INTRODUCTION TO SUPERSYMMETRY AND SUPERGRAVITATION

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1 - NOTATION - SPINOR SPACE REPRESENTATION OF LORENTZ GROUP

The Lie algebra of the generators $M^{\ell m}$ of homogeneous Lorentz Group (HLG), ($\ell = 0,1,2,3$) is given by

$$[M_{\ell m}, M_{pq}] = i\,(\eta_{\ell p}M_{mp} - \eta_{mp}M_{\mu q} + \eta_{mq}M_{mp} - \eta_{\ell q}M_{mp})$$

where $M_{\ell m} = -M_{m\ell}$ and $\eta_{\ell m} = \mathrm{diag}(-1,1,1,1)$ is the metric.

A realization of this algebra is obtained in terms of the elements $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ of Clifford algebra over Minskowski space; they satisfy

$$\{\gamma^\ell, \gamma^m\}_+ = 2\,\eta^{\ell m}$$

In fact

$$M_{\ell m} = -i\,\sigma_{\ell m}$$

where

$$\sigma_{\ell m} = \frac{1}{4}[\gamma_\ell, \gamma_m] = \frac{1}{4}(\gamma_\ell\gamma_m - \gamma_m\gamma_\ell)$$

The representation of γ^ℓ by 4 x 4 matrices is an irreducible representation of the Clifford algebra. Thus we obtain a representation of H.L.G. by 4 x 4 complex matrices $S(\Lambda)$

$$S(\Lambda) = e^{\frac{1}{8}\lambda_{\ell m}[\gamma^\ell, \gamma^m]} \;;\; \det S(\Lambda) = 1$$

where $\lambda_{\ell m} = (\Lambda_{\ell m} - \eta_{\ell m})$ and $\Lambda^\ell m$ is Lorentz transformation matrix: $\eta_{\ell m}\Lambda^\ell_p\, \Lambda^m{}_q = \eta_{pq}$.

The corresponding representation space is 4-dimensional complex space called Dirac Spinor Space. $S(\Lambda)$ acts on 4-Spinors $\Psi(x)$ which transform as:

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$$

$$\Psi \xrightarrow{\Lambda} S(\Lambda)\,\Psi$$

We note also the identity:

$$\gamma^\ell = \Lambda^\ell{}_m \cdot S(\Lambda)\,\gamma^m\, S^\lambda(\Lambda)$$

showing that γ^ℓ are "invariant" matrices and index ℓ is a 4-vector index w.r.t H.L.G. transformations. A convenient representation for γ^ℓ is the Weyl representation defined by:

$$\gamma^\ell = i\begin{pmatrix}0 & \sigma^\ell\\ -\sigma^\ell & 0\end{pmatrix}$$

where in terms of Pauli matrices $\vec{\sigma}$

$\sigma_\ell = (I,\vec{\sigma})$; $\sigma^\ell = (-I,\vec{\sigma}) = \eta^{\ell m}\sigma_m$

Explicitly

$$\gamma^0 = -i\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} \quad , \quad \gamma^k = i\begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}$$

and

$$\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} = \gamma_0\gamma_1\gamma_2\gamma_3$$

The charge conjugation matrix C is taken to the

$$C = -\gamma^0\gamma^2 = \begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix} = -C^T$$

We note

$$\gamma^0\gamma^{\mu +}\gamma^0 = \gamma^\mu \quad , \quad \gamma^0\gamma_5^+\gamma^0 = -\gamma_5$$

In Weyl representation $S(\Lambda)$ takes the form:

$$S(\Lambda) = \begin{pmatrix}S_1(\Lambda) & 0\\ 0 & S_1^{+-1}(\Lambda)\end{pmatrix}$$

where S_1, S_1^{+-1} are 2 x 2 matrices:

$$S_1(\Lambda) = \exp\left[\frac{i}{2}\vec{\sigma}.\vec{a} - \frac{1}{2}\vec{\sigma}.\vec{b}\right]$$

$$S_1^{+-1}(\Lambda) = \exp\left[\frac{i}{2}\vec{\sigma}.\vec{a} + \frac{1}{2}\vec{\sigma}.\vec{b}\right]$$

Here $\vec{a}$, $\vec{b}$ are real and parametrize the Lorentz transformation.

Thus the representation is reducible and $S_1(\Lambda)$ and $S_1^{+-1}(\Lambda)$ are 2-dimensional inequivalent representations of $SL(2,C) \approx SO(3,1)$, HLG group. They give rise to the representations

$D(\frac{1}{2}, 0)$ with generators $\{-\frac{i}{2}\vec{\sigma}, -\frac{1}{2}\vec{\sigma}\}$

and

$D(0, \frac{1}{2})$ with generators $\{-\frac{i}{2}\vec{\sigma}, +\frac{1}{2}\vec{\sigma}\}$.

We note that $S_1 \sim S_1^{-1T}$ and $S_1^* \sim S_1^{+-1}$,($\sim$: equivalent) since

$$S_1 = \sigma_2\, S_1^{-1T}\, \sigma_2$$

$$S_1^* = \sigma_2\, S_1^{-1T}\, \sigma_2$$

We write in the Weyl representation $\Psi \in [D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})]$ as

$$\Psi = \begin{pmatrix}(\chi_\alpha)\\ (\bar{\psi}^{\dot\beta})\end{pmatrix} \equiv \begin{pmatrix}\chi\\ \bar{\psi}\end{pmatrix} \quad , \quad \begin{matrix}\alpha = 1,2\\ \dot\alpha = \dot 1,\dot 2\end{matrix}$$

so that under HLG we have the transformations $\chi' = S_1\chi$ or $\chi'_\alpha = S_{1\alpha}{}^\beta\chi_\beta$ if $S_1 \equiv (S_{1\alpha}{}^\beta)$ and $\bar{\psi}' = (S_1^{+-1})\psi$ which implies $\bar{\psi}'^{\dot\alpha} = (S_1^{*-1})_{\dot\beta}{}^{\dot\alpha}\,\bar{\psi}^{\dot\beta}$.

From $\chi' = S_1\chi = \sigma^2 S_1^{-1T} \sigma^2\chi$ we find $(\sigma_2\chi')^T = (\sigma_2\chi) S_1^{-1}$. This leads to the invariant $(\sigma_2\chi')^T \eta' = (\sigma_2\chi)^T\eta$ if $\eta' = S_1\eta$. It is suggested then to define χ^α

$$(\chi^\alpha) = i\sigma_2(\chi_\alpha)$$

or

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta \quad \text{where} \quad (\epsilon^{\alpha\beta}) \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Similarly

$$(\sigma_2\bar{\Psi}')^T = (\sigma_2\bar{\Psi})^T S_1^+$$

and we define

$$(\bar{\Psi}_{\dot\alpha}) = -i\sigma_2(\bar{\Psi}^{\dot\alpha})$$

or

$$\bar{\Psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\, \bar{\Psi}^{\dot\beta} \quad \text{where} \quad (\epsilon_{\dot\alpha\dot\beta}) = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We verify that

$$\chi'^\alpha = S_1^{-1}{}_\beta{}^\alpha\, \chi^\beta$$

$$\bar{\Psi}'_{\dot\alpha} = S^*_{1\dot\alpha}{}^{\dot\beta}\, \Psi_{\dot\beta}$$

The $\epsilon_{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$ are defined as inverses, viz,

$$\epsilon_{\alpha\sigma}\, \epsilon^{\sigma\beta} = \delta_\alpha^\beta \qquad \text{and} \qquad \epsilon_{\dot\alpha\dot\sigma}\, \epsilon^{\dot\sigma\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}$$

Thus

$$(\epsilon_{\alpha\beta}) = (\epsilon_{\dot\alpha\dot\beta}) = -i\sigma_2$$

and

$$(\epsilon^{\alpha\beta}) = (\epsilon^{\dot\alpha\dot\beta}) = +i\sigma_2$$

Clearly, $\chi^\alpha\eta_\alpha$, $\bar{\Psi}^{\dot\alpha}\, \xi_{\dot\alpha}$ are Lorentz invariant. We write

$$\chi\eta = \chi^\alpha\eta_\alpha = -\chi_\alpha\eta^\alpha$$

$$\bar{\Psi}\bar{\xi} = \bar{\Psi}_{\dot\alpha}\xi^{\dot\alpha} = -\bar{\Psi}^{\dot\alpha}\xi_{\dot\alpha}$$

We show easily that $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$, $\epsilon^{\dot\alpha\dot\beta}$ and δ_α^β are Lorentz invariant tensors. Now

$$S(\Lambda) = \left(\begin{array}{c|c} (S_{1\alpha}{}^\beta) & 0 \\ \hline 0 & (S_1^{-1T}{}_{\dot\alpha}{}^{\dot\beta}) \end{array}\right)$$

$$S^{-1}(\Lambda) = \left(\begin{array}{c|c} (S_1^{-1}{}_\alpha{}^\beta) & 0 \\ \hline 0 & (S^+_{1\dot\alpha}{}^{\dot\beta}) \end{array}\right)$$

$$S\gamma^m S^{-1} = i \begin{pmatrix} 0 & S_1\sigma^\ell S_1^+ \\ S_1^{-1+}(-\sigma_\ell)S_1^{-1} & 0 \end{pmatrix}$$

From $\gamma^\ell = \Lambda^\ell{}_m\, S(\Lambda)\gamma^m\, S^{-1}(\Lambda)$ in order to match the indices in 2-component notation we must write:

$$\sigma^{\ell} = (\sigma^{\ell}{}_{\alpha\dot\beta}) = (-I, \vec\sigma)$$

$$-\sigma_{\ell} = (\bar\sigma^{\ell\dot\alpha\beta}) = (-I, -\vec\sigma)$$

Clearly, $\sigma^{\ell}{}_{\alpha\dot\beta}$, $\bar\sigma^{\ell\dot\alpha\beta}$ are invariant under Lorentz transformation as is obvious from their definitions

$$\gamma^{\ell} = i\begin{pmatrix} 0 & (\sigma^{\ell}{}_{\alpha\dot\beta}) \\ (\bar\sigma^{\ell\dot\alpha\beta}) & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^{\ell} \\ -\sigma_{\ell} & 0 \end{pmatrix}$$

From $\Psi^{\alpha} = \varepsilon^{\alpha\beta}\Psi_{\beta}$, $\Psi_{\alpha} = \varepsilon_{\alpha\beta}\Psi^{\beta}$ and similar relations for dotted indices we also derive

$$(\sigma^{\ell}{}_{\alpha}{}^{\dot\beta}) = \varepsilon^{\dot\beta\dot\gamma}\sigma^{\ell}{}_{\alpha\dot\gamma} = -\sigma^{\ell}{}_{\alpha\dot\gamma}\,(i\sigma_2)^{\dot\gamma\dot\beta} = -(i\sigma^{\ell}\sigma_2)_{\alpha}{}^{\dot\beta}$$

$$(\bar\sigma^{\ell\dot\alpha}{}_{\beta}) = -(i\sigma_{\ell}\sigma_2)^{\dot\alpha}{}_{\beta}$$

Also

$$\bar\sigma^{\ell}{}_{\dot\alpha\beta} = \sigma^{\ell}{}_{\beta\dot\alpha} \qquad \text{(Hermiticity)}$$

and

$$\bar\sigma^{\ell}{}_{\dot\alpha\beta} = (\sigma^{\ell}{}_{\alpha\dot\beta})^{*} = -\sigma_2\sigma_{\ell}\sigma_2$$

The completeness relation is given by

$$\sigma^{m}{}_{\alpha\dot\beta}\,\sigma_{m}{}^{\alpha'\dot\beta'} = -2\,\delta_{\alpha}^{\alpha'}\,\delta_{\dot\beta}^{\dot\beta'}$$

We may then express any 4-vector V^{ℓ} in terms of 2-component notation:

$$V_{\alpha\dot\alpha} = V^{\ell}\,\sigma_{\ell\alpha\dot\alpha} \qquad \text{(Penrose)}$$

$$V^{\ell} = -\frac{1}{2}\,\bar\sigma^{\dot\alpha\alpha}\,V_{\alpha\dot\alpha}$$

If V^{ℓ} are real, then $V_{\alpha\dot\alpha}$ is a Hermitian matrix. For

$$F_{\ell m} = -F_{m\ell}$$

$$\sigma^{\ell}{}_{\alpha\dot\alpha}\,\sigma^{m}{}_{\beta\dot\beta}\,F_{\ell m} = \varepsilon_{\alpha\beta}\,F_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,F_{\alpha\beta}$$

where $F_{\alpha\beta}$, $F_{\dot\alpha\dot\beta}$ are symmetric and if $F_{\ell m}$ is real, they are complex conjugate of each other.

For Dirac adjoint we obtain

$$\Psi^{+} = (\chi_{\dot\alpha}^{*}, \Psi^{\dot\alpha *}) = (\bar\chi_{\dot\alpha}, \Psi^{\alpha})$$

$$\bar\Psi = \Psi^{+}\gamma^{0} = -\Psi^{+}\gamma_{0} = (-i)(\bar\chi_{\dot\alpha}, \Psi^{\alpha})\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = -i(\Psi^{\alpha}, \bar\chi_{\dot\alpha})$$

Let

$$\Psi = \begin{pmatrix} \chi_{\alpha} \\ \bar\Psi^{\dot\alpha} \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} \xi_{\alpha} \\ \bar\eta^{\dot\alpha} \end{pmatrix}$$

then the bilinears are expressed as:

$$\bar\Psi\phi = -i\,(\Psi\xi + \bar\chi\bar\eta)$$

$$\bar{\Psi}\,(-i\gamma_5)\phi = -i(\Psi\xi - \bar{\chi}\bar{\eta}) \quad \text{etc.}$$

In Weyl representation we note

$$\frac{1}{2}\,(1 - i\gamma_5)\Psi = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix} ,$$

$$\frac{1}{2}\,(1 + i\gamma_5)\Psi = \begin{pmatrix} 0 \\ \bar{\Psi}_{\dot\alpha} \end{pmatrix} ,$$

$$\bar{\Psi}^{\dot\alpha} = \xi^{\dot\alpha\dot\beta}\bar{\Psi}_{\dot\beta} = \left[i\sigma_2\,(\Psi^*_\beta) \right]^{\dot\alpha}$$

and the Majorana spinor takes the form

$$\Psi_M \equiv \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \qquad ; \qquad \bar{\Psi}^T_M = -i \begin{pmatrix} \chi^\alpha \\ \bar{\chi}_{\dot\alpha} \end{pmatrix}$$

so that

$$\Psi^C_M \equiv C\,\bar{\Psi}^T_M = \begin{pmatrix} -i\sigma_2(\chi^\alpha) \\ +i\sigma_2(\chi_{\dot\alpha}) \end{pmatrix} = \begin{pmatrix} \xi_{\alpha\beta}\chi^\beta \\ \xi^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} = \Psi_M$$

The <u>Charge Conjugation matrix C</u> in a representation satisfying $(\gamma^0)^2 = -I, \gamma^0\gamma^{\ell+}\gamma^0 = \gamma^\ell$ satisfies

$$C^{-1}\,\gamma^\ell\,C = -\,\gamma^{\ell T}$$

$$C^T = -C$$

The charge conjugate spinor Ψ^C is defined to be $\Psi^C = C\,\bar{\Psi}^T = C\,\gamma^{0T}\,\Psi^*$. If Ψ satisfies Dirac equation for a particle of charge e then Ψ^C satisfies the Dirac equation for a particle with charge (-e). In Weyl representation $C = -\,\gamma^0\gamma^2$.

We remark that we may obtain the so called Majorana representation for γ-matrices by a unitary transformation of the matrices in Weyl representation:

$$\gamma^\ell_{Maj} = U\,\gamma^\ell_{Weyl}\,U^+$$

$$U = \frac{1}{\sqrt{2}}\,(I + i\,\gamma^2_{Weyl})$$

The representation has the convenient properties:

$$\gamma^0_M\gamma^\ell_M\gamma^0_M = \gamma^{\ell+}_M = \gamma^{\ell T}_M$$

$$C_M = -\,\gamma^{0T}_M$$

so that charge conjugate coincides with complex conjugation.

A <u>Majorana Spinor</u> Ψ_M is defined (in any representation) to be a Dirac spinor satisfying the Majorana condition: $\Psi^C_M = \Psi_M$ leading to $\bar{\Psi}_M = -\,\Psi^T\,C^{-1}$. In Weyl representation it is essentially defined by a 2-spinor χ_α and thus has four real components.

We will assume that the spinor components anticommute, viz, $\Psi_\alpha\phi_\beta = -\phi_\beta\Psi_\alpha$. The <u>Complex Conjugation</u> operation is defined such that the order of anti-commuting factors is reversed. Thus

$$(\bar{\Psi}\;O\;\phi)^* = \phi^*_\beta\,(\gamma^0 O)^+_{\beta\alpha}\,\Psi_\alpha = \bar{\phi}\,\gamma^0\,O^+\,\gamma^0\,\Psi$$

where we used $(\gamma^0)^2 = -I$ and $\gamma^{0+} = -\,\gamma^0$.

