STOCHASTIC METHODS IN COSMOLOGY

M.NOVELLO
Departamento de Relatividade e Partículas - DRP
Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150 - Rio de Janeiro, RJ (22290)
BRASIL

I MADELUNG-FLUID DESCRIPTION OF THE SCHÖDINGER FIELD

In the so-called hydrodynamical (or Madelung-fluid) description of quantum mechanics[1], a fluid density $\rho(\vec{x}, t)$ and a fluid velocity $\vec{v}(\vec{x}, t)$ defined by

$$\vec{v} = \frac{\nabla S}{m} \tag{1a}$$

where S correspond to the action of the system, are assumed to satisfy a law of conservation of probability (or matter),

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0 \tag{1b}$$

in such a way that the dynamics is expressed by

$$\frac{\partial S}{\partial t} + H_M = 0 \quad , \tag{1c}$$

with

$$H_M = H_{classical} + H_{diffusion} = \left(\frac{p^2}{2m} + V\right) - \frac{\hbar^2}{2m}\left[\frac{1}{\sqrt{\rho}}\left(\nabla^2 \sqrt{\rho}\right)\right] = H_o - \frac{\hbar^2}{2m}\left(e^{-R}\, \nabla^2\, e^{R}\right) \quad , \tag{2}$$

in which

$$\rho(\vec{x}, t) \equiv e^{2R(x,\, t)} \tag{3}$$

(Notes from a series of lectures concerning Stochastic Methods and some of their applications in Cosmology, prepared and revised by L.A.R.Oliveira, Research Assistant, DRP-CBPF).

The diffusion term in eq(2) is sometimes called a "ghost" part. Eq (1b) then implies

$$\frac{d\rho}{\partial t} + \mathrm{div}(\rho\vec{v}) = 2\dot{R}\, e^{2R} + 2e^{2R}\vec{v}\cdot\nabla R + \mathrm{div}\,\vec{v}\cdot e^{2R} = 0$$

or, using the definition eq(1a),

$$\dot{R} + \frac{\nabla R\cdot\nabla S}{m} + \frac{\nabla^2 S}{2m} = 0 \quad . \tag{4}$$

Let us now consider a complex function $\psi = \psi[\rho, S]$, defined in terms of ρ and S as

$$\psi(\vec{x}, t) = \sqrt{\rho(\vec{x}, t)}\ \exp\left[\frac{i}{\hbar} S\ (\vec{x}, t)\right] ; \tag{5}$$

then

$$\nabla\psi = \psi\ \nabla R + \frac{i}{\hbar}\ \psi\nabla S \tag{6a}$$

and also

$$\nabla^2\psi = \psi\left[\nabla^2 R + \frac{i}{\hbar}\nabla^2 S + (\nabla R)^2 + \frac{2i}{\hbar}\nabla R\cdot\nabla S - \frac{1}{\hbar^2}(\nabla S)^2\right] \tag{6b}$$

Hence, when we calculate the operation

$$(i\hbar\ \partial/\partial t - H_o)\psi \tag{7}$$

where $H_o = \left(-\frac{h^2}{2m}\nabla^2 + V\right)$ is the Hamiltonian of a particle of mass m submitted to a potential V, we therefore obtain

$$(i\hbar\frac{\partial}{\partial t} - H_o)\psi = i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - V\psi =$$

$$= i\hbar\left[\frac{\partial R}{\partial t} + \frac{1}{2m}\nabla^2 S + \frac{1}{m}\nabla R\cdot\nabla S\right]\psi + \tag{8}$$

$$+ \left[-\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}e^{-R}\nabla^2 e^R - \frac{1}{2m}(\nabla S)^2 - V\right] ,$$

since

$$\nabla^2 R + (\nabla R)^2 = e^{-R}\nabla^2 e^R$$

The first term in eq(8) vanishes due to eq(16b) or eq(4), while the second vanishes due to eq(1c). Thus, the complex function $\psi(\vec{x}, t)$ is shown to satisfy Schödinger's equation for a particle in a potential in result of the defining properties of the Madelung fluid.

II STOCHASTIZATION PROCEDURE

A. BASIC DEFINITIONS

Let M_n be a n-dimensional vector space and let $x_i(t)$ be an arbitrary point of M_n, $(i = 1, 2, ..., n)$, with t being a continuous parameter, $(0 \leq t < \infty)$.

A diffusion process $q_i(t)$ in M_n is defined by the following properties:

i) $q_i(t)$ is a Markovian process - this means that in the irregular, aleatory process $q_k(t_o) \equiv x_k(t_o)$, the variables $q_k(t)$ for any $t > t_o$ are independent of the behaviour of $q_k(s)$ for any $s \leq t_o$, that is, the process has no memory.

ii) there exists a density $\rho(\vec{x}, t)$ such that the expectation value $E[F]$ of any function $F(\vec{x}(t), t)$ on M_n is given by

$$E\,[F(\vec{x}(t), t] = < F(\vec{x}, t) >_t \equiv \int \rho(\vec{x}, t)\, F(\vec{x}, t)\, d^n x \quad . \tag{9}$$

iii) there are two functions $\vec{v}_{(\pm)}(\vec{x}, t)$, called drift velocities, and a constant ν (of dimensions $[L^2\, T^{-1}]$) such that for any function $F(\vec{x}, t)$, defined by the process $q_i(t)$, we can write

$$F(\vec{q}(t+\Delta t), t+\Delta t) = F(\vec{q}(t), t) + \frac{\partial F}{\partial t}\Delta t + (\vec{v}_{(+)} \cdot \nabla F)\Delta t +$$

$$+ \left[\vec{W}(t+\Delta t) - \vec{W}(t)\right] \cdot \nabla F + \frac{1}{2}\left[W^i(t+\Delta t) - W^i(t)\right] \cdot$$

$$\cdot \left[W^j(t+\Delta t) - W^j(t)\right] \partial_{ij}F + o(\Delta t) \ , \qquad (10a)$$

$$F(\vec{q}(t), t) = F(\vec{q}(t-\Delta t), t-\Delta t) + \frac{\partial F}{\partial t}\Delta t + (\vec{v}_{(-)} \cdot \nabla F)\Delta t +$$

$$+ \left[\vec{W}(t) - \vec{W}(t-\Delta t)\right] \cdot \nabla F - \frac{1}{2}\left[W^i(t) - W^i(t-\Delta t)\right] \cdot$$

$$\cdot \left[W^j(t) - W^j(t-\Delta t)\right] \partial_{ij}F + o(\Delta t) \ . \qquad (10b)$$

In eqs. (10a) and (10b) above, each stochastic variable $W^i(t)$ is called a <u>Wiener process</u> (which corresponds to the mathematical content of Brownian motion), satisfying

$$E\left[W^i(t+\Delta t) - W^i(t)\right] \equiv E\left[dW^i(t)\right] = 0 \ , \qquad (11a)$$

$$E\left[dW^i(t)\, dW^j(t)\right] \equiv 2\nu\, \delta^{ij}\, dt \ . \qquad (11b)$$

In general, the process sketched above, when applied to a particle of mass m which diffuses in a medium performing a Brownian motion, will impose $\nu \sim \frac{1}{m}$, so it follows that $\nu =$ = [action]/m . We will see later on that Nelson's interpretation of stochastic mechanics as a representation for quantum mechanics yields $\nu = \hbar/2m$.

