INITIAL VALUE PROBLEM POSITIVITY OF ENERGY

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INTRODUCTION

We review in these lectures the foundations of the solution of the initial value problem in General Relativity and some recent advances.

In the first chapter we establish the 3+1 splitting of Einstein equations, and we explain how the "temporal gauge" (zero shift), together with an appropriate choice of lapse enables one to write, out of these equations, a well posed, causal, hyperbolic system to determine the evolution of initial data satisfying the constraints.

In the second chapter we show how the constraints can be solved through the conformal method, initiated by Lichnerowicz and developped by Choquet-Bruhat and York.

In the third chapter we review the recent proofs of positivity of the gravitational energy due to Schoen and Yau, and to Witten.

NOTATIONS

Space time M: 4-dimensional C^∞ manifold

Space time metric g or $g_{\mu\nu}$, signature $(-,+,+,+)$, $\mu,\nu = 0,1,2,3$

Space time covariant derivative ∇ or ∇_μ

$$\mathrm{Riem}(g): \nabla_\alpha \nabla_\beta u_\lambda - \nabla_\beta \nabla_\alpha u_\lambda = R_{\alpha\beta,\lambda}{}^{\mu}\, u_\mu$$

$$\mathrm{Ricc}(t): R_{\alpha\beta} = R_{\alpha\lambda,\beta}{}^{\lambda} = \partial_\lambda \Gamma_\alpha{}^\lambda{}_\beta - \partial_\alpha \Gamma_\beta{}^\lambda{}_\lambda + \Gamma_\alpha{}^\lambda{}_\beta \Gamma_\lambda{}^\mu{}_\mu - \Gamma_\alpha{}^\lambda{}_\mu \Gamma_\beta{}^\mu{}_\lambda$$

$$R(g) = R^\alpha_\alpha$$

$$\Gamma_\alpha{}^\lambda{}_\beta = \frac{1}{2} g^{\lambda\mu}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}) \quad , \quad \partial_\alpha = \frac{\partial}{\partial x^\alpha}$$

Field equations:

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

S: Einstein tensor $(S_{\mu\nu})$, T: stress energy tensor $(T_{\mu\nu})$

I - CAUCHY PROBLEM

1. 3+1 Formulation of the Einstein Equations

We consider a 4-dimensional C^∞ manifold M which has the topology $S \times \mathbb{R}$. We denote by $(x,t) \in S \times \mathbb{R}$ the points of M. We shall always take local coordinates adapted to the product structure, $x^i = 1,2,3$ coordinates on S and $x^0 = t$. We consider on M a pseudo riemannian metric g which induces a non degenerate metric $\overline{g}$ on each submanifold S_t ($x^0 = t$). In local coordinates we have

$$g_{ij} = \overline{g}_{ij} \quad , \quad \overline{g}^{ij} \text{ inverse matrix of } \overline{g}_{ij} \tag{1-1}$$

and the identity[1]

$$ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu = (g^{00})^{-1}(dx^0)^2 + \bar{g}_{ij}(dx^i+\beta^i dx^0)(dx^j+\beta^j dx^0) \quad (1\text{-}2)$$

where

$$\beta_i = g_{0i} \quad , \quad \beta^i = \bar{g}^{ij}\beta_i = -(g^{00})^{-1}\, g^{0i} \qquad (1\text{-}3)$$

We suppose, to specify the computation[2], that the S_t are space--like, that is $g^{00} > 0$ and we set

$$\alpha = (-g^{00})^{-1/2} \qquad (1\text{-}4)$$

α is (up to a sign) the projection of the vector tangent to the curve $\{x\} \times \mathbb{R}$ on the unit normal n to S_t: it is called the lapse: β, called the shift, is the projection of this vector on the tangent plane to S_t. We have if n is the unit normal to S_t and $\tau = \partial/\partial x^0$ the tangent to the time line

$$n_0 = -\alpha \ , \ n^0 = \alpha^{-1} \ , \ n_i = 0 \ , \ n^i = -\alpha^{-1}\beta^i \ , \ \tau.n = n_0 \ ,$$
$$\beta^\lambda = \tau^\lambda \ (g^\lambda_\alpha + n_\alpha n^\alpha)$$

The lapse and shift are thus linked with the choice of the transversal curves to the S_t's. The geometry of one "slice" S_t, as submanifold of (M,g), is characterized by its induced metric $\bar{g}$ and its "extrinsic curvature" K, symmetric two-tensor on S_t proportional to the projection on the tangent plane to S_t of the

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(1) consequence of $g_{00} = \dfrac{1 - g_{0i}g^{0i}}{g^{00}}$ and $g^{0i} = -g^{0i}\bar{g}^{ij}g_{0j}$

(2) The computation can be carried if $g^{00} \neq 0$, that is $g(\nu,\nu) \neq 0$ when ν is a normal to S_t, thus if S_t is not tangent to the isotropic cone of the metric. Note that $\alpha = 0$ (excluded by the hypothesis $g^{00} > 0$) means that the line $\{x\} \times \mathbb{R}$ is tangent to S_t, in contradiction with the hypothesis $V = S \times \mathbb{R}$.

Lie derivative of g with respect to a vector field n which coincides on S with its unit normal. The projection operator is given by a contraction with the 2-tensor

$$\pi^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} + n^{\alpha}n_{\beta} \quad , \quad g^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} \qquad (1\text{-}5)$$

and we have [3]

$$K = -\frac{1}{2}\,\pi \mathcal{L}n \quad , \quad \text{i.e.} \quad K_{ij} = -\frac{1}{2}(\nabla_i n_j + \nabla_j n_i) = -\alpha\,\Gamma^0_{ij} \qquad (1\text{-}6)$$

$$K_{i0} = -\alpha\, n^j K_{ij} \quad , \quad K_{00} = \alpha^2\, n^i n^j K_{ij} \qquad (1\text{-}7)$$

a simple computation gives that (1-6) is equivalent to ($\overline{\nabla}$ is the covariant derivative in the metric $\overline{g}$):

$$\frac{\partial \overline{g}}{\partial t} = -2\alpha K + \mathcal{L}_{\beta}(\overline{g}) \quad , \quad \text{i.e.} \quad \frac{\partial g_{ij}}{\partial t} = -2\alpha K_{ij} + \overline{\nabla}_i \beta_j + \overline{\nabla}_j \beta_i \quad (1\text{-}8)$$

The Einstein equations $S_{\lambda\mu} = T_{\lambda\mu}$ can be formulated in terms of $\overline{g}$, K, α and β. The computation is easy when the shift is zero: this hypothesis is no restriction on the space time, since the family of 3-manifolds S_t admit always orthogonal trajectories, tangent to the non vanishing vector field n. We shall also see that this choice of "gauge" is coherent with the proof of an existence existence theorem for solutions. Anyway we can also deduce from computations done with zero shift intrinsic relations (cf (1-12), (1-13)). When $\beta = 0$ the metric reduces to

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(3) K is a tensor on S, $K \in (\otimes TS)^2$

$$K^{i0} = K^{00} = 0$$

$$K_{ij} = g_{i\alpha} g_{j\beta} K^{\alpha\beta} = \overline{g}_{ih}\overline{g}_{jm}K^{hm} \quad .$$

$$ds^2 = - {}^2(dx^0)^2 + g_{ij}\, dx^i dx^j$$

and (1-8) to:

$$\frac{\partial \overline{g}}{\partial t} = - 2\, \alpha\, K \quad , \quad \text{i.e.} \quad \frac{\partial g_{ij}}{\partial t} = - 2\, \alpha\, K_{ij} \qquad (1\text{-}9)$$

while (overlined quantities are relative to $\overline{g}$)

$$\Gamma_i{}^k{}_j = \overline{\Gamma}_i{}^k{}_j$$

$$\Gamma_0{}^i{}_j = -\alpha K^i{}_j \quad , \quad \Gamma_i{}^0{}_j = -\frac{1}{\alpha} K_{ij}$$

$$\Gamma_0{}^0{}_i = \frac{1}{\alpha}\, \partial_i \alpha \quad , \quad \Gamma_0{}^i{}_0 = \alpha\, \partial^i \alpha \quad , \quad \Gamma_0{}^0{}_0 = \frac{1}{\alpha}\, \partial_0 \alpha \; .$$

We deduce from these formulas and the definition of Ricc(g), after some simplifications:

$$R_{ij} \equiv \overline{R}_{ij} - \alpha^{-1}\, \partial_0\, K_{ij} - 2\, K_{ih}\, K^h_j + K_{ij} K^h_h - \alpha^{-1}\, \overline{\nabla}_i \partial_j\, \alpha \, , \qquad (1\text{-}10_a)$$

$$R_{0i} \equiv \alpha(- \overline{\nabla}_h\, K_i{}^h + \partial_i\, K^h_h) \quad , \qquad (1\text{-}10_b)$$

$$R_{00} \equiv \alpha^2 (\alpha^{-1}\, \overline{\nabla}^i\, \partial_i\, \alpha + \alpha^{-1}\, \partial_0 K^h_h - K_i{}^j\, K_j{}^i) \quad , \qquad (1\text{-}10_c)$$

and using

$$\partial_0 g^{ij} = 2\, \alpha\, K^{ij}$$

the scalar curvature

$$R \equiv g^{00} R_{00} + g^{ij} R_{ij} \equiv R + (K^h_h)^2 + K_i{}^j K_j{}^i - 2\alpha^{-1}\, \partial_0 K^h_h - 2\alpha^{-1} \overline{\nabla}_i \partial^i\, \alpha$$

We deduce from these identities the following ones

$$S_{00} \equiv R_{00} - \frac{1}{2}\, g_{00}\, R \equiv \frac{\alpha^2}{2}\, (\overline{R} - K_i{}^j K_j{}^i + (K^h_h)^2) \qquad (1\text{-}11_a)$$

$$S_{0i} \equiv R_{0i} \equiv \alpha(- \bar{\nabla}_h K_i^{\ h} + \partial_i K_h^h) \qquad (1\text{-}11_b)$$

which depend only on $\bar{g}$ and K on S_t, and α.

We remark that $\alpha^{-2} S_{00}$ is a scalar function on S_t expression in the chosen "gauge" (zero shift) of the scalar

$$S_{\perp\perp} \equiv S_{\alpha\beta} n^\alpha n^\beta \quad , \quad n^0 = \alpha^{-1} \quad , \quad n^i = 0$$

which, in an arbitrary shift, can also be written

$$S_{\perp\perp} \equiv S^{\alpha\beta} n_\alpha n_\beta \equiv S^{00} \alpha^2$$

while $\alpha^{-1} S_{0i}$ are the components of a covariant vector $S_\perp$ on S_t, which can be written, for an arbitrary shift

$$(S_\perp)_i = n_\alpha S^\alpha{}_\beta \pi^\beta{}_\lambda$$

that is

$$(S_\perp)_i = - \alpha S_i^0$$

We have thus obtained a first set from the Einstein equations (1-10) called constraints because they contain no second derivatives of the unknown (remark also that the left hand sides depend only on $\bar{g}$ and K on S_t), which we write, in coordinate free notation:

$$\bar{R} - K.K + (\mathrm{tr}\ K)^2 = 2\rho \quad , \quad \rho = T_{\perp\perp} \qquad (1\text{-}12_a)$$

$$\bar{\nabla}.K - \bar{\nabla}\ \mathrm{tr}\ K = j \qquad j = -T_\perp \qquad (1\text{-}12_b)$$

The identity $(1\text{-}10_a)$ can also be written intrinsically with the help of the projection operator π on S:

$$\pi\,\mathrm{Ricc}(g) \equiv - \pi \mathcal{L}_n K + \mathrm{Ricc}(\bar{g}) - 2K.K + K\ \mathrm{tr}\ K - \overline{\mathrm{Hess}}(\alpha)$$

since we have:

$$(\pi \mathcal{L}_n K)_{ij} = \pi_i^\alpha \pi_j^\beta (\mathcal{L}_n K)_{\alpha\beta} = (\mathcal{L}_n K)_{ij}$$

$$= n^\lambda \partial_\lambda K_{ij} + K_{i\mu} \partial_j n^\mu + K_{j\mu} \partial_i n^\mu$$

thus, for a zero shift, in agreement with formula ($1\text{-}10_a$)

$$(\pi \mathcal{L}_n K)_{ij} = \alpha^{-1} \partial_0 K_{ij} \quad .$$

For a non zero shift we have (cf. (1-7))

$$K_{i0} = -\alpha n^j K_{ij} \quad , \quad n^j = -\alpha^{-1} \beta^j$$

thus

$$(\pi \mathcal{L}_n K)_{ij} = \alpha^{-1} \partial_0 K_{ij} + n^h \partial_h K_{ij} + K_{ih} \partial_j n^h + K_{nh} \partial_i n^h$$

$$- \alpha K_{ih} n^h \partial_j \alpha^{-1} - \alpha K_{jh} n^h \partial_i \alpha^{-1}$$

$$= \alpha^{-1} \partial_0 K_{ij} - \alpha^{-1} (\overline{\mathcal{L}}_\beta K)_{ij} \quad ,$$

where $\overline{\mathcal{L}}_\beta$ is the Lie derivative with respect to the vector field β on S_t. The "evolution" part of Einstein equations can therefore be written, in the general case

$$\frac{1}{\alpha} \frac{\partial K_{ij}}{\partial t} = \overline{R}_{ij} - 2K_{ih}K_j{}^h + K_{ij}K_h^h - \alpha^{-1} \overline{\nabla}_i \partial_j \alpha$$
$$- \alpha^{-1} (\overline{\mathcal{L}}_\beta K)_{ij} - (T_{ij} - \frac{1}{2} g_{ij} T) \quad . \qquad (1\text{-}13)$$

We see that the constraints (1-12) are relations between $\overline{g}$ and K and the sources, while (1-13) are equations which determine $\partial_0 K$ on S_t, when $\overline{g}$, K, and the lapse and shift are given on S_t. No equation determines the derivatives of these last auantities transversal to S_t: this non uniqueness of the solution corresponds to the invariance by diffeomorphisms of the equations.

If we give arbitrarily α and β on some domain $S \times I$, $I \subset \mathbb{R}$, together the source T and if we give $\overline{g}$ and K on S_0, $0 \in I$, the equations (1-13) determines $\partial_0 K$ on S_0 (we know already $\partial_0 \overline{g}$ from (1-8)), and all derivatives of K and $\overline{g}$ are obtained by successive derivations (if T initial data, α and β are C^∞): such a computation does not prove the existence of a solution (except eventually for analytic data) – moreover it must be checked that a solution of (1-13) on $S \times I$, with initial data satisfying the constraints (1-12) on S_0 satisfies the equations on $S \times I$ is not true for an arbitrary T: it can be proved using the Bianchi identities, in the analytic case with arbitrary lapse and shift, in vacuum[4] (T = 0). Non zero sources have to satisfy the so-called "conservation equations"

$$\nabla_\lambda T^{\lambda\mu} = 0 \quad .$$

2. Cauchy Problem

A solution of Einstein equation $G_{\lambda\mu} = T_{\lambda\mu}$ must satisfy, on each submanifold S_t the equations, called constraints

$$\overline{R} - K.K + \mathrm{tr}\, K = 2\rho \qquad (2\text{-}1_a)$$

$$\overline{\nabla}.K - \nabla\, \mathrm{tr}\, K = j \qquad (2\text{-}1_b)$$

(4) This result is already contained in the 1922 paper of Elie Cartan, which proved that the Einstein equations in vacuum, written as an exterior differential system, are in involution.

where $\rho = T_{\perp\perp}$ and $j = -T_{\perp}$ are respectively the densities of energy and momentum of the sources.

The problem we will adress ourselves in ths paragraph is: suppose these necessary conditions satisfied on a given 3-manifold S_0 by given $\overline{g}$ and K, are they sufficient for the existence of a space time (V,g), satisfying Einstein equations and admitting S_0 as an imbedded submanifold on which it induces the metric $\overline{g}$ and the extrinsic curvatura K ? We shall show that the answer is yes in vacuum, and extend the result to simple models for the sources.

To simplify the equations to solve we look for a space time $V = S \times I$, g with zero shift: the time lines $\{x\} \times t$ will be orthogonal to the S_t (this is no restriction and has some analogy with the "temporal gauge" of Yang-Mills theory).
The identity (1-8) becomes:

$$\frac{\partial \overline{g}_{ij}}{\partial t} = -2 \alpha K_{ij} \tag{2-2}$$

and the identity ($1\text{-}10_a$) gives, for a solution of vacuum Einstein equations

$$\frac{\partial K_{ij}}{\partial t} = \alpha(\overline{R}_{ij} + K_{ij}K^h_h - 2K_i{}^m K_{mj}) - \overline{\nabla}_i\overline{\nabla}_j\alpha \quad . \tag{2-3}$$

We can give arbitrarily α, the equations (2-1) and (2-2) determine then the derivatives $\frac{\partial \overline{g}}{\partial t}$ and $\frac{\partial K}{\partial t}$ on $S_0 = S \times \{0\}$ in terms of the initial data $\overline{g}$ and K on S_0) and, formally, all their derivatives by successive derivations of these equations (effectively if the initial data are C^∞, all derivatives which appear being again C^∞ and products defined). However such a computation does not give the existence of a solution, except eventually for analytic data, which is in any case very unsatisfactory since the analytic hypothesis is in contradiction with the finite propagation velocity that we expect for the gravitational relativistic field.

To make appear out of the equation (2-3) equations wich resemble some "wave" equation we shall combine them with the constraints which read here

$$\bar{R} - K.K + (\mathrm{tr}\, K)^2 = 0 \quad , \quad \text{i.e.} \quad \bar{R} - K_{ij}K^{ij} + (K^i_i)^2 = 0 \qquad (2\text{-}4_a)$$

$$\bar{\nabla}.K - \nabla\, \mathrm{tr}\, K = 0 \quad , \quad \bar{\nabla}_i K^{ij} - \bar{\nabla}^j K \qquad (2\text{-}4_b)$$

<u>Lemma 1</u> with the choice of lapse

$$\alpha = (\det \bar{g})^{1/2}\, a^{-1/2} \qquad (2\text{-}5)$$

(a is some given tensor density on S, independent of t) the system constituted of (2-2) and

$$\partial_0 R_{ij} - \alpha^2(\bar{\nabla}_i S_{j0} + \bar{\nabla}_j S_{i0}) = 0 \qquad (2\text{-}6)$$

is a quasilinear hyperbolic system for the unknown $\bar{g}$ (properly riemannian metric) and K.

<u>Proof</u>: From the definitions of the connexion and the Ricci tensor one deduces the formulas (Lichnerowicz 1961), setting $\partial_0 \bar{g}_{ij} = \bar{g}'_{ij}$,

$$\partial_0 \bar{\Gamma}^k_{ij} = \frac{1}{2}\, \bar{g}^{kh}\, (\bar{\nabla}_i \bar{g}'_{jh} + \bar{\nabla}_j \bar{g}'_{ih} - \bar{\nabla}_h \bar{g}'_{ij}) \qquad (2\text{-}7)$$

$$\partial_0 \bar{R}_{ij} = \frac{1}{2}\, \bar{g}^{kh}\, \{\bar{\nabla}_k(\bar{\nabla}_i \bar{g}'_{jh} + \bar{\nabla}_j \bar{g}'_{ih}) - \bar{\nabla}_i \bar{\nabla}_j \bar{g}'_{hk}\} \qquad (2\text{-}8)$$

using on the one hand the relation (2-2), on the other hand the Ricci identity we obtain:

$$\partial_0 \bar{R}_{ij} \equiv \bar{\nabla}^h \bar{\nabla}_h (\alpha K_{ij}) - \bar{\nabla}_i \bar{\nabla}_h (\alpha K^h_j) - \bar{\nabla}_j \bar{\nabla}_h (\alpha K^h_i) + \bar{\nabla}_i \bar{\nabla}_j (\alpha K^h_h)$$

$$+ \alpha \bar{R}_{ihj}{}^k K^h_k - \alpha \bar{R}_{hi} K^h_j - \alpha \bar{R}_{hj} K^h_i$$

that is

$$\partial_0 \bar{R}_{ij} \equiv \alpha [\bar{\nabla}_h \bar{\nabla}^h K_{ij} - \bar{\nabla}_i \bar{\nabla}_{(h} K_{j)}{}^h + \bar{\nabla}_i \bar{\nabla}_j K^h_h] + f_{ij} \qquad (2\text{-}9)$$

$$f_{ij} \equiv 2(\bar{\nabla}_h \alpha)(\nabla^h \alpha) K_{ij} + (\bar{\nabla}_h \nabla^h \alpha) K_{ij} - (\bar{\nabla}_i \alpha) \bar{\nabla}_h K_j{}^h - (\bar{\nabla}_h \alpha) \bar{\nabla}_i K^h_j -$$

$$(\bar{\nabla}_i \bar{\nabla}_h \alpha) K_j{}^h + (\bar{\nabla}_{(i} \alpha) \bar{\nabla}_{j)} K^h_h + (\bar{\nabla}_i \bar{\nabla}_j \alpha) K^h_h + \alpha (R_{ihj}{}^k K_k{}^h - R_{h(i} K_{j)}{}^h)$$

$$(2\text{-}10)$$

Inserting the identities ($1\text{-}11_b$) we obtain:

$$\partial_0 \bar{R}_{ij} \equiv \alpha \bar{\nabla}_h \bar{\nabla}^h K_{ij} + \alpha^2 \bar{\nabla}_{(i} S_{j)0} - \alpha \bar{\nabla}_i \bar{\nabla}_j K^h_h + f_{ij} \qquad (2\text{-}11)$$

We have therefore, using ($1\text{-}10_a$), setting $\alpha' = \partial_0 \alpha$, $K'_{ij} = \partial_0 K_{ij}$;

$$\Box \equiv \alpha^{-2} \partial_0^2 - \bar{\nabla}_h \bar{\nabla}^h$$

$$\partial_0 R_{ij} - \alpha^2 \bar{\nabla}_{(i} S_{j)0} \equiv -\alpha \Box K_{ij} + \frac{\alpha'}{\alpha^2} K'_{ij} - \alpha \bar{\nabla}_i \bar{\nabla}_j \operatorname{tr} K + f_{ij} - \qquad (2\text{-}12)$$

$$\frac{1}{\alpha} \nabla_i \nabla_j \alpha' + \frac{\alpha'}{\alpha^2} \bar{\nabla}_i \bar{\nabla}_j \alpha - \bar{\Gamma}^{k'}_{ij} \bar{\nabla}_k \alpha + K'_{ij} K^h_h +$$

$$K_{ij} K'^h_h - 2 K'_{im} K_j{}^m - 2 K_{im} K'^m_j$$

We see on (2-12) that $\partial_0 R_{ij} - \alpha^2 \bar{\nabla}_{(i} S_{j)0}$ contains no third derivatives of the $\bar{g}$'s. It will contain second derivatives of the K' only through the operator $\Box$, and no third derivatives of α, if we choose α such that

$$\alpha^2 \operatorname{tr} K + \alpha' = 0 \tag{2-13}$$

since we have

$$\alpha \overline{\nabla}_i \overline{\nabla}_j K_h^h = \alpha^{-1} \nabla_i \nabla_j (\alpha^2 K_h^h) + m_{ij}$$

with

$$m_{ij} \equiv -2(\overline{\nabla}_i \alpha)\overline{\nabla}_j K_h^h - 2(\overline{\nabla}_j \alpha)\overline{\nabla}_i K_h^h - 2\alpha^{-1}(\overline{\nabla}_i \alpha)\overline{\nabla}_j(\alpha K) - 2(\overline{\nabla}_i \overline{\nabla}_j \alpha)\ K_h^h$$

with the choice (2-13) the expression (2-12) reduces to

$$\partial_0 R_{ij} - \alpha^2 \overline{\nabla}_{(i} S_{j)0} \equiv -\alpha \Box K_{ij} + f_{ij} + n_{ij} \tag{2-14}$$

where f_{ij} is given by (2-10) and n_{ij} by:

$$n_{ij} = -m_{ij} + K_{ij} K'^h_h - K_h^h \overline{\nabla}_i \overline{\nabla}_j \alpha - \overline{\Gamma}^{k'}{}_{i\ j}\ \overline{\nabla}_k \alpha$$
$$- 2\ K'_{im} K_j{}^m - 2\ K_{im} K'^m_j \quad .$$

We remark that by (2-1) we have

$$\frac{(\det \overline{g})'}{\det \overline{g}} = -2\ \alpha\ K_i^i$$

the relations (2-13) is therefore

$$\frac{(\det \overline{g})'}{\det \overline{g}} = -2\ \alpha' = 0$$

and its general solution for the scalar α is

$$\alpha = (\det \overline{g})^{1/2}\ a^{-1/2} \tag{2-15}$$

with a an arbitrary, positive, scalar density on S (independant of x^0).

Remark The gauge condition (with zero shift)

$$\alpha' + \alpha^2 \operatorname{tr} K = 0$$

expresses that the submanifold S_t satisfy the harmonicity condution

$$\nabla^\lambda \nabla_\lambda x^0 \equiv g^{\lambda\mu}\Gamma^0_{\lambda\mu} = 0 \quad .$$

The identity (2-14) shows that, with the choice (2-15) for α and $\beta = 0$ the equations in vacuum $R_{\alpha\beta} = 0$ imply

$$\partial_0 R_{ij} - \alpha^2 \overline{\nabla}_{(i} S_{j)0} \equiv -\alpha \Box \ K_{ij} + f_{ij} + n_{ij} = 0 \qquad (2\text{-}16)$$

these equations together with the definition (2-2) of K are a third order system for g, with principal operator $\Box\ \partial_0$, a hyperbolic operator if $\alpha \neq 0$ and $\overline{g}$ is properly riemannian. The explicit expression of the lower order terms, given by (2-14), is irrelevant for the local existence theorem, but is important for numerical computations, and could probably be used for the study of global properties and the arising of singularities.

Lemma 2 Let $\overline{g}$ and K verify the hyperbolic system (2-2),(2-16), and α be given by (2-13), then Einstein tensor $S^{\alpha\beta}$ corresponding to the metria

$$-\alpha^2 (dx^0)^2 + \overline{g}_{ij}\, dx^i dx^j \quad , \quad \alpha = (\det \overline{g})^{1/2}\, a^{-1/2} \qquad (2\text{-}17)$$

verifies a linear, homogeneous hyperbolic system.

Proof: By the Bianchi identities we have

$$\nabla_\alpha S^{\alpha\beta} \equiv 0 \tag{2-18}$$

which can be written, modulo linear terms in $S^{\alpha\beta}$

$$\partial_0 S^{00} + \overline{\nabla}_i S^{i0} \approx 0 \tag{2-18$_a$}$$

$$\partial_0 S^{j0} + \overline{\nabla}_i S^{ij} \approx 0 \tag{2-18$_b$}$$

The equations (2-16) says that the metric (2-17) verifies the equations

$$\partial_0 R_{ij} - \alpha^2 (\overline{\nabla}_i S_{j0} + \overline{\nabla}_j S_{i0}) = 0 \quad .$$

We have, in the case of zero shift

$$R^{00} \equiv 2\, S^{00} - \frac{1}{\alpha^2}\, g_{hk}\, R^{hk}$$

thus

$$S^{ij} \equiv R^{ij} - g^{ij}(g_{hk}R^{hk} - \alpha^2 S^{00})$$

and the equations (2-16) imply, modulo linear terms in $S^{\alpha\beta}$

$$\partial_0 S^{ij} \approx -\alpha^2(\overline{\nabla}^i S^{j0} + \overline{\nabla}^j S^{i0}) + 2\,\alpha^2\, g^{ij}\, \overline{\nabla}_h\, S^{h0} + \alpha^2\, g^{ij}\, \partial_0\, S^{00}$$

from which we deduce, modulo linear terms in $S^{\alpha\beta}$, by the Bianchi identity (2-18$_b$)

$$\partial_0 S^{ij} \approx -\alpha^2(\overline{\nabla}^i S^{j0} + \overline{\nabla}^j S^{i0}) + \alpha^2\, g^{ij}\, \overline{\nabla}_h\, S^{h0} \quad . \tag{2-19}$$

From which we deduce, modulo linear terms in $S^{\alpha\beta}$, $\overline{\nabla}_i S^{\alpha\beta}$

$$\overline{\nabla}_i \partial_0 S^{ij} \approx -\alpha^2 \overline{\nabla}_i \overline{\nabla}^i S^{j0} \quad .$$

The Bianchi identities (2-18$_b$) imply therefore, modulo linear terms in $S^{\alpha\beta}$ and $\overline{\nabla}_i S^{\alpha\beta}$

$$\Box S^{j0} \approx 0 \quad . \tag{2-20}$$

The system $(2\text{-}18_a)$, (2-19), (2-20) is a linear homogeneous syste for the $S^{\alpha\beta}$ which can be shown to be hyperbolic by derivating equation (2-20) with respect to x^0. We then obtain a third order equation

$$\Box \partial_0 S^{j0} \approx 0$$

where the symbol ≈ 0 means modulo linear terms in $S^{\alpha\beta}$, their first derivatives and the second derivatives of only S^{j0} (we use $(2\text{-}18_a)$ and (2-19) to eliminate second derivatives of S^{00} and S^{ij}).

We can now give a general local existence theorem for the solution of the Cauchy problem under some regularity hypothesis on the data. In order to formulate these hypothesis in an intrinsic way we endow the manifold S (supposed C^∞) with a given properly riemannian C^∞ metric[(5)] e, for instante the euclidean metric if S is $\mathbb{R}^3$. We denote by ∂ or ∂_i the covariant derivative in this metric e. We say that tensor field on S is in the Sobolev space H_s if it is in the closure with respect to the H_s norm of the vector space of C^∞ tensor fields (of the same type) with compact support. The H_s norm of a tensor field f on S is

$$\|f\|_{H_s} = \left(\int \sum_{k=0}^{s} |\partial^k f|^2 \, dS\right)^{1/2}$$

$|\partial^k f|$ denotes the e-norm of the tensor field $\partial^k f$ at a point $x \in S$ and dS the e-volume element of S: if f is a 2-tensor $|f|^2 = f_{ij} f_{kh} e^{ik} e^{jh}$.

---

(5) the scalar density a in (2-15) can be determinant of e but not necessarily.

The existence theorem which follows is a consequence of the hyperbolicity proven in lemmas 1 and 2 and of the standard results on quasi linear hyperbolic systems (cf. J. Leray 1952, P. Dionne 1954) as improved (concerning regularity) in Hughes, Kato and Marsden 1976, Choquet-Bruhat, Christodoulou et Francaviglia 1978 (second order case), Choquet-Bruhat 1983 (third order case).

<u>Theorem</u> Let $(\bar{g},K)$ be two symmetric 2-tensors on a C^∞ riemannian manifold (S,e) such that

$\bar{g}$ is a continuous and bounded properly riemannian metric

and $\partial\bar{g} \in H_{s-1}$, $K \in H_{s-1}$, $s \geq 3$

$(\bar{g},K)$ satisfy the constraints .

Then there exists a space time (V,g), $V = S \times I$, $I \subset \mathbb{R}$ which admits $(S,\bar{g},K)$ as initial data and satisfies the vacuum Einstein equations.

We have formulated "globally in space" the Cauchy problem, and have obtained the existence of a solution on $S \times I$, $I \subset \mathbb{R}$.

By introducing weights in the Sobolev spaces of the data Christodoulou and O'Murchadha 1981 have been able to prove the existence of the solution in domains of $\mathbb{R}^4$ which allow for boosts of the initial surface $S = \mathbb{R}^3$ if these data are asymptotically euclidean in an appropriate way (cf section II).

Another kind, in a sense opposite, generalization valid for all hyperbolic systems and essential for causality is the "localisation" of the Cauchy problem. The spaces H_s^{loc} are defined as spaces of tensor fields on S such their restriction to each bounded open ball Ω of each coordinate chart belongs to $H_s(\Omega)$, closure of C^∞ tensor fields on $\bar{\Omega}$ with respect to the $H_s(\Omega)$ norm. The existence theorem is still valid if these spaces replace the spaces H_s in the hypothesis and in the conclusion the domain $S \times I$ is replaced by a neighborhood U of S in $S \times \mathbb{R}$. Moreover it can be proved that the solution at a point $x \in U$ depends only on the Cauchy data given on $S \cap \mathscr{E}(x)$ where $\mathscr{E}(x)$ denotes the past of x, determined by the isotropic cones of g: such a property is interpreted as a "finite propagation speed" of the gravitational field.

3. Cauchy Problem with Sources

In the presence of sources with stress energy tensor $T_{\alpha\beta}$ the Einstein equations are

$$S_{\alpha\beta} = T_{\alpha\beta} \tag{3-1}$$

they imply (cf Bianchi identities (2-18)) the "conservation laws" or "equations of motion" for the sources

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad . \tag{3-2}$$

Eventually (3-1) and (3-2) have to be completed by "constitutive equations", for instance an "equation of state" in the case of a perfect fluid. In that case we have (ρ and p proper energy density and pressure)

$$T_{\alpha\beta} = (\rho+p)\, u_\alpha u_\beta + p\, g_{\alpha\beta} \quad , \quad u^\alpha u_\alpha = -1 \quad . \tag{3-3}$$

The equation of state is the datum of a function

$$\rho = f(p)$$

To have a more realistic model one introduces also the density of proper entropy S. The equation of state is then

$$\rho = f(p,S) \tag{3-4}$$

and the equations (3-2) have to be completed by some thermodynamical law. In the absence of dissipative phenomena this law is usually taken to be ("adiabatic" flow)

$$u^\alpha \partial_\alpha S = 0 \tag{3-5}$$

which can be shown (Taub 1957, Lichnerowicz 1967) to be equivalent to the conservation of baryon number

$$\nabla_\alpha (r\, u^\alpha) = 0 \quad , \quad r = f^{-1}(\rho + p)$$

modulo the thermodynamic equation

$$TdS = df - r^{-1}\, dp \quad .$$

The equations of motion of a perfect fluid, deduced from (3-2) are

$$(\rho+p)\; u^\alpha \nabla_\alpha u_\beta + (g^\alpha{}_\beta + u^\alpha u_\beta)\partial_\alpha p = 0 \tag{3-6}$$

$$(\rho+p)\nabla_\alpha\; u^\alpha + u^\alpha \partial_\alpha \rho = 0 \quad . \tag{3-7}$$

The system of equations (3-5), (3-6), (3-7), with the relation (3-4), can be proved (Choquet-Bruhat 1957) to be hyperbolic with outer characteristic inside the isotropic cone of the metric g if the equation of state is such that [6]

(6) If $\frac{\partial \rho}{\partial p} = 1$ the fluid is said to be "incompressible", the acoustic waves have the same propagation velocity as gravitation. The system is only hyperbolic in a generalized sense, the existence theorem is valid in a more restricted class of functions.

$$\frac{\partial \rho}{\partial p} < 1 \quad .$$

This system coupled with the Einstein equations (3-1) can also be proved to be causal, by using the gauge of §2 (cf a proof using harmonic coordinates in Choquet-Bruhat 1962).

II - Solution of the Constraints

1. Introduction

We have seen in §1 that – at least for simple models of sources – to every solution of the constraints on an initial 3-manifold S:

$$R(\overline{g}) - K.K + (\mathrm{tr}\ \underline{K})^2 = 2\rho \tag{1.1}$$

$$\overline{\nabla}.K - \overline{\nabla}\ \mathrm{tr}\ \underline{K} = j \tag{1.2}$$

corresponds a space time (M,g), essentially unique. A fundamental problem is therefore to construct gravitational initial data on S-metric $\overline{g}$ and 2-tensor K – and non gravitational data – scalar $\rho = T_{\perp\perp}$ (proper energy density – and vector j (proper momentum density) – that satisfy the four equations (1-1) and (1-2). The unknown are largely redundant (they are still twelve, when are j are given), but the equations are non linear and only global solutions on S are maeningful. Moreover we know that the equations (1-1) and (1-2) have different meaning: (1-2) is essentially linked with the invariance of the theory by isometries of S, while (1-1), called the hamiltonian constraint, has a deeper meaning, still not well understood, but which seems to contain the essential of the dynamics of the Einstein theory. The conformal method, which we describe in this section, provides a decoupling of the two sets of constraints, (1-1) and (1-2), and writes (1-1) as a (non linear) elliptic partial differential on S for one scalar function.

2. The Conformal Method

It was remarked by Lichnerowicz (1944) that if we set

$$\overline{g} = \phi^4 \gamma \quad , \quad \text{i.e.} \quad \overline{g}_{ij} = \phi^4 \gamma_{ij} \quad , \qquad (2\text{-}1)$$

where γ is a given riemannian metric on S and ϕ some positive scalar function we have the identity

$$R(\overline{g}) \equiv \phi^{-5}(R(\gamma) - 8\Delta_\gamma \phi) \qquad (2\text{-}2)$$

where Δ_γ is the scalar laplacian in the metric γ . He proved moreover that if $\mathrm{tr}\, K = 0$ and if we define A by:

$$A^{ij} = \phi^{10} K^{ij} \qquad (2\text{-}3)$$

the momentum constraint can be expressed as a linear system on A which does not contain ϕ, when γ is known. The proof goes as follows.

The connections $\Gamma(\gamma)$ and $\Gamma(\overline{g})$ of the two conformally related metrics γ and $\overline{g}$ are found to be related by

$$(\overline{g}) - \Gamma(\gamma) = C \qquad (2\text{-}4)$$

with the tensor C given by

$$C_j{}^i{}_k = 2\phi^{-1} (\delta^i_j \partial_k \phi + \delta^i_k \partial_j \phi - \gamma^{ih} \gamma_{jk} \partial_h \phi) \quad .$$

From this identity the relation (2-1) results by a straightforward computation. On the other hand we have if D denotes the covariant derivative in the metric γ, for an arbitrary tensor B

$$\overline{\nabla}_i B^{ij} \equiv D_i B^{ij} + C_i{}^i{}_h B^{hj} + C_i{}^j{}_h B^{ih}$$
$$\equiv D_i B^{ij} + 10\, \phi^{-1}\, \partial_h\, \phi\, B^{hj} - 2\, \phi^{-1}\, \gamma^{jh}\, \partial_h\, \phi\, \mathrm{tr}\, B \tag{2-5}$$

while

$$D_i(\phi^{-10} A^{ij}) \equiv \phi^{-10}(D_i A^{ij} - 10\, \phi^{-1}\, A^{ij}\partial_i\phi) \tag{2-6}$$

thus, if K^{ij} is given by (2-3)

$$\overline{\nabla}_i K^{ij} \equiv \phi^{-10}\, D_i A^{ij} - 2\phi^{-1}\gamma^{jh}\partial_h\, \phi\, \mathrm{tr}\, K \quad . \tag{2-7}$$

which proves that if tr K = 0 and j = 0 the momentum constraint is equivalent to

$$D_i A^{ij} = 0 \qquad , \qquad \gamma_{ij} A^{ij} = 0$$

If we replace K in terms of A and ϕ we obtain a semi-linear elliptic equation for ϕ

$$8\Delta_\gamma \phi - R(\gamma) - \phi^{-7}\, A_{ij} A^{ij} + 2\rho\phi^5 = 0 \quad . \tag{2-8}$$

The system (2-7), (2-8) has been studied (Y. Choquet-Bruhat 1956) and written as an elliptic system (1971) J. York has shown that the decoupling can be obtained in a more general case, namely when tr K = constant, and even when $j \neq 0$ by an appropriate conformal scaling of the sources, as follows: set again

$$\overline{g} = \phi^4 \underline{\gamma} \qquad , \qquad \text{i.e.} \quad \overline{g}_{ij} = \phi^4 \gamma_{ij}$$

but now[7]

$$K^{ij} = \phi^{-10} A^{ij} + \frac{1}{3} \bar{g}_{ij} \tau \qquad (2\text{-}9)$$

where A is a symmetric traceless 2-tensor

$$\text{i.e.} \qquad \text{tr } A = A^{ij}\gamma_{ij} = 0 \quad , \quad \text{tr } K = \tau \quad ,$$

We deduce from (2-5) and (2-6)

$$\bar{\nabla}_i \, (\phi^{-10} A^{ij}) = \phi^{-10} D_i A^{ij}$$

and we have therefore

$$\bar{\nabla}_i K^{ij} - \bar{g}_{ij} \partial_i \text{ tr } K = \phi^{-10} D_i A^{ij} - \frac{2}{3} \bar{g}^{ij} \partial_i \tau$$

If we set

$$j = \phi^{-10} v \quad , \quad \text{i.e.} \quad j^i = \phi^{-10} v^i$$

we see that the momentum constraint is equivalent to

$$D_i A^{ij} - \frac{2}{3} \phi^6 \gamma^{ij} \partial_i \tau = v^j \quad , \quad \text{tr } A = \gamma^{ij} A_{ij} = 0 \qquad (2\text{-}10)$$

these are linear equations for A when γ is given, not containing ϕ, as announced, if τ = constant.