For Majorana spinors we may derive the following symmetry relation

$$\bar{\xi}\, O\, \eta = \bar{\eta}\, (C^{-1}\, O\, C)^T\, \xi$$

Thus we obtain:

$$(\bar{\Psi}\phi) = (\bar{\phi}\Psi) = -(\bar{\Psi}\phi)^*$$

$$(\bar{\Psi}\gamma^\ell\phi) = -(\bar{\phi}\gamma^\ell\Psi) = -(\bar{\Psi}\gamma^\ell\phi)^*$$

$$(\bar{\Psi}\gamma_5\phi) = (\bar{\phi}\gamma_5\Psi) = -(\bar{\Psi}\gamma_5\phi)^*$$

$$(\bar{\Psi}\gamma_5\gamma^\ell\phi) = (\bar{\phi}\gamma_5\gamma^\ell\Psi) = -(\bar{\Psi}\gamma_5\gamma^\ell\phi)^*$$

$$(\bar{\Psi}\sigma^{\ell m}\phi) = -(\bar{\phi}\sigma^{\ell m}\Psi) = (\bar{\Psi}\sigma^{\ell m}\phi)^*$$

In 2-component notation, for example,

$$\chi\sigma^m\bar{\Psi} \equiv \chi^\alpha\sigma^m_{\alpha\dot{\beta}}\,\bar{\Psi}^{\dot{\beta}} = -\bar{\Psi}^{\dot{\beta}}\,\sigma^m_{\alpha\dot{\beta}}\,\chi^\alpha = -\bar{\Psi}_{\dot{\beta}}\,\bar{\sigma}^{m\dot{\beta}\alpha}\,\chi_\alpha = -\bar{\Psi}\,\bar{\sigma}^m\,\chi$$

and

$$(\chi\sigma^m\bar{\Psi})^* = -\bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^{m\dot{\beta}\alpha})^*\,\Psi_\beta = \Psi_\beta\,\sigma^{m\beta\dot{\alpha}}\,\chi_{\dot{\alpha}} = \Psi\,\sigma^m\,\bar{\chi} = -\bar{\chi}\,\bar{\sigma}^m\,\Psi$$

It follows that for Majorana spinor ξ, $\bar{\xi}\,\gamma^\ell\,\xi = \bar{\xi}\,\sigma^{\ell m}\,\xi = 0$, and only axial vector, scalar and pseudoscalar survive.

The parity and time-inversion operations can also be realized on four spinors:

$$\Psi'(\chi') = S(\Lambda)\;\Psi(\chi)$$

where we require that $S(\Lambda)$ corresponding to these discrete operations satisfies

$$\gamma^\ell = \Lambda^\ell{}_m\; S(\Lambda)\;\gamma^m\; S^{-1}(\Lambda)$$

as well as the relations corresponding to the equations

$$\Lambda_S^{-1}\,\Lambda_R\,\Lambda_S = \Lambda_t^{-1}\,\Lambda_R\,\Lambda_t = \Lambda_R \quad ,$$

$$\Lambda_S^{-1}\,\Lambda_L\,\Lambda_S = \Lambda_t^{-1}\,\Lambda_L\,\Lambda_t = \Lambda_L^{-1}$$

Here Λ_S, Λ_t are space reflection and time inversion matrices and Λ_R,Λ_L correspond to space rotations and pure Lorentz transformation. For example, for space reflection

$$\Psi'(-x^0,-\vec{x}) = i\eta\;\gamma^0\;\Psi(x^0,\vec{x})$$

where η is intrinsic parity, $\eta = \pm 1, \pm i$. It follows that $\Psi'^C(x^0,-\vec{x}) = -i\eta^*\gamma^0\Psi^C(x^0,\vec{x})$. For Majorana spinor we must have $\eta = \pm\, i$.

Finally, we mention the Fierz rearrangement theorem. If

$$I_\alpha = (M\Psi)_\alpha\;(\bar{\lambda}\;N\;\chi)$$

where M, N are operators and Ψ,λ,χ are anticommuting Majorana spinors then we may rewrite

$$I_\alpha = -\frac{1}{4}\sum_A\;(\bar{\lambda}\;O^A\;\Psi)(M\;O_A\;N\;\chi)_\alpha$$

where

$$O^A = \{I,\;\gamma^\ell,\;\sigma^{\ell m},\;\gamma_5\gamma^\ell,\;\gamma_5\}$$

$A = 1 \ldots 16$, is the complete set satisfying $Tr \ (O^A O^B) = 4\ \delta^{AB}$.

2 - SUPERSYMMETRY

2.1 - GLOBAL SYPERSYMMETRY

Supersymmetry is the symmetry between bosonic and fermionic variables. In the contex of Lagrangian field theory models it is the symmetry between bosonic and fermionic fields. Consider a model with one scalar field A(X) and a Majorana spinor field χ described by the action

$$S = \int L\ d^4x = \int d^4x \left[-\frac{1}{2} (\partial_\ell A)(\partial^\ell A) - \frac{i}{2} \bar{\chi}\ \gamma \cdot \partial\ \chi \right]$$

The action is invariant under global s.s transformations:

$$\delta A = i\ \bar{\varepsilon}\ \chi \equiv (\bar{\varepsilon}\ Q)\ A$$

$$\delta\chi = (\not\partial\ A)\ \varepsilon \equiv (\bar{\varepsilon}\ Q)\ \chi$$

where ε is a constant Majorana spinor, $\not\partial = \gamma^\ell \partial_\ell$ and Q are (Majorana spinor) generators of supersymmetry transformations.

We remark that dimension $L \equiv [L] = 4$. It follows from the form of (kinetic terms) Lagrangian that $[A] = 1$, $[\chi] = 3/2$. Thus there is dimensional gap between hosonic field A and fermionic field which is filled in by ε with $[\varepsilon] = -\frac{1}{2}$ in the bose-fermi transformation δA. The transformation $\delta\chi$ then must involve a derivative as follows by dimensional arguments.

Consider now the commutators:

$$[\delta(\varepsilon_2), \delta(\varepsilon_1)]\ A = 2i\ \bar{\varepsilon}, \gamma^\ell\ \varepsilon_2(\partial_\ell A) = [\bar{\varepsilon}_2 Q, \bar{\varepsilon}, Q]\ A$$

The fermionic charges Q anticommute with constant parameters ε_1, ε_2 and we easily derive

$$\{Q_\alpha, Q_\beta\}_+ A = 2i\ (\gamma^\ell C)_{\alpha\beta}\ \partial_\ell\ A = -\ 2(\gamma^\ell C)_{\alpha\beta}\ P_\ell$$

where $P_\ell = -i\ \partial_\ell$ is the translation operator. The s.s charges close into space-time translations through an anticommutator. In other words if we are to have a closure of the algebra involving generators of s.s transformations, translations must also be included in the algebra. When the global s.s is lifted to a local s.s e.g. $\varepsilon = \varepsilon(x)$, we will involve translations over "distance" $[\bar{\varepsilon}_1(x)\ \gamma^\ell\ \varepsilon_2(x)]$ which depends on the coordinate x. In a sense we may expect in the case of local s.s. the appearence of general coordinate transformations.

In the case under consideration we do not obtain the closure for the χ commutator

$$[\delta(\varepsilon_2), \delta(\varepsilon_1)]\ \chi \neq 2i\ (\bar{\varepsilon}_1\ \gamma^\ell \varepsilon_2)\ \partial_\ell\ \chi$$

even when we use equations of motion (on-shell).

2.2 - ON-SHELL WESS-ZUMRINO MODEL

In this model the algebra of generators closes on shell. The model uses two spin zero fields and one spin 1/2 Majorana field. The Lagrangian

$$L = -\frac{1}{2} (\partial_\ell\ A)^2 - \frac{1}{2} (\partial_\ell\ B)^2 - \frac{i}{2} \bar{\chi} \not\partial\ \chi$$

is invariant under global (rigid) s.s. transformations:

$$\delta A = i\ \bar{\varepsilon}\ \chi$$
$$\delta B = i\ \bar{\varepsilon}\ \gamma_5\ \chi$$
$$\delta\chi = \not\partial\ (A + B\ \gamma_5)\ \varepsilon$$

In fact

$$\delta L = -\frac{i}{2}(\partial_\ell K^\ell)$$

$$K^\ell = \bar{\varepsilon}\,\delta^\ell\,\partial\!\!\!/\,(A + B\,\gamma_5)\,\chi$$

The equations of motion are

$$\Box A = \Box B = \partial\!\!\!/\,\chi = 0$$

and

$$[\delta(\varepsilon_2),\ \delta(\varepsilon_1)]\ (\text{Field}) \overset{\Omega}{=} 2i\ (\bar{\varepsilon}_1\ \gamma^\ell\ \varepsilon_2)\ \partial_\ell\ (\text{Field})$$

Here $\overset{\Omega}{=}$ indicates that we have used equations of motion to obtain the equality. Thus the super-multiplet (A,B,χ) furnishes a representation on-shell of the s.s. algebra (*)

$$\{Q_\alpha,\ Q_\beta\} = -2\ (\gamma^\ell C)_{\alpha\beta}\ P_\ell$$

$$[P_\ell,\ Q_\alpha] = 0$$

From the definition of Q as a Dirac spinor we also have

$$[Q_\alpha,\ M_{\ell m}] = i\ (\sigma_{\ell m})_{\alpha\beta}\ Q_\beta$$

The algebra generated by Q_α, P_ℓ, $M_{\ell m}$ is supersymmetric extension of Poincaré algebra. It has "odd" elements Q_α and "even" elements P_ℓ, $M_{\ell m}$. It is a superalgebra with Z_2 grading such that

(i) The bracket of two generators is always antisymmetric except for two "odd" elements when it is symmetric.

(ii) The "odd" generators form a representation of the ordinary Lie algebra spanned by the "even" elements.

(iii) The Jacobi identities are modified due to grading.

2.3 - N-EXTENDED SUPER-POINCARÉ ALGEBRA: SP_4^N

(Sohnius, Haag and Lopuszanski, Nucl.Phys., B88, 61 (1975)).

In 4-dimensions the most general supersymmetry algebra, consistent with Poincaré invariance and certain requirements of the properties of S-matrix coming from a relativistic quantum field theory has the following structure:

$$\{Q^i_\alpha,\ Q^{j}_\beta\} = (\gamma^\ell C)_{\alpha\beta}\ P_\ell\ \delta^{ij} + C_{\alpha\beta}\ U^{ij} + (\gamma_5 C)_{\alpha\beta}\ V^{ij}$$

$$[U^{ij},Q^{i'}_\alpha] = [V^{ij},Q^{i'}_\alpha] = [U^{ij},V^{i'j'}] = [U^{ij},U^{i'j'}] = [V^{ij},V^{i'j'}] = 0$$

$$[Q^i_\alpha,\ M_{\ell m}] = i\ (\sigma_{\ell m})_\alpha^{\ \beta}\ Q^i_\beta$$

(*) N.B.: Under space translations $\delta x^\ell = \xi^\ell$ a field $\phi(x)$ transforms as $\delta\phi(x) = -\xi^\ell\partial_\ell\phi(x)$, etc. Writing $P_\ell = -i\partial_\ell$ as the generator of space translations $\delta x^\ell = i(\xi\cdot P)\,x^\ell$, $\delta\phi(x) = -i(\xi\cdot P)\phi(x)$. If then follows, for example, $[\delta(\varepsilon),\delta(\xi)]A(x) = -i[\xi\cdot P,\bar{\varepsilon}Q]A(x) = 0$ leading to $[P^\ell,Q_\alpha] = 0$.

$$[Q^i_\alpha, P_\ell] = 0$$

$$[M_{\ell m}, M_{pq}] = i\,(\eta_{\ell p} M_{mq} - \ldots + \ldots - \ldots)$$

$$[M_{\ell m}, P_n] = i\,(\eta_{\ell n} P_m - \eta_{mn} P_\ell)$$

$$[P_\ell, P_m] = 0$$

U^{ij} and V^{ij} where $i,j = 1 \ldots N$ are internal symmetry indices are present only for $N > 1$. Also $U^{ij} = - U^{ji}, V^{ij} = - V^{ji}$ are P_4 invariant operators. U, V belong to the centre of SP_4^N and called "Central Charges". They have important consequences on the structure of the representations acting on one particle states. Central Charges have dim = + 1; they occur only in field theories where there is a dimensional parameter, say,

(a) a mass parameter in L.

(b) the energy scale introduced via spontaneous breakdown of internal symmetry.

In the absence of central charges the SP_4^N algebra has a U(N) symmetry. This symmetry is relativistic: $[U(N), P_4] = 0$.

The N-charges Q^i_α are Majorana spinors

$$Q^i = \begin{pmatrix} Q^i_\alpha \\ \bar{Q}^{\dot\alpha}_i \end{pmatrix}$$

We note that

$$\{Q_\alpha, Q_\beta\} = (\gamma^\ell C)_{\alpha\beta} P_\ell = (\not{P} C)_{\alpha\beta}$$

implies

$$\{Q_\alpha, \bar{Q}_\beta\} = - (\gamma^\ell)_{\alpha\beta} P_\ell = - (\not{P})_{\alpha\beta}$$

where

$$\bar{Q} = Q^t \gamma^0 \qquad \text{and} \qquad \bar{Q}^T_\alpha = C^{-1}_{\alpha\beta} Q_\beta \qquad \text{and} \qquad \{\bar\xi Q, \bar\eta Q\} = - (\bar\xi \gamma^\ell \eta) P_\ell$$

Thus Q_α in a sense are "square roots" of Dirac (operator) equation. On tracing with γ^0 follows the important relation:

$$H = - P_0 = \frac{1}{4} \sum_{\alpha=1}^{4} (Q_\alpha Q^*_\alpha + Q^*_\alpha Q_\alpha)$$

This holds even in the presence of central charges.

The global or rigid s.s is the square of translation operator and one expects that local s.s should be the square root of general relativity. This result is essentially the outerme of different dimensions of boson and fermion fields. For local s.s case one expects

$$[\delta\varepsilon_1(x), \delta\varepsilon_2(x)]\, B \sim [\bar\varepsilon_2(x)\, \gamma^\ell \varepsilon_1(x)]\, \partial_\ell\, B + \ldots$$

One would thus be led to translations over distance $\bar\varepsilon_2(x)\, \gamma^\ell\, \varepsilon_1(x)$ which differ from point to point. This is the idea of general coordinate transformations. Thus local s.s should lead to gravity. This is, however, only a heuristic argument.

2.4 - OFF-SHELL WESS-ZUMINO MODEL: ($SP_4^{N=1}$) Auxiliary Fields

We would like to have off-shell closure of s.s algebra. It was slowly realized that in a s.s field theory we should have equal numbers of bosonic and fermionic components. In on-shell model discussed in 2.2 we have

A, B 2 bosonic

$\bar{\chi}^{\dot{\alpha}}, \chi_\alpha, \alpha = 1,2$ 4 fermionic

We need to add 2 bosonic fields F, G. Consider

$$L_{kim} = -\frac{1}{2}(\partial_\ell A)^2 - \frac{1}{2}(\partial_\ell B) = -\frac{i}{2}\bar{\chi}\not\partial\chi + \frac{1}{2}F^2 + \frac{1}{2}G^2$$

which is invariant under rigid s.s transformations

$$\delta A = i\bar{\varepsilon}\chi$$

$$\delta B = i\bar{\varepsilon}\gamma_5\chi$$

$$\delta\chi = \{F + \gamma_5 G + \not\partial(A + B\gamma_5)\}\varepsilon$$

$$\delta F = i\bar{\varepsilon}\not\partial\chi$$

$$\delta G = i\bar{\varepsilon}\gamma_5\not\partial\chi$$

We may add also invariant L_{mass} and L_{int} terms:

$$L_{mass} = m\,(FA + GB - \frac{i}{2}\bar{\chi}\chi)$$

$$L_{int} = g\,[F(A^2 - B^2) + 2G\,A\,B - i\,\bar{\chi}\,(A - B\gamma_5)\,\chi]$$

The "auxiliary fields" F and G allow us to close the algebra without using equations of motion (off-shell):

$$[\delta_2, \delta_1]\,\chi = 2i(\bar{\varepsilon}_1\gamma^\ell\varepsilon_2)\,\partial_\ell\,\chi$$

The $\not\partial\chi$ term is now cancelled due to the contributions from F, G fields. The super-multiplet (A, B, χ, F, G) called scalar or chiral multiplet realizes the s.s (global) algebra off-shell.

The fields A, B, χ give rise to a representation on single particle states of s.s algebra (see latter) - the lowest spin representation of s.s with N = 1.

The auxiliary fields F, G satisfy purely algebraic equations of motion. They are important to keep the s.s transformation laws linear and allow off-shell closure of the algebra. This in turn allows us to build a tensor calculus. Quantum rules and super-Feynman rules can be build (using superfield formulation).

Rigid (or global) s.s invariance implies conserved noether current:

$$J_\ell = \not\partial\,(A - B\gamma_5)\,\gamma_\ell\,\chi - (F + G\gamma_5)\,\gamma_\ell\,\chi$$

$$\partial_\ell J^\ell_\alpha = 0 \qquad ; \qquad Q_\alpha = \int d^3x\, J^0_\alpha$$

The field equations for auxiliary fields are

$$F + mA + g(A^2 - B^2) = 0$$

$$G + mB + 2g\,AB = 0$$

Eliminating F, G we obtain the Lagrangian

$$L = -\frac{1}{2}(\partial_\ell A)^2 - \frac{1}{2}(\partial_\ell B)^2 - \frac{1}{2}m^2(A^2+B^2)$$

$$-\frac{i}{2}\bar{\chi}(\partial\!\!\!/ + m)\chi - g\,m\,A\,(A^2+B^2)$$

$$-\frac{g^2}{2}(A^2+B^2)^2 - i\,g\,\bar{\chi}(A - B\gamma_5)\chi$$

which is supersymmetric extension of ϕ^3, ϕ^4 or Yukawa theory.

The reulting s.s of the Lagrangian not only implies the equality of masses for A, B and χ but also a precise relationship between the interaction terms and is responsible for high degree of renormalizability.

We note in passing that the multiplet (A,χ) initially considered has nothing to do with s.s it is not a representation of s.s on or off-shell. In fact, not even the on-shell states have the equal number of bose and fermi degrees of freedom required to form an irreducible representation of supersymmetry.

2.5 - $(1,\frac{1}{2})$ GLOBAL S.S MULTIPLET: (γ,ν)

Another example of a s.s multiplet is given by

$$L = -\frac{1}{4}(\partial_\ell V_m - \partial_m V_\ell)^2 - \frac{1}{2}\bar{\lambda}\,\partial\!\!\!/\,\lambda + \frac{1}{2}D^2$$

is invariant under global s.s transformations

$$\delta V_\ell = -\bar{\varepsilon}\,\gamma_\ell\,\lambda$$

$$\delta\lambda = F_{\ell m}\,\sigma^{\ell m}\,\varepsilon + i\,\gamma_5\,D\,\varepsilon$$

$$\delta D = i\,\bar{\varepsilon}\,\gamma_5\,\partial\!\!\!/\,\lambda$$

Then

$$[\delta_1,\delta_2]\,\lambda = 2\,\bar{\varepsilon}_2\,\gamma^\ell\,\varepsilon,\ \partial_\ell\,\lambda$$

But if we eliminate D first by using field equations for it and insert in L (eq. D = 0) one finds

$$[\delta_1,\ \delta_2] = 2\,\bar{\varepsilon}_2\,\gamma^\ell\,\varepsilon_1\,\partial_\ell\,\lambda + \frac{1}{2}(\bar{\varepsilon}_1\,\gamma^\ell\,\varepsilon_2)\,\gamma_\ell\,\partial\!\!\!/\lambda + (\bar{\varepsilon}_1\,\sigma^{\ell m}\,\varepsilon_2)\,\sigma_{\ell m}\,\partial\!\!\!/\lambda$$

For boson fields, even in absence of auxiliary fields the algebra closes due to dimensional considerations.

3 - REALIZATION OF S.S ALGEBRA ON COORDINATE SUPERSPACE

3.1 - METHOD OF INDUCED REPRESENTATION

Consider first the realization of Pincaré Group P on Minskowski space. P is defined by

$$P = \{(a_\ell,\Lambda)\,|\,(a',\Lambda')(a,\Lambda) = (a' + \Lambda' a,\ \Lambda'\Lambda)\}$$

where

a_ℓ : four-vector of translations
$\Lambda = (\Lambda^\ell m)$: Lorentz rotations, $\Lambda^T\,\eta\,\Lambda = \eta$

and a_ℓ, $\Lambda^\ell m$ are real. P has subgroups

$T_4 = \{(a,I) | (a',I)(a,I) = (a' + a,I)\}$

$L = \{(0,\Lambda) | (0,\Lambda')(0,\Lambda) = (0,\Lambda'\Lambda)\}$

and

$T_4 < P \quad , \quad L \subset P$

Since $(a,\Lambda) = (a,I)(0,\Lambda)$ the coset decomposition w.r.t $L \subset P$ (L is closed) is $P/L = \{(a,I)L\}$. The coset space P/L may be used for a realization of the group P as a transformation group.

Consider an element of P/L, say, $(\chi_\ell, I)L$. Let $(a,\Lambda) \in P$ be any arbitrary element, then,

$$\begin{aligned}(a,\Lambda)(\chi,I)\,L &= (a + \Lambda\chi,\Lambda)\,L \\ &= (a + \Lambda\chi,I)(0,\Lambda)\,L \\ &= (a + \Lambda\chi,I)\,L\end{aligned}$$

Thus

$$(\chi, I)L \xrightarrow{(a,\Lambda)} (a + \Lambda\chi,I)\,L \equiv (\chi',I)\,L$$

e.g., any $(a,\Lambda) \in P$ induces a transformation on the coset space P/L parametrized by (χ,I)

$$\chi'^{\ell} = a^{\ell} + \Lambda^{\ell}{}_m\, \chi^m$$

Thus we have a realization of P on the coset space parameters. We loosely write $P/L = \{\chi_\ell\}$. The whole Poincaré group is realized on the space of 4 coordinates χ_ℓ. Even though P itself is characterized by 10 essential parameters $(a_\ell, \Lambda^{\ell}{}_m)$.

Consider now the Super-Poincaré group $SP_4^{N=1}$. A general element of SP_4 may be written as

$$e^{-i(\bar{\varepsilon}Q+\xi.P+ \frac{1}{2}\lambda.M)}$$

where

ξ_ℓ : translations
ε_α : Supersymmetry transformations
$\lambda_{\ell m}$: Lorentz rotations.

Also $L \subset SP_4$ is a closed subgroup. The coset space SP_4/L consists of the elements of the form

$$e^{-i(\bar{\theta}Q+\chi.P)}L$$

as is clear from

$$e^A\, e^B = e^{A+B+ \frac{1}{2}(A,B)+\dots}$$

and the commutation relations of Q,P,M. Considering left multiplication on a fixed element of coset space we obtain a realization of s.s transformation on $(\chi_\ell,\theta_\alpha)$ coordinates or parameters.

$$g_L(x',\theta') \equiv e^{-i(\bar{\varepsilon}Q+\xi.P+ \frac{1}{2}\lambda.m)}\, e^{-i(\bar{\theta}Q+\chi.P)}\, L$$

Set $\lambda = 0$ for convenience, $A = -i(\bar{\varepsilon}Q+\xi.P)$, $B = -i(\bar{\theta}Q+x.P)$ we find

$$[A,B] = (\bar{\varepsilon}\,\gamma^{\ell}\,\theta)\,P_\ell \quad , \quad [[A,B],B] = 0 \quad ,$$

$$g_L(x',\theta') = e^{-i\,[(\bar{\varepsilon} + \bar{\theta})\,Q + (\xi + x).P] + \frac{1}{2}\,\bar{\varepsilon}\,\gamma^{\ell}\,\theta\,P_\ell}\, L$$

Thus, the induced representation on superspace coordinates is given by

$$x'_\ell = x_\ell + \xi_\ell + \frac{i}{2}(\bar\varepsilon\,\gamma_\ell\,\theta)$$

(Salam and Strathedee)

$$\theta'_\alpha = \theta_\alpha + \varepsilon_\alpha$$

We note:

$$\delta^L_P\, x_\ell = \xi_\ell \quad , \quad \delta^L_P\,\theta_\alpha = 0$$

$$\delta^L_Q\, x_\ell = \frac{i}{2}\bar\varepsilon\,\gamma_\ell\,\theta \quad , \quad \delta^L_Q\,\theta_\alpha = \varepsilon_\alpha \quad , \quad \delta^L_Q\,\bar\theta_\alpha = \bar\varepsilon_\alpha$$

We remind $[i\,\bar\varepsilon\,\gamma_\ell\,\theta]^* = i\,\bar\varepsilon\,\gamma_\ell\,\theta$ is real though nilpotent, $(\bar\varepsilon\,\gamma_\Omega\,\theta)^5 = 0$. We loosely write $\frac{SP}{L} = \{x^\ell,\theta_\alpha\}$. Next set

$$\varepsilon = \xi = 0 \quad , \quad \lambda \neq 0:$$

Using

$$e^A\, e^B = e^{B+[A,B]+\dots}\, e^A$$

we derive

$$(x'.P) = (x.P) + \frac{1}{2}\lambda_{\ell m}\,(x^\ell\,P^m - x^m\,P^\ell)$$

$$\theta'_\alpha = \bar\theta_\alpha - \frac{1}{2}\lambda_{\ell m}\,(\bar\theta\,\sigma^{\ell m})_\alpha$$

so that for infinitesimal transformations

$$\delta^L_M\, x^\ell = -\lambda^\ell_m\, x^m$$

$$\delta^L_M\,\bar\theta_\alpha = -\frac{1}{2}\lambda_{\ell m}\,(\bar\theta\,\sigma^{\ell m})_\alpha$$

$$\delta^L_M\,\theta_\alpha = +\frac{1}{2}\lambda_{\ell m}\,(\sigma^{\ell m}\,\theta)_\alpha$$

We may have as well used right multiplication on the right cosets

$$g_R(x'',\theta'') = L\, e^{-i(\bar\theta Q + x.P)}\; e^{-i(\bar\varepsilon Q + \xi.P + \frac{1}{2}\lambda.M)}$$

We get

$$x''_\ell = x_\ell + \xi_\ell - \frac{i}{2}\bar\varepsilon\,\gamma_\ell\,\theta$$

$$\theta''_\alpha = \theta_\alpha + \varepsilon_\alpha$$

$$x''\,.\,P = x\,.\,P - \frac{1}{2}\lambda_{\ell m}\,(x^\ell P^m - x^m\,P^\ell)$$

$$\bar\theta''_\alpha = \bar\theta_\alpha + \frac{1}{2}\lambda_{\ell m}\,(\bar\theta\,\sigma^{\ell m})_\alpha$$

which lead to

$$\delta^R_Q\,\theta_\alpha = \delta^L_Q\,\theta_\alpha \quad , \quad \delta^R_Q\, x_\ell = -\,\delta^L_Q\, x_\ell$$

$$\delta^R_P\, x_\ell = \delta^L_P\, x_\ell \quad , \quad \delta^R_M\, x^\ell = -\,\delta^L_M\, x^\ell$$

$$\delta^R_P\,\theta_\alpha = \delta^L_P\,\theta_\alpha = 0 \;, \quad \delta^R_M\,\theta_\alpha = -\,\delta^L_M\,\theta_\alpha$$

Thus s.s. transformations (rigid) may be realized as transformation group over the superspace coordinates (x_ℓ,θ_α); the 8 coordinates spanning a superspace (Salam and Strathdee)

$$[x_\ell, x_m] = [x_\ell,\theta_\alpha] = 0 \quad , \quad \{\theta_\alpha,\theta_\beta\} = 0$$

e.g. $(\theta_1)^2 = 0$. θ_α are called Grassmann coordinates.

The s.s group of Wess and Zumino may be thought of as arising from an extended space-time. θ_α we remind are Majorana spinors and transform like Dirac spinor under Lorentz transformations.

3.2 - GENERATORS IN COORDINATE REPRESENTATION

We may now obtain a representation of generators acting on superspace coordinates. We have

$$\delta^L_P\, x^\ell = \delta^R_P\, x^\ell = \xi^\ell = \xi^m \frac{\partial}{\partial x^m} x^\ell \equiv i\ (\xi\ .\ P^L) x^\ell = i\ (\xi\ .\ P^R)\ x^\ell$$

so that

$$P^L_\ell = P^R_\ell = -i\ \ \partial_\ell$$

Also

$$\delta^L_P(\xi)\ \theta_\alpha = i(\xi.P)\ \theta_\alpha = \delta^R_P(\xi)\ \theta_\alpha = 0$$

For Lorentz transformations:

$$\delta^L_M(\lambda)\ \ (x^\ell,\theta_\alpha) = (-\ \lambda^\ell_m\ x^m,\ \frac{1}{2}\ (\lambda.\sigma\theta)_\alpha)$$

From commutation relations (M,P) and these variations we find:

$$M^{\ell m} = (x^\ell P^m - x^m P^\ell) - i(\sigma^{\ell m}\theta)_\alpha \frac{\partial}{\partial\theta_\alpha}$$

if

$$\delta^L_M\ (x^\ell,\theta_\alpha) \equiv \frac{i}{2}\ (\lambda.M)\ \ (x^\ell,\theta_\alpha)$$

For the case of s.s transformations

$$\delta^L_Q\, x^\ell = \frac{i}{2}\ \bar\varepsilon\ \gamma^\ell\ \theta = \frac{i}{2}\ \bar\varepsilon_\alpha(\gamma^m\theta)_\alpha\ \frac{\partial}{\partial x^m}\ x^\ell$$

$$\delta^L_Q\ \theta_\alpha = \varepsilon_\alpha = \varepsilon_\beta\ \frac{\partial}{\partial\theta_\beta}\ \theta_\alpha = -\bar\varepsilon_\sigma C_{\sigma\beta}\ \frac{\partial}{\partial\theta_\beta}\ \theta_\alpha \qquad ;\ \theta = C\theta^{-T}$$

$$\theta^T = \bar\theta C^T$$
$$= -\bar\theta C$$

$$\delta^L_Q(x^\ell,\theta_\alpha) = \bar\varepsilon_{\alpha'}\left[-C_{\alpha'\beta}\ \frac{\partial}{\partial\theta_\beta} + \frac{i}{2}\ (\gamma^m\theta)_{\alpha'}\ \frac{\partial}{\partial x^m}\right](x^\ell,\theta_\alpha) \equiv i(\bar\varepsilon.Q^L)(x^\ell,\theta_\alpha)$$

where

$$Q^L_\alpha = i\ C_{\alpha\beta}\ \frac{\partial}{\partial\theta_\beta} + \frac{1}{2}\ (\gamma^m\theta)_\alpha\ \frac{\partial}{\partial x^m} = \left[i\,C\frac{\partial}{\partial\theta} + \frac{1}{2}\ (\gamma^m\theta)\ \frac{\partial}{\partial x^m}\right]_\alpha$$

We may check $[Q_\alpha, M_{\ell m}] = i(\sigma_{\ell m})_{\alpha\beta}Q_\beta$. From $\delta^R_Q(x^\ell,\theta_\alpha) = \delta^L_Q(-x^\ell,\theta_\alpha)$ it follows:

$$Q^R_\alpha = \left[iC\ \frac{\partial}{\partial\theta} - \frac{1}{2}\ (\gamma^m\theta)\ \frac{\partial}{\partial x^m}\right]_\alpha$$

We may derive a commutation relation between Q^L and Q^R as follows:

$$\delta^L_Q(\eta)\ \gamma^R_Q(\varepsilon)\ x^\ell = \delta^L_Q(\eta)\ \ (-\ \frac{i}{2}\ \bar\varepsilon\ \gamma^\ell\ \theta) = -\ \frac{i}{2}\ (\bar\varepsilon\ \gamma^\ell\ \eta)$$

$$\delta^R_Q(\varepsilon)\ \delta^L_Q(\eta)\ x^\ell = \delta^R_Q(\varepsilon)\ \ (\frac{i}{2}\ \bar\eta\ \gamma^\ell\ \theta) = \frac{i}{2}\ \bar\eta\ \delta^\ell\ \varepsilon = -\ \frac{i}{2}\ (\bar\varepsilon\ \gamma^\ell\ \eta)$$

$$\delta^L_Q(\eta)\ \delta^R_Q(\varepsilon)\ \theta_\alpha = \delta^R_Q(\varepsilon)\ \delta^L_Q(\eta)\ \theta_\alpha = 0$$

Thus we may write

$$\delta_Q^L(\eta)\ \delta_Q^R(\varepsilon) = \delta_Q^R(\varepsilon)\ \delta_Q^L(\eta)$$

or

$$(\bar{\eta}\ .\ Q^L)(\bar{\varepsilon}\ .\ Q^R) = (\bar{\varepsilon}\ .\ Q^E)\ (\bar{\eta}\ .Q^L)$$

$$-\ \bar{\eta}_\alpha\ \bar{\varepsilon}_\beta\ Q^L_\alpha\ Q^R_\beta = \bar{\varepsilon}_\beta\ Q^R_\beta\ \bar{\eta}_\alpha\ Q^L_\alpha = \bar{\eta}_\alpha\ \bar{\varepsilon}_\beta\ Q^R_\beta\ Q^L_\alpha$$

We obtain

$$\{Q^L_\alpha,\ Q^R_\beta\}_+ = 0$$

We verify directly

$$\{Q^L_\alpha, Q^L_\beta\} = -\ i\ (\gamma^\ell C)_{\alpha\beta}\ \partial_\ell = (\gamma^\ell C)\ P_\ell$$

and it follows by inspection

$$\{Q^R_\alpha, Q^R_\beta\} = -\ (\gamma^\ell C)\ P_\ell$$

3.3 - SCALAR SUPERFIELD $\phi(x,\theta)$

Over superspace may be defined by

$$\phi'\ (x',\theta') = \phi(x,\theta)$$

Then for infinitesimal transformations

$$\delta\phi(x,\theta) \equiv \phi'\ (x,\theta) - \phi(x,\theta) = -\left[\delta x^\ell \frac{\partial}{\partial x^\ell} + \delta\theta_\alpha \frac{\partial}{\partial\theta_\alpha}\right]\phi(x,\theta) + \ldots$$

Note that we always use left or ordinary derivative and $\delta\theta_\alpha$ must be kept on the left of $\frac{\partial}{\partial\theta_\alpha}$. Also $\delta x^\ell \equiv \delta^L x^\ell$, $\delta\theta_\alpha \equiv \delta^L\theta_\alpha$, etc.

$$\delta_P(\xi)\ \phi(x,\theta) = -\ \xi^\ell\ \partial_\ell\ \phi = -i(\xi.P)\ \phi(x,\theta)$$

$$\delta_M(\lambda)\ \phi(x,\theta) = -\ \frac{i}{2}\ (\lambda.M)\ \phi(x,\theta)$$

$$\delta_Q(\varepsilon)\ \phi(x,\theta) = -\ \frac{i}{2}\ (\bar{\varepsilon}\ \gamma^\ell\ \theta)\ \partial_\ell\ \phi - \varepsilon_\alpha \frac{\partial}{\partial\theta_\alpha}\ \phi$$

Using

$$\varepsilon_\alpha = -\ \bar{\varepsilon}_\beta\ C_{\beta\alpha}$$

$$\varepsilon_\alpha \frac{\partial}{\partial\theta_\alpha} = -\ \bar{\varepsilon}_\beta\ C_{\beta\alpha} \frac{\partial}{\partial\theta_\alpha}$$

we may rewrite the last expression as

$$\delta_Q(\varepsilon)\ \phi = \bar{\varepsilon}_\alpha \left[C_{\alpha\beta} \frac{\partial}{\partial\theta_\beta} - \frac{i}{2}\ (\gamma^\ell\theta)_\alpha \frac{\partial}{\partial x^\ell}\right]\phi = -\ i(\bar{\varepsilon}\ Q)\ \phi$$

Thus

$$\delta\phi(x,\theta) = -i\left[(\xi.P) + \frac{1}{2}\ (\lambda.M) + \bar{\varepsilon}\ Q\right]\ \phi(x,\theta)$$

corresponding to

$$\delta(x^\ell,\theta_\alpha) = +i\left[\xi.P + \frac{1}{2}\ \lambda.M + \bar{\varepsilon}Q\right]\ (x^\ell,\theta_\alpha)$$

Here $Q_\alpha \equiv Q^L_\alpha$, $M_{\ell m} = M^L_{\ell m}$.

Since θ_α are anticommuting parameters we may expand $\phi(x,\theta)$ in terms of θ_α obtaining a finite series expansion:

$$\phi(x,\theta) = A(x) + \bar{\theta}\psi(x) + \frac{1}{4}\left[\bar{\theta}\theta\ F(x) + \bar{\theta}\ \gamma_5\ \theta\ G + \bar{\theta}\ \gamma_\ell\gamma_5\theta\ A^\ell(x) + \bar{\theta}\theta\bar{\theta}\chi(x)\right] + \frac{1}{32}\ (\bar{\theta}\theta)^2\ D(x)$$

Here $A(x)$, $\psi(x)$, $F(x)$, $A^\ell(x)$, $\chi(x)$, $G(x)$, $D(x)$ are the component fields of the scalar superfield $\phi(x,\theta)$. We remind $\bar{\theta}\ \gamma^\ell\ \theta = 0$ and $\bar{\theta}\ \sigma_{\ell m}\ \theta = 0$. Superfields are very useful in writing Lagrangians and introducing interactions just as we do in ordinary field theory.

The transformations of component fields may be worked out straightforwardly by comparing the coefficients of θ's on the two sides. Superfields are a necessity to work out superfield propagators and a systematic quantization of the theory.

It is easy to pass over to 2-component formulation:

$$Q = \underset{1,2,3,4}{(Q_\alpha)} = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix} \qquad \alpha = 1,\ 2 \qquad \dot{\alpha} = \dot{1},\ \dot{2}$$

$$\theta = \begin{pmatrix} \theta_\alpha \\ \bar{\theta}^{\dot{\alpha}} \end{pmatrix} \qquad \text{Majorana Spinors}$$

$$-\ i(\bar{\varepsilon}Q) = -(\varepsilon^\alpha\ Q_\alpha + \varepsilon_{\dot{\alpha}}\ \bar{Q}^{\dot{\alpha}})$$

From

$$\gamma^0 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad , \quad \bar{\varepsilon} = \varepsilon^t\ \gamma^0 \quad ,$$

$$C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \quad ; \quad \gamma^\ell = i\begin{pmatrix} 0 & (\sigma^\ell_{\alpha\dot{\beta}}) \\ (\bar{\sigma}^{\ell\dot{\alpha}\beta}) & 0 \end{pmatrix}$$

$$iC = \begin{pmatrix} -\varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

we easily derive

$$Q_\alpha = -\ \varepsilon_{\alpha\beta}\ \frac{\partial}{\partial\theta_\beta} + \frac{i}{2}\ (\sigma^\ell_{\alpha\dot{\beta}}\ \bar{\theta}^{\dot{\beta}})\ \frac{\partial}{\partial x^\ell}$$

$$\bar{Q}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\ \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} + \frac{i}{2}\ (\bar{\sigma}^{\ell\dot{\alpha}\beta}\ \theta_\beta)\ \frac{\partial}{\partial x^\ell}$$

Here $\alpha,\beta = 1,2$, $\dot{\alpha},\dot{\beta} = 1,2$ is clear from the context. Care has to be taken in raising and lowering indices:

$$\chi_\alpha = \varepsilon_{\alpha\beta}\ \chi^\beta \text{ but } \frac{\partial}{\partial\theta^\alpha} = -\ \varepsilon_{\alpha\beta}\ \frac{\partial}{\partial\theta_\beta} \quad \text{etc.}$$

Thus

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\ \sigma^\ell_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\ \partial_\ell.$$

3.3 - COVARIANT DERIVATIVES (SALAM AND STRATHDEE)

Spinor covariant derivative D_α must be defined since $\partial/\partial\theta_\alpha$ is not covariant under s.s transformations. It is by definition such that $\phi(x,\theta)$ and $(\bar{\eta}D)\phi(x,\theta)$ transform under s.s transformations in like fashion, viz,

$$\delta_Q\phi = - i\ (\bar{\varepsilon}Q)\ \phi \qquad ; \quad \varepsilon,\eta \quad \text{constants.}$$

$$\delta_Q \left[(\bar{\eta}D)\phi(x,\theta)\right] = - i\ (\bar{\varepsilon}Q)(\bar{\eta}D)\phi(x,\theta)$$

But

$$\delta_Q \left[(\bar{\eta}D)\phi\right] \equiv \bar{\eta}D\phi'(x,\theta) - (\bar{\eta}D)\phi(x,\theta) = (\bar{\eta}D)\delta\phi(x,\theta) = -i\ (\bar{\eta}D)(\bar{\varepsilon}Q)\phi(x,\theta)$$

Thus

$$\left[\bar{\varepsilon}Q,\bar{\eta}D\right] = 0$$

or

$$\{Q_\alpha,D_\beta\}_+ = 0$$

We may thus identify D_α with Q^R_α above, $(Q_\alpha \equiv Q^L_\alpha)$.

For the covariant derivative corresponding to $\partial/\partial x^\ell$ we note:

$$\delta_P\phi = - i\ (\xi\ .\ P)\ \phi$$

we require

$$\left[\xi\ .\ P\ ,\ \xi^\ell\ D_\ell\right] = 0 \qquad \text{or} \qquad \left[P^\ell,\ D_m\right] = 0$$

Clearly, $D_\ell = P_\ell$ since $(P_\ell,\ P_m) = 0$. The spinor covariant derivative

$$D_\alpha \equiv Q^R_\alpha = iC\ \frac{\partial}{\partial\theta} - \frac{1}{2}\ (\gamma^\ell\theta)\ \frac{\partial}{\partial x^\ell}$$

differs by a sign in the second term compared to Q_α. In 2-component notation

$$D_\alpha = - \varepsilon_{\alpha\beta}\ \frac{\partial}{\partial\theta_\beta} - \frac{i}{2}\ \sigma^\ell_{\alpha\dot{\beta}}\ \bar{\theta}^{\dot{\beta}}\ \frac{\partial}{\partial x^\ell}$$

$$\bar{D}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\ \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} \quad \frac{i}{2}\ \bar{\sigma}^{\ell\dot{\alpha}\beta}\ \theta_\beta\ \frac{\partial}{\partial x^\ell}$$

Covariant derivatives are necessary to impose covariant constraints on super-fields. Thus Wess-Zumino model is obtained by imposing on the scalar superfield the constraints $D_\alpha\phi = \bar{D}_{\dot{\alpha}}\phi = 0$.

It is clear that

$$\{D_\alpha,\ D_\beta\} = - (\gamma^\ell C)_{\alpha\beta}\ P_\ell$$

$$\{D_\alpha,\ \bar{D}_\beta\} = + (\gamma^\ell)_{\alpha\beta}\ P_\ell$$

$$\left[P_\ell,\ D_\ell\right] = 0$$

We may pass to 2-component notation:

$$\gamma^\ell C = \begin{pmatrix} 0 & -i\sigma^\ell \sigma_\alpha \\ -i\sigma_\ell \sigma_\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\ell{}_\alpha{}^{\dot\beta} \\ \bar\sigma^{\ell\dot\alpha}{}_\beta & 0 \end{pmatrix}$$

$$\{Q_\alpha, \bar Q^{\dot\beta}\} = \sigma_\alpha^{\ell\dot\beta}\, P_\ell$$

$$\{Q_\alpha, Q_\beta\} = \{\bar Q^{\dot\alpha}, \bar Q^{\dot\beta}\} = 0$$

It follows

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = \sigma^\ell_{\alpha\dot\beta}\, P_\ell = \sigma^\ell_{\dot\beta\alpha}\, P_\ell$$

In the presence of central charges we have

$$\{Q^i_\alpha, \bar Q^j_{\dot\beta}\} = \sigma^\ell_{\alpha\dot\beta}\, P_\ell\, \delta^{ij}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \varepsilon_{\alpha\beta}\, (-V^{ij} + iU^{ij}) = \varepsilon_{\alpha\beta}\, Z^{ij}$$

$$\{\bar Q^i_{\dot\alpha}, \bar Q^j_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta}\, (V^{ij} + iU^{ij})$$

For the covariant derivatives

$$\{D^i_\alpha, \bar D^j_{\dot\beta}\} = -\, \sigma^\ell_{\alpha\dot\beta}\, P_\ell\, \delta^{ij}$$

$$\{D^i_\alpha, D^j_\beta\} = 0 = \{\bar D^i_{\dot\alpha}, \bar D^j_{\dot\beta}\}$$

Covariant derivatives do not anti-commute like Grassmann variables:

$$\{\theta_{\alpha i}, \theta_{\beta j}\}_+ = \{\bar\theta^i_{\dot\alpha}, \theta^j_{\dot\beta}\} = \{\theta_{\alpha i}, \bar\theta^j_{\dot\beta}\} = 0$$

$$[x_\ell, x_m] = 0 \qquad , \qquad [x_\ell, \theta_{\alpha i}] = [x_\ell, \bar\theta_{\dot\alpha i}] = 0$$

We remark that

$$Q_\alpha = D_\alpha - (\gamma^\ell \theta)_\alpha \frac{\partial}{\partial x^\ell}$$

$$\delta_Q(\varepsilon)\, \phi(x,\theta) = -i \left[\bar\varepsilon D - (\bar\varepsilon\, \gamma^\ell\, \theta) \frac{\partial}{\partial x^\ell} \right] \phi(x,\theta)$$

Also

$$\bar\varepsilon D = -\, i(\varepsilon D + \bar\varepsilon \bar D) \quad ,$$

$$\bar\varepsilon\, \gamma^\ell\, \theta = -i(\varepsilon^\alpha, \bar\varepsilon_{\dot\alpha}) \cdot i \begin{pmatrix} 0 & \sigma^\ell_{\alpha\dot\beta} \\ \bar\sigma^{\ell\dot\alpha\beta} & 0 \end{pmatrix} \cdot \begin{pmatrix} \theta_\alpha \\ \bar\theta^{\dot\alpha} \end{pmatrix}$$

$$= \varepsilon^\alpha\, \sigma^\ell_{\alpha\dot\beta}\, \bar\theta^{\dot\beta} + \bar\varepsilon_{\dot\alpha}\, \bar\sigma^{\ell\dot\alpha\beta}\, \theta_\beta$$

$$= \varepsilon\, \sigma^\ell\, \bar\theta - \theta_\beta\, \bar\sigma^{\ell\dot\alpha\beta}\, \bar\varepsilon_{\dot\alpha}$$

$$= \varepsilon\, \sigma^\ell\, \bar\theta - \theta^\beta\, \sigma^\ell_{\beta\dot\alpha}\, \bar\varepsilon^{\dot\alpha}$$

$$= (\varepsilon\, \sigma^\ell\, \bar\theta - \theta\, \sigma^\ell\, \bar\varepsilon)$$

in 2-component notation. Thus for $\phi = \phi(x,\theta,\bar\theta)$

$$\delta_Q(\varepsilon)\phi = -\left[\varepsilon D + \bar\varepsilon\bar D + i(\theta\sigma^\ell\bar\varepsilon - \varepsilon\,\sigma^\ell\,\bar\theta)\,\partial_\ell\right]\phi(x,\theta,\bar\theta)$$

Because of anti-commutation relations of D_α any product of D can be reduced to a linear combination of 16 independent operators:

$$I,\ D_\alpha,\ \bar DD,\ \bar D\,\gamma_5\,D,\ \bar D\,\gamma_\ell\gamma_5\,D,\ \bar DDD_\alpha,\ (\bar DD)^2$$

Since Q_α's are linear differential operators we may show that

(i) linear combination of superfields is again a superfield.

(ii) product of superfields is again a superfield.

Superfields form a highly reducible representation of s.s algebra. We may reduce it by imposing covariant constraints which do not restrict their x-dependence through differential equations in x-space.

Since D and Q commute (anti) it is very convenient to define component fields of $\phi(x,\theta)$ by applying to $\phi(x,\theta)$ the D, $\bar D$'s and then setting $\theta = 0$. The order of D's is important but the difference involves extra terms which are just ordinary x-derivatives of lower dimension component fields. $\delta_Q(\varepsilon)$ on component fields thus defined is readily obtained.

3.4 - GENERAL REMARKS

Wess-Zumino supersymmetry group introduced to incorporate boson-fermion symmetry thus can be visualized as arising from a supergauge transformation on a superspace with coordinates (x^ℓ,θ_α) - (Salam and Strathdee) - in the same way as Poincaré group symmetry corresponds to Poincaré transformation group on coordinates x^ℓ. The supergauge transformations

$$x'^\ell = x^\ell + \xi^\ell + \frac{i}{2}\,\bar\varepsilon\,\gamma^\ell\,\theta$$

plus Lorentz rotations

$$\theta'_\alpha = \theta_\alpha + \varepsilon_\alpha$$

is a subgroup of general coordinate transformations of (x^ℓ,θ_α). Ignoring Lorentz rotations we have

$$dx'^\ell = dx^\ell + \frac{i}{2}\,\bar\varepsilon\,\gamma^\ell\,d\theta$$

$$d\theta'_\alpha = d\theta_\alpha$$

$$(dx'^\ell - \frac{i}{2}\,\bar\theta'\,\gamma^\ell\,d\theta') = (dx^\ell - \frac{i}{2}\,\bar\theta\,\gamma^\ell\,d\theta)$$

Thus line element invariant under Poincaré group and supergange group is

$$ds^2 = (dx^\ell - \frac{i}{2}\,\bar\theta\,\gamma^\ell\,d\theta)^2 + d\bar\theta\,(K + L\,\gamma_5)\,d\theta$$

For K = L = 0 we obtain a singular metric. Writing

$$Z^A \equiv (x^\ell,\theta_\alpha)$$

$$ds^2 = dZ^A\,g_{AB}(Z)\,dZ^B$$

we obtain, $T = (K - L\,\gamma_5)$,

$$g_{\ell m} = g_{m\ell} = \eta_{\ell m}$$

$$g_{\ell\alpha} = -\,g_{\alpha\ell} = -\,(\bar\theta\,\gamma_\ell)_\alpha\,\frac{i}{2}$$

$$g_{\alpha\beta} = - g_{\beta\alpha} = - (\bar\theta \gamma^\ell)_\alpha (\bar\theta \gamma_\ell)_\beta (\tfrac{i}{2})^2 + (T \gamma_0)_{\alpha\beta}$$

where we use Majorana representation for γ matrices: $C = \gamma_0^T = - \gamma_0$ and $\eta_{\ell m} = \mathrm{diag}(1,-1,-1,-1)$.

The inverse metric $g_{AC}\, g^{CB} = \delta_A^B$ is

$$g^{\ell m} = g^{m\ell} = \eta^{\ell m} - (\tfrac{i}{2})^2 f \bar\theta \gamma^\ell T \gamma^m \theta$$

$$g^{\ell\alpha} = g^{\alpha\ell} = - f (\tfrac{i}{2}) (T \gamma^\ell \theta)_\alpha$$

$$g^{\alpha\beta} = - g^{\beta\alpha} = f (T \gamma_0)_{\alpha\beta}$$

when

$$f = \frac{1}{(K^2 + L^2)}$$

The geometry, though flat in space-time, is curved in other sectors.

The first attempts to unify gravity with other fields were made along these lines by Arnowit, Nath and Zumino. They formulated a Super-Riemannian geometry over superspace with coordinates $Z^A = (x^\ell, \theta_\alpha)$. The hope was that $R_{AB}(x,\theta)$ considered as a superfield may lead to a unified dynamical theory of gravity with other fields.

With the definitions of super-trace and super-determinant developed by these authors it is possible to formulate a Superconformal group of Z^A. The algebra of the infinite-dimensional general covariance group over superspace can be shown to be the closure of the algebras of its three finite parameter subgroups: the special linear group, the superconformal group, and a four parameter supergauge group to desentangle the ordinary special conformal transformations. The closure of the algebras is under the modified commutators, presently called graded commutator

$$\{M_1, M_2\} = M_1 M_2 - (-1)^{M_1 M_2} M_2 M_1$$

where $M_1 (M_2)$ appearing in $(-1)^{M_1 M_2}$ indicates total number of Fermi indices appearing in the operators $M_1 (M_2)$. The algebra defined here satisfies modified Jacobi identities and can be shown to be Lie-admissible as well as Jordon admissible. This property allows to classify supersymmetry algebras and extend to the supersymmetry algebra the techniques of Lie algebra.

Grassmann anticommuting variables and supersymmetry have also found their utility in formulating a new pseudo-classical description of particles with spin. Casalluoni, Berezin and Marinov gave such a description for S = 1/2 particle. The generalization for any spin was subsequently given (N.C.,19,239 ('77); Phip.Rev.D15, 3568 (77)c/Nivaldo).

3.5 - INTEGRATION OVER GRASSMANN VARIABLES

In order to write action in terms of superfields we need to define $\int_\theta$. Berezin's book already had it. For a single Grassmann variable θ we have

$$\int_\theta (\mathrm{const}) = 0$$

$$\int_\theta \theta = 1$$

$$\int_\theta f(\theta) = \frac{\partial}{\partial\theta} f(\theta)$$

$$\int_\theta \frac{\partial f}{\partial\theta} (\theta) = 0$$

$$\int_\theta f(\theta) \frac{\partial}{\partial\theta} g(\theta) = \pm \int_\theta \frac{\partial f}{\partial\theta} g(\theta)$$

where (+) sign is to be used if f is "odd".

We also note that

(i) $\int_\theta$ is linear (fermionic) "odd" operator

(ii) $\delta(\theta-\theta') = (\theta-\theta')$

(iii) $\int_{\theta 1} \int_{\theta 2} f = - \int_{\theta 2} \int_{\theta 1} f$

We also collect the expressions for super-trace and superdeterminant of supermatrix of the form:

$$M = \begin{pmatrix} A & \Gamma \\ \Delta & B \end{pmatrix}$$

where

$$A = M_{bb} \quad , \quad \Gamma = M_{bf}$$

$$B = M_{ff} \quad , \quad \Delta = M_{fb}$$

Then

$$\text{Sup tr } M = \text{Tr } A - \text{Tr } B$$

$$S - \text{Tr } (M_1 M_2) = S - T_r \, (M_2 M_1)$$

$$S - \det (M) = e^{\text{Tr } \ln M}$$

$$S - \det (M_1 M_2) = (S - \det (M_1)) \, (S - \det (M_2))$$

$$S - \det (M) = \frac{\det (A - \Gamma B^{-1} \Delta)}{\det B} = \frac{\det A}{\det (B - \Delta A^{-1} \Gamma)}$$

also for $M = (I + X)$, X infinitesimal,

$$\delta(S\text{-det } M) = (S\text{-det}(M))(S\text{-Tr}(M^{-1} \, \delta M))$$

4 - MASSLESS REPRESENTATION OF EXTENDED SUPERSYMMETRY

To construct a supergravity theory we need to know with which fermionic fields (and other fields) we must combine the spin 2 field to obtain a bose-fermi symmetry. We will thus look at the spin content of the representation one particle states of the supersymmetry algebra.We will only consider massless case. A general discussion has recently given by Ferrara et al.

Since central charges have dimension one, in the massless case they will be absent.

$$\{Q^i_\alpha, \bar{Q}^j_\beta\} = - (\gamma^\ell)_{\alpha\beta} \, P_\ell \, \delta^{ij}$$

$$[M_{\ell m}, Q^i_\alpha] = - i \, (\sigma_{\ell m})_{\alpha\beta} \, Q^i_\beta$$

$$[Q^i_\alpha, P_\ell] = 0$$

The symmetric form of the first relation implies that spinor charges transform under an internal symmetry e.g.

$$[T^a, Q^i_\alpha] = - (t^a)^{ij} \, Q^j_\alpha$$

$$[T^a_5, Q^i_\alpha] = - \gamma_5 \, (s^a)^{ij} \, Q^j_\alpha$$

where T^a, a Lorentz scalar, and T^a_5, a pseudo-scalar are internal symmetry generators. Majorana condition requires that $(t^a)^{ij}$ be pure imaginary while $(S^a)^{ij}$ pure real. Both t^a and S^a are Hermitian N x N matrices.

In 2-component notation

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = \sigma^\ell_{\alpha\dot\beta} \quad P_\ell \, \delta^{ij}$$

For massless case we choose the basis of 1-particle states with standard momentum $P^\ell = (w,0,0,w)$ e.g. $\{|P\ldots\rangle\}$. On these states

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = (-\sigma_{\alpha\dot\beta} + \sigma^3_{\alpha\dot\beta})\, w\, \delta^{ij}$$

$$= (I_{\alpha\dot\beta} + \sigma^3_{\alpha\dot\beta})\, w\, \delta^{ij}$$

$$= 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w\, \delta^{ij}$$

Hence

$$\{Q^i_1, \bar{Q}^j_{\dot 1}\} = 2\, w\, \delta^{ij}$$

and

$$\{Q^i_2, \bar{Q}^i_{\dot 2}\} = 0 \Rightarrow Q^i_2 \bar{Q}^i_{\dot 2} + \bar{Q}^i_{\dot 2} Q^i_2 = 0$$

Since L.H.S. is positive definite we get $Q^i_2 = 0 = \bar{Q}^i_{\dot 2}$. Q^i_2 creates zero norm states and should be ignored in counting physical states. Q^i_1 span a Clifford algebra of N-complex elements.

The "little algebra" corresponding to this basis with standard momentum $\mathring{P}$ is the sub-algebra of SP^N_4 which leaves $|P_1\ldots\rangle$ (or $\mathring{P}$) invariant. It is generated by Q_α, $\bar{Q}^{\dot\alpha}$, T^a, T^a_5, J_3, $M^{01}-J_2$, $M^{02}+J_1$. The last three are the generators of $E_2 = SO(2) \otimes T_2$ subgroup (in fact Pauli Lubanski vector $w_\sigma(p) = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} p^\rho$, $w.p = 0$, reduces to $w_0 = -w_3 - w M^{12} = wJ_3$, $w_1 \sim (M^{20} - M^{23})\, w$, $w_2 - (-M^{01} + M^{13})\, w$). To have discrete helicity states we must set translation operators of T_2, viz w_1, w_2 to zero. The physically acceptable "little algebra" is given by the supersymmetric extension of E_2, viz, Q^i_1, $\bar{Q}^i_{\dot 1}$, T^a, T^5_a, J_3. From the commutation relation of J_3 with Q^i_1 and $\bar{Q}^i_{\dot 1}$ it follows that Q^i_1 and $\bar{Q}^i_{\dot 1}$ are ladder operators, one raising the helicity by 1/2 while the other lowers the helicity by 1/2.

The helicity content may be easily obtained as below:

Let λ be maximum helicity. The possible helicity states are

Lowering chain:

$$\lambda \to (\lambda - \tfrac{1}{2}) \to \ldots \to (\lambda - \tfrac{k}{2}) \ldots \to (\lambda - \tfrac{N}{2})$$

Multiplicity:...$\binom{N}{K}$....1

Raising chain:

$$(-\lambda + \tfrac{N}{2}), \quad , (-\lambda + \tfrac{K}{2}), \ldots \leftarrow (-\lambda + \tfrac{1}{2}) \leftarrow -\lambda$$

Multiplicity:...$\binom{N}{K}$ ....

Total multiplicity = Total number of helicity

$$\text{states} \qquad = 2 \sum_0^N \binom{N}{K} = 2.2^N$$

However, when

$$\lambda - \frac{N}{2} = - \lambda \quad \text{or} \quad \lambda = \frac{N}{4}$$

the two chains coincide and

Total Multiplicity = 2^N

We give some illustrations:

N = 1 :

(a) $|\lambda_{max}| = 2$

$$2 \rightarrow \frac{3}{2} \qquad\qquad -\frac{3}{2} \leftarrow -2$$

Particle content: $(S = 2) + (S = \frac{3}{2})$, Multiplicity = 4

Field theoretic representation:

e_μ^ℓ one gravitaton

Ψ_μ one gravitino

Here μ is world or curved space index.

(b) $|\lambda_{max}| = \frac{3}{2}$

$$\frac{3}{2} \rightarrow \frac{1}{2} \qquad\qquad -\frac{1}{2} \leftarrow -\frac{3}{2}$$

Particle content: $(S = \frac{3}{2}) + (S = \frac{1}{2})$, Multiplicity = 4

Fields: Ψ_μ, A_μ

(c) $|\lambda_{max}| = 1$

$$1 \rightarrow \frac{1}{2} \qquad\qquad -\frac{1}{2} \leftarrow -1$$

Particle content: $(S = 1) + (S = \frac{1}{2})$

Fields: A_μ, λ

(d) $|\lambda_{max}| = \frac{1}{2}$

$$\frac{1}{2} \rightarrow 0 \qquad\qquad 0 \leftarrow -\frac{1}{2}$$

Particle content: $2(S = 0) + (S = \frac{1}{2})$

Fields: A, B, χ

N = 4, $|\lambda_{max}| = 1$:

$1 \rightarrow \frac{1}{2} \rightarrow 0 \rightarrow -\frac{1}{2} \rightarrow -1$ Lowering chain

$1 \leftarrow \frac{1}{2} \leftarrow 0 \leftarrow -\frac{1}{2} \leftarrow -1$ Raising chain

Multiplicity:

1 4 6 4 1

Particle content: $(S = 1) + 4\ (S = \frac{1}{2}) + 6\ (S = 0)$

Total multiplicity = 2^N = 16.

$N = 8,\ |\lambda_{max}| = 2$:

Particle content: $(S = 2) + 8(S = \frac{3}{2}) + 28(S = 1) + 56\ (S = \frac{1}{2}) + 70\ (S = 0)$

Multiplicity: $2^N = 256$

$N = 7,\ |\lambda_{max}| = 2$:

Particle content: $(S = 2) + (7 + 1)(S = \frac{3}{2}) + (21 + 7)\ (S = 1) + (35 + 21)(S = \frac{1}{2}) + (35+35)(S=0)$

Multiplicity: $2 \cdot 2^N = 256$

If we require one gravitation we must stop at N = 8. Since particle content of N = 7, 8 coincide we have 7 different super-gravity theories. For N = 9 we get

N = 9: $(S = \frac{5}{2})+(9+1)(S = 2)+(36+9)(S = \frac{3}{2})+(84+36)S(1)+(126 + 84)S(\frac{1}{2})+(124+126)(S = 0)$

	S=2	3/2	1	1/2	0	MULTIPLICITY	OBSERVATIONS
N = 1	1	1				4	Simple Super-gravity
		1	1			4	Spin 3/2 multiplet
			1	1		4	Vector multiplet
				1	2	4	Wess-Zumino Scalar multiplet
N = 2	1	2	1			8	0(2) Supergravity
		1	2	1		8	
			1	2	2	8	0(2) Super-Yang-Mills
				1	2	4	
N = 3	1	3	3	1		16	0(3) S.G.
		1	3	3	1+1	16	
			1	3+1	3+3	16	0(3) S-Y.M.
N = 4	1	4	6	4	1+1	32	0(4) S.G.
		1	4	6+1	4+4	32	
			1	4	6	16	0(4) S-Y.M.
N = 5	1	5	10	10+1	5+5	64	
		1	5+1	10+5	10+10	64	
N = 6	1	6	15+1	20+6	15+15	128	
		1	6	15	20	64	
N = 7	1	7+1	21+7	35+21	35+35	256	0(7) S.G.
N = 8	1	8	28	56	70	256	0(8) S.G.

EXTENDED SUPERSYMMETRY MASSLESS REPRESENTATIONS

5 - LINEARIZED SIMPLE SUPERGRAVITY LAGRANGIAN

5.1 - LOCAL SYMMETRY, COMPENSATING GAUGE FIELDS, NOETHER COUPLING TECHNIQUE

In Sec. 2 we constructed Lagrangians invariant under rigid or global supersymmetry transformations and where the fields carried a representation of rigid supersymmetry. We would now like to promote the symmetry to a local one so that the fields carry the representation of local sypersymmetry. We will illustrate the procedure by considering well known simple examples.

Consider the Lagrangian of a complex scalar field:

$$L_0 = (\partial_\ell \phi^t)(\partial^\ell \phi) - m^2 \phi^t \phi$$

which is invariant under $\phi(x) \to \phi'(x) = e^{-i\alpha}\phi(x)$, α = const. The corresponding Noether current is

$$j_N^\ell = ie\left[(\partial^\ell \phi^t)\phi - (\partial^\ell \phi)\phi^t\right] \quad , \quad \partial_\ell j^\ell = 0.$$

For local symmetry, $\alpha = \alpha(x)$, the kinetic term in L_0 is not invariant, since,

$$\partial_\ell \phi'(x) = e^{-i\alpha(x)}\left[\partial_\ell \phi - i(\partial_\ell \alpha)\phi\right]$$

We must introduce a compensating gange field $A_\ell(x)$ and the covariant derivative $D_\ell(A) \equiv (\partial_\ell - ieA_\ell)$ such that $(D_\ell \phi)' = e^{-i\alpha(x)}(D_\ell \phi)$. The transformation property of the gauge field then follows to be

$$A'_\ell(x) = (A_\ell(x) - \frac{i}{e}\partial_\ell \alpha)$$

and the Lagrangian invariant under local U(1) transformation may be written as

$$L = (D_\ell \phi)^+(D^\ell \phi) - m^2\phi^t\phi - \frac{1}{4}(\partial_\ell A_m - \partial_m A_\ell)^2 = L_0 + j_N^\ell A_\ell + e^2(\phi^+\phi)(A^\ell A_\ell) - \frac{1}{4}(\partial_\ell A_m - \partial_m A_\ell)^2$$

The additional terms needed are a coupling of the gauge field with the Noether current, a contact interaction term and the kinetic term corresponding to the (massless) gauge field.

The necessity of such interaction terms in general may also be seen as follows. Consider

$$L = L\left[\phi, \partial_\ell \phi\right]$$

where ϕ is a field multiplet. Then

$$\delta L = \delta\phi \frac{\partial L}{\partial \phi} + \delta(\partial_\ell \phi)\frac{\partial L}{\partial(\partial_\ell \phi)} = -\delta\phi\left[-\frac{\partial L}{\partial \phi} + \partial_\ell \frac{\partial L}{\partial(\partial_\ell \phi)}\right] + \partial_\ell\left[\delta\phi \frac{\partial L}{\partial(\partial_\ell \phi)}\right]$$

where

$$\delta\phi \simeq \phi'(x) - \phi(x)$$

Global invariance implies

$$\delta L = \partial_\ell(\varepsilon \Lambda^\ell(x)) \quad , \quad \varepsilon = \text{const.}$$

so that the conserved Noether current is

$$\varepsilon\, j_N^\ell = \left[\delta\phi \frac{\partial L}{\partial\, \partial_\ell \phi} - \varepsilon \Lambda^\ell\right]$$

$$\varepsilon\, \partial_\ell\, j_N^\ell \triangleq 0$$

When $\varepsilon = \varepsilon(x)$ clearly

$$\delta L = \partial_\ell(\varepsilon(x)\, A^\ell(x)) + (\partial_\ell \varepsilon)\, S^\ell(x)$$

$$\partial_\ell(\varepsilon(x)\, j_N^\ell) + (\partial_\ell \varepsilon)\, S^\ell$$

or

$$j_N^\ell = S^\ell$$

Thus

$$\delta L = \partial_\ell(\varepsilon(x)\, A^\ell) + (\partial_\ell \varepsilon)\, j_N^\ell$$

and to compensate for $(\partial_\ell \varepsilon)\, j_N^\ell$ term we must add to the theory a gauge field A_ℓ and an interaction term of the form $j_N^\ell A_\ell$. Additional contact type of interaction terms may be required to make the complete Lagrangian locally invariant.

Consider next the case of SU(2) Yang-Mills theory described in terms of the fields A_ℓ^i (i = = 1,2,3, the isospin index). The linearized (free) Lagrangian

$$L_0 = -\frac{1}{4}\ \vec{F}_{\ell m} \cdot \vec{F}^{\ell m} \qquad , \qquad \vec{F}_{\ell m} = (\partial_\ell \vec{A}_m - \partial_m \vec{A}_\ell)$$

is invariant under global SU(2) transformations $\delta\vec{A}_\ell = \vec{\alpha} \times \vec{A}_\ell$ and local abelian gauge transformations $\delta\vec{A}_\ell = \partial_\mu \vec{\Lambda}$. We will reconstruct by Noether procedure the non-linear interacting theory. When SU(2) becomes local e.g. $\alpha = \alpha(x)$ we have $\delta\vec{A}_\ell = \vec{\alpha}(x) \times \vec{A}_\ell$, $\delta L_0 = (\partial_\ell \vec{\alpha}) \cdot \vec{J}^\ell$ where $\vec{J}_\ell = -\vec{A}_m \times \vec{F}^{\ell m}$ is the Noether current. The Lagrangian $L' = L_0 - \frac{g}{2}\ \vec{J}^\ell \cdot \vec{A}_\ell$ is locally invariant to order g^0 if we combine the initially independent local and rigid transformations of the linearized theory together and identify $\vec{\Lambda} = \frac{1}{g}\ \vec{\alpha}(x)$ so that

$$\delta\vec{A}_\ell = \vec{\alpha}(x) \times \vec{A}_\ell(x) + \frac{1}{g}\, (\partial_\ell \vec{\alpha})$$

We continue this step by step process of amending Lagrangian and transformations order by order in g untill we have a locally invariant Lagrangian. In the present case

$$\delta L' = -\, g(\vec{A}_\ell \times \vec{A}_m) \cdot (A^m \times \partial^\ell \vec{\alpha}) \quad .$$

We find

$$L'' = L' + \frac{g}{4}\, (\vec{A}_\ell \times \vec{A}_m) \cdot (\vec{A}^m \times \vec{A}^\ell) = -\frac{1}{4}\, \vec{F}_{\ell m} \cdot \vec{F}^{\ell m}$$

where

$$\vec{F}_{\ell m} = (\partial_\ell \vec{A}_m - \partial_m \vec{A}_\ell) - g\, \vec{A}_\ell \times \vec{A}_m$$

is inv ariant to order g. In fact L" is invariant under the above local transformation to all orders in g and we recober the usual Y - M theory. We find for the commutator

$$[\delta_1, \delta_2]\, \vec{A}_\ell = \vec{\alpha}_2 \times (\frac{1}{g}\, \partial_\ell\, \vec{\alpha}_1 + \vec{\alpha}_1 \times \vec{A}_\ell) - (1 \leftrightarrow 2) = \frac{1}{g}\, \partial_\ell(\vec{\alpha}_2 \times \vec{\alpha}_1) + (\vec{\alpha}_2 \times \vec{\alpha}_1) \times \vec{A}_\ell = \delta_{12}\, \vec{A}_\ell$$

Thus we have a set of local transformations which have a closing algebra, and which have the Lagrangian L invariant.

5.2 - LINEARIZED S-G LAGRANGIAN (ON SHELL)

From the s.s. algebra we find that if we promote the global symmetry to local symmetry we will end up with general coordinate transformations. To preserve the local s.s. invariance in a field theory already invariant under rigid s.s. we will be obliged to add compensating gauge field $\Psi_\ell(x)$ just like in q.e.d. to assure local U(1) invariance we must add gauge field $A_\ell(x)$. Thus a massless spin 3/2 Majorana field $\Psi_\ell(x)$ must enter the theory if we require local s.s. invariance. Moreover, this field couples to the Noether current of global s.s. But Ψ_ℓ will itself require a massless companion of spin 1 or spin 2 to preserve the supersymmetry.

Since gravity (s = 2) is necessarily coupled to the stress tensor of all matter,it is natural to take (2, 3/2) as N = 1 supergravity multiplet. A (3/2, 1) supermultiplet as the gauge field not only forfeits connection to gravity but does not give a consistent nonlinear gauge theory since neither the real 3/2 field nor any other real matter fields can couple minimally to photon (s = 1). The natural place of this multiplet is as a "matter" multiplet coupled to (2, 3/2) where it leads to O(2) extended supergravity. In view of the discussion in Sec. 5.1 we may write the linearized supergravity Lagrangian L^0 as the sum of Fierz-Pauli spin 2 (linearized gravity) and Rarita-Schwinger spin 3/2 Lagrangians:

$$L^0 = L_{FP}\,(h^{\ell m}) + L_{RS}(\Psi_\ell)$$

$$L_{FP}(h^{\ell m}) = -\frac{1}{4}\left[h_{\ell m,n}\, h^{\ell m,n} - 2h_\ell h^\ell + 2h^\ell h_{,\ell} - h_{,\ell}\, h^{,\ell}\right]$$

$$L_{RS} = -\frac{i}{2}\,\varepsilon^{\ell mnp}\,\bar{\Psi}_\ell\,\gamma_5\,\gamma_m\,\partial_n\,\Psi_p$$

where

$$h_{\ell m} = h_{m\ell}\ ,\ h_\ell = \partial_m\, h^m_\ell\ ,\quad h = h^\ell{}_\ell$$

and a comma indicates ordinary derivative.

L^0 is invariant under two separate Abelian gauge transformations

$$\delta h_{\ell m} = \partial_\ell\, \xi_m + \partial_m\, \xi_\ell(x)$$

$$\delta\Psi_\ell = \partial_\ell\, \alpha(x)$$

and under rigid (global) supersymmetry transformations (non-Abelian):

$$\delta h_{\ell m} = i\,\bar{\varepsilon}\,\gamma_\ell\,\Psi_m + i\,\bar{\varepsilon}\,\gamma_m\,\Psi_\ell$$

$$\delta\Psi_\ell = (\partial_n\, h_{\ell m})\,\sigma^{nm}\,\varepsilon \qquad\qquad \varepsilon\ :\ \text{constant.}$$

If we now require local s.s. invariance $\varepsilon \to \varepsilon(x)$ we must add an interaction term of the form $K\,\bar{\Psi}_\ell{}^\alpha J^\ell_\alpha$ when J^ℓ_α is Noether current corresponding to global s.s. invariant and K the gravitational coupling constant. However, with this term added we also require a term $K\, h_{\ell m}\, T^{\ell m}$, since J^ℓ_α under s.s. transforms into the stress tensor of the system, to ensure the local s.s. of the interacting Lagrangian. The final S-G Lagrangian is invariant under a single non-Abelian s.s. gauge transformation.

The linearized supergravity differs from the Wess-Zumino model. One has to take into consideration the gauge transformations in addition to rigid s.s. transformations in order to obtain a closed algebra. The commutator

$$[\sigma(\varepsilon_1),\ \delta(\varepsilon_2)] \qquad \text{(field)}\quad .$$

(field) gives rise to, on using equations of motion (on shell), space-time translation. The commutator of a gauge transformation and a supersymmetry transformation on the field should vanish or give rise to a gauge transformation on the (same) field. One may show that the most general rigid s.s. transformation consistent with these requirements is as given above.

The requirement of local super symmetry invariance for the Lagrangian of the simple supergravity multiplet leads necessarily to interacting field theory and unifies gravitational field with a spin 3/2 field. The difficulties of such a unification are the highly non-linear nature of general relativity and the difficulty of coupling higher spin fields in a consistent way. It was shown by Freedman, Van Nienwenhuizen and Ferrara and Desér and Zumino that the sum of the Einstein action and that for a massless,(most) minimally coupled, Rarita-Schwinger.Majorana field fudfils the consistency criteria. This is due to the requirement of local s.s. invariance.

We make some remarks on the off-shell formulation. On shell $h_{\ell m}$ and Ψ_μ each have two helicities. However, off-shell $h_{\ell m} = h_{m\ell}$ has 10 degrees of fredom minus 4 gauge degrees of freedom giving 6 bosonic degrees of freedom. The field Ψ^α_ℓ off shell has 16 minus 4 gauge degrees of freedom, giving 12 fermionic degrees of freedom. Assuming the existence of a minimal formulation we need 6 bosonic degrees to balance the bosonic and fermionic degrees of freedom. We also require that the auxiliary fields do not propagate e.g. there is no kinetic term in the Lagrangian corresponding to them and carry dimension 2 so that they appear as squares like in Wess-Zumino model.

5.3 - Spin 3/2 Rarita-Schwinger Lagrangian

$$L_{RS} = -\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,\bar\Psi_\mu \gamma_5\,\gamma_\nu\,\partial_\rho\,\Psi_\sigma$$

Spinorial gauge invariances (local)

$$\delta\Psi_\sigma = \partial_\sigma\,\varepsilon(x)$$

Equations of motion:

$$\frac{\delta\, L_{RS}}{\delta\,\bar\Psi_\mu} = 0 \quad \Rightarrow R^\mu \equiv \varepsilon^{\mu\nu\rho\sigma}\,\gamma_5\,\gamma_\nu\,\partial_\rho\,\Psi_\sigma = 0$$

under gauge transformation $\delta R^\mu = 0$.

Gauge fixing: choose the gange $\gamma\cdot\Psi = 0$ (under gauge transformations $\delta(\gamma\cdot\Psi) = \partial\!\!\!/\varepsilon$. This can be solved to obtain the gauge $\gamma\cdot\Psi = 0$). Now we have identities

$$\partial\!\!\!/\Psi_\mu - \partial_\mu(\gamma\cdot\Psi) = R_\mu - \frac{1}{2}\gamma_\mu\,(\gamma\cdot R)$$

$$\partial\!\!\!/(\gamma\cdot\Psi) - (\partial\cdot\Psi) = 2\sigma^{\mu\nu}\,\partial_\mu\,\Psi_\nu = \frac{1}{2}\gamma\cdot R$$

Thus in the gauge chosen the equations of motion $R^\mu = 0$ lead to:

$$\left.\begin{array}{l}(\partial\cdot\partial)\,\Psi_\nu = 0\\ \partial_\mu\,\Psi^\mu = 0\end{array}\right\} \Rightarrow \Box\,\Psi_\mu = 0$$

This is the usual way of writing R.S. equations. We remark that $L_{1/2}$ may also be written in an analogous form:

$$L_{1/2} = \varepsilon^{\mu\nu\alpha\beta}\,\bar\lambda\,\,\gamma_5\,\gamma_\mu\,\gamma_\nu\,\gamma_\alpha\,\partial_\beta\,\lambda \equiv -\frac{1}{2}\,\bar\lambda\,\partial\!\!\!/\,\lambda$$

The mass term takes the form

$$m\,\varepsilon^{\mu\nu\alpha\beta}\,\bar\lambda\,\gamma_5\,\gamma_\mu\,\gamma_\nu\,\gamma_\alpha\,\gamma_\beta\,\lambda$$

Thus we must check that there is no helicity 1/2, but only 3/2 (massless cases) in L_{RS}.

Coupling to E.M. field complex R.S. field:

$$\varepsilon^{\mu\nu\rho\sigma}\,\gamma_5\,\gamma_\nu\,(\partial_\rho - i\,e\,A_\rho)\,\Psi_\sigma = 0$$

Then

$$(\partial_\mu - i\,e\,A_\mu)\cdot\varepsilon^{\mu\nu\rho\sigma}\,\gamma_5\,\gamma_\nu\,(\partial_\rho - i\,e\,A_\rho)\,\Psi_\sigma = 0$$

Using

$$[\partial_\mu - i\,e\,A_\mu,\,\partial_\rho - i\,e\,A_\rho] \sim F_{\mu\rho}$$

we find

$$F^{\mu\nu}\gamma_\mu \Psi_\nu = 0$$

Extra constraint requires that either $\Psi_\nu = 0$ or that photon be a gauge excitation. This inconsistency is a rule in all higher spin field couplings.

6 - TETRAD FORMULATION OF ORDINARY GRAVITY

6.1 - TETRAD FORMULATION

Since we must work with spinors we must use tetrad formulation. Spinors can only be introduced locally in the tangent space.

We specify a local frame by giving tetrad fields $e^\mu_\ell(x)$ and $e^\ell_\mu(x)$. The Greek index indicates the curved space (world) vector index while the Roman index indicates a vector index in local tangent space. Given a vector $A^\mu(x)$ its component reffered to the local frame are

$$A^\ell(x) = e^\ell_\mu(x)\ A^\mu(x)$$

The components $A^\ell(x)$ are world scalars but they transform as a four-vector with respect to local transformations which rotate the local tetrad frame. We assume that e^μ_ℓ are linearly independent and $e^\mu_\ell\, e^\ell_\nu = \delta^\mu_\nu$. It follows then $e^\mu_m\, e^\ell_\mu = \delta^\ell_m$.

Clearly, $A^\mu B_\mu = e^\mu_\ell A^\ell e^m_\mu B_m = A^\ell B_\ell$. The vierbein or tetrads themselves have mixed indices transforming as a world vector and a local vector. To be definite we will assume the local tangent space group to be Lorentz group L. There are two invariances involved: the invariance w.r.t. general coordinate transformations and the invariance w.r.t. local Lorentz gauge transformations. Thus we require, in order to define covariant derivatives two sets of affinities or connections.

Consider local vector field $A^\ell(x)$. Under local Lorentz rotations $\Lambda^\ell_m(x)$:

$$A'^\ell(x) = \Lambda^\ell_m(x)\ A^m(x)$$

$$(\partial_\mu A^\ell)' = \partial_\mu A'^\ell(x) = \Lambda^\ell{}_m(\partial_\mu A^m) + (\partial_\mu \Lambda^\ell{}_m)\ A^m$$

and $(\partial_\mu A^\ell)$ does not transform as A^ℓ due to the presence of the second term. We define covariant derivative

$$D_\mu A^\ell \equiv (\partial_\mu + \Gamma_\mu)\ A^\ell \equiv D^\ell_{\mu m}\ A^m$$

such that

$$(D_\mu A^\ell)' = \Lambda^\ell_m(x)\ (D_\mu A^m)$$

or

$$(\partial^\ell_m \partial_\mu + \Gamma'^\ell_{\mu m})\ \Lambda^m_m(x)\ A^n(x) = \Lambda^\ell_m(x)\ (\delta^m_n \partial_\mu + \Gamma^m_{\mu n})\ A^n$$

or

$$\Gamma'^\ell_{\mu m} \Lambda^m{}_n = \Lambda^\ell_m \Gamma^m_{\mu n} - \delta^\ell_m \partial_\mu \Lambda^m{}_n$$

or

$$\Gamma'^\ell_\mu{}_m = \Lambda^\ell_{m'} \Gamma^{m'}_\mu{}_n (\Lambda^{-1})^n{}_m - (\partial_\mu \Lambda^\ell{}_n)(\Lambda^{-1})^n{}_m$$

Writing

$$\Gamma_\mu = (\Gamma^\ell_{\mu m}) \quad , \quad \Lambda = (\Lambda^\ell{}_m)$$

$$\Gamma'_\mu = \Lambda\ \Gamma_\mu\ \Lambda^{-1} - (\partial_\mu \Lambda)\Lambda^{-1}$$

is the transformation law of $\Gamma^\ell_{\mu m}$. We will adopt the usual notation $\Gamma_\mu \rightarrow \omega_\mu$ so that

$$\omega'_\mu = \Lambda\ \omega_\mu\ \Lambda^{-1} - (\partial_\mu \Lambda)\ \Lambda^{-1}$$

and

$$D_\mu\ A^\ell = \partial_\mu\ A^\ell + \omega^\ell_{\mu m}\ A^m$$

Requiring that

$$D_\mu (A_\ell\ B^\ell) = \partial_\mu\ (A_\ell\ B^\ell)$$

it follows

$$D_\mu\ A_\ell = \partial_\mu\ A_\ell - \omega^m_{\mu\ell}\ A_m$$

Consider next the Dirac field $\Psi(x)$. It is defined to transform under local Lorentz tran- formations as follows:

$$\Psi'(x) = S(\Lambda(x))\ \Psi(x)$$

when $S(\Lambda)$ constitutes a representation of the Lorentz group. The covariant derivative $D_\mu = (\partial_\mu + \Gamma_\mu)$ is required to satisfy

$$(D_\mu\ \Psi(x))' = S(\Lambda(x))\ (D_\mu\ \Psi(x))$$

or

$$(\partial_\mu + \Gamma'_\mu)\ \Psi'(x) = (\partial_\mu + \Gamma'_\mu)\ S(\Lambda(x))\ \Psi(x) = S(\Lambda(x))\ (\partial_\mu + \Gamma_\mu)\ \Psi(x)$$

or

$$\Gamma'_\mu\ S\ \Psi = S\ \Gamma_\mu\ \Psi - (\partial_\mu\ S)\ \Psi$$

or

$$\Gamma'_\mu = S(\Lambda(x))\ \Gamma_\mu\ S^{-1}(\Lambda(x)) - (\partial_\mu\ S(\Lambda(x)))\ S^{-1}\ (\Lambda(x))$$

Consider infinitesimal transformation:

$$S(\Lambda(x)) \simeq I + \frac{1}{2}\ \lambda^{\ell m}(x)\ \sigma_{\ell m}$$

$$\Gamma'_\mu \simeq \Gamma_\mu + \frac{1}{2}\ \lambda^{\ell m}(x)\ [\sigma_{\ell m}, \Gamma_\mu(x)] - \frac{1}{2}\ \sigma_{\ell m}(\partial_\mu\ \lambda^{\ell m})$$

where

$$\Lambda^\ell{}_m(x) \approx \delta^\ell{}_m + \lambda^\ell{}_m(x) \quad ; \quad \lambda_{\ell m} = -\lambda_{m\ell}$$

Also

$$\omega'^{\ell}_{\mu\ m} = \omega^\ell_{\mu m} + \lambda^\ell{}_n\ \omega^n_{\mu m} - \omega^\ell_{\mu n}\ \lambda^n{}_m - \partial_\mu\ \lambda^\ell{}_m$$

or

$$\delta\omega_\mu = [\lambda, \omega_\mu] - \partial_\mu \lambda$$

We derive for

$$\omega_\mu^{[\ell m]} = \frac{1}{2}(\omega_\mu^{\ell m} - \omega_\mu^{m\ell})$$

$$\omega'^{[\ell m]}_\mu = \omega_\mu^{[\ell m]} + \lambda^\ell_n\, \omega_\mu^{[nm]} - \omega_\mu^{[\ell n]}\, \lambda^m_n - \partial_\mu\, \lambda^{\ell m}$$

and

$$\frac{1}{4}\lambda^{\ell m}\omega_\mu^{[\ell' m']}\left[\sigma_{\ell m}, \sigma_{\ell' m'}\right]$$

$$= \frac{1}{4}\lambda^{\ell m}\,\omega_\mu^{[\ell' m']}\,(\eta_{m\ell'}\sigma_{\ell m'} - \eta_{\ell\ell'}\,\sigma_{mm'} - \eta_{mm'}\,\sigma_{\ell\ell'} + \eta_{\ell m'}\,\sigma_{m\ell'})$$

$$= \frac{1}{2}\left[\omega_\mu^{[\ell m']}\,\lambda^{\ell'}{}_{\ell}\;\;\sigma_{\ell' m'} + \omega_\mu^{[\ell' m]}\,\lambda_m{}^{m'}\;\sigma_{m'\ell'}\right]$$

We may thus write

$$\Gamma_\mu = \frac{1}{2}\sigma_{\ell m}\,\omega_\mu^{[\ell m]} = \frac{1}{4}\sigma_{\ell m}(\omega_\mu^{\ell m} - \omega_\mu^{m\ell}) = \frac{1}{2}\sigma_{\ell m}\,\omega_\mu^{\ell m}$$

Γ_μ sees only the antisymmetric part of $\omega_\mu^{\ell m}$; we do not have to impose, a priori, any symmetry property on $\omega^\ell_{\mu m}$.

We remind that the matrices γ^ℓ are invariant under local transformations:

$$\gamma^\ell = \Lambda^\ell{}_m(x)\; S(\Lambda(x))\; \gamma^m\; S^{-1}(\Lambda(x))$$

$S\,\gamma^m\,S^{-1}$ represents the transformation of operators acting on Ψ which transforms as $\Psi \to S\Psi$ and $\Lambda^\ell{}_m(x)$ is due to vector (local) nature of the index "ℓ". If we define $\gamma^\mu_{(x)} = e^\mu_\ell(x)\,\gamma^\ell$ then under local Lorentz transformations:

$$\gamma'^\mu(x) = (e^\mu_\ell\,\gamma^\ell)' = e^\mu_m\,\Lambda^m_\ell\,\Lambda^\ell_n\,S\,\gamma^n\,S^{-1} = e^\mu_n\,S\,\gamma^n\,S^{-1} = S(\Lambda(x))\gamma^\mu{}_{(x)}\;S^{-1}(\Lambda(x))$$

Under general coordinate transformations:

$$\Psi(x) \to \Psi(x)$$

$$\gamma^\mu(x) \to \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\gamma^\nu(x)$$

$$\gamma^\ell \to \gamma^\ell$$

$$\omega_\mu \to \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\omega_\nu(x)$$

e.g. the index μ in $\omega_\mu^{\ell m}$ or Γ_μ is tensorial. However, the indices ℓ, m here are non-tensorial w.r.t. L; ω_μ is a connection.

For world vectors A^μ (which is scalar w.r.t. local transformation) we have to introduce covariant derivative through space-time connections $\Gamma^\lambda_{\mu\rho}$

$$A^\mu{}_{;\nu} = \partial_\nu\, A^\mu + \Gamma^\mu_{\alpha\nu}A^\alpha$$

Requiring

$$(A^\mu\, B_\mu)_{;\lambda} = \partial_\lambda(A^\mu\, B_\mu)$$

leads to

$$A_{\mu;\nu} = \partial_\nu A_\mu - \Gamma^\alpha_{\mu\nu} A_\alpha$$

The affinities $\Gamma^\lambda_{\mu\rho}$ transform as

$$\Gamma'^{\mu}_{\nu\rho}(x') = \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\rho} \Gamma^\sigma_{\lambda\beta}(x) + \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\rho} \frac{\partial x'^\mu}{\partial x^\beta}$$

e,g, non-tensorially. It is easily verified that

$$D_\mu \delta^\ell_m = 0 \quad ,$$

$$\delta^\mu_{\nu;\lambda} = 0$$

The complete covariant derivative (indicated by ;) of tetrad field is thus written as

$$e^\ell_{\mu;\lambda} = \partial_\lambda e^\ell_\mu - \Gamma^\alpha_{\mu\lambda} e^\ell_\alpha + \omega^\ell_{\lambda m} e^m_\mu$$

$$e^\mu_{\ell;\lambda} = \partial_\lambda e^\mu_\ell + \Gamma^\mu_{\alpha\lambda} e^\alpha_\ell - \omega^m_{\lambda\ell} e^\mu_m$$

These definitions are consistent with $A^\mu = e^\mu_\ell A^\ell$ etc. For the metric tensor $\eta_{\ell m}$ in the tangent space

$$\eta_{\ell m;\lambda} = \partial_\lambda \eta_{\ell m} - \omega^n_{\lambda\ell} \eta_{nm} - \omega^n_{\lambda m} \eta_{\ell n} = -(\omega_{\lambda m\ell} + \omega_{\lambda\ell m}) = -2\,\omega_{\lambda(\ell m)}$$

Thus the symmetric part of $\omega^{\ell m}_\mu$ is seen by the covariant derivative of the metric in tangent space.
Covariant derivative of γ^ℓ requires a bit of care. We write

$$\Psi'^\alpha(x) = S(\lambda)^\alpha{}_\beta \Psi^\beta(x)$$

α,β are spinor indices. Then Ψ_α carrying the contragradient representation transforms as

$$\Psi'_\alpha(x) = (S^{-1T}(\Lambda))^\alpha{}_\beta \Psi_\beta = (S^{-1}(\Lambda))^\beta{}_\alpha \Psi_\beta$$

and $(\Psi^\alpha \xi_\alpha)$ is invariant under L. We may also write explicitly,

$$\gamma^\ell = ((\gamma^\ell)^\alpha_\beta)$$

$$(\gamma^\ell)^\alpha_\beta = \Lambda^\ell{}_m S(\Lambda)^\alpha{}_\sigma S^{-1}(\Lambda)^\delta_\beta \gamma^{m\sigma}{}_\delta$$

e.g. $\gamma^{\ell\alpha}{}_\beta$ is an invariant tensor. Now

$$D_\mu \Psi = (\partial_\mu + \Gamma_\mu) \Psi$$

or

$$\Psi^\alpha_{;\mu} = \partial_\mu \Psi^\alpha + \Gamma^\alpha_{\mu\beta} \Psi^\beta$$

From

$$(\Psi^\alpha \phi_\alpha)_{;\mu} = \partial_\mu (\Psi^\alpha \phi_\alpha)$$

we obtain

$$\Psi_{\alpha;\mu} = \partial_\mu \Psi_\alpha - \Gamma^\beta_{\mu\alpha} \Psi_\beta$$

Thus for complete covariant derivative of γ^{ℓ}:

$$\gamma^{\ell\alpha}{}_{\beta;\lambda} = \partial_\lambda \gamma^{\ell\alpha}{}_{\beta} + \omega^{\ell}_{\lambda m} \gamma^{m\alpha}{}_{\beta} + \Gamma^{\alpha}_{\lambda\sigma} \gamma^{\ell\sigma}{}_{\beta} - \Gamma^{\sigma}_{\lambda\beta} \gamma^{\ell\alpha}{}_{\sigma}$$

$$\gamma^{\ell}{}_{;\lambda} = \partial_\lambda \gamma^{\ell} + \omega^{\ell}_{\lambda m} \gamma^{m} + [\Gamma_\lambda, \gamma^{\ell}]$$

We show easily

$$\gamma^{\mu}{}_{;\lambda} = \partial_\lambda \gamma^{\mu} + \Gamma^{\mu}_{\rho\lambda} \gamma^{\rho} + [\Gamma_\lambda, \gamma^{\mu}]$$

From the relation

$$[\sigma^{\ell m}, \gamma^{n}] = \eta^{nm} \gamma^{\ell} - \eta^{n\ell} \gamma^{m}$$

we derive

$$[\omega_{\ell m} \sigma^{\ell m}, \gamma^{n}] = \omega_{\ell}{}^{n} \gamma^{\ell} - \omega^{n}{}_{m} \gamma^{m}$$

Thus

$$[\Gamma_\mu, \gamma^{n}] = \frac{1}{2} (\omega_{\mu m}{}^{n} - \omega^{n}_{\mu m}) \gamma^{m}$$

and we obtain a useful identity:

$$\omega^{\ell}_{\lambda m} \gamma^{m} + [\Gamma_\lambda, \gamma^{\ell}] = \frac{1}{2} (\omega_{\lambda m}{}^{\ell} + \omega^{\ell}_{\lambda m}) \gamma^{m}$$

It follows

$$\gamma^{\ell}{}_{;\lambda} = \partial_\lambda \gamma^{\ell} + \frac{1}{2} (\omega^{\ell m}_{\lambda} + \omega^{m\ell}_{\lambda}) \gamma_m = \frac{1}{2} (\omega^{\ell m}_{\lambda} + \omega^{m\ell}_{\lambda}) \gamma_m = \omega^{(\ell m)}_{\lambda} \gamma_m$$

since γ^{ℓ} are constant matrices. From

$$\gamma^{\ell} \gamma^{m} + \gamma^{m} \gamma^{\ell} = 2 \eta^{\ell m}$$

We rederive

$$\eta^{\ell m}{}_{;\lambda} = 2 \omega^{(\ell m)}_{\lambda}$$

The covariant derivative of γ^{ℓ} thus sees the symmetric part of $\omega^{\ell m}_{\lambda}$. Except for the facility in using traces of γ matrices all the relevant information is already contained in $e^{\mu}_{\ell}, \eta_{\ell m}$ and their covariant derivatives.

The space time metric tensor $g_{\mu\nu}(x)$ may be defined as

$$g_{\mu\nu}(x) = e^{\ell}_{\mu} e^{m}_{\nu} \eta_{\ell m}$$

$$g^{\mu\nu}(x) = e^{\mu}_{\ell} e^{\nu}_{m} \eta^{\ell m}$$

Also

$$\eta_{\ell m} = g_{\mu\nu}(x) e^{\mu}_{\ell} e^{\nu}_{m}$$

$$\eta^{\ell m} = g^{\mu\nu} e^{\ell}_{\mu} e^{m}_{\nu}$$

These relations are consistent as regards covariant differentiations. One also checks the consistency of

$$\{\gamma^\mu(x),\ \gamma^\nu(x)\} = 2\ g^{\mu\nu}(x)$$

and check

$$\{\gamma^\mu,\gamma^\nu\}_{;\lambda} = 2\ g^{\mu\nu}{}_{;\lambda} \qquad \text{etc.}$$

6.2 - CURVATURE TENSORS

Corresponding to the (Lorentz) gauge connection $\omega_\lambda = (\omega^\ell_{\lambda m})$ we may introduce a field strength or curvature which is gauge covariant. This is similar to the case of Yang-Mills theory or ordinary electrodynamics. We find that

$$P_{\lambda\rho}(\omega) = \partial_\lambda\ \omega_\rho - \partial_\rho\ \omega_\lambda + [\omega_\lambda,\omega_\rho]$$

transforms as

$$P_{\Lambda\ell} \xrightarrow[L]{} \Lambda\ P_{\lambda\rho}\ \Lambda^{-1}$$

We will write

$$R^\ell{}_{m\lambda\rho}(\omega) \equiv [P_{\lambda\rho}(\omega)]^{\ell}_{\ m}$$

The space-time connections $\Gamma^\lambda_{\mu\rho}$ give rise to space-time curvature tensor

$$R^\mu{}_{\nu\lambda\rho}(\Gamma) = \partial_\lambda\ \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\beta\lambda}\ \Gamma^\beta_{\nu\rho} - (\lambda \longleftrightarrow \rho)$$

The two are connected by the following equations:

$$e^\ell_{\mu;\lambda\rho}(\Gamma,\omega) - e^\ell_{\mu;\rho\lambda}(\Gamma,\omega) + (\Gamma^\alpha_{\lambda\rho} - \Gamma^\alpha_{\rho\lambda})\ e^\ell_{\mu;\alpha}(\Gamma,\omega) = R^\alpha{}_{\mu\lambda\rho}(\Gamma)\ e^\ell_\alpha - R^\ell{}_{m\lambda\rho}(\omega)\ e^m_\mu$$

If we impose the condition $e^\ell_{\mu;\lambda}(\Gamma,\omega) = 0$ we obtain

$$R^\alpha{}_{\mu\lambda\rho}(\Gamma) = R^\ell{}_{m\lambda\rho}(\omega)\ e^m_\mu\ e^\alpha_\ell$$

We note also

$$[D_\lambda,D_\rho]\ e^\ell_\mu = R^\ell{}_{m\lambda\rho}(\omega)\ e^m_\mu$$

when

$$D_\rho\ e^\ell_\mu = D^\ell_{\rho m}\ e^m_\mu = (\partial_\rho\ e^\ell_\mu + \omega^\ell_{\rho m} e^m_\mu\ ).$$

Note that the indices on curvature tensors are all tensorial unlike those on connections.

Tetrad condition $e^\ell_{\mu;\lambda}(\Gamma,\omega) = 0$ establishes a relation between Γ and ω. It should be possible to rewrite Einstein action in terms of e^ℓ_μ and $\omega^\ell{}_{\mu m}$. Note that

$$g_{\mu\nu;\lambda}(\Gamma) = \eta_{\ell m;\lambda}\ e^\ell_\mu\ \ e^m_\nu$$

If

$$e^\ell_{\mu;\lambda} = 0$$

we get

$$g_{\mu\nu;\lambda}(\Gamma) = \eta_{\ell m;\lambda}\ e^\ell_\mu\ e^m_\nu = -\ 2\ \omega_{\lambda(\ell m)}\ e^\ell_\mu\ \ e^m_\nu$$

In Einstein-Cartan theory, $g_{\mu\nu;\lambda}(\Gamma) = 0$ implies that $\omega^{\ell m}_\lambda = -\ \omega_\lambda{}^{m\ell}$. This may also be verified from explicit expression of $\omega^{\ell m}_\lambda$ in this case. If $\omega_\lambda{}^{(\ell\ m)} = 0$ the E.C. geometry is obtained only if $e^\ell_{\mu;\lambda} = 0$.

We may consider more generaly the tetrad condition:

$$e^{\ell}_{\mu;\lambda}(\bar{\Gamma},\omega) = - K^{\ell}_{\lambda m}\, e^{m}_{\mu}$$

Substituting this in the equation above relating $R^{\alpha}_{\mu\lambda\rho}(\Gamma)$ and $R^{\ell}{}_{m\lambda\rho}(\omega)$ we easily establish that if K_λ satisfies (independent of $\bar{\Gamma}$!!)

$$P_{\lambda\rho}(K) + [\omega_\lambda, K_\sigma] - [\omega_\rho, K_\lambda] = 0$$

the following relation is satisfied:

$$R^{\alpha}_{\mu\lambda\rho}(\bar{\Gamma}) = R^{\ell}{}_{m\lambda\rho}(\omega)\; e^{m}_{\mu}\; e^{\alpha}_{\ell}$$

But the relation just preceeding this is equivalent to $R^{\ell}{}_{m\lambda\rho}(\omega+K) = R^{\ell}{}_{m\lambda\rho}(\omega)$. The expression of $\bar{\Gamma}$ is easily shown to be

$$\bar{\Gamma}^{\alpha}_{\mu\lambda} = \Gamma^{\alpha}_{\mu\lambda} + L^{\alpha}_{\mu\lambda} \qquad ; \qquad L^{\alpha}_{\mu\lambda} = K^{\ell}_{\lambda m}\; e^{m}_{\mu}\; e^{\alpha}_{\ell}$$

Thus the sets of connections ($\bar{\Gamma}$, $\bar{\omega} \equiv \omega + K$) and (Γ,ω) give rise to the same curvature tensors and correspond to curvature tensor copies. See more details in "Curvature Tensor Copies in Affine Geometry", P.P.Srivastava, C.B.P.F.preprint N.F.-049/81.

6.3 - KIBBLE - SCIAMA FORMULATION OF EINSTEIN-CARTAN THEORY (SKETCH)

Assume tetrad condition $e^{\ell}_{\mu;\lambda}(\Gamma,\omega) = 0$, then,

$$\begin{aligned} R(g,\Gamma) &= \delta^{\sigma}_{\mu}\; g^{\nu\rho} R^{\mu}(\Gamma)_{\nu\rho\sigma} = g^{\nu\rho} R_{\nu\rho}(\Gamma) \\ &= \delta^{\sigma}_{\mu}\; g^{\nu\rho}\; e^{\mu}_{\ell}\; e^{m}_{\nu}\; R^{\ell}{}_{m\rho\sigma}(\omega) \\ &= e^{\mu\ell}\; e^{\nu m}\; R_{\ell m\nu\mu}(\omega) \equiv R\,(e,\omega) \end{aligned}$$

and

$$\begin{aligned} R(e,\omega) &= e^{\lambda m}\; e^{\rho\ell}\; R_{m\ell\rho\lambda}(\omega) \\ &= \frac{1}{2} H^{\lambda\rho}_{m\ell}(e)\; R^{m\ell}_{\rho\lambda}(\omega) = H^{\lambda\rho}_{m\ell}(e)\; . \left[\omega^{\ell m}_{\lambda,\rho} + \omega^{n\ell}_{\lambda}\; \omega^{m}_{\rho n}\right] \end{aligned}$$

where

$$H^{\lambda\rho}_{m\ell}(e) = (e^{\lambda}_{m}\; e^{\rho}_{\ell} - e^{\rho}_{m}\; e^{\lambda}_{\ell})$$

Now

$$g_{\mu\nu} = e^{\ell}_{\mu}\; \eta_{\ell m}\; e^{m}_{\nu}$$

$$\det\,(g_{\mu\nu}) = \det\,(e^{\ell}_{\mu}) - \det\,(\eta_{\ell m})\; .\; \det\,(e^{m}_{\nu})$$

From

$$e^{\ell}_{\mu}\; e^{\mu}_{m} = \delta^{\ell}_{m}$$

$$\det\,(e^{\ell}_{\mu})\; .\; \det\,(e^{\mu}_{m}) = \det\,(\delta^{\ell}_{m}) = 1$$

Also

$$\det\,(g^{\mu\rho})\; \det\,(g_{\rho\nu}) = \det\,(\delta^{\mu}_{\nu}) = 1$$

Call

$$e = \det (e^{\ell}_{\mu})$$

$$g = \det (g_{\mu\nu})$$

$$\eta = \det (\eta_{\ell m}) \quad ; \quad \eta^2 = 1$$

Then

$$g = \eta \cdot e^2 \quad , \quad e = \sqrt{\eta g}$$

$$\det (g^{\mu\nu}) = \frac{1}{g} \quad , \quad \det (e^{\mu}_{\ell}) = \frac{1}{e}$$

We remind g(x) is scalar density of weight + 2 while $\frac{1}{g}$ is scalar density of weight - 2 so that

$$e_{\mu\nu\lambda\rho} = \sqrt{\eta g}\ \varepsilon_{\mu\nu\lambda\rho} \qquad e^{\mu\nu\lambda\rho} = \frac{1}{\sqrt{\eta g}}\ \varepsilon^{\mu\nu\lambda\rho}$$

are tensors. (d^4x) is a scalar density of weight (-1).

From

$$e\ \varepsilon_{\mu\nu\rho\sigma} = e^m_{\mu}\ e^n_{\nu}\ e^p_{\rho}\ e^q_{\sigma}\ \ \varepsilon_{mnpq}$$

we derive

$$e\ \varepsilon_{\mu\nu\rho\sigma}\ e^{\rho}_{p'}\ e^{\sigma}_{q'} = e^m_{\mu}\ e^n_{\nu}\ \varepsilon_{mnp'q'}$$

and hence

$$e^m_{\mu}\ e^n_{\nu}\ \varepsilon_{mnpq}\ \varepsilon^{\mu\nu\rho\sigma} = 2\ e(e^{\rho}_{p}\ e^{\sigma}_{q} - e^{\rho}_{q}\ e^{\sigma}_{p}) = 2\ e\ H^{\rho\sigma}_{pq}(e)$$

Thus for scalar density (weight +1) $\sqrt{\eta g}\ R(g,\Gamma)$ we get

$$\sqrt{\eta g}\ R(g,\Gamma) = \frac{1}{4}\ \varepsilon^{\mu\nu\rho\sigma}\ \varepsilon_{mnpq}\ e^m_{\mu}\ e^n_{\nu}\ R^{pq}_{\rho\sigma}(\omega)$$

$$= e\ H^{\lambda\rho}_{m\ell}(e)\ [\omega^{\ell m}_{\lambda\ ,\rho} + \omega^{n\ell}_{\lambda}\ \omega^{m}_{\rho n}]$$

$$\equiv e\ R(e,\omega)$$

Here $\varepsilon^{\mu\nu\rho\sigma}$, ε_{mnpq} are permutation symbols; the first is a tensor density of weight + 1, the other has weight zero. For the variations we note

$$\delta g_{\mu\nu} = 2(\delta e^{\ell}_{\mu})\ e_{\ell\nu}$$

$$\delta e^{\ell}_{\mu} = -\ e^m_{\mu}\ e^{\ell}_{\rho}(\delta\ e^{\rho}_{m})$$

$$\delta g_{\mu\nu} = -\ 2\ e_{\ell\nu}\ e^m_{\mu}\ e^{\ell}_{\rho}\ (\delta\ e^{\rho}_{m}) = -\ 2\ g_{\nu\rho}\ e^m_{\mu}\ (\delta\ e^{\rho}_{m})$$

$$\to -\ (g_{\mu\rho}\ e^m_{\nu}\ \delta\ e^{\rho}_{m} + g_{\nu\rho}\ e^m_{\mu}\ \delta\ e^{\rho}_{m})$$

The action is

$$S = S_G + S_M = \int L_G\ d^4x + \int L_M\ d^4x \qquad M = \text{matter}$$

For integral spin fields $\tau^{\mu\nu}\ (= \tau^{\nu\mu})$ is defined by

$$\delta_g\ S_M = -\ \frac{1}{2} \int \tau^{\mu\nu}(\delta\ g_{\mu\nu})\ d^4x = \int g_{\mu\rho}\ e^{\ell}_{\nu}\ \tau^{\mu\nu}(\delta\ e^{\rho}_{\ell})\ d^4x$$

Define:

$$\tau_{\mu}^{\ell} = g_{\rho\mu}\, e_{\nu}^{\ell}\, \tau^{\rho\nu}$$

then

$$\tau^{\sigma\lambda} \equiv \tau^{\sigma\ell}\, e_{\ell}^{\lambda} = g^{\sigma\mu}\, e_{\ell}^{\lambda}\, \tau_{\mu}^{\ell}$$

For spinor fields, L_M contains e_{μ}^{ℓ} and $\omega_{\mu}^{\ell m}$. We define Spin-density $\zeta_{\ell m}^{\mu}$ by

$$\delta_{\omega}\, S_M = \frac{1}{2} \int \zeta_{\ell m}^{\mu}\, \delta\omega_{\mu}^{\ell m}\, d^4x$$

or

$$\delta_{\omega}\, L_M = \frac{1}{2}\, \zeta_{\ell m}^{\mu}\, \delta\omega_{\mu}^{\ell m}$$

In the usual formulation of integral spin case in ordinary gravity only covariant curls, e.g. $(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})$, appear. The space time connections $\Gamma_{\mu\nu}^{\lambda}$ do not appear in L_M. The matter couples to gravity through most minimal interaction. For spinor fields $\omega_{\mu}^{\ell m}$ crops up in the covariant derivative $D_{\mu} = (\partial_{\mu} + \frac{1}{2}\, \omega_{\mu}^{\ell m}\, \sigma_{\ell m})$. Again for spin 3/2 Ψ_{μ} field appearing in supergravity theory it is the covariant curl $(D_{\mu}\Psi_{\nu}-D_{\nu}\Psi_{\mu})$ that appears in the corresponding L_M rather than $\Psi_{\nu;\mu} - \Psi_{\mu;\nu}$ which will involve space-time connections $\Gamma_{\mu\nu}^{\lambda}$. This most minimal coupling does give rise to a consistent theory. In the L_M the $\omega_{\mu}^{\ell m}$ term appears in the form

$$\frac{\partial\, L_M}{\partial(\partial_{\mu}\Psi)}\, \Gamma_{\mu}\Psi = \frac{1}{2}\, \frac{\partial\, L_M}{\partial(\partial_{\mu}\Psi)}\, \omega_{\mu}^{\ell m}\, \sigma_{\ell m}\, \Psi$$

so that

$$\delta_{\omega}\, S_M = \frac{1}{2} \int d^4x \left[\frac{\partial\, L_M}{\partial(\partial_{\mu}\Psi)}\, \sigma_{\ell m}\, \Psi\right] \delta\omega_{\mu}^{\ell m}$$

and

$$\zeta_{\ell m}^{\mu} = \frac{\partial\, L_M}{\partial(\partial_{\mu}\Psi)}\, \sigma_{\ell m}\, \Psi \qquad \text{spin-density}$$

Thus $\zeta_{\ell m}^{\mu}$ is generalization of the spin density of Ψ that appears in special relativity in association with the density of orbital angular momentum.

6.4 - PALATANI FORMULATION IN TETRAD FORMULATION

The gravitational Lagrangian is

$$L_G = \frac{1}{8K^2}\, \varepsilon^{\mu\nu\rho\sigma}\, \varepsilon_{mnpq}\, e_{\mu}^{m}\, e_{\nu}^{n}\, R_{\rho\sigma}^{pq}(\omega)$$

We vary e_{μ}^{ℓ} and $\omega_{\mu}^{m\ell}$ independently to obtain equations of motion: $\delta_e S=0$ gives

$$e\, [R_{\sigma}^{n} - \frac{1}{2}\, e_{\sigma}^{n}\, R] = K^2\, \tau_{\sigma}^{n}$$

when

$$R_{\sigma}^{n} = R^{n\ell}(\omega)_{\rho\sigma}\, e_{\ell}^{\rho}$$

$$\frac{\delta\, L_M}{\delta\, e_n^{\sigma}} = \tau_{\sigma}^{n}$$

$$R = R(e,\omega)$$

Now

$$R_{\sigma\rho}(\Gamma) = R^{\mu}{}_{\sigma\rho\mu}(\Gamma) = R^{m\ell}{}_{\rho\lambda}(\omega)\ e_m^{\lambda} e_{\sigma\ell} = R^{\ell m}{}_{\lambda\rho}\ e^{\lambda}{}_m\ e_{\sigma\ell} = R_{\rho}^{\ell}\ e_{\sigma\ell}$$

where

$$R_{\rho}^{\ell} = e^{\sigma\ell}\ R_{\sigma\rho}$$

We obtain

$$(R_{\rho\sigma} - \frac{1}{2}\ g_{\rho\sigma}\ R) = \frac{K^2}{\sqrt{\eta g}}\ \tau_{\rho\sigma}$$

It resembles the usual theory except that $R_{\mu\nu}$ and $\tau_{\mu\nu}$ are not necessarily symmetric.

Next connder $\delta_{\omega} S = 0$ we note that

$$\delta_{\omega}\ R^{m\ell}{}_{\rho\lambda}(\omega) = [D_{\rho}\ \delta\omega_{\lambda}^{m\ell} - D_{\lambda}\ \delta\omega_{\rho}^{m\ell}]$$

corresponding to the Palatini identity in ordinary tensorial formulation of gravity. It is only D_{μ} rather than full covariant derivative which appears. Note also that it follows from the transformation properties of connections that (the difference) $\delta\omega_{\mu}^{\ell m}$ is tensorial on all its indices. Thus

$$D_{\rho}(\delta\omega_{\lambda}^{m\ell}) = \partial_{\rho}(\delta\omega_{\lambda}^{m\ell}) + \omega_{\rho n}^{m}(\delta\omega_{\lambda}^{n\ell}) + \omega_{\rho n}^{\ell}(\delta\omega_{\lambda}^{mn})$$

Equations of motion ate derived to be

$$\frac{1}{2}\ \varepsilon_{pqm\ell}\ \varepsilon^{\mu\nu\lambda\rho}\ e_{\nu}^{q}\ [D_{\rho}e_{\mu}^{p} - D_{\mu}e_{\rho}^{p}] = -\ K^2\ \zeta_{m\ell}^{\lambda}$$

This looks like a dynamical equation. However, if we impose the tetrad condition (it does not follow in Palatini formulation) the equation reduces to an algebraic equation. Tetrad condition gives

$$D_{\rho}\ e_{\mu}^{\ell} = \Gamma_{\mu\rho}^{\alpha}\ e_{\alpha}^{\ell}$$

Defining torsion,

$$2S_{\mu\rho}^{\alpha} = (\Gamma_{\mu\rho}^{\alpha} - \Gamma_{\rho\mu}^{\alpha})$$

we obtain

$$\varepsilon_{pqm\ell}\ \varepsilon^{\mu\nu\lambda\rho} e_{\nu}^{q}\ e_{\alpha}^{p}\ S_{\mu\rho}^{\alpha} = -\ K^2\ \zeta_{m\ell}^{\lambda}$$

which lead to the algebraic relation:

$$S_{\rho\lambda}^{\nu} + \delta_{\rho}^{\nu}\ S_{\lambda} - \delta_{\lambda}^{\nu}\ S_{\rho} = \frac{K^2}{2}\ S_{\rho\lambda}^{\nu}$$

where

$$S_{\rho\lambda}^{\nu} = \frac{\zeta_{\rho\lambda}^{\nu}}{e} = \frac{\zeta_{\rho\lambda}^{\nu}}{\sqrt{\eta g}} \qquad \text{is Spin-tensor}$$

and

$$\zeta_{\lambda\rho}^{\nu} \equiv e_{\lambda}^{m}\ e_{\rho}^{n}\ \zeta_{mn}^{\nu} \quad , \quad S_{\rho} = S_{\rho\alpha}^{\alpha}$$

Torsion tensor is essentially the spin tensor of fields other than gravitation.

$$S_{\rho\lambda}^{\nu} = \frac{K^2}{2}\ (S_{\rho\lambda}^{\nu} + \frac{1}{2}\ \delta_{\rho}^{\nu}\ S_{\lambda} - \frac{1}{2}\ \delta_{\lambda}^{\nu}\ S_{\rho})$$

where

$$S_\rho = S^\alpha_{\rho\alpha} = -\frac{K^2}{4} S_\rho \quad , \quad S_\rho = S^\alpha_{\rho\alpha}$$

7 - LAGRANGIAN FOR SIMPLE SUPERGRAVITY

7.1 - NOETHER COUPLING METHOD

Noether coupling method mentioned in section 5.1 was used in the linearised supergravity theory of Section 5.2 to build step by step the first non-linear Simple Supergravity Lagrangian in the second order formulation by Freedman, van Nieuwenherizen and Ferrara. The local supersymmetry transformation and the gauge transformations become knitted together as in the case of Yang-Mills theory. A first order formulation was given by Desar and Zumino. In the first order formulation the action is simply the generally covariant and locally Lorentz covariant form of the linearized action (K = 1).

$$S_{SG} = S_G + S_{RS}$$

$$L_G = \frac{1}{2} e\ R(e,\omega) = \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{mnpq}\ e^m_\mu\ e^n_\nu\ R^{pq}{}_{\rho\sigma}(\omega)$$

$$L_{RS} = -\frac{i}{2} \varepsilon^{\lambda\mu\nu\rho} \bar{\Psi}_\lambda \gamma_5 \gamma_\mu D_\mu \Psi_\rho$$

where

$$e = \det e^\ell_\mu \quad , \quad R(e,\omega) = e^\mu_\ell\ e^\nu_m\ R^{\ell m}{}_{\mu\nu}(\omega) \quad , \quad D_\mu = \partial_\mu + \frac{1}{2} \omega^{\ell m}_\mu \sigma_{\ell m}$$

Note that only the curl enters L_{RS}. This allows us to use $D_\nu \Psi_\rho$ instead of using the complete covariant derivative $\Psi_{\rho;\nu}$ which involves an additional term dependent on space-time connection $\Gamma^\lambda_{\mu\nu}$. This is analogous to the coupling of Maxwell field in ordinary gravity where

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

which is a covariant curl and gauge invