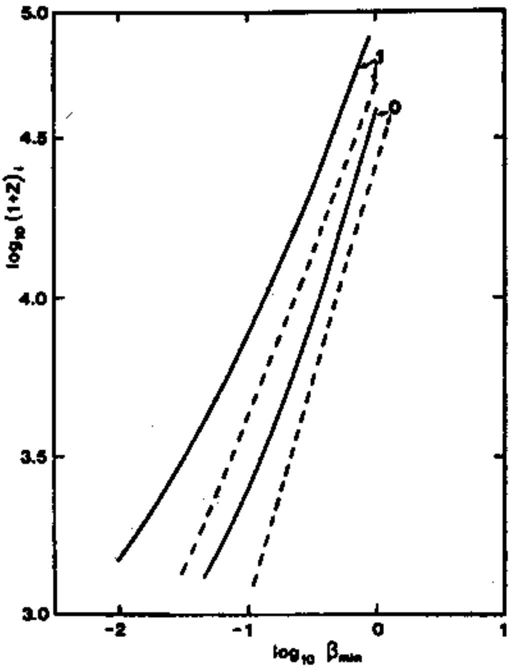

Fig. 7: The minimum β for a given epoch. The curves have $\epsilon = 0$. The abscissa gives $\log \beta_{min}$ where $\beta < \beta_{min}$ guarantees collapse of the void. The solid curves have $\alpha = 0.01$, the dashed curves have $\alpha = 0.001$. The curves are labelled by $(d\chi/d\eta)_i$. The curve labelled "1" has $(d\chi/d\eta)_i = 1 - 10^{-6}$.

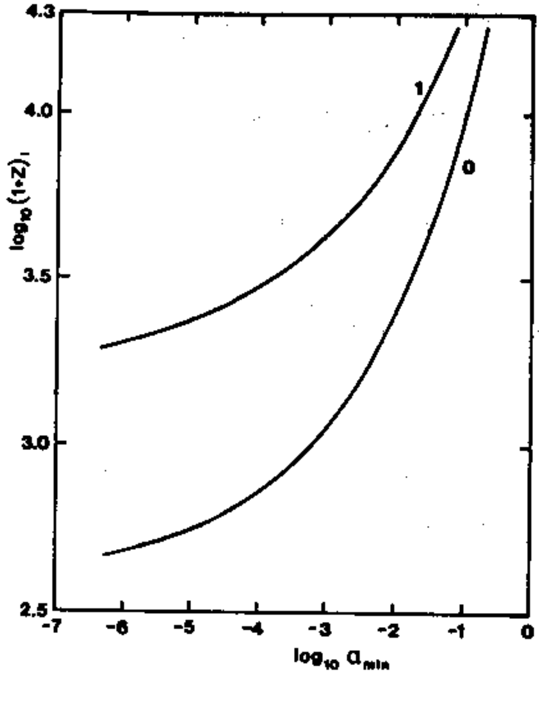

Fig. 8: The minimum α for a given epoch. The curves have $\varepsilon = 0$ and $\beta = 0.1$. The abscissa gives $\log_{10}\alpha_{min}$ where $\alpha < \alpha_{min}$ guarantees eventual collapse of the void. The curves are labelled by $(d\chi/d\eta)_i$. The curve labelled "1" has $(d\chi/d\eta)_i = 1 - 10^{-6}$.

Appendix A: Dimensionless Equations for Vacuum Voids

All quantities are evaluated at Σ.

For $k = 0$

$$\frac{d\chi}{d\eta} = \frac{(\Psi\{1+\Psi-(\frac{2(2\sqrt{\nu}+\mu\eta)\chi}{(4\sqrt{\nu}+\mu\eta)\eta})^2\})^{1/2} - \frac{2(2\sqrt{\nu}+\mu\eta)\chi}{(4\sqrt{\nu}+\mu\eta)\eta}}{(1+\Psi)}, \tag{A.1}$$

and

$$\frac{dM}{d\eta} = (\frac{16\nu+12\sqrt{\nu}\mu\eta+3\mu^2\eta^2}{2(4\sqrt{\nu}+\mu\nu)\eta)})\frac{\chi^2 d\chi/d\eta}{\sqrt{1-(d\chi/d\nu)^2}} - 2\varepsilon M(\frac{2(2\sqrt{\nu}+\mu\eta)}{(4\sqrt{\nu}+\mu\eta)\eta} + \frac{1}{\chi}\frac{d\chi}{d\eta}), \tag{A.2}$$

where

$$\Psi = (\frac{m-\alpha m_i}{M})^2 + 4(\frac{m+\alpha m_i}{(4\sqrt{\nu}+\mu\eta)\eta\chi}) + (\frac{2M}{(4\sqrt{\nu}+\mu\eta)\eta\chi})^2 - 1, \tag{A.3}$$

and

$$m = \frac{(2\sqrt{\nu}+\mu\eta)^2\chi^3}{2(4\sqrt{\nu}+\mu\eta)\eta}. \tag{A.4}$$

We have transformed $\eta \rightarrow a_0\eta$ and $\chi \rightarrow a_0\chi/c$. Note that M, m, χ and η have the dimensions of time.

For $k = \pm 1$

$$\frac{d\chi}{d\eta} = \frac{(\Psi\ (\cos(h)^2\chi + \Psi\ - A^2\sin(h)^2\chi))^{1/2} - A\sin(h)\chi\ \cos(h)\chi}{\cos(h)^2\chi+\Psi}, \tag{A.5}$$

with

$$A = \frac{2\sqrt{\nu}\ \cos(h)\eta + \mu a_0\ \sin(h)\eta}{2\sqrt{\nu}\ \sin(h)\eta + \mu a_0 K(1-\ \cos(h)\eta)}, \tag{A.6}$$

where $K \equiv k/\mid k\mid$ and

$$\frac{dM}{d\eta} = \frac{B\ \sin(h)^2\chi d\chi/d\eta}{\sqrt{1-(d\chi/d\eta)^2}} - 2\varepsilon M\ \{\ \cot(h)\chi\frac{d\chi}{d\eta} + A\}, \tag{A.7}$$

with

$$B = a_0^2(\frac{(2\sqrt{\nu}\ \cos(h)\eta + \mu a_0\ \sin(h)\eta)^2}{2\sqrt{\nu}\ \sin(h)\eta + \mu a_0 K(1-\ \cos(h)\eta)} + K\{2\sqrt{\nu}\ \sin(h)\eta + \mu a_0 K(\frac{1}{-} -\ \cos(h)\eta)\}) \tag{A.8}$$

where

$$\Psi = (\frac{m - \alpha m_i}{M})^2 + (\frac{m + \alpha m_i}{a \ \sin(h)\chi}) + (\frac{M}{2a \ \sin(h)\chi})^2 - 1, \tag{A.9}$$

and

$$m = \frac{1}{2} a \ \sin(h)^3 \chi (A^2 + K). \tag{A.10}$$

Here we have transformed $\chi \rightarrow \sqrt{|k|}\,\chi$, $\eta \rightarrow c\sqrt{|k|}\eta$, $\nu \rightarrow \nu/c^2 \mid k \mid$, $m \rightarrow \sqrt{|k|}\, m/c^2$, and $M \rightarrow \sqrt{|k|} M$ so that χ, η, ν, m and M are dimensionless. For equations (A.9) and (A.10) a is given as in equation (4.12), but in terms of the transformed quantities.

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5. The Transition from Minkowski space to Schwarzschild Spacetime

In this final section I would like to return to the study of boundary surfaces. The particular problem of interest here is the transition from Minkowski space to the Schwarzschild spacetime (and vice versa) by way of a radiation field approximated by the Vaidya metric. This transition involves a *null* boundary surface separating the Minkowski and Schwarzschild spacetimes. (Recent general treatments of junction conditions at null surfaces have been given by Redmount 1985, and by Clarke and Dray 1987.) The problem considered here, though highly idealized, gives rise to a number of interesting questions. For example, in the radiating counterpart to the familiar Oppenheimer-Snyder collapse, what is the "endstate"? Alternatively, if one transforms Minkowski space into Schwarzschild by way of an ingoing radiation field, how "strong" are the "shell-focusing" singularities which develop?

5.1 Boundary Surface Collapse to zero mass

I would like to consider the collapse of a radiating object whose history has been contrived to give zero mass at some event in its history. Exterior to the object I consider a high frequency (eikonal) approximation to a unidirectional radial flow of unpolarized radiation. The associated exterior metric is the Vaidya metric (e.g. Linquist, Schwartz and Misner 1965) which in single-null coordinates takes the form

$$ds_+^2 = 2cdrdw - (1 - \frac{2m(w)}{r})dw^2 + r^2 d\Omega^2, \tag{5.1}$$

where for $c = +1$, the radiation field is considered "ingoing" (m is monotone increasing, w the advanced time), and for $c = -1$ the field is "outgoing" (m monotone decreasing, w the retarded time). The associated energy-momentum tensor is

$$\left.\begin{aligned} &T_{\alpha\beta} = \Xi \delta^w_\alpha \delta^w_\beta = \Xi k_\alpha k_\beta, k_\alpha k^\alpha = 0 \\ &\text{where} \\ &\Xi \equiv \frac{c}{4\pi r^2}\frac{dm}{dw}. \end{aligned}\right\} \tag{5.2}$$

k_α is tangent to the trajectories $w = const$. In these notes I will consider the radiating counterpart to the familiar Oppenheimer-Synder collapse scenario (see Lake and Hellaby 1981). Interior to Σ we take a flat Robertson-Walker metric generated by a perfect fluid.

We start with the ansatz

$$p = \epsilon\rho \; , \; \epsilon = const. \tag{5.3}$$

The junction conditions (1.3) and (1.4) now give the exterior history of Σ as

$$r_\Sigma = \frac{-3\varepsilon(1+\varepsilon)}{(1+3\varepsilon)^2} w \tag{5.4}$$

with

$$m = -\frac{6\varepsilon^3(1+\varepsilon)w}{(1+3\varepsilon)^4}. \tag{5.5}$$

Note that this history is a special case. Writing the interior metric as

$$ds_-^2 = a^2(t)(d\mathscr{r}^2 + \mathscr{r}^2 d\Omega^2) - dt^2, \tag{5.6}$$

for the history (5.4) with (5.5) we have chosen the final condition

$$\mathscr{r}_\Sigma(t=0) = 0. \tag{5.7}$$

This evolution has been chosen to give $m = 0$ at a finite value of w (by translation, $w = 0$) since we are interested in a "naked" endstate. In particular, $\mathscr{r} = 0$ is singular at $w = 0$ (e.g. $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges along r_Σ) and naked, since $w = 0$ reaches "scri", for physically reasonable values of ε (e.g. $\varepsilon = 1/3$). A collapse history of this type was, apparently, first pointed out by Bondi (1964), and the first detailed example was given by Demianski and Lasota (1968) (see Steinmüller, King and Lasota 1975).

Unfortunately, as can be seen from equation (5.5), $m \to \infty$ as $w \to -\infty$, and so the initial conditions can not be considered "regular". This deficiency can be overcome by using the model of a mixture of noninteracting dust and blackbody radiation as in section 4. Then we obtain the exterior history

$$r_\Sigma = \frac{\sqrt{\nu}\eta^2(4\sqrt{\nu}+\mu\eta)}{6(2\sqrt{\nu}+\mu\eta)} \tag{5.8}$$

and

$$m = \frac{4\nu^{3/2}\eta^2}{27(4\sqrt{\nu}+\mu\eta)(2\sqrt{\nu}+\mu\eta)} \tag{5.9}$$

with

$$\left.\begin{aligned} w = &-\frac{(3\mu^2\eta^2 + 24\mu\sqrt{\nu}\eta + 28\nu)\eta}{36\mu} + \frac{2\nu^{3/2}\eta}{\mu(\mu\eta + 2\sqrt{\nu})} \\ &-\frac{4\nu^{3/2}}{\mu^2}\ln(\frac{\mu\eta}{2\sqrt{\nu}}+1) + \frac{128\nu^{3/2}}{27\mu^2}\ln(\frac{3\mu\eta}{8\sqrt{\nu}}+1). \end{aligned}\right\} \tag{5.10}$$

The constants ν and μ are as given in section 4. Note that $m_{init.} = 4\nu^{3/2}/27\mu^2$ (for $\eta \to \infty$). Moreover, near $w = 0$ the history is indistinguishable from the case $\varepsilon = 1/3$.

The essential features of the collapse scenario discussed above are not changed by the inclusion of a (constant) bulk viscosity (Lake 1982). One can conclude that there is a class of radiation collapse histories, with regular initial conditions, which give rise to a naked singularity. (It should be noted however, that alternate equations of state can lead to very different histories for Σ (see Zhang and Lake 1982, and Santos 1984).) Two questions to be answered are, how "singular" is the endstate, and how long does it last? The first question is answered in section 5.4 below. We now look at the second question.

It was first pointed out by Unruh (1985) that with $m \propto w$ there exist coordinates in which Minkowski space is a C^1 continuation of the Vaidya metric across the terminal null cone $w = 0$. The following demonstration of this continuation is given by Lake and Hellaby (1985). Write

$$m = \begin{cases} m_+, w > 0 \\ m_-, w < 0, \end{cases} \tag{5.11}$$

and for the outgoing ($c = -1$) metrics (5.1) let

$$r(u, w) \equiv \frac{u - w}{2} + \frac{2m(w)w}{u} \tag{5.12}$$

define u. Then

$$\left.\begin{aligned} ds^2 = (4\frac{m(w)w}{u^2} - 1)dudw + r^2 d\Omega^2 \\ -4(\frac{m'(w)w + m(w)}{u} - \frac{m(w)u}{u(u-w) + 4m(w)w})dw^2. \end{aligned}\right\} \tag{5.13}$$

It follows that the sole C^1 condition on $g_{\alpha\beta}$ at $w = 0$ is

$$(m) = 0. \tag{5.14}$$

Since conditions (1.1) and (1.2) are satisfied with (5.14), we conclude that with (5.14) $w = o$ is a *null boundary surface*.

Whereas Minkowski space is a C^1 continuation of the Vaidya metrics considered above, it is *not* the unique continuation since any metric with $m_+(0) = 0$ will do. On the null surface $w = 0$ we have the components

$$R_{w\theta w\theta} = 2m'(0), \tag{5.15}$$

and

$$G^{u}_{w} = -\frac{32m'(0)}{r^2(u,0)}. \qquad (5.16)$$

The first is proportional to the luminosity at spatial infinity, and the second is proportional to the energy flux observed on $w = 0$ by any timelike trajectory crossing it. With $m_+ = 0$ and $m_- \propto w$ then the metric is C^1 at $w = 0$ but the flux and luminosity are C^{0-}.

In conclusion we can say that the end state of radiating collapse to zero mass can be made *instantaneously* singular by demanding that $m = 0$ for $w > 0$. This demands, for example, that the flux be discontinuous at $w = 0$. Note that if the metric is C^2 at $w = 0$ then the endstate is not instantaneously singular and the flux is continuous. For the collapse scenarios considered here then one can phrase the cosmic censorship hypothesis as - why must the metrics be C^{2-} at $w = 0$?

I find the indeterminacy arrived at above somewhat unsatisfactory. How are we to determine that the metrics must be C^{2-}? One way out of this situation is summarized in Appendix A following an argument of Waugh and Lake (1986a). The importance of backscattered radiation (via a radial null test field) in the Vaidya metric near zero mass is examined and it is found that the backscattered radiation for an outgoing Vaidya metric becomes blueshifted without bound. This argues that the Vaidya metric *cannot* physically model the late stages of radiating collapse to zero mass. To my knowledge it is this instability argument which saves the cosmic censorship hypothesis for the examples given in this section.

5.2 Collapse of a pure radiation field

As a result of the foregoing, we are lead to consider the collapse of a pure radiation field. Kuroda (1984) and Papapetrou (1985) have considered the transition

$$m = \begin{cases} 0, w < 0 \\ \lambda w, 0 < w < w_0 \\ \lambda w_0, w > w_0, \end{cases} \qquad (5.17)$$

where the null surfaces $w = o$ and $w = w_0$ are boundary surfaces. Here λ is a constant, not to be confused with the affine parameter. (Though the metric looks C^{1-} at $w = 0$ and at $w = w_0$, as shown above, this is a coordinate effect. The metric is C^{2-} at $w = 0$ and at $w = w_0$.) Whereas the ingoing null geodesics satisfy $w = const.$, the outgoing null geodesics have

$$\frac{dr}{dw} = \frac{1}{2}(1 - \frac{2\lambda w}{r}) \qquad (5.18)$$

for $0 < w < w_0$. Equation (5.18) can be analyzed following standard techniques (e.g. Nemytskii and Stepanov 1960). One finds that for $0 < \lambda \leq 1/16$ the origin $r = w = 0$ is a nodal singularity. For $\lambda > 1/16$ it is a spiral point. For $0 < \lambda \leq 1/16$ then the transition (5.17) is accompanied by a naked singularity at the origin. The nature and strength of this singularity is examined in the following two sections. It should be noted that due to the nodal nature of the singular origin, the singularity is certainly *not* instantaneous.

The transition (5.17) is not entirely smooth in the sense that it is not C^2. The question to be answered then is, is it possible to construct a smooth (C^2) transition from Minkowski space to Schwarzschild space by way of an ingoing Vaidya field in a finite interval of advanced time? (Kuroda 1984 does give a source function $m(w)$ which grows like w^2 near $w = o$ and shows that $r = w = 0$ remains singular but visible from infinity. The resultant field, however, never settles down to the Schwarzschild solution and m grows without bound.) The following example of a smooth transition from Minkowski to Schwarzschild has been given previously (Lake 1986).

Consider a *particular* solution to

$$\frac{dr}{dw} = \frac{1}{2}(1 - \frac{2m(w)}{r}), \tag{5.19}$$

say $r = r(w)$. Along this geodesic introduce the (non-affine) parameter p where

$$dp \equiv \frac{dw}{r(w)}. \tag{5.20}$$

In terms of p, equation (5.19) reduces to

$$\frac{d^2w}{dp^2} - \frac{1}{2}\frac{dw}{dp} + m(p) = 0. \tag{5.21}$$

As a result,

$$r(p) = \frac{1}{2}e^{p/2}\{\beta - 2\int m(p)e^{-p/2}dp\} \tag{5.22}$$

with

$$w(p) = \alpha + e^{p/2}\{\beta - 2\int m(p)e^{-p/2}dp\} + 2\int m(p)dp, \tag{5.23}$$

where α and β are constants. Notice that a variation in β does *not* produce a congruence in a given background field $m(w)$, but rather a collection of different background fields! An exception is the Schwarzschild field where

$$\left.\begin{aligned} w(p) &= \alpha + \beta e^{p/2} + 2m(2+p) \\ &\text{and} \\ r(p) &= \frac{1}{2}\beta e^{p/2} + 2m, \end{aligned}\right\} \tag{5.24}$$

m constant. (This covers the ingoing Eddington-Finkelstein patch of the Kruskal-Szekeres diagram.)

Now suppose

$$m(p) = -\gamma\, p\, e^{p/2}, \tag{5.25}$$

where γ is a constant. Scaling by γ, the subsequent analysis here is dimensionless. (This is equivalent to setting $\gamma = 1$.) Set $\alpha = 0$ so that from equations (5.22) and (5.23) we have

$$w(p) = e^{p/2}(p^2 - 4p + 8 + \delta), \tag{5.26}$$

and

$$r(p) = \frac{1}{2}e^{p/2}(p^2 + \delta), \tag{5.27}$$

where δ is a (dimensionless) constant. (This is the history of the particular backscattered ray.) Note that $r > 2m$ for $p > -\infty$ with $\delta > 4$. Moreover,

$$\frac{dm}{dw} = -\frac{(p+2)}{(p^2+\delta)} \tag{5.28}$$

so

$$m = m' = w = 0 \ as \ p \to -\infty, \tag{5.29}$$

and

$$m = 2/e, m' = 0, w = (20+\delta)/e \quad for \quad p = -2. \tag{5.30}$$

In summary, the following gives a smooth (C^2, manifestly C^1) transition from Minkowski to Schwarzschild:

For $w \leq 0, m = 0$.

For $0 \leq w \leq (20+\delta)/e$; $m = -pe^{p/2}$, $p \leq -2$ and $r(p) = \frac{1}{2}e^{p/2}(p^2+\delta)$ along the particular ray (say $\mathcal{N}$).

For $w \geq (20+\delta)/e; m = 2/e, p \geq -2$ and $r = \frac{1}{2}(\delta - 4)e^{p/2} + 4/e$ along $\mathcal{N}$. Note that if $\delta > 4$, $\mathcal{N}$ propagates from $r = w = 0$ to "scri". If $\delta = 4$, $\mathcal{N}$ propagates from $r = w = 0$ to $2m$ and stays there. (This is the marginally naked case.) The origin is "singular" since $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ along $\mathcal{N} \sim e^{-2p}p^{-10}$ as $p \to -\infty$ (but see section 5.4 below). The singularity is persistent for $\delta > 4$ since $\mathcal{N}$ hits scri and not future timelike infinity.

5.3 Regular coordinates for the Vaidya metric

The coordinates used in the metric (5.1) do not give a complete picture of the spacetime. Consider, for example, the outgoing case $c = -1$. If $m = 0$ for finite w a nodal singularity can develop at the origin (see above). What, for example, is the character of this node? If $m \not\to 0$ it follows, by consideration of the backscattered rays, that the coordinates are incomplete (just as the Eddington-Finkelstein coordinates give an incomplete picture of the Schwarzschild spacetime).

More useful, but less intuitive, coordinates for the Vaidya metric have been given by Israel (1967). Define u by

$$dw \equiv \frac{cdu}{U(u)}, \tag{5.31}$$

such that

$$dU(u) = \frac{du}{4m(v)}, \tag{5.32}$$

and define t by

$$r \equiv 2m(u) + U(u)t. \tag{5.33}$$

Then, the metric (5.1) takes the form

$$ds^2 = 2dudt + (4m'(u)/U(u) + t^2/2mr)du^2 + r^2 d\Omega^2. \tag{5.34}$$

In Israel coordinates (u, θ, ϕ, t) the radial null geodesics are given by

$$\left.\begin{aligned} &u = const., \\ &\text{and} \\ &\frac{dt}{du} = -\left(\frac{2m'}{U} + \frac{t^2}{4m(2m+U(u)t)}\right). \end{aligned}\right\} \tag{5.35}$$

The behaviour of the second class must, in general, be obtained by numerical integration. (Moreover, if $m' \neq 0$, these trajectories consist of two distinct types (Pim and Lake

1985). Those that begin *and* end at $r = 0$, and those that hit $r = 0$ only once. For the first type it follows that r is maximal for $t = 0$ and $r_{max} = 2m(u)$.) These coordinates have seen too little application in the literature. (The only application of Israel coordinates for the Vaidya metric of which I am aware is the consideration of radiating shells by Pim and Lake 1985.)

Whereas the Israel coordinates work beautifully in Schwarzschild (see, for example, Israel 1966), they can break down in Vaidya. For example, if $m = \lambda w$, it follows that $U(u) = u^{1/(1+4\lambda)}$ and so g_{ww} is irregular at $m = 0$ for finite t. What is needed for Vaidya is a set of manifestly regular coordinates, and to this end the best choice is a set of double-null coordinates. A little experimentation with the metric (5.1) shows that a search for transformations to double-null coordinates is fruitless. In what follows I retreat to a consideration of Einstein's equations *ab initio* in double-null coordinates. (This has been done for Schwarzschild by Synge 1974.) Further details of this analysis are given by Waugh and Lake (1986). I will concentrate here on the characteristics of the spacetime for linear mass functions in order to unfold the nodes mentioned above.

The spherically symmetric metric in double-null coordinates is

$$ds^2 = -2f(u,v)dudv + r^2(u,v)d\Omega^2, \tag{5.36}$$

where the coordinates (u, v) are not assumed to be related to any previous coordinates used in these notes. The algebra associated with the metric (5.36) is reproduced in Appendix B. We take the associated energy momentum tensor to be of the form

$$T_{\alpha\beta} = \frac{h(u,v)}{8\pi}k_\alpha k_\beta \quad, k_\alpha k^\alpha = 0 \tag{5.37}$$

where $k^\alpha = (\dot{u}, 0, 0, 0)$ for flow along v-direction, and $k^\alpha = (0, 0, 0, \dot{v})$ along the u-direction, $\dot{} \equiv d/d\lambda$ for affine λ. (In what follows we consider, without loss of generality, a flow along the v-direction only.) From the form (5.37) the Einstein equations reduce to

$$\left.\begin{aligned} 2(f_1 r_1/r - r_{11})/r &= 0 \\ 2(r_1 r_4 + r r_{14})/f + 1 &= 0 \\ 2(f_4 r_4/f - r_{44})/r &= h(u,v) \\ (f_1 f_4/f - f_{14})/f - 2r_{14}/r &= 0, \end{aligned}\right\} \tag{5.38}$$

which in turn follow from

$$\left.\begin{aligned} f &= 2B(v)r_1 \\ r_4 &= -B(v)(1 - 2A(v)/r) \\ h &= -4B(v)A_4/r^2, \end{aligned}\right\} \tag{5.39}$$

where A and B are "arbitrary" functions of v. From the definition of $m(\frac{1}{2}r^3 R_{\theta\phi}{}^{\theta\phi})$ we have

$$m = \frac{r}{2} + r\frac{r_1 r_4}{f} \tag{5.40}$$

so that

$$A(v) = m. \tag{5.41}$$

There is no loss in generality in taking

$$B = -\frac{m_4}{2\mid m_4 \mid} \equiv -\frac{c}{2}, m_4 \neq 0. \tag{5.42}$$

This reduces v to the proper time in the rest frame at infinity. In summary, given $m(v)$ such that $m' \neq 0$, the solution to

$$\frac{\partial r}{\partial v} = \frac{c}{2}(1 - \frac{2m(v)}{r}) \tag{5.43}$$

gives the metric (5.36) via

$$f = -c\frac{\partial r}{\partial u}, \tag{5.44}$$

with the associated energy momentum tensor

$$T_{\alpha\beta} = \frac{c}{4\pi r^2}\frac{dm}{dv}\delta_\alpha^v \delta_\beta^v. \tag{5.45}$$

(Note that r is *not* a coordinate, it is $\sqrt{g_{\theta\theta}}$.) The Schwarzschild case $m_4 = 0$ is treated (once again!) in Appendix C. In what follows we consider

$$m = c\lambda v, \lambda = const. > 0. \tag{5.46}$$

With the form (5.46) let

$$g \equiv r/cv, \tag{5.47}$$

so that from equation (5.43)

$$\ln cv + \int \frac{g dg}{g^2 - g/2 + \lambda} = D(u), \tag{5.48}$$

where D is "an arbitrary" function of u. From equation (5.44) then

$$f = \frac{-cD_1(r^2 - \frac{1}{2}cvr + \lambda v^2)}{r}. \tag{5.49}$$

We must choose $D(u)$ so as to remove zeros in f. This done, $r(u,v)$ follows from equation (5.48). Three cases must now be considered separately:

i) $\lambda > 1/16$

With $\lambda > 1/16$, it follows from (5.49) that f is without zeros ($D_1 \neq 0$). If, for example, we take

$$D = -cu \tag{5.50}$$

then from (5.49) we have simply

$$f = \frac{r^2 - \frac{1}{2}cvr + \lambda v^2}{r}. \tag{5.51}$$

The function $r(u,v)$ follows implicitly from equations (5.48) and (5.50) and is given by

$$-cu = \frac{1}{2}\ln | r^2 - \frac{1}{2}cvr + \lambda v^2 | + \frac{1}{\delta}\arctan(\frac{4r - cv}{cv\delta}) \tag{5.52}$$

where $\delta \equiv \sqrt{16\lambda - 1}$. The spacetime diagram is shown in figure 1(a).

ii) $\lambda = 1/16$

We now have

$$f = -cD_1(4r - cv)^2, \tag{5.53}$$

where, from equation (5.48),

$$D(u) = \frac{cv}{cv - 4r} + \ln | \frac{4r - cv}{4} | \equiv L. \tag{5.54}$$

With the choice

$$D = c(\frac{1}{u} - u), \tag{5.55}$$

we have

$$f = \frac{(4r - cv)^2}{32r}\{L^2 + 4 + cL\sqrt{L^2 + 4}\} \tag{5.56}$$

which, on the horizon ($4r = cv > 0, u = 0$) reduces to

$$f(u = 0, v > 0) = \frac{cv}{4}. \tag{5.57}$$

The spacetime diagram is shown in figure 1(b).

iii) $\lambda < 1/16$

We now find

$$f = -\frac{cD_1(r - {}_0r/4)(r - {}_1r/4)}{r} \tag{5.58}$$

where

$${}_0r \equiv cv(1 - \Delta), \tag{5.59}$$

and

$${}_1r \equiv cv(1 + \Delta), \tag{5.60}$$

with $\Delta \equiv \sqrt{1 - 16\lambda}$. From equation (5.48) we now have

$$D(u) = \ln \mid (r - {}_0r/4)^{(\Delta - 1)/2\Delta}(r - {}_1r/4)^{(\Delta+1)/2\Delta} \mid . \tag{5.61}$$

There are two regions to be considered. With

$$D(u) = \frac{1 + \Delta}{2\Delta} \ln \mid cu \mid \tag{5.62}$$

and $cu > 0$ for ${}_1r/4 > r$ we find

$$f = \frac{1 + \Delta}{2\Delta r}(r - {}_0r/4)^{2/(1+\Delta)} \tag{5.63}$$

so that the choice (5.62) is useful for $r >_0 r/4$. With

$$D(u) = \frac{\Delta - 1}{2\Delta} \ln \mid cu \mid \tag{5.64}$$

and $cu > 0$ for ${}_0r/4 > r$ we find

$$f = \frac{1 - \Delta}{2\Delta r}({}_1r/4 - r)^{2/(1-\Delta)} \tag{5.65}$$

and so the choice (5.64) is useful for $r <_1 r/4$. This covers the complete spacetime. The diagram is shown in figure 2.

It is clear from figures 1(b) and 2 that the "node" $r = w = 0$ has a rather more detailed structure. It is what Eardley and Smarr (1979) have called a "shell focusing singularity". Hiscock, Williams and Eardley (1982) have also given double null coordinates for the Vaidya metric, but valid only for $\lambda > 1/16$. (They do, however, obtain the Penrose diagrams given above.)

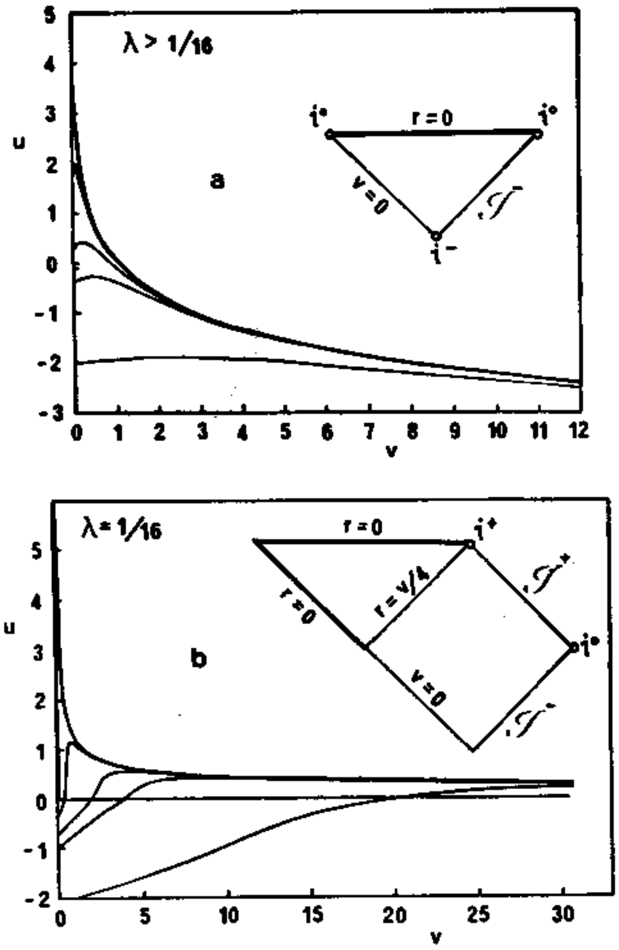

Fig. 1: (a) $u - v$ diagram for a linear mass function $m = c\lambda v$ with $\lambda > 1/16$ and $c = +1$ (ingoing field). The curves represent surfaces of constant $r(0, 0.1, 1/2, 1$ and $5)$. The future is the right and up. The Penrose diagram is inserted. Note that the outgoing case $(c = -1)$ is obtained by reflection about a horizontal axis. (b) As in (a) but for $\lambda = 1/16$. The u-axis $(u > 0)$ is a "shell focusing" singularity.

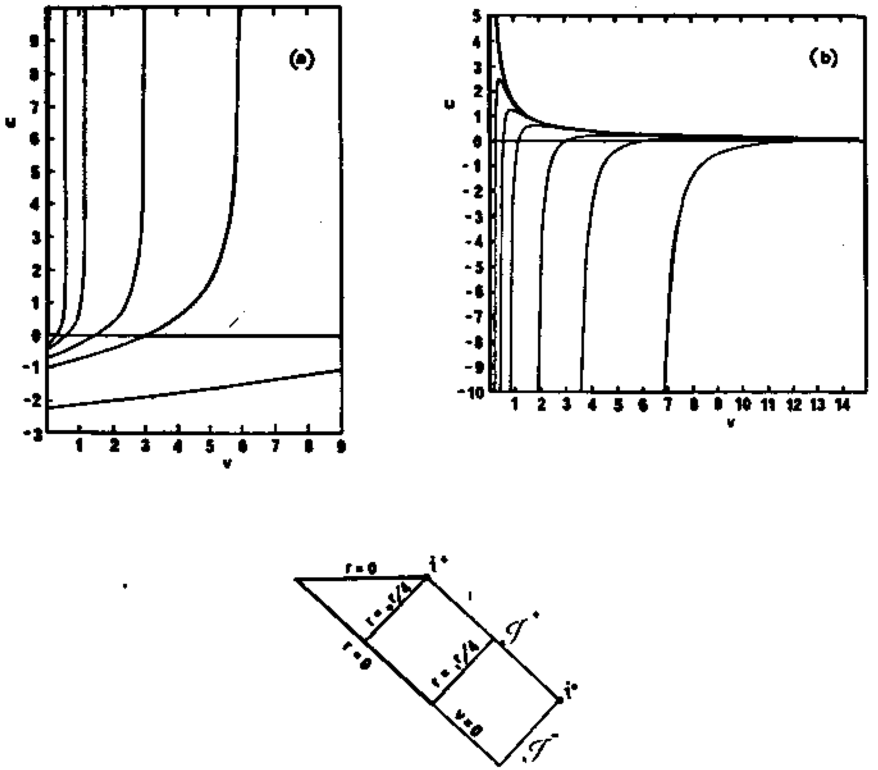

Fig. 2: $u - v$ diagram for a linear mass function $m = c\lambda u$ with $\lambda < 1/16$ and $c = +1$ (ingoing field). (The particular case shown has $\lambda = 1/18$.) (a) A portion of the spacetime obtained from (5.62) ($r >_0 r/4, r =_0 r/4$ at $u \to \infty$). The curves represent surfaces of constant r, where the values of r shown are (from top to bottom) $0.1, 0.2, 1/2, 1$, and 5. $r = 0$ is given by the positive u axis. (b) A portion of the spacetime obtained from (5.64) ($r <_1 r/4, r =_1 r/4$ at $u \to -\infty$). The curves represent surfaces of constant r, where the values of r are shown (from top to bottom) are $r = 0, 0.05, 0.1, 0.2, 1/2, 1$, and 2. The u axis also gives $r = 0$. In both diagrams the future is the right and up. The Penrose diagram is shown. Again, the outgoing case ($c = -1$) is obtained by reflection about a horizontal axis.

5.4 Strengths of singularities in the Vaidya spacetime

We now examine the "strength" of singularities in the Vaidya spacetime. The argument given here follows the work of Rajagopal and Lake (1987) which generalizes the analysis of Hollier (1986).

Along a null geodesic, affinely parametrized by λ, with 4-tangent ℓ^α let

$$\psi(\lambda) \equiv R_{\alpha\beta}\ell^\alpha \ell^\beta, \tag{5.66}$$

where $R_{\alpha\beta}$ is the Ricci tensor, and the geodesic terminates at $\lambda = 0$ (by choice of λ). Define the limiting focusing condition (LFC)

$$\lim_{\lambda \to 0} \lambda\psi > 0, \tag{5.67}$$

and the strong LFC (SLFC)

$$\lim_{\lambda \to 0} \lambda^2\psi > 0. \tag{5.68}$$

Clarke and Królak (1986) have shown that the SLFC is equivalent to termination in a strong curvature singularity in the sense of Tipler (see, e.g., Tipler, Clarke and Ellis 1980).

It is convenient to return to the single null coordinates of the metric (5.1). Whereas k^α is associated with the radiation field ($w = const.$), ℓ^α is tangent to the backscattered field which evolves according to equations (A.5) and (A.6). We are concerned here with the strength of the shell focusing singularity $r = m = w = 0$ at $\lambda = 0$ for an ingoing Vaidya field ($c = +1$). (We take $d\lambda > 0, \lambda \geq 0$.) Whereas $\psi = 0$ for k^α, we find

$$\psi = \frac{8}{\lambda^2}\frac{dm}{dw} \tag{5.69}$$

for the backscattered field. As a result, it follows that the SLFC holds *only* as long as

$$\frac{dm}{dw}\,|_{w=0}\, (\equiv \mu) > 0, \tag{5.70}$$

that is $m \sim \mu w$ as $w \to 0$. Moreover, SLFC holds for all geodesics which terminate at the node $r = w = 0$. (Hollier 1986 has shown that with $m = \mu w$, the Cauchy horizon satisfies the SLFC.)

Since $m \sim \mu w$ as $w \to 0$ is a strong condition to impose on $m(w)$, it is of interest to examine the (not strong) character of the node for a more general function. Suppose

$$m \sim \varepsilon w^n, n > 1 \tag{5.71}$$

as $w \to 0$, where $\epsilon = const. > 0$. Then, from the general form (5.69), we have

$$\psi \sim \frac{8n\epsilon w^{n-1}}{\lambda^2} \tag{5.72}$$

as $w \to 0$. Equation (5.72) shows that some details of the geodesic history are required (i.e. $w(\lambda)$ as $\lambda \to 0$).

Following standard techniques (e.g. Nemytskii and Stepanov 1960) it can be shown that equations (A.5) and (A.6) have the regular critical direction

$$w = 2r. \tag{5.72}$$

A single null geodesic (the Cauchy horizon) leaves a node tangent to this direction. Along the Cauchy horizon then we have

$$\psi \sim \frac{8n\epsilon}{\lambda^{3-n}} \tag{5.73}$$

as $w \to 0$ so that the past most point of a shell focusing singularity satisfies the LFC *only* for $1 < n \leq 2$. In contrast, for the remaining geodesics which reach the node we find that

$$\psi \sim \frac{2}{\epsilon(n-1)\lambda^2(-\ln\lambda)} \tag{5.74}$$

as $\lambda \to 0$. As a result, the LFC *is* satisfied along the remaineder of the shell focusing singularity.

The limiting form (5.71) is itself very restrictive since, as seen for example in section 5.2, m need *not* have an expansion near the node. However, for the form (5.25) we find that the limit (5.74) holds, but with $n = 1/\epsilon = 2$.

As a final remark, note that for the form (5.74) ψ grows faster than $1/\lambda^{\beta}$ for *all* $\beta < 2$. That is, the remainder of the singularity *just* fails to be "strong".

Appendix A: Backscattered Radiation in the Vaidya Metric near Zero Mass

The argument summarized in this Appendix follows that given by Waugh and Lake (1986). The Vaidya metric (5.1) consists of the radiation field $w = const.$, and the "backscattered" test field which satisfies

$$\frac{dr}{dw} = \frac{c}{2}(1 - 2\frac{m(w)}{r}). \tag{A.1}$$

We subject this "test" field to the following test: if, for a "generic" observer, $\nu_e >> \nu_0$ for the backscattered field, then the field is considered insignificant. If $\nu_e << \nu_0$, the "test" field is significant and ought to be included in $T_{\alpha\beta}$ (which it is *not* for the Vaidya metric). In what follows we consider backscattered trajectories along which $r > 2m(w)$. We choose w such that $r = m = 0$ at $w = 0$ and take $w < 0$ for $c = +1$ (ingoing field) and $w > 0$ for $c = -1$ (outgoing field).

Write $\chi = u^\alpha \ell_\alpha$, ℓ_α tangent to the backscattered rays. We find $\chi \propto \dot{w}$, $\equiv d/d\,\lambda$ for affine λ, where

$$\dot{w} = exp(-c\int \frac{m}{r^2}dw) = r\exp(-c\int \frac{dw}{2r}). \tag{A.2}$$

First suppose that $r(w) \sim \delta cw$ as $w \to 0$ along the ray (i.e. $m \sim \frac{1}{2}(1 - 2\delta)\delta cw, 0 < \delta < 1/2, \delta \equiv | r'(0) |$). Then

$$\dot{w} \sim (cw)^{-\epsilon}, \epsilon \equiv (1 - 2\delta)/2\delta \tag{A.3}$$

as $w \to 0$. As a result, for $c = -1$, $\chi_0 \sim | w |^{-\epsilon}$ as $w \to 0$ and so $\nu_e << \nu_0$. That is, the backscattered field is not ignorable near the endstate of collapse and so the Vaidya approximation is not physically reasonable. In contrast, for $c = +1$, (ingoing "real" field), $\chi_e \sim w^{-\epsilon}$ as $w \to 0$ so $\nu_e >> \nu_0$. As a result, the outgoing backscattered field is ignorable. These results agree with what one might intuitively expect.

For $r(w) \sim \delta(cw)^n$, $n > 1$, as $w \to 0$ (i.e. $n \sim \delta(cw)^n/2$, $\delta \equiv | r^n(0) | /n!$) we find

$$\dot{w} \sim \exp\{\frac{1}{2(n-1)\delta(cw)^{n-1}}\} \tag{A.4}$$

as $w \to 0$. Equation (A.4) leads to exactly the same conclusions as equation (A.3).

Clearly $m(w)$ need *not* have an expansion about $w = 0$ (see, e.g., section 5.2). A more general argument is, therefore, required. The null geodesic equations for the backscattered field can be reduced to the set

$$\frac{dr}{d\lambda} = \frac{r - 2m(w)}{\lambda}, \tag{A.5}$$

and

$$\frac{dw}{d\lambda} = \frac{2r}{c\lambda}, \tag{A.6}$$

for affine λ with $d\lambda > 0$, $c\lambda \geq 0$, and $w = r = m = 0$ at $\lambda = 0$. If follows from equation (A.5) that

$$\ln \mid \lambda \mid = \int_a^r \frac{dt}{t - 2m(t)} \tag{A.7}$$

where $r = a$ at $\mid \lambda \mid = 1$. From the weighted mean value theorem we have

$$\int_a^r \frac{dt}{t - 2m(t)} = \frac{1}{\psi} \ln(\frac{r}{a}) \tag{A.8}$$

where $\psi = 1 - 2m(\delta)/\delta$ and $r \leq \delta \leq a$ so that $0 \leq \psi < 1$. As a result

$$r = a \mid \lambda \mid^{\psi} \tag{A.9}$$

and so from equation (A.6)

$$\dot{w} = 2a \mid \lambda \mid^{\psi - 1} . \tag{A.10}$$

(Note that with $\psi = 0$, $r = a = 2m$, the Schwarzschild horizon.) Equation (A.10) leads to exactly the same conclusions as obtained above.

Appendix B: The algebra associated with a general spherically symmetric metric in double-null coordinates*

For the metric (5.36) the Christoffel symbols of the second kind are

$$\begin{aligned} \Gamma^1_{11} &= f_1/f \\ \Gamma^1_{22} = \sin^{-2}\theta\Gamma^1_{33} &= rr_4/f \\ \Gamma^2_{33} &= -\sin\theta\cos\theta \\ \Gamma^2_{12} = \Gamma^3_{13} &= r_1/r \\ \Gamma^2_{42} = \Gamma^3_{43} &= r_4/r \\ \Gamma^3_{23} &= \cos\theta/\sin\theta \\ \Gamma^4_{22} = \sin^{-2}\theta\Gamma^4_{33} &= rr_1/f \\ \Gamma^4_{44} &= f_4/f. \end{aligned} \tag{B.1}$$

The Riemann-Christoffel tensor as calculated from (B.1) is

$$\begin{aligned} R_{1212} = \sin^{-2}\theta R_{1313} &= r(f_1r_1/f - r_{11}) \\ R_{1224} = \sin^{-2}\theta R_{1334} &= rr_{14} \\ R_{1414} &= -f_{14} + f_1f_4/f \\ R_{2323} &= r^2\sin^2\theta(1 + 2r_1r_4/f) \\ R_{2424} = \sin^{-2}\theta R_{3434} &= r(f_4r_4/f - r_{44}). \end{aligned} \tag{B.2}$$

The Ricci tensor then reduces to

$$\begin{aligned} R_{11} &= 2(f_1r_1/f - r_{11})/r \\ R_{22} = \sin^{-2}\theta R_{33} &= 2(r_1r_4 + rr_{14})/f + 1 \\ R_{44} &= 2(f_4r_4/f - r_{44})/r \\ R_{14} &= (f_1f_4/f - f_{14})/f - 2r_{14}/r. \end{aligned} \tag{B.3}$$

From the components (B.3) we find that the Ricci scalar is given by

$$R = R^\alpha_\alpha = 2(\{f_{14} - f_1f_4/f\}/f^2 + \{1 + 2(r_1r_4 + 2rr_{14})/f\}/r^2) \tag{B.4}$$

* Only nonvanishing terms are given. The coordinates are ordered as (u, θ, ϕ, v).

and that

$$\begin{aligned} R^\beta_\alpha R^\alpha_\beta = 2(2f^2 r_{14} + r(f f_{14} - f_1 f_4))^2/r^2 f^6 \\ +2(f + 2r r_{14} + 2r_1 r_4)^2/r^4 f^2 \\ +8(f r_{11} - r_1 f_1)(f_{44} - r_4 f_4)/r^2 f^4. \end{aligned} \qquad (B.5)$$

From (B.3) and (B.4) it follows that the components of the Einstein tensor are

$$\begin{aligned} G_{11} = 2(f_1 r_1/f - r_{11})/r \\ G_{22} = \sin^{-2}\theta G_{33} = r^2(f_1 f_4/f - f_{14})/f^2 - 2r r_{14}/f \\ G_{44} = 2(f_4 r_4/f - r_{44})/r \\ G_{14} = (f + 2(r_1 r_4 + r r_{14}))/r^2. \end{aligned} \qquad (B.6)$$

The Weyl tensor is given by

$$\begin{aligned} C_{1224} = r^2(f_1 f_4 - f f_{14})/6f^2 - f/6 + (r r_{14} - r_1 r_4)/3 \\ C_{1334} = \sin^2\theta C_{1224} \\ C_{1414} = \frac{2f}{r^2} C_{1224} \\ C_{2323} = \frac{-2r^2}{f}\sin^2\theta C_{1224}. \end{aligned} \qquad (B.7)$$

As usual, the Weyl scalar is given by

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = K - 2R^\beta_\alpha R^\alpha_\beta + \frac{1}{3}R^2, \qquad (B.8)$$

where K is the Kretschman scalar ($R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$). For the metric (5.36) we find

$$\begin{aligned} K = 4(f + 2r_1 r_2)^2/r^4 f^2 + 16(f r_{11} - r_1 f_1)(f r_{44} - r_4 f_4)/r^2 f^4 \\ +16 r_{14}^2/r^2 f^2 + 4(f_1 f_4 - f f_{14})^2/f^6. \end{aligned} \qquad (B.9)$$

Appendix C: Yet another derivation of the Schwarzschild metric in null coordinates*

With $m_4 = 0$ we retain $B(v)$. The Einstein equations are

$$\frac{\partial r}{\partial v} = -B(v)(1 - \frac{2m}{r}), \tag{C.1}$$

$m = const.$, and

$$f = 2B(v)\frac{\partial r}{\partial u}. \tag{C.2}$$

Equation (C.1) gives

$$\int \frac{dr}{1 - 2m/r} = -\int B dv + C(v). \tag{C.3}$$

As a result,

$$f = 2BC_1(1 - \frac{2m}{r}). \tag{C.4}$$

Let

$$B = -\frac{2m}{v}, \tag{C.5}$$

and

$$C_1 = \frac{2m}{u}. \tag{C.6}$$

It then follows that

$$(1 - r/2m)e^{r/2m} = uv, \tag{C.7}$$

where $uv > 0$ for $r < 2m$, and

$$f = \frac{16m^3}{r}e^{-r/2m}. \tag{C.8}$$

* Compare Synge (1974).

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