Note also that the variance (sometimes called the co-variance), given by

$$\mathrm{cov}\ (X, Y) = E\left[(X - E[X])\ (Y - E[Y])\right], \tag{12}$$

if X and Y are two aleatory variables, in our case is

$$\mathrm{cov}\ (dW^i, dW^j) = E\ (dW^i\ dW^j)\ , \tag{13}$$

due to eq(11a). Remark that according to the (co)variance properties of Wiener processes, $\sigma(W^i) = \sqrt{\Delta t}$ - a feature which is at the basis of the (strange) properties of stochasticity.

The first dramatic consequence of the existence of a random-type motion $\vec{q}(t)$ is that we cannot deal with continuous curves which admit derivatives everywhere along the trajectory; we cannot define $\frac{d\vec{q}}{dt}(t)$. We thus have to face the question of generalizing the concept of derivative for a stochastic motion (observe that a similar problem on how to extend the notion of derivative to a curved spacetime occurs when we pass from the Lorentz group to MMG, in General Relativity).

In order to elaborate this extension, we start by conside ring that the basic principle to be used arises from a generalization of the Langevin equation to the stochastic regime:

$$d\vec{x}(t) = \vec{v}_{(+)}\ (\vec{x}(t),\ t)\ dt + d\vec{W}(t)\ , \tag{14}$$

in which the "differentials" $d\vec{x}$ are to be understood in the sense of finite differences $\vec{x}(t + \Delta t) - \vec{x}(t)$, and $\vec{W}(t)$ is a Wiener process. In this case, $E[d\vec{x}] = \vec{v}_{(+)}$.

Keeping these remarks in mind, we now introduce the concepts of mean forward and backward derivatives[2]:

$$(D_{(+)}F)\ (\vec{x},\ t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\Big[\Big(F(\vec{q}(t + \Delta t),\ t + \Delta t) - F(\vec{q}(t),\Delta t)\Big) \Big|_{\vec{q} = \vec{x}} \Big] \tag{15a}$$

$$(D_{(-)}F)\ (\vec{x},\ t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\Big[\Big(F(\vec{q}(t),\ t) - F(\vec{q}(t - \Delta t),\ t - \Delta t) \Big) \Big|_{\vec{q} = \vec{x}} \Big] \tag{15b}$$

In virtue of eqs(10a), (10b) and (14) we can write

$$(D_{(+)}F)(\vec{x},\ t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{\partial F}{\partial t} \Delta t + (\vec{v}_{(+)} \cdot \nabla F)\Delta t + \nu(\nabla^2 F)\Delta t\right]$$

$$(D_{(-)}F)(\vec{x},\ t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{\partial F}{\partial t} \Delta t + (\vec{v}_{(-)} \cdot \nabla F)\Delta t - \nu(\nabla^2 F)\Delta t\right] ,$$

and so we arrive at the final expressions for the derivatives, respectively

$$(D_{(+)}F)(\vec{x},\ t) = \frac{\partial F}{\partial t} + \vec{v}_{(+)} \cdot \nabla F + \nu\nabla^2 F \tag{16a}$$

$$(D_{(-)}F)(\vec{x},\ t) = \frac{\partial F}{\partial t} + \vec{v}_{(-)} \cdot \nabla F - \nu\nabla^2 F\ . \tag{16b}$$

C. THE FOKKER-PLANCK EQUATION

We shall leave to the reader the proof that, for any functions F and G, in the stochastic world holds the relation

$$\frac{d}{dt} E[F \cdot G] = E\left[F \cdot D_{(+)}G\right] + E\left[D_{(-)}F \cdot G\right] . \tag{17}$$

Setting G = 1, we obtain the special case

$$\frac{d}{dt} E[F] = E\left[D_{(-)}F\right] = E\left[D_{(+)}F\right] . \tag{18}$$

Exploiting eqs (16a) and (16b), we have

$$\frac{d}{dt} \int \rho F \, d^4x = \int \left(\frac{\partial \rho}{\partial t} \cdot F + \rho \frac{\partial F}{\partial t} \right) d^4x =$$

$$= \int \rho \left(\frac{\partial F}{\partial t} + \vec{v}_{(-)} \cdot \nabla F - \nu \nabla^2 F \right) d^4x \; ;$$

hence, if F is arbitrary, results

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}_{(-)}) + \nu \nabla^2 \rho = 0 \; , \tag{19}$$

and also

$$\frac{d}{dt} \int \rho F \, d^4x = \int \rho \left(\frac{\partial F}{\partial t} + \vec{v}_{(+)} \cdot \nabla F + \nu \nabla^2 F \right) d^4x$$

so that, analogously,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v}_{(+)} \right) - \nu \nabla^2 \rho = 0 \; . \tag{20}$$

Let us define now a <u>current velocity</u> $\vec{v}$,

$$\vec{v} \equiv \frac{\vec{v}_{(+)} + \vec{v}_{(-)}}{2} \tag{21a}$$

and an <u>osmotic velocity</u> $\delta\vec{v}$,

$$\delta\vec{v} \equiv \frac{\vec{v}_{(+)} - \vec{v}_{(-)}}{2} \tag{21b}$$

such that adding up eqs(19) and (20) we obtain the <u>Fokker-Planck equation</u>,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad , \tag{22}$$

while subtraction yields

$$- \nabla \cdot (\rho \delta \vec{v}) + \nu \, \nabla \cdot \nabla \rho = 0$$

wherefrom

$$\delta \vec{v} = \frac{\nu \nabla \rho}{\rho} \quad . \tag{23}$$

We are thus led to introduce the following two concepts of stochastic differentiation procedure : a so-called drift (or systematic) <u>derivative</u>, given by the operator

$$D \equiv \frac{D_{(+)} + D_{(-)}}{2} \quad , \tag{24a}$$

and a so-called <u>stochastic derivative</u>, given by

$$\delta D \equiv \frac{D_{(+)} - D_{(-)}}{2} \quad , \tag{24b}$$

which operating on an arbitrary function F yield, respectively,

$$DF = \frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F \quad , \tag{25a}$$

$$(\delta D) F = \delta \vec{v} \cdot \nabla F + \nu \nabla^2 F \quad . \tag{25b}$$

Some useful relations that we leave to the reader to show are

$$D\vec{q} = \vec{v} \tag{26a}$$

$$(\delta D) \, \vec{q} = \delta \vec{v} \tag{26b}$$

and

$$\frac{D_{(+)} D_{(-)} + D_{(-)} D_{(+)}}{2} = DD - (\delta D)(\delta D) \quad . \tag{26c}$$

III NELSON'S STOCHASTIC INTERPRETATION

Nelson[3] has proposed a stochastic interpretation of the particle description of the Madelung fluid relying on the following hypothesis:

Let $\vec{q}(t)$ be a stochastic process with probability density $\rho(\vec{x}, t)$ and drift velocities $\vec{v}_{(\pm)}$. Suppose that there exists a function $S_{(+)}(\vec{x}, t)$ such that

$$\vec{v}_{(+)} = \frac{\nabla S_{(+)}}{m} \quad ; \tag{27}$$

then, due to eq(23), we can write

$$\vec{v}_{(-)} = \vec{v}_{(+)} - 2\delta\vec{v} = \vec{v}_{(+)} - 2\,\nu \frac{\nabla\rho}{\rho} =$$

$$= \frac{1}{m} \nabla \left(S_{(+)} - 2m\,\nu \ln\rho \right) = \frac{1}{m} \nabla S_{(-)} \quad ,$$

so that we obtain, by analogy,

$$\vec{v}_{(-)} = \frac{\nabla S_{(-)}}{m} \quad , \tag{28}$$

where

$$S_{(-)} \equiv S_{(+)} - 2m\,\nu \ln\rho \ . \tag{29}$$

Introducing the functions

$$S \equiv \frac{S_{(+)} + S_{(-)}}{2} \tag{30a}$$

and

$$\delta S \equiv \frac{S_{(+)} - S_{(-)}}{2} \tag{30b}$$

we then conclude that

$$\vec{v} = \frac{\nabla S}{m} \tag{31a}$$

and

$$\delta\vec{v} = \frac{\nabla(\delta S)}{m} \tag{31b}$$

according to eqs(21a) and (21b).

We can now proceed to build a dynamics for the quantity $S(\vec{x}, t)$, introducing the so-called Nelson-Newton equation:

$$m\,\vec{a} = -\nabla U\ , \tag{32}$$

in which U stands for a potential and the acceleration $\vec{a}$ must be understood in the symmetrized form

$$\vec{a} = \left[\frac{D_{(+)}\,D_{(-)} + D_{(-)}\,D_{(+)}}{2}\right]\vec{q} = \left[DD - (\delta D)(\delta D)\right]\vec{q} \tag{33}$$

The origin of this symmetrization procedure rests on the fact that we are interested in obtaining a time-reversible theory — just as Schrödinger's is. We remark also that in general one should write

$$\vec{a} = \left[DD - \mu\,(\delta D)(\delta D)\right]\vec{q} \tag{34}$$

for a given linear combination coefficient μ, or else, which is the same,

$$\vec{a} = \frac{1}{4}\left\{(1+\mu)\left[D_{(+)}D_{(-)} + D_{(-)}D_{(+)}\right] + (1-\mu)\left[D_{(+)}D_{(-)} - D_{(-)}D_{(+)}\right]\right\}\vec{q}\ . \tag{35}$$

Nelson's choice, in order to provide for his stochastic approach to quantum mechanics, was to take $\mu = 1$. Another possible choice, which corresponds to a pure Brownian motion, is $\mu = -1$.

Due to eqs(26a) and (26b), eq(33) can be rewritten as

$$\vec{a} = D\vec{v} - (\delta D)\,\delta\vec{v}\ . \tag{36}$$

Well,

$$Dv^k = \frac{\partial v^k}{\partial t} + (\vec{v}\cdot\nabla)\,v^k \quad ,$$

and reminding the fact that $\nabla(\vec{u}\cdot\vec{u}) = 2(\vec{u}\cdot\nabla)\,\vec{u}$, we obtain

$$Dv^k = \frac{\partial v^k}{\partial t} + \frac{1}{2}\nabla(\vec{v}\cdot\vec{v}) = \frac{\partial v^k}{\partial t} + \frac{1}{2m^2}\nabla(\nabla S\cdot\nabla S)\,, \tag{37}$$

using eq(31a). On the other hand, according to eqs (30b) and (31b), we have

$$(\delta D)\ \delta\vec{v} = (\delta D)\left(\frac{\nabla(\delta S)}{m}\right) = (\delta D)\ \nabla(\nu\ln\rho) = 2\nu(\delta D)(\nabla R)\,, \tag{38}$$

where we have put $\rho = e^{2R}$, just as in eq(3). Using eq(25b) and eq(23), this is

$$(\delta D)\,\delta\vec{v} = 2\nu\Big[(\delta\vec{v}\cdot\nabla)\nabla R + \nu\,\nabla^2(\nabla R)\Big] =$$

$$= 2\nu^2\nabla\,(\nabla R\cdot\nabla R) + 2\nu^2\ \nabla(\nabla^2 R)\quad . \tag{39}$$

Then the acceleration $\vec{a}$ turns out to be

$$\vec{a} = \frac{1}{m}\nabla\left(\frac{\partial S}{\partial t}\right) + \frac{1}{2m^2}\nabla(\nabla S\cdot\nabla S) - 2\nu^2\nabla(\nabla R\cdot\nabla R) -$$

$$- 2\nu^2\ \nabla(\nabla^2 R)\ ;$$

since $\nabla^2 e^R = e^R\left[\nabla^2 R + \nabla R\cdot\nabla R\right]$, this is equal to

$$\vec{a} = \frac{1}{m}\nabla\left(\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - 2\nu^2 m\, e^{-R}\,(\nabla^2 e^R)\right)\,. \tag{40}$$

If now we make use of the Nelson-Newton equation eq(32), we obtain for the potential U the expression

$$U = -\frac{\partial S}{\partial t} - \frac{(\nabla S)^2}{2m} + 2\nu^2 m\, e^{-R}(\nabla^2 e^R)\ , \tag{41}$$

wherefrom

$$- \frac{\partial S}{\partial t} = \left(\frac{p^2}{2m} + U \right) - 2\nu^2 m \, e^{-R} \, (\nabla^2 e^R) \; ,$$

where the last term corresponds precisely to the additional term characteristic of the Madelung—fluid picture of quantum mechanics (see eqs (2) or (8)), provided that

$$\nu = \frac{\hbar}{2m} \; . \tag{42}$$

Following Nelson, we can then split the acceleration $\vec{a}$ into two terms, a <u>current acceleration</u>, typical of a system of particles with velocity field $\vec{v}$, and an <u>osmotic acceleration</u>, typical of hydrodynamics:

$$\begin{aligned} \vec{a} &= \vec{a}_{current} + \vec{a}_{osmotic} = \\ &= \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) v \right] - \left[\frac{1}{2} \nabla (\delta\vec{v} \cdot \delta\vec{v}) + 2\nu^2 \nabla (\nabla^2 R) \right] \; . \end{aligned} \tag{43}$$

Let g_{ij} be a (positive definite) Riemannian metric so that

$$ds^2 = g_{ij}\, dx^i\, dx^j \; (> 0) \tag{44}$$

We will define a stochastic process $x^k(t)$, where t is a parameter, on this Riemannian manifold as a generalization of the previous, Euclidean case (section II above), and depart from

$$dx^k(t) = v^k_{(+)}(x^k(t),\, t)\, dt + dW^k(t) \;, \tag{45}$$

which represents a n-dimensional generalization of the Langevin equation. In the same way, the definition of a Wiener process can be achieved by a natural extension[4]:

$$E\left[dW^i\, dW^j\right] = 2\nu\, g^{ij} dt \tag{46}$$

$$E\left[dW^i\right] = -2\nu\, \Gamma^i_{jk}\, g^{ik}\, dt \tag{47}$$

in which we now use the expectation

$$E[F] = \int \rho(\vec{x},\, t)\, F(\vec{x},\, t)\, \sqrt{g}\, d^n x \;. \tag{48}$$

Eq (47) is indeed the most immediate proposal for the extension of eq(11a) to the Riemannian regime; on the other hand, since now the transport of the components w^j of a given vector $\vec{w}$ is to be given by

$$dw^j = -\, \Gamma^j_{lk}\, dw^l\, dx^k \tag{49}$$

where Γ^j_{lk} are the affine connections of the manifold, we can see that in the case of a "pure" Wiener process $\vec{W}$, the use of eq(45) gives

$$E\left[dW^j\right] = E\left[-\Gamma^j_{\ell k} dW^\ell dx^k\right] =$$

$$= -\Gamma^j_{\ell k} \; E\left[dW^\ell dW^k\right] \quad , \tag{50}$$

so that from eq(47) follows, for consistency, eq(47b). It is straightforward to show that for an arbitrary scalar $F(\vec{x}, t)$ we have

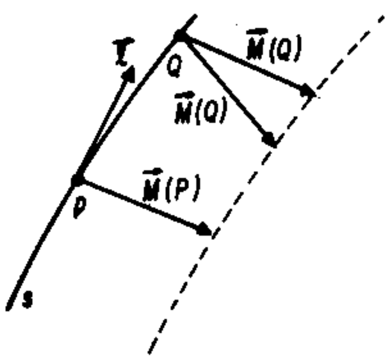
$$D_{(\pm)}F = \frac{\partial F}{\partial t} + v^i_{(\pm)} \nabla_i F \pm \nu\nabla^2 F \quad , \tag{51}$$

where now the Laplacian ∇^2 is to be given by

$$\nabla^2 F = \frac{1}{\sqrt{g}} \left(\sqrt{g}\; F_{,i}\; g^{ij}\right)_{,j} \quad , \tag{52}$$

with the notation $\partial_i F = F_{,i}$.

There are some problems, however, when we envisage to apply such definitions of the operators $D_{(\pm)}$ to objects like tensors. The reason is that for a given vector M^i, the term $M^i(\vec{x}, + \Delta\vec{x}) - M^i(\vec{x})$ is not conveniently defined, since it is not a tensor. This difference shall be evaluated on a given point; to do so, we need the concept of Lie transportation[5] (we follow here, with some minor distinctions, Guerra's solution[4]).



$\delta x^k = \ell^k \;\delta s$, s = curve parameter

Fig. 1

Define

$$D_{(+)}M^k = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\Big[-\tilde{M}^k(\vec{x} + \delta\vec{x}, t + \Delta t) + + M^k(\vec{x}, + \delta\vec{x}, t + \Delta t)\Big] \tag{53}$$

in which $\tilde{\vec{M}}$ has been Lie-transported from P to Q [see fig.1]. Now, from the theory of Lie transport it follows that

$$\tilde{M}^\ell(\lambda + \delta\lambda) = M^\ell(\lambda) + \frac{\delta M^\ell}{\delta S}\Big|_{\lambda=0}(\delta\lambda) + \frac{\delta^2 M^\ell}{\delta\lambda^2}\Big|_{\lambda=0}(\delta\lambda)^2 + \ldots =$$

$$= M^\ell(\lambda) + 0 + R^\ell_{ijk}\,(\ell^i\,\delta\lambda)(\ell^k\,\delta\lambda)\,M^j + \ldots =$$

$$= M^\ell(\lambda) + R^\ell_{ijk}\,\delta x^i\,\delta x^k\,M^j + \ldots \tag{54}$$

Hence

$$D_{(+)}M^\ell = E\Big[- M^\ell - R^\ell_{ijk}\,\delta x^i\,\delta x^k\,M^j + \frac{\partial M^\ell}{\partial t}\Delta t + \frac{\partial M^\ell}{\partial x^k}\delta x^k + + \frac{1}{2}\frac{\partial^2 M^\ell}{\partial x^i \partial x^k}\,\delta x^i\,\delta x^k + \ldots\Big] \tag{55}$$

and using eq(46) and eq(47) we obtain

$$D_{(+)}M^\ell = -R^\ell_{ijk}\,g^{ik}\,M^j + \frac{\partial M^\ell}{\partial t} + v^k_{(+)}\,M^\ell_{,k} + \nu\nabla^2 M^\ell \tag{56}$$

or in general

$$D_{(\pm)}M^\ell = \frac{\partial M^\ell}{\partial t} + v^k_{(\pm)}\,\nabla_k\,M^\ell \pm \nu\nabla^2 M^\ell \mp \nu R^\ell_k\,M^k\,, \tag{57}$$

in which ∇_k and ∇^2 are to be understood in the general covariance sense. Dynamics for (ρ, v) follows the same lines as above (Section III).

V STOCHASTIZATION OF THE ELECTROMAGNETIC FIELD

In 1979 Guerra and Loffredo[6] presented a simple and convincing way of treating Maxwell's equations for the electromagnetic field in a stochastic manner. Since we intent to use a similar approach to treat the case of the gravitational field, we shall review briefly their procedure.

On a three-dimensional basis, Maxwell's equations can be written as

$$\frac{\partial \vec{E}}{\partial t} = \text{rot}\ \vec{B} \quad , \quad \text{div}\ \vec{B} = 0 \tag{58a,b}$$

$$\frac{\partial \vec{B}}{\partial t} = -\ \text{rot}\ \vec{E} \quad , \quad \text{div}\ \vec{E} = 0\ . \tag{59a,b}$$

Or, in a covariant four-dimensional way, as

$$F^{\mu\nu}{}_{;\nu} = 0 \quad , \quad F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{60a,b}$$

where $A_\mu(x)$ is an electromagnetic four-potential. If we adopt the Lagrangian of this field to be

$$L = -\frac{1}{4} \int \sqrt{g}\ F_{\mu\nu}\ F^{\mu\nu}\ d^4x \quad , \tag{61}$$

then we obtain for the corresponding Hamiltonian the expression

$$H = \int (E^2 + B^2)\ d^4x \quad . \tag{62}$$

It will be convenient to decompose vectors $\vec{E}$ and $\vec{B}$ in a suitable basis $\left\{\vec{u}_n(\vec{x})\right\}$ of complete (real) vector functions, chosen so as to satisfy appropriately the following conditions of definition, normalization and completeness:

$$\mathrm{div}\ \vec{u}_n(\vec{x}) = 0 \tag{63a}$$

$$\nabla^2\vec{u}_n + \frac{1}{m_n}\vec{u}_n = 0 \tag{63b}$$

$$\int \vec{u}_n(\vec{x}) \cdot \vec{u}_m(\vec{x})\ d^3x = \delta_{nm} \tag{64}$$

$$\sum_n u_n^i(\vec{x})\ u_n^j(\vec{y}) = \delta^{ij}\ \delta^3(\vec{x} - \vec{y}) \quad . \tag{65}$$

Thus, we can set

$$\vec{B}(\vec{x}, t) = \sum_n \vec{u}_n(\vec{x})\ q_n(t) \quad , \tag{66a}$$

$$\vec{E}(\vec{x}, t) = - \sum_n \mathrm{rot}\ \vec{u}_n(\vec{x})\ m_n\ \dot{q}_n(t) \quad . \tag{66b}$$

Using such decomposition in the expression for H, we get

$$H = \sum_n \frac{1}{2} \frac{Pn^2}{m_n} + \frac{1}{2} q_n^2 \quad , \tag{67}$$

where $P_n \equiv m_n\ \dot{q}_n$, and where we first restrict our universe to a box and then let the dimensions of the box go to infinity. This system reduces to the dynamics of n uncoupled harmonic oscilators, each one obeying

$$\ddot{q}_n + \frac{1}{m_n} q_n = 0 \quad . \tag{68}$$

Now, we will let functions $q_n(t)$ be "stochastized", that is, to be such that each obeys a Langevin-like equation,

$$dq_n(t) = v_n^{(+)}(q, t)\ dt + dW_n(t) \quad , \tag{69}$$

in which $dW_n(t)$ is a Wiener variable (see eq(11) above), and also satisfies the equation for the "stochastic harmonic oscillator", that is, eq(68) expressed in the Nelson-Newton system, which reads

$$\frac{1}{2}\left[D_{(+)}\ v_n^{(-)} + D_{(-)}\ v_n^{(+)} \right] + \frac{1}{m_n} q_n = 0 \quad . \tag{70}$$

Then,

$$\vec{B} = (\vec{x},\ t) = \Sigma\ \vec{u}_n(\vec{x})\ q_n(t) \tag{71}$$

becomes a <u>stochastic variable</u>. Hence, it follows that

$$d\vec{B} = \Sigma\ \vec{u}_n(\vec{x})\ dq_n(t) = \Sigma\ \vec{u}_n(\vec{x}) \left(v_n^{(+)}\ dt + dW_n \right) , \tag{72}$$

and also that

$$D_{(\pm)}\ \vec{B} = \Sigma\ \vec{u}_n(\vec{x})\ v_n^{(\pm)} \ . \tag{73}$$

Let us define the "true" (i.e., not stochastic) variables

$$\vec{E}_{(\pm)}(\vec{x},\ t) \equiv - \Sigma\ m_n \text{ rot } \vec{u}_n(\vec{x})\ v_n^{(\pm)} \quad ; \tag{74}$$

then

$$\begin{aligned} \text{rot } \vec{E}_{(\pm)} &= - \Sigma\ m_n \text{ rot rot } \vec{u}_n(\vec{x})\ v_n^{(\pm)} = \\ &= \Sigma\ m_n\ \nabla^2\ \vec{u}_n(\vec{x})\ v_n^{(\pm)} = \Sigma\ m_n\ \left(\frac{1}{m_n} \right) \vec{u}_n(\vec{x})\ v_n^{(\pm)} = \\ &= - \Sigma\ \vec{u}_n(\vec{x})\ v_n^{(\pm)} \end{aligned} \tag{75}$$

(where we have used eq(63) and the vector relation

$$\text{rot rot } \vec{M} = \text{grad (div } \vec{M}) - \nabla^2\ \vec{M}\ ) \quad .$$

Using eq(75) into eq(73) we have that

$$D_{(\pm)}\ \vec{B} = - \text{ rot } \vec{E}_{(\pm)} \tag{76}$$

and thus

$$d\vec{B} = - \text{ rot } \vec{E}_{(+)} + d\vec{W} \quad , \tag{77}$$

where $dW = \Sigma\, \vec{u}_n(\vec{x})\, dW_n(t)$. Now,

$$D_{(+)}\vec{E}_{(-)} = - \Sigma\, m_n \,\mathrm{rot}\, \vec{u}_n(\vec{x})\, D_{(+)}\, v_n^{(-)} \tag{78a}$$

$$D_{(-)}\vec{E}_{(+)} = - \Sigma\, m_n \,\mathrm{rot}\, \vec{u}_n(\vec{x})\, D_{(-)}\, v_n^{(+)} \tag{78b}$$

so that using eq(70) there results in the Nelson-Newton scheme

$$\frac{1}{2}\left[D_{(+)}\vec{E}_{(-)} + D_{(-)}\vec{E}_{(+)}\right] = -\Sigma\, m_n \,\mathrm{rot}\, \vec{u}_n(\vec{x}) \,\cdot$$

$$\cdot\ \frac{1}{2}\left[D_{(+)}\, v_n^{(-)} + D_{(-)}\, v_n^{(+)}\right] = -\Sigma\, m_n \,\mathrm{rot}\, \vec{u}_n(\vec{x})\, (-\frac{1}{m_n})\, q_n =$$

$$= \Sigma\, \mathrm{rot}\, \vec{u}_n(\vec{x})\, q_n(t) = \mathrm{rot}\, \vec{B} \quad , \tag{79}$$

and therefore the equation for $\frac{\partial\vec{E}}{\partial t}$ in the Nelson-Newton approach results to be

$$\frac{1}{2}\left[D_{(+)}\,\vec{E}_{(-)} + D_{(-)}\,\vec{E}_{(+)}\right] = \mathrm{rot}\,\vec{B}\ . \tag{80}$$

Remark, however, that while $\vec{B}$ is a stochastic variable, $\vec{E}$ is not!

VI THE CASE OF THE GRAVITATIONAL FIELD

There are, in the literature, two formal ways (at least) to describe gravity:

a) Einstein's manner, by means of Einstein's equations

$$R^{\mu}_{\nu} - \frac{1}{2} R \, \delta^{\mu}_{\nu} = -k \, T^{\mu}_{\nu} \tag{81}$$

b) Jordan's manner, by means of an equation involving Weyl's conformal tensor $W^{\alpha\beta\mu\nu}$, plus initial conditions:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu} \, . \tag{82}$$

We shall consider here Jordan's way[7].

Let us define "electric" and "magnetic" components of Weyl's conformal tensor by

$$E_{\mu\nu} = -W_{\mu\alpha\nu\beta} \, V^{\alpha} \, V^{\beta} \tag{83a}$$

$$B_{\mu} = -W^{*}_{\mu\alpha\nu\beta} \, V^{\alpha} \, V^{\beta} \tag{83b}$$

where W^{*} is the dual of Weyl's tensor W .

Using this decomposition into eq(82) one can show[8] that "Maxwell-like" equations are obtained:

$$\frac{D}{Dt} \overleftrightarrow{E} = \mathrm{rot} \, \overleftrightarrow{B} \tag{84a}$$

$$\frac{D\overleftrightarrow{B}}{Dt} = - \, \mathrm{rot} \, \overleftrightarrow{E} \tag{84b}$$

$$\mathrm{div} \, \overleftrightarrow{B} = 0 \tag{84c}$$

$$\mathrm{div} \, \overleftrightarrow{E} = 0 \quad , \tag{84d}$$

NOTE: eqs(84) are only formal expressions, in which rot and div are operators which generalize in four-dimensional curved (Riemannian) space the usual $\vec{\nabla}$ operator in three dimensions, and so are to be understood in a covariant sense[9]. They are obtained from Bianchi identities for the (decomposed) conformal Weyl tensor W *via* projection operations; for instance, eq(84a) in full reads

$$h^{\varepsilon\alpha} h^{\lambda\delta} E_{\alpha\lambda;\delta} + \eta^{\varepsilon}_{\beta\mu\nu} V^{\beta} B^{\nu\lambda} \sigma^{\mu}_{\lambda} + 3B^{\varepsilon\nu} \omega_{\nu} = 0 \quad , \qquad (85)$$

in which V_{μ} is the four-velocity field associated to a perfect fluid, $h_{\mu\nu}$ is the corresponding projector, and where

$$\sigma_{\mu\nu} = V_{\mu;\nu} + V_{\nu;\mu} - \frac{1}{3} V^{\alpha}{}_{;\alpha} \; h_{\mu\nu} \qquad (86)$$

$$\omega_{\mu\nu} = V_{\mu;\nu} - V_{\nu;\mu} \quad , \quad \omega^{\tau} = \frac{1}{2} \eta^{\alpha\beta\rho\tau} \omega_{\alpha\beta} V_{\rho} \qquad (87)$$

$$\eta^{\alpha\beta\mu\nu} = - \frac{1}{\sqrt{-g}} \; \varepsilon^{\alpha\beta\mu\nu} \quad , \qquad (88)$$

$\varepsilon^{\alpha\beta\mu\nu}$ being the Levi-Civita symbol (which is not a true tensor; however, the dual object $\eta^{\alpha\beta\mu\nu}$ is!). For a complete review of these "*quasi-Maxwellian" equations of gravitation, along with* further comments, the reader is referred to ref.[9].

Eqs(84) have been shown by Lichnerowicz[10] to be completely equivalent to *Einstein's dynamics. One could argue, then,* that if this is the case one should be able to find a potential such that $\overleftrightarrow{E}$ and $\overleftrightarrow{B}$ could be reduced to derivatives of this potential. *Indeed, this has been done by* Lanczos[11]. Let $A_{\alpha\beta\mu}$ be a (true) tensor such that

$$A_{\alpha\beta\mu} = - A_{\beta\alpha\mu} \qquad (89a)$$

$$\overset{*}{A}_{\alpha\beta\mu} \, g^{\alpha\mu} = 0 \iff A_{\alpha\beta\mu} + A_{\beta\mu\alpha} + A_{\mu\alpha\beta} = 0 \; . \qquad (89b)$$

Then we can write, for any Riemannian space, that

$$W_{\alpha\beta\mu\nu} = -A_{\alpha\beta\mu;\nu} + A_{\alpha\beta\nu;\mu} - A_{\mu\nu\alpha;\beta} + A_{\mu\nu\beta;\alpha} -$$

$$- \frac{1}{2} (A_{\nu\alpha} + A_{\alpha\nu}) g_{\beta\mu} - \frac{1}{2} (A_{\beta\mu} + A_{\mu\beta}) g_{\alpha\nu} +$$

$$+ \frac{1}{2} (A_{\alpha\mu} + A_{\mu\alpha}) g_{\beta\nu} + \frac{1}{2} (A_{\beta\nu} + A_{\nu\beta}) g_{\alpha\mu} +$$

$$+ \frac{2}{3} A^{\sigma\lambda}{}_{\sigma;\lambda} \; (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) \quad , \tag{90}$$

where

$$A_{\mu\nu} \equiv A_{\mu}{}^{\alpha}{}_{\nu;\alpha} - A_{\nu}{}^{\alpha}{}_{\alpha;\mu} \quad . \tag{91}$$

Note that although the $A_{\alpha\beta\mu}$ should be given as functions of the geometry (i.e., of the metric $g_{\mu\nu}$), an explicit analytical relation is unknown (and cannot be, in general, obtained). In the case of "weak" field only, where

$$g_{\mu\nu} \sim \eta_{\mu\nu} + \varepsilon \, \psi_{\mu\nu} \qquad (\varepsilon^2 << \varepsilon) \; , \tag{92}$$

it has been possible[11] to find one such relation:

$$A_{\alpha\beta\mu} = \frac{1}{4} \left[\psi_{\alpha\mu,\beta} - \psi_{\mu\beta,\alpha} + \frac{1}{6} \psi_{,\alpha} \eta_{\mu\beta} - \frac{1}{6} \psi_{,\beta} \eta_{\mu\alpha} \right] , \tag{93}$$

expressed in the so-called Lanczos gauge in which

$$A_{\alpha\beta\mu} \, g^{\alpha\mu} = 0 \tag{94a}$$

$$A_{\alpha\beta}{}^{\lambda}{}_{;\lambda} = 0 \tag{94b}$$

(It is a direct consequence of eq(90) that there are <u>10</u> degrees of freedom to be fixed in $A_{\alpha\beta\mu}$, since $W_{\alpha\beta\mu\nu}$ has only <u>10</u> independent components and a general $A_{\alpha\beta\mu}$ has <u>20</u>; fixing the gauge, via eqs(94), we reduce this freedom to the required 10 independent conditions).

Let us now set forth the dynamics; we choose for the Langrangian density, as in Maxwell's theory, the expression

$$\mathcal{L}_o = \sqrt{-g}\ (E^2 - B^2) \quad , \tag{95}$$

where

$$E^2 \equiv E_{\mu\nu}\ E^{\mu\nu}\ ,\ B^2 = B_{\mu\nu}\ B^{\mu\nu}\ . \tag{96}$$

It is a straightforward exercise to show that Jordan's eqs. arise from this Langrangian. Then, we can find the corresponding Hamiltonian for the gravitational field:

$$\mathcal{H} = \sqrt{-g}\ (E^2 + H^2) \tag{97}$$

(We observe that there are extra terms in the general expression of $\mathcal{H}$, which conspurcate its apparent positivity and are due to the fact that the space is curved, so there are effects on the equation of motion of an arbitrary observer that moves with velocity V_μ; these can be used to describe the evolution of the gravitational field in terms of the quantities $E_{\mu\nu}$ and $B_{\mu\nu}$[12]).

Therefore, we are able to use a similar method to that chosen for the Maxwell field in order to find a stochastic realization of the gravitational field (a question left to the reader: should $B_{\mu\nu}$ be stochastic and $E_{\mu\nu}$ not, just as in Maxwell's case ?).

VII STOCHASTIC PROCESSES IN COSMOLOGY

There have been, up to now, very few applications of stochastic techniques in Cosmology. Among these, we can quote the works of Guinzburg et al.[13] and Novello[14]. These authors treat, in a phenomenological way, the geometry of spacetime as a stochastic process; the russians work in Einstein's formulation of gravitation and Novello, in Jordan's. The essential idea of both approaches is to take the metric $g_{\mu\nu}$ as a stochastic variable. However, this poses a lot of problems, which are circunvented in the phenomenological point of view adopted by the authors. Basically, they set

$$g_{\mu\nu} = < g_{\mu\nu} > + \delta g_{\mu\nu} \tag{98}$$

where < > denotes a mean value; this assumption implies[13] that Einstein's equations are slightly changed,

$$R^{\mu}{}_{\nu} - \frac{1}{2} R\delta^{\mu}{}_{\nu} = -kT^{\mu}{}_{\nu} + \Phi^{\mu}{}_{\nu} \quad , \tag{99}$$

in which $\Phi^{\mu}{}_{\nu}$ is some (complicate) functional of $\delta g_{\mu\nu}$. How should one proceed from here ? The trick is to develop $\Phi^{\mu}{}_{\nu}$ in terms of the mean metric $< g_{\mu\nu} >$, or in terms of powers of $< R^{\mu}{}_{\nu} >$. In Novello's treatment, for instance, this implies the use of a decomposition in which a "stochastic" current term $Q^{\alpha\beta\mu}$ is written as a functional of $< E_{\mu\nu} >$ and $< B_{\nu\mu} >$, thereby resulting

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu} + Q^{\alpha\beta\mu} \quad , \tag{100}$$

with

$$Q^{\alpha\beta\mu} \equiv \sum_k c_k P_k^{\alpha\beta\mu} \left[< E_{\mu\nu} > , < B_{\mu\nu} > \right] \quad (101)$$

One of the main consequences of these stochastic approaches was[13] to modify the behaviour of the gravitational field in the neighborhood of a singularity; another, to speed up the time evolution of (standard) inhomogeneous perturbations in the process of galaxy formation[15].

Here we intend to present another and very distinct stochastization procedure, based on two very recent papers by Gruszczak et al.[16] and Novello and Oliveira[17]. The first one deals with the stochastization of a given Friedmann geometry by allowing a stochastic behaviour for the equation of state $p = p(\rho)$. They set

$$p = \frac{1}{3} \rho + \frac{\beta}{R^2} \hat{n}(t) \quad , \quad (102)$$

in which $\hat{n}(t)$ is a "white noise" such that

$$< \hat{n}(t) > = 0 \quad , \quad (103a)$$

$$< \hat{n}(t), \hat{n}(t + \tau) > = \frac{1}{2} N_o \, \delta(\tau) \quad . \quad (103b)$$

The most important consequence of the introduction of such probabilistic perturbation is the removal of the inevitability of the appearance of singularities — the "radius" of the Universe extends indefinitely to infinity (even for closed models $\epsilon = 1$!). Let us now turn our attention to a more specific model[17]. Before entering the explicit calculations of probabilistic effects on given cosmological models, though, we shall point out some few remarks concerning the meaning of this enterprise.

Although one of the most basic tenets concerning the Universe, nowadays, be that of its unity — associated to the image of totalization which we are used to picture it with — recently there have been speculations involving more complex structures in which totalities like (e.g.) De Sitter cosmos are allowed to exist as separate, distinct entities. Some authors have in fact proposed to think of such <u>mini-cosmos</u> as nuclei of (elementary) particles. The main known effort in this direction is due to Markov[18], who analyzed the case of "friedmons" — mini-Friedmann-like universes.

Of course, in order to seriously consider one such idea, one should provide for a specific mechanism of interaction among these worlds. It seems that the simplest way to achieve this is to split these collections of mini-cosmos into individual systems, each one perceiving the remaining systems as a perturbative effect of random character — as in a stochastic process. Let us try to sketch now this procedure:

A De Sitter universe is characterized by a Riemannian line element described (in the Gaussian/co-moving system of coordinates) by

$$ds^2 = dt^2 - R^2(t)\ d\Omega^2 \quad , \tag{104}$$

where

$$d\Omega^2 = d\chi^2 + f(\chi)^2\ [\ d\theta^2 + \sin^2\theta\ d\psi^2] \ , \tag{105a}$$

$$f(\chi) = \sin\chi,\ \sinh\chi \text{ or } \chi \tag{105b}$$

Einstein's equations for the "radius" R(t) are, in the vacuum,

$$\frac{3\dot{R}^2}{R^2} + \frac{3\varepsilon}{R^2} = -\Lambda \tag{106a}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}-\frac{\epsilon}{R^2}=-\Lambda \qquad , \tag{106b}$$

($\epsilon = 0, \pm 1$). If $\epsilon = -1$ ("open" case) these equations reduce to the unique relation

$$\dot{R}^2+\frac{\Lambda}{3}R^2=1 \qquad , \tag{107}$$

which is nothing but the condition of "conservation of energy" for an harmonic oscillator. Indeed, from eq(107) we can easily obtain the tradidional equation

$$\ddot{R}+\frac{\Lambda}{3}R=0 \quad . \tag{108}$$

Setting $\underline{p = \dot{R}}$ for the associate unit-mass momentum, we have for the corresponding Hamiltonian H the expression

$$H=\frac{p^2}{2(m)}+\frac{(m)\omega^2}{2}q^2=\frac{p^2}{2}+\frac{\omega^2q^2}{2} \qquad , \tag{109}$$

with $R = q$, $\dot{R} = p$, $m = 1$, $\omega^2 = \frac{\Lambda}{3}$.

The general classical solution of eq(107) is

$$q_{c\ell}(t)=q_o\cos\omega t+\frac{p_o}{\omega}\sin\omega t \tag{110a}$$

$$p_{c\ell}(t)=p_o\cos\omega t-\omega q_o\sin\omega t \tag{110b}$$

with (q_o, p_o) being the initial data. The Schrödinger equation corresponding to this classical harmonic oscillator structure,

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2(m)}\nabla^2\psi+(m)\frac{\omega^2}{2}x^2\psi \qquad , \tag{111}$$

admits solutions of the form $\psi = \sqrt{\rho}\exp\left[\frac{i}{\hbar}S\right]$ such that

$$\rho=\frac{1}{\sqrt{2\pi}\,\sigma}\exp-\frac{1}{2}\left(\frac{x-x_{c\ell}}{\sigma}\right)^2 \qquad , \tag{112}$$

$$S=xp_{c\ell}-\frac{1}{2}p_{c\ell}x_{c\ell}-\frac{\hbar\omega t}{2} \qquad , \tag{113}$$

along with (Gaussian) mean value and variance given by

$$< x > = x_{c\ell} \tag{114}$$

$$\sigma^2 = (< x^2 > - < x >^2) = \frac{\hbar}{2(m)\omega} \tag{115}$$

Recalling the Madelung-fluid description of Quantum Mechanics (Section I) we can see that using the phase given by eq(113), eq. (1a) implies

$$v = \left| \frac{\nabla S}{(m)} \right| = \frac{p_{c\ell}}{(m)} \tag{116}$$

for the norm of the associate fluid velocity $\vec{v}$; also, using the density given by eq(112), the osmotic velocity $\delta\vec{v}$ (eqs(21)) turns out to have the norm

$$\delta v = \nu \left| \frac{\nabla\rho}{\rho} \right| = -\omega \, (x - < x >) \, , \tag{117}$$

with the usual diffusion coefficient $\nu = \hbar/2(m)$. Then Nelson's forward and backward drift velocities $v_{(\pm)}$ (eqs(27), (28)) are found to be

$$v_{(+)} = v + \delta v = \frac{p_{c\ell}}{(m)} - \omega \, (x - < x >) \quad , \tag{118a}$$

$$v_{(-)} = v - \delta v = \frac{p_{c\ell}}{(m)} + \omega \, (x - < x >) \quad , \tag{118b}$$

so the pertinent Langevin equation can be expressed as

$$dq(t, W) = v_{(+)}(q(t), t) \, dt + d\,W(t) =$$

$$= \left[\frac{p_{c\ell}}{(m)} - \omega \, (x - < x >) \right] dt + d\,W(t) \quad , \tag{119}$$

in which $W(t)$ is a <u>Wiener process</u> (see eqs(11)).

Therefore, for De Sitter case (eq.(107)), we obtain

$$dR(t, W) = \left[\dot{R}_{c\ell}(t) - \omega(R(t) - R_{c\ell}(t)) \right] dt + dW(t) \tag{120}$$

The manifestly Gaussian nature of the problem implies[2]

$$E[R(t, W)] = R_{c\ell}(t) \tag{121a}$$

$$E[R^2(t, W)] = R^2_{c\ell}(t) + \sigma^2 = R^2_{\ell}(t) + \frac{\sqrt{3}}{2}\,\frac{\hbar}{\sqrt{\Lambda}}\ , \tag{121b}$$

in which we made use of eq(103) and eq(115), and E[ ] stands for an expectation value procedure.

We can see from these expressions that the net effect of the "environment" is to preclude the colapse of the model: the classical (i.e., non-stochastic) singularity disappears, since the minimum of the average of the square radius $R^2(t)$ (which is the important quantity to be constructed and put into the line element ds^2 of the geometry) results proportional to $\Lambda^{-1/2}$. This means that if Λ is small, the radius will be great. Indeed, a rapid calculation (in units $m = \hbar = c = 1$) gives $\hbar/\sqrt{\Lambda} \sim 10^{-9}$cm, so that $R_{min} \sim 10^{-2}$cm, much greater than Planck's lenght (on this instance, see also the work of Trautman[19]). Remark, however, that we have used the present, very small, value of Λ, and also that we are able to fix arbitrarily the value of Λ for each bubble or minicosmos, thereby resulting many sorts of minimum radii.

VIII REFERENCES

[1] F.Madelung, Zs. f. Phys. 40 (1926), 322

[2] F.Guerra, Phys. Reps. 77, nº 3 (1981), 263-312

[3] E.Nelson, "Dynamical Theories of Brownian Motion" (Princeton University Press, Princeton, 1967)

[4] D.Dohrn and F.Guerra, Lett. N. Cim 22, nº 4 (1978), 121

[5] R.Gilmore, "Lie Groups, Lie Algebras and some of their Applications" (Wiley/Interscience, N.York, 1974)

[6] F.Guerra and M.I.Loffredo, Lett. N. Cim. 27 (1980), 41

[7] M.Novello and J.M.Salim, Fund.Cosm.Phys. 8 nº 3 (1983), 201, and references therein .

[8] G.F.R.Ellis, "Relativistic Cosmology", in "Proceedings of the International School of Physics Enrico Fermi, Course XLVII" (Academic Press, N.York, 1971), pag. 104

[9] J.M.Salim, PhD thesis, CBPF (1980)

[10] A.Lichnerowicz, "Théorie Relativiste de la Gravitation et de l'Electromagnetisme" (Masson, Paris, 1955)

[11] C.Lanczos, Rev.Mod.Phys. 34, nº 3 (1962), 379

[12] M.Novello, CBPF Preprint NF-054/83 (1983)

[13] V.L.Guinzburg, D.A.Kirzhnits and A.A. Lyubushin, JETP 33 (1941), 242

[14] M.Novello, Rev. Bras. Fis. 8 (1978), 442

[15] M.Novello, Lett.N.Cim. 24, nº 3 (1979), 72

[16] J.Gruszczak, M.Heller and M.Szydłowski, Phys.Letts.100A, nº 2 (1984), 82

[17] M.Novello and L.A.R.Oliveira, Phys.Letts. 109A, nº 9 (1985), 454.

[18] M.A.Markov, JETP 24, nº 3 (1967), 584

[19] A.Trautman, Bull. Acad. Polon. Sci.,Ser. Sci. Math.Astr. et Phys. 20 (1972) 185; 503; 895 .