York argued for the conformal weight given to j by taking radiation, for instance electromagnetic, as a source. He also argued that the proper scaling for the scalar product part ρ of the source is:

$$\rho = \phi^{-8} q \qquad (2\text{-}11)$$

(7) If we lower indices of K with $\bar{g}$ and of A with γ the relation (2-9) reads

$$K_{ij} = \phi^{-2} A_{ij} + \frac{1}{3} \bar{g}_{ij} \tau$$

where q is a given scalar on S.

The hamiltonian constraint then reads

$$8\Delta_j\phi - R(\gamma) - \phi^{-7}\, A.A + \frac{1}{3}\,\phi^5\tau^2 + 2q\phi^{-3} = 0\ . \qquad (2\text{-}12)$$

This semi-linear elliptic equation for the unknown ϕ, when A, τ, q are known, have better stability properties (cf. Choquet--Bruhat and York 1980) that the equation (2-8) with unscaled source.

In the next two paragraphs we treat the uncoupled system (2-10), (2-12), when τ = constant.

3. Solutions of the Momentum Constraint

When τ = constant, and j is given, the momentum constraint obtained after conformal rescalings is equivalent to the linear system

$$D_i\, A^{ij} = v^j\ , \qquad (3\text{-}1_a)$$

$$\text{tr}\ A = 0\ . \qquad (3\text{-}1_b)$$

The general solution of this (undetermined) system is obtained by what is called a "York splitting" which we now describe. The general solution of $(3\text{-}1_b)$ is

$$A_{ij} = B_{ij} - \frac{1}{3}\,\gamma_{ij}\ \text{tr}\ B \qquad (3\text{-}2)$$

where B is an arbitrary symmetric 2-tensor.

Replacing in $(3\text{-}1_a)$ we find:

$$L(B)^j \equiv D_i\ B^{ij} - \frac{1}{3}\,\gamma^{ij}\,\partial_i\ \text{tr}\ B = v^j\ . \qquad (3\text{-}3)$$

The left hand side is a differential linear operator L from symmetric 2-tensors into vectors, whose dual L^*, acting from vectors into 2-tensors is defined by the equality of the integrals, for all u and B with compact support:

$$\int u_j L^j(B) d\mu(\gamma) \equiv \int u_j (D_i B^{ij} - \frac{1}{3} \gamma^{ij} \partial_i \operatorname{tr} B) \, d\mu(\gamma) =$$

$$= \int B^{ij} L^*_{ij}(u) \, d\mu(\gamma)$$

$d\mu(\gamma)$ is the volume element of S in the metric γ. Performing an integration by part we find

$$\begin{aligned} L^*_{ij}(u) &\equiv -\frac{1}{2} (D_i u_j + D_j u_i) + \frac{1}{3} \gamma_{ij} D_h u^h \\ &\equiv -\frac{1}{2} (\mathcal{L}_u(\gamma) - \frac{1}{3} \gamma \operatorname{tr} \mathcal{L}_u(\gamma)) \end{aligned} \tag{3-4}$$

where $\mathcal{L}_u(\gamma)$ is the Lie derivative of γ with respect to u.

The operator L^* has "injective symbol" $\sigma(L^*)$: that is if we replace the derivation D by a covariant vector ξ we obtain:

$$\sigma_{ij}(L^*) \quad - \frac{1}{2} (\xi_i u_j + \xi_j u_i) + \frac{1}{3} \gamma_{ij} \xi_h u^h$$

and we have that

$$\sigma(L^*) = 0 \quad \text{for} \quad u \neq 0 \text{ implies } \xi = 0 \, .$$

For such operators, and for an appropriate functional space E of tensor fields over S one has the following general "splitting" theorem (generalized from Fredholm cf Berger - Elbin 1969)

$$E \equiv \ker L^* \oplus \operatorname{range} L \tag{3-5}$$

where the sum is L^2-orthogonal. Thus (3-3) has a solution B (equivalently $v \in$ range L) if and only if v is L^2-orthogonal to ker L^*. We note that $-2L^*(u)$ determines the action of the infinitesimal diffeomorphism generated by u on the tensor density of weight $-2/3$ associated with γ (and which is the same for all metrics conformal to γ):

$$\tilde{\gamma} = (\det \gamma)^{-1/3} \gamma \quad . \tag{3-6}$$

Indeed the Lie derivative of $\tilde{\gamma}$ with respect to u is

$$L_u \tilde{\gamma} \equiv (\det \gamma)^{-1/3} (L_u \gamma - \frac{\gamma}{3} \operatorname{tr} L_u \gamma) \equiv -2(\det \gamma)^{-1/3} L^*(u) \tag{3-7}$$

By (3-3) ker L^* is identical with the set of vector fields u such that:

$$L_u \gamma - \frac{\gamma}{3} \operatorname{tr} L_u \gamma = 0 \quad , \tag{3-8}$$

that is to the set of conformal Killing vectors of γ.

The equation (3-3) admits solutions (i.e. $v \in$ range L) if and only if v is L^2-orthogonal to every conformal Killing vector of γ. We can formulate this as a theorem:

<u>Theorem 1</u>. The momentum constraint (3-1) admits solutions for every v if the manifold (S,γ) admits no conformal Killing vector field.

2. If the manifold (S,γ) admits a conformal Killing vector u the momentum constraint (3-1) admits solutions if and only if

$$\int_S \gamma_{ih} u^i v^h \, d\mu(\gamma) = 0 \quad . \tag{3-9}$$

Remarks 1. A conformal Killing vector of (S,γ) is also a conformal Killing vector of the conformal manifold $(S,\bar{g})$, $u^i = \bar{u}^i$. We have $\bar{g}_{ih} = \phi^4 \gamma_{ih}$, thus $d\mu(\bar{g}) = \phi^6\, d\mu(\gamma)$, and $v^h = \phi^{10} j^h$. The relation (3-9) is therefore a conformal invariant (as seen by O'Murchadha and York 1974), it can also be written, independantly of ϕ:

$$\int_S \bar{g}_{ih}\, u^i\, j^h\, d\mu(\bar{g}) = 0$$

2. The number of linearly independant conformal Killing vectors on an $n = 3$ dimensional manifold is $\frac{1}{2}(n+1)(n+2) = 10$: this maximum number is attained, in the case of compact manifolds, by S^3. For non compact manifolds the functional spaces to be used for the splitting theorem to be valid impose fall off properties at infinity of the unknown. The euclidean space $\mathbb{R}^3$ admits no conformal Killing vector tending to zero at infinity. The same can be proved to be true (O'Murchadha and Christodoulou 1981) for asymptotically euclidean manifolds, and the appropriate functional spaces. The momentum constraint (3-1) has then solutions for every v.

3. The space of solutions of (3-1) (equivalently,(3-3)) is, as usual, the vectorial sum of ker L and a particular solution which O'Murchadha and York seek under the form

$$\underline{B} = \mathcal{L}_W(\gamma) - \frac{\gamma}{3}\, \mathrm{tr}\, \mathcal{L}_W(\gamma) \quad ,$$

inspired by the dual form of the splitting theorem

$$E = \ker L \oplus \mathrm{range}\, L^* \quad .$$

They obtain for the vector W (which they call a "vector potential") the second order elliptic system

$$L(B)^j \equiv (LL^*W)^j \equiv D_i D^i w^j + \frac{2}{3} D^j D_i w^i + R^j{}_i w^i = v^i \quad .$$

By the previous arguments this second order operator on w has as kernel (in a given functional space of vector fields over S) the conformal Killing vectors of γ (equivalently, $\bar{g}$), since

$$\ker LL^* = \ker L^*$$

as follows from the equality of the L^2 scalar products

$$(LL^* w, u) = (L^* w, L^* u) \quad .$$

The existence of w is insured under the condition (3-3) on v (which is no restriction if γ admits no conformal Killing fields), w is not unique, but $B = \mathcal{L}_w(\gamma) - \frac{\gamma}{3} \operatorname{tr} \mathcal{L}_w(\gamma)$ is unique (since w is determined up to addition of a conformal Killing vectors).

4. Hamiltonian Constraint

We have seen in §2 that after conformal rescaling the hamiltonian constraint becomes a semi-linear elliptic equation for ϕ, called Lichnerowicz equation, which we write

$$\Delta\phi = P(\phi) \tag{4-1}$$

with

$$P(\phi) = c\phi - a\phi^{-7} + b\phi^5 - d\phi^{-3}$$

Δ is the Laplace operator on functions, in the metric γ, $c = \frac{1}{8} R(\gamma)$, $a = \frac{1}{8} A.A$, $b = \frac{1}{12} \tau^2$, $d = \frac{1}{4} \rho$.

We note that, by their definition, we have $a \geq 0$, $b \geq 0$. We suppose also that the source have a positive energy density, that is $d \geq 0$.

We treat in this paragraph the case where S is compact, that is closed without boundary. A consequence of the equation (1-1), whose left hand side is a divergence, is then

$$\int_S P(\phi)d\mu(\gamma) = 0 \quad . \qquad (4-2)$$

It is therefore necessary, for (1-1) to have a solution ϕ, that $P(\phi)$ changes sign on S, thus since we want this solution to be positive, the polynomial (with $z = \phi^4$ and a, b, c given functions of $x \in S$)

$$Q_x(z) \equiv \phi^7 P(\phi) \equiv bz^3 + cz^2 - dz - a \qquad (4-3)$$

must have at least one positive root for some point $x \in S$. Note that $Q(0) \leq 0$. A general method of proof of existence of solutions of second order quasi linear elliptic equations on compact manifolds which applies to (4-1) (cf. Y. Choquet-Bruhat 1972) was given by Choquet-Bruhat and Leray (1972) using the Hölder spaces $C^{k,\alpha}$. In the case of a 3-dimensional manifold the Sobolev spaces H_s can be also used. We have seen $H_4 \subset C^2$ and H_2 is an algebra. The following theorem gives a sufficient condition (not necessary) for the Lichnwrowicz equation to have a positive solution on S.

Theorem 1. If $\gamma \in H_4$ is a properly riemannian metric on S (compact); a, b, c, d $\in H_2$, and there exists two constants ℓ and m such that on S

$$P(\ell) < 0 \quad \text{and} \quad P(m) > 0 \quad \text{with} \quad 0 < \ell < m \tag{4-4}$$

then the equation (4-1) admits at least one solution $\phi \in H_4$, $\phi > 0$ on S.

2. If $a \geq 0$, $b \geq 0$, $d \geq 0$ suficient conditions for (4-4) to be satisfied are either

i) $b > 0$, $a+d \geq 0$ and $a+d \neq 0$ with $c < 0$ whenever $a+d = 0$

ii) $b = 0$, $a+d > 0$ and $c > 0$.

Proof 1. It rests on the Leray-Schauder degree theory (cf. for instance Choquet-Bruhat and DeWitt-Morette 1982 p. 563): if the degree of a mapping f from an open set Ω of a Banach B into B , $u' \to v = f(u)$ is not zero at v_0, then by the definition $v_0 \in$ range f, i.e. the equation $v_0 = f(u)$ as at least one solution $u \in \Omega$. The Leray-Schauder (1934) fundamental theorem says that if F_t is a homotopy of compact maps $\overline{\Omega} \to B$, with $\overline{\Omega}$ the closure of a bounded set of B with boundary $\partial\Omega$, such that $(Id - F_f)(u) \neq v_0$ foa all $u \in \partial\Omega$, then all the maps $Id - F_t$, $0 \leq t \leq 1$ have the same degree at v_0. The map Id denotes the identity, which has degree 1 at v_0 is $v_0 \in \Omega$. When $v_0 = 0$ (origin of B) a solution of $(Id - F_t)(u) = 0$ is a fixed point of the mapping F_t.

A solution of the non linear elliptic equation (4-1) is a fixed point of the mapping $F: v \to u$ defined by solution of the linear elliptic equation (we write $\Delta u - u$ and not Δu in the left hand side, because Δ is not invertible on a compact manifold, having a non vanishing kernel)

$$\Delta u - u = P(v) - v \quad .$$

The $v \to u$ mapping is well defined on the open set $v > 0$ of H_2, due to the following:

Lemma 1. On a compact riemannian manifold (S,γ) with $\gamma \in H_2$, the linear elliptic equation

$$\Delta u - u = f$$

with $f \in H_2$ has one and only one solution $u \in H_4$ that satisfies, for positive constants C_1, C_2 depending on (S,γ) the inequalities

$$\|u\|_{H_4} \leq C_1 \|f\|_{H_2} \quad . \qquad (4\text{-}5_a)$$

$$\|u\|_{H_2} \leq C_2 \|f\|_{L_2} \quad . \qquad (4\text{-}5_b)$$

One uses also the fact that H_2 is an algebra when S is 3-dimensional and that $v > 0$ is an open set of H_2 due to the topological inclusion $H_2 \subset C^0$.

The mapping F is compact i.e. maps bounded sets of H_2 into relatively compact sets due to the inequality $(4\text{-}5_a)$, and the fact that a bounded set of H_4 is relatively compact in H_s, $s < 4$.

The homotopic family F_t, with $F_1 = F$ can be defined by, c being a constant

$$\Delta u - u = t(P(v)-v) + (t-1)_c \quad , \qquad 0 \leq t \leq 1 \quad , \qquad (4\text{-}6)$$

where F_0 is the constant map $v \to u = c$. It can be defined by the resolution of the equation (independent of v)

$$\Delta u - u = -c \quad . \qquad (4\text{-}7)$$

The map F_0 has one fixed point, $u = c$[Id - F_0 has degree 1 at zero].

The problem is to find a bounded open set Ω of H_2, containing c such that the mappings F_t have no fixed point on $\partial\Omega$, that is the equations

$$\Delta u - u = t(P(u)-u) + (t-1)c \tag{4-8}$$

have no solution on $\partial\Omega$.

The inequality ($4\text{-}5_b$) is not sufficient to give the result, but the maximum principle will lead to the required estimate.

Lemma 2. (Maximum principle) If u is C^2 on S and at a point $x \in S$, $\Delta u < 0$ ($\Delta u > 0$), then u cannot attain a minimum (maximum) at x.

Note that (lemma 1) a solution u of (4-8) is in H_4, therefore in C^2, if it is in H_2. The hypothesis (4-4) allows us to apply the lemma 2: there are two numbers $\ell > 0$ and $m > \ell$ such that $P(\ell) < 0$ and $P(m) > 0$ at every point $x \in S$. We choose for instance $c = \frac{\ell+m}{2}$, then if at a point $x \in S$ we have $u(x) = \ell$ the equation (4-8) implies, at this point

$$\Delta u = t\, P(\ell) + (1-t)\, \frac{\ell-m}{2} < 0 \quad , \quad 0 \le t \le 1$$

and

$$\Delta u = t\, P(m) + (1-t)\, \frac{m-\ell}{2} > 0 \quad , \quad 0 \le t \le m \quad .$$

A solution u of (4-8) cannot therefore attain a minimum equal to ℓ, nor a maximum equal to m.

We choose for Ω a bounded open set of H_2 of the type:

$$\|u\|_{H_2} < K \quad , \tag{$4\text{-}10_a$}$$

$$\ell < u < m \quad . \tag{$4\text{-}10_b$}$$

To show that (4-8) admits no solution on $\partial\Omega$ we suppose that $u \in \overline{\Omega}$; that is

$$\|u\|_{H_2} \le K \quad , \qquad (4\text{-}11_a)$$

and

$$\ell \le u \le m \qquad (4\text{-}11_b)$$

and we show that u must then be in Ω. The fact that $\ell \le u \le m$ implies $\ell < u < m$ has just been proved. We now remark that it also imples

$$\|t(P(u)-u) + (1-t)\frac{\ell+m}{2}\|_{L_2} \le K_1 \quad , \quad 0 \le t \le 1 \quad ,$$

where K_1 is a constant which depends only on S, γ, the coefficients a, b, c, d, ℓ and m. The inequality $(4\text{-}5_b)$ gives then, for all solutions of (4-8)

$$\|u\|_{H_2} \le C_2 K_1 \quad .$$

It is therefore sufficient to choose the K of $(4\text{-}10_a)$ such that $K > C_2 K_1$ to insure that all solutions of (4-8) in $\overline{\Omega}$ satisfy the strict inequality $(4\text{-}10_a)$.

2. Under the hypothesis (i), or (ii) the polynomial Q (cf (4-3)) has one and only one strictly positive root, for every $x \in S$.

<u>Remark</u> 1) In the case $b > 0$, that is for solutions $(\overline{g},K)$ which will satisfy tr K = constant $\neq 0$, it is possible to show existence of $\overline{g}$, conformal to a given metric γ, with only the hypothesis that $a+d \ge 0$ and $a+d \neq 0$ on an open subset U of M, that is that A and ρ are not both identically zero on S, by making a conformal transformation which makes the scalar curvature negative

on S\U (O'Murchadha and York 1973).

If A and ρ are both zero on S the equation reduces to

$$\Delta\phi - c\phi - b\phi^5 = 0 \quad , \quad c = \frac{1}{8} R(\gamma) \quad , \qquad (4\text{-}12)$$

which expresses that the metric $g = \phi^4\gamma$ has for scalar curvature the negative constant $-\frac{b}{8}$: it is clear (cf. (4-3)) that if $b > 0$ the equation (4-12) has no positive solution if $c \geq 0$. It has been proved that a sufficient condition for the existence of a positive solution is

$$\int_S R(\gamma)d\mu(\gamma) \leq 0 \quad , \quad R(\gamma) \not\equiv 0 \quad .$$

2) $b = 0$, that is solutions for which the initial manifold S will be maximal (tr K = 0) in the space-time: the conditions for existence of a solution are more restrictive in this case, in accordance with the fact that space times with closed space-like sections do not in general possess a maximal such section. If $R(\gamma) \leq 0$ the Lichnerowicz equation (4-1) has no positive solution (cf. (4-3)), for any $A \geq 0$ and $\rho \geq 0$ (except the trivial one $\phi = 1$ if $A \equiv \rho \equiv R(\gamma) = 0$).

III - Positivity of the Gravitational Energy

It is a consequence of the equivalence principle that there can be no pointwise intrinsic definition of a local density of gravitational energy in General Relativity. However it is possible to define the energy of a gravitational field, that is of a hyperbolic metric (pseudo riemannian with signature (-,+,+,+)) with respect to another given hyperbolic metric, sometimes called background metric. A local positive density of energy can be defined for high frequency gravitational waves propagating in a given background, for instance the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right) dt^2 + 2du\, dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

and there exists a relation between this energy and the variation of the "mass" m(u) (cf Choquet-Bruhat 1969). On the other hand if this background metric is the Minkowski metric, and the gravitational field is asymptotically minkowskian at spatial infinity, it is possible (cf Arnowitt, Deser, Misner 1962) to define on a space-like slice a global energy, which is conserved by time evolution. It is also possible, in this asymptotically minkowskian case, to define a conserved global linear 3-momentum of the gravitational field, relative to a space like slice. It has been conjectured that this energy is non negative for all solutions of Einstein equations with sources satisfying a positive energy condition, and that it is zero only for a vanishing gravitational field, that is for the Minkowski metric itself. It has even been conjectured that the A.M.M energy-momentum 4-vector is time like and future directed; its length $m \geq 0$ is called the mass of the gravitational field; $m = 0$ implies that the space time is Minkowski. The expressions for the components of A.D.M 4-momentum are

$$P^0 = \int_S (\partial_i g_{ij} - \partial_j g_{ii})\, dS_j \quad \text{(energy)} \qquad (0\text{-}1)$$

$$P^{\ell} = -2 \int_S (\delta^{\ell j} K^i{}_i - K^{\ell j})\, dS_j \qquad \text{(linear 3-momentum)} \qquad (0\text{-}2)$$

with S the 2-sphere at infinity of a space like slice, where the metric has the asymptotic behaviour

$$g_{\alpha\beta} - \eta_{\alpha\beta} = 0(r^{-1}) \qquad , \qquad \partial_\alpha g_{\beta\lambda} = 0(r^{-2}) \qquad (0\text{-}3)$$

($\eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$ is the Minkowski metric, $\alpha,\beta,\ldots = 0,1,2,3$, $i,j,\ldots = 1,2,3$)
$K_{ij} = -|g^{00}|^{-1/2}\, \Gamma^0_{ij}$ is the extrinsic curvature of the slice.

The positive mass conjecture (with our signature convention) is that $P^\lambda U_\lambda$ is positive for all solutions of Einstein equations with sources which satisfy a positive energy condition, and every time-like, past directed, vector U_λ at space-like infinity.

The positive mass conjecture has been proved to be true along the years in a number of special cases. The positivity has been proved for all space times in a finite neighborhood of Minkowski space time by methods of functional analysis by Choquet-Bruhat and Marsden 1976. The full positive mass theorem has been proved by Schoen and Yau 1979, using minimal 2-surfaces imbedded in the space like slice in a way which resembles the use of the Raychaudhuri equation in the proof of the singularity theorems. A completely different proof has been given by E. Witten, using spinors, a method which originates from supergravity.

We shall review the fundamental features of these two proofs. To simplify the exposition we take the space time to be diffeomorphic to $\mathbb{R}^4$, and the background metric to be the standard Minkowski metric. The results can be extended to manifolds with more complicated topologies (see the original papers or the review article Choquet-Bruhat 1983).

1. The Proof of Schoen and Yau

The basic theorem of Schoen and Yau (1979) about the positive energy of asymptotically minkowskian space times is purely geometrical: it says that an asymptotically euclidean 3-manifold with metric g of the type:

$$g_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij} + h_{ij} \quad , \quad h_{ij} = O(r^{-2}) \qquad (1\text{-}1)$$

cannot have both $R(g) \geq 0$ and $m < 0$. It thus applies to space--times which satisfy the weak energy condition and admit a maximal slice (tr K = 0). They treat the general case of asymptotically minkowskian space times in subsequent papers (1981 a, 1981 b).

The proof of the basic theorem relies on the consideration of minimal (with respect to area) 2-dimensional surfaces of the 3-space (S,g). Indeed the existence of such a minimal surface Σ yields by considering the second variation of the area A a functional inequality

$$A''(\Sigma) \geq 0 \quad . \qquad (1\text{-}2)$$

The analogue of the inequality (1-2), for 1-dimensional submanifolds which minimize (maximize in the case of a hyperbolic metric) the arc length has been the source of many important global properties in riemannian geometry – and is the cornerstone of the singularity theorems in General Relativity, when one uses for realistic sources, the fundamental equation discovered by

Raychaudhuri. Like the second variation around a geodesic, the second variation around a minimal (or maximal) submanifold has a remarkable simple expression in geometrical terms. However the existence of these manifolds, with the relevant properties is difficult to prove, and it is why the Schoen and Yau proof, though conceptually simple, is technically very intricate.

The second variation of the area of S, $A''(\Sigma)$, is a quadratic form on functions of class C^2 on S which reads for functions with compact support

$$A''(\Sigma)(f,f) = -\int_S f(\Delta_\Sigma f + f\,\mathrm{Ricc}(\nu,\nu) + f|b|^2)d\Sigma =$$
$$= \int_\Sigma (|\nabla f|^2 - f^2(\mathrm{Ricc}(\nu,\nu) + |b|^2)d\Sigma \tag{1-3}$$

where the laplacian Δ_Σ, the norm $|\;|$ and $d\Sigma$ are taken in the metric induced on Σ by g, $\mathrm{Ricc}(\nu,\nu)$ is the Ricci tensor of g contracted with the unit normal ν to Σ, b is the extrinsic curvature of Σ in (S,g). One has the identity (cf an analogous one in the hamiltonian constraint), if $\mathrm{tr}\, b = 0$ (verified since Σ is a minimal surface)

$$\mathrm{Ricc}(\nu,\nu) - R = -\frac{1}{2}|b|^2 - R_\Sigma$$

where R_Σ is the scalar (gaussian) curvature of Σ , thus (1-3) implies

$$\int_\Sigma (R + \frac{1}{2}|b|^2 - R_\Sigma)\, f^2\, dS = -A''(\Sigma)(f,f) \le 0 \tag{1-4}$$

Schoen and Yau prove that $m \ge 0$ in (1-2) ab absurdo (with some more, reasonable, hypothesis on the smoothness of h_{ij} and the fall off at infinity of its derivatives). They suppose that $m < 0$, and they use this assumption to prove the existence of a complete area minimizing surface Σ, which lies between two parallel euclidean planes. They also use the hypothesis $m < 0$ to construct a metric conformal to g_{ij} which has a strictly positive riemannian scalar curvature outside of a compact set. Then they use the formula (1-4) applied to this last metric, and the Gauss-Bonnet theorem with boundary to arrive at a contradiction. These proofs rely on dedicate estimates and deep results of differential geometry in the large.

2. Witten's Proof[8]

We define as usual on a space time, 4-manifold V with hyperbolic metric $g_{\alpha\beta}$, a 4-spinor ψ , and its covariant derivative

$$\nabla_\lambda \psi \equiv \partial_\lambda \psi + \sigma_\lambda \psi_\rho$$

where σ_λ is the connection 1-form on spinors, namely

$$\sigma_\lambda = - \frac{1}{4} \omega_\lambda{}^a{}_b \gamma_a \gamma^b$$

with $\omega_\lambda{}^a{}_b$ the riemannian connection (a, b orthonormal frame indices) and γ_a , a = 0,1,2,3, a set of standard Dirac matrices

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}\, I \quad , \quad \eta_{ab} = \mathrm{diag}(-1,1,1,1)$$

γ^0 antihermitian and γ^i hermitian.

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(8) This proof has been completed, and simplified by Parker and Taubes 1982, Nester 1981, Reula 1982.

We denote by $\overline{\psi}$ the cospinor (Dirac adjoint)

$$\overline{\psi} = \overset{\vee}{\psi} \gamma^0 \qquad (\overset{\vee}{\psi} \text{ hermitian conjugate})$$

We set

$$\xi = \frac{1}{4!} \eta_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d \qquad \text{(spinor-cospinor)}$$

We remark that, with our definition of $\overline{\psi}$, $\overline{\psi}\gamma^\lambda\psi$ is real, time lime and past directed.

It has been remarked very early in the development of supergravity (Deser and Teitelboim 1977) that the commutator of two supersymmetry transformations is a Lie derivative of the tetrad, and thus also of the metric, that is, if

$$\delta_\alpha e_a^\lambda = \overline{\alpha}\gamma_a\psi_a \quad , \quad \delta_a\psi_\lambda = \nabla_\lambda\alpha \tag{2-1}$$

we have, if α is an anticommuting spinor

$$(\delta_{\alpha_2}\delta_{\alpha_1} - \delta_{\alpha_1}\delta_{\alpha_2})\, e_a^\lambda = \nabla_a\xi^\lambda \quad , \quad \text{with} \quad \xi^\lambda = \overline{\alpha}_1\gamma^\lambda\alpha_2 \tag{2-2$_a$}$$

$$(\delta_{\alpha_2}\delta_{\alpha_1} - \delta_{\alpha_1}\delta_{\alpha_2})\, g_{\lambda\mu} = \nabla_\lambda\xi_\mu + \nabla_\mu\xi_\lambda = (L_\xi g)_{\lambda\mu} \tag{2-2$_b$}$$

Thus, in some sense, the gravitational energy, "generator of time translations at infinity", should be the square of supercharge, generating supersymmetry transformation. In fact, in quantum supergravity the hamiltonian H may be written, at least formally

$$H = (1/\hbar)\ \mathrm{tr}\ Q^2 \tag{2-3}$$

where Q is the supercharge. It has also been suggested (Grisaru 1978) that it might be possible to obtain the positive energy theorem in General Relativity by taking a classical limit, $\hbar \to 0$, of (2-3). These statements have not been proved, but it is possible to get inspiration from the formulas (2-2) to guess an expres for the 4-momentum of an asymptotically minkowskian space-time – which we shall check to coincide with the A.D.M definition. We shall also show that this expression is indeed the one used by E. Witten in his positivity proof. We proceed as follows (for another derivation, in the hamiltonian formalism see Horowitz and Strominger 1982, Deser 1982, or in the covariant formalism see Hull 1983).

Let α be a constant spinor in Minkowski space time, and let ϕ_ρ be a spin 3/2 field in an asymptotically minkowskian space time with metric $g_{\alpha\beta}$. The charge $Q(\alpha)$ of the field ϕ_ρ, relative to α and the space like slice Σ is up to a constant factor given by the integral

$$Q(\alpha) = \int_S \overline{\alpha}\ \xi\ \varepsilon^{\lambda\mu\nu\rho}\ \gamma_\mu\ \partial_\nu\ \phi_\rho\ d\Sigma_\lambda$$

where

$$d\Sigma_\lambda = \frac{1}{3!}\ \varepsilon_{\lambda\alpha\beta\gamma}\ dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

which can also be written as an integral at infinity

$$Q(\alpha) = \int_\Sigma \overline{\alpha}\ \xi(\gamma_\mu\phi_\rho - \gamma_\rho\phi_\mu)\ dx^\mu \wedge dx^\rho$$

by the second supersymmetry variation (at $\psi_\lambda = 0$, we had $\phi_\lambda = \delta_\alpha\psi_\lambda$) we obtain:

$$\delta_\alpha Q(\alpha) = \int_\Sigma \overline{\alpha}\ \xi(\gamma_\mu\nabla_\rho\alpha - \gamma_\rho\nabla_\mu\alpha)\ dx^\mu \wedge dx^\rho \quad .$$

We see that $4\,\delta_\alpha Q(\alpha)$ is the integral over Σ of the real 2-form ω defined on space time by

$$\omega = 4\, \bar{\alpha}\, \xi\, \gamma_\lambda\, \nabla_\rho \alpha - \nabla_\rho \bar{\alpha}\, \xi\, \gamma_\lambda \alpha$$

which can also be written

$$\omega = \frac{1}{2} E^{\sigma\alpha}\, dS_{\sigma\alpha} \qquad (2\text{-}4)$$

with $dS_{\sigma\alpha}$ the 2-dimensional volume element corresponding to the metric $g_{\alpha\beta}$

$$dS_{\sigma\alpha} = \frac{1}{2} \eta_{\sigma\alpha\lambda\mu}\, dx^\lambda \wedge dx^\mu \quad , \quad \eta_{\sigma\alpha\lambda\mu} = \sqrt{|g|}\, \epsilon_{\sigma\alpha\lambda\mu} \quad (2\text{-}5)$$

and $E^{\sigma\alpha}$ the real antisymmetric tensor:

$$E^{\sigma\alpha} = 2\, \eta^{\sigma\alpha\delta\beta} (\bar{\alpha}\, \xi\, \gamma_\delta\, \nabla_\beta \alpha - \nabla_\beta \bar{\alpha}\, \xi\, \gamma_\delta \alpha) \quad , \qquad (2\text{-}6)$$

ω is the Witten's 2-form.

We now give the relation between the Witten 2-form and A.D.M 4-momentum. Since the mathematics in Witten's proof are fairly straightforward, we give precise definitions under which it works.

We define a Hilbert space H of fields of 4-spinors over Σ (diffeomorphic to $\mathbb{R}^{-3}$) as the closure of the vector space of such fields which are infinitely differentiable and with compact support, in the norm

$$\|\alpha\|_H = \left\{ \int_{\mathbb{R}^3} \{\sigma^{-2} (\tilde{\alpha}\alpha) + \delta^{ij}\, \partial_i \tilde{\alpha}\, \partial_j \alpha\} d^3x \right\}^{1/2}$$

(we denote by $\partial_i \alpha$ the partial derivative, and set $\sigma = (1+r^2)^{1/2}$). If the metric $g_{\alpha\beta}$ is twice continuously differentiable and satisfies on Σ the asymptotic conditions

$$g_{\alpha\beta} - \eta_{\alpha\beta} = 0(r^{-1}) \quad , \quad \partial_\lambda g_{\alpha\beta} = 0(r^{-2}) \quad , \quad \partial^2_{\lambda\mu} g_{\alpha\beta} = 0(r^{-3}) \; .$$

We have the theorem:

<u>Theorem</u>: If on Σ we have

$$\alpha = \psi_0 + \psi_1$$

with ψ_0 smooth and constant on Σ and $\psi_1 \in H$, then

$$\int_S \omega = P_\lambda U^\lambda \quad \text{with} \quad U^\lambda = \overline{\psi}_0 \gamma^\lambda \psi_0 \quad . \qquad (2$$

<u>Proof</u>: we have, $\omega = \omega_0 + \omega_1$ with on Σ ($x^0 = 0$) :

$$\omega_0 = 4(\overline{\psi}_0 \xi \gamma_i \nabla_j \psi_0 - \nabla_j \overline{\psi}_0 \xi \gamma_i \psi_0)\, dx^i \wedge dx^j \qquad (2\text{-}9)$$

$$\omega_1 = 4(\overline{\psi}_0 \xi \gamma_i \nabla_j \psi_1 - \nabla_j \overline{\psi}_1 \xi \gamma_i \psi_0 + \overline{\psi}_1 \xi \gamma_i \nabla_j \alpha - \nabla_j \alpha \xi \gamma_i \psi_1) dx^i \wedge dx^j \qquad (2\text{-}10)$$

We prove first that

$$\int_S \omega_1 = 0 \quad . \qquad (2\text{-}11)$$

Indeed, by definition (K_n sequence of increasing compact sets covering Σ)

$$\int_\Sigma \omega_1 = \lim_{n=\infty} \int_{\partial K_n} \omega_1 = \lim_{n=\infty} \int_{K_n} d\omega_1 = \int_S d\omega_1 \qquad (2\text{-}12)$$

we have, on S

$$d\omega_1 = 8 \,\mathrm{Re}\, \{\nabla_k \overline{\psi}_0 \xi \gamma_i \nabla_j \psi_1 + \overline{\psi}_0 \xi \gamma_i \nabla_k \nabla_j \psi_1 + \nabla_k \overline{\psi}_1 \xi \gamma_i \nabla_j \alpha + \overline{\psi}_1 \xi \gamma_i \nabla_k \nabla_j \alpha\}\, dx^k \wedge dx^i \wedge dx^j \quad . \qquad (2\text{-}13)$$

The second covariant derivatives, antisymmetrized, are replaced by the Riemann tensor. An inspection of the various terms added in (2-13) shows that they are all integrable on S, and that, given $\overline{\psi}_0$, their integral on S depends continuously on the H norm of ψ_1. We can therefore obtain $\int_S d\omega_1$ as the limit of integrals of exact differentials of forms with compact support (by approximating ψ_1 in H by spinors with compact support); such integrals are zero on Σ, and the same is true of their limit.

Second, we prove that

$$\int_\Sigma \omega_0 = P_\lambda U^\lambda \quad , \qquad (2\text{-}14)$$

where U^λ is the time like or null vector in the lorentzian frame at infinity given by

$$U^\lambda = \overline{\psi}_0 \gamma^\lambda \psi_0 \qquad (2\text{-}15)$$

while P^λ is the A.D.M 4-momentum.

Indeed, by hypothesis ψ_0 is constant on Σ, that is $\partial_i \psi_0 = 0$. Thus,

$$\omega_0 = - \overline{\psi}_0 (\xi \gamma_i \gamma_a \gamma_b + \gamma_a \gamma_b \xi \gamma_i) \psi_0 \omega_j{}^{ab}\, dx^i \wedge dx^j$$

which gives the formula (2-14) by using the multiplication properties of Dirac matrices, and the value of the connection in the orthonormal frame (cf Choquet-Bruhat and DeWitt-Morette 1982 p.308)

$$\theta^\alpha = dx^\alpha + a^\alpha{}_\beta\, dx^\beta$$

where

$$a^\alpha{}_\beta = \frac{1}{2} \eta^{\alpha\lambda} (h_{\lambda\beta} + f_{\lambda\beta}) \quad \text{with} \quad f_{\lambda\beta} = O(r^{-2}) \quad , \quad \partial_\alpha f_{\lambda\beta} = O(r^{-3}) .$$

Proof of Positivity

From the Stokes theorem and the expression of ω we deduce

$$\int_S \omega = \int_\Sigma d\omega = \int_\Sigma \nabla_\alpha E^{\sigma\alpha} d\Sigma_\sigma \quad , \quad d\Sigma_\sigma = \frac{1}{3!} \eta_{\sigma\beta\lambda\mu}\, dx^\beta \wedge dx^\lambda \wedge dx^\mu \tag{2-16}$$

with (due to the antisymmetry in α, β this expression is real)

$$\nabla_\rho E^{\sigma\rho} = 8\, \eta^{\sigma\rho\delta\beta}(\nabla_\rho \bar{\alpha}\xi\gamma_\delta \nabla_\beta \alpha + \bar{\alpha}\xi\gamma_\delta \nabla_\rho \nabla_\beta \alpha)$$

we transform the first term by the Dirac algebra property

$$\eta^{\sigma\alpha\delta\beta}\xi\gamma_\delta = \frac{1}{2}(\gamma^\sigma\gamma^{\alpha\beta} + \gamma^{\alpha\beta}\gamma^\sigma)$$

and the *second* by the Ricci identity on spinors (of Lichnerowicz 1964)

$$\nabla_\alpha \nabla_\beta \alpha - \nabla_\beta \nabla_\alpha \alpha = -\frac{1}{4} R_{\alpha\beta}{}^{\lambda\mu}\gamma_{\lambda\mu}\alpha \tag{2-17}$$

we use also

$$S^\sigma{}_\lambda \equiv R^\sigma{}_\lambda - \frac{1}{2}\delta^\sigma{}_\lambda R \equiv \frac{1}{4}\eta^{\sigma\alpha\delta\beta}\eta_{\delta\mu\nu\lambda}R_{\alpha\beta,}{}^{\mu\nu}$$

and we obtain

$$\nabla_\alpha E^{\sigma\alpha} = -4\nabla_\alpha \bar{\alpha}(\gamma^\sigma\gamma^{\alpha\beta} + \gamma^{\alpha\beta}\gamma^\sigma)\nabla_\beta \alpha + 2S^\sigma{}_\lambda \bar{\alpha}\gamma^\lambda \alpha \tag{2-18}$$

Einstein's equations, $S^\sigma{}_\lambda = T^\sigma{}_\lambda$ can be used to write the second term as

$$2T^\sigma{}_\lambda u^\lambda$$

We know that $T^\sigma{}_\lambda n_\sigma u^\lambda$ is non negative if n is time like and the sources satisfy the dominant energy condition. We shall now study the contribution of the first term in the integral (2-17), when S is the space like manifold $x^0 = 0$. It reduces then to (one has $\gamma^0\gamma^{0i} + \gamma^{0i}\gamma^0 = 0$).

$$2\nabla_i\bar{\alpha}(\gamma^0\gamma^{ij}+\gamma^{ij}\gamma^0)\nabla_j\alpha = 4\nabla_i\bar{\alpha}\gamma^{ij}\nabla_j\alpha$$
$$= 4g^{ij}\nabla_i\tilde{\alpha}\nabla_j\alpha + 4\nabla_i\tilde{\alpha}\gamma^i\gamma^j\alpha \tag{2-19}$$

We have proved the identity (valid if $S^0{}_\lambda$ is integrable on S)

$$U^\lambda P_\lambda = \int_\Sigma \omega = \int_S (2g^{ij}\nabla_i\tilde{\alpha}\nabla_j\alpha + G^0{}_\lambda\bar{\alpha}\gamma^\lambda\alpha + 2\nabla\tilde{\alpha}\gamma^i\gamma^j\nabla_j\alpha)\, d\Sigma \ . \tag{2-20}$$

We deduce from (2-20) that $U^\lambda P_\lambda \geq 0$ if α satisfies the Witten's equation

$$D\alpha = \gamma^i\nabla_i\alpha = 0 \ . \tag{2-21}$$

It can be proved, using a general theorem of Choquet-Bruhat and Christodoulou 1981, that this equation has a solution α, with $\alpha = \psi_0+\psi_1$, ψ_0 constant $\psi_1 \in H$, under the hypothesis made on $g_{\alpha\beta}$.

The identity (2-20) gives, when α satisfies (2-21) and $S_{\lambda\mu} = T_{\lambda\mu}$, the manifestly positive expression for $P_\lambda U^\lambda$:

$$P_\lambda U^\lambda = 2\int_\Sigma (|\nabla\alpha|^2 + T^\sigma{}_\lambda n_\sigma u^\lambda)\, d\Sigma \tag{2-22}$$

Note that the integrand in (2-22) does not give a positive density of local energy for the gravitational field since α, solution of (2-21), is a non local functional of the initial data (g k)

The equality (2-22) shows that $P^\lambda = 0$ if and only if the space time is empty ($T_{\lambda\mu} = 0$) and $\nabla_i \alpha = 0$ for every $\alpha = \psi_0 + \psi_1$ solution of (2-21). From the Ricci identity, and the possibility of deformation of S, one deduces that the Riemann tensor of $g_{\alpha\beta}$ vanishes – thus the space time is flat. It has also been demonstrated (Ashtekar and Horowitz 1982, Taub 1983), that P^λ cannot be a null vector (i.e. $m = 0$ implies flatness of space time, as conjectured).

Interesting recent developments extend the previous results to space time containing black-holes, or background geometries other than Minkowski, for instance de Sitter and anti de Sitter space times.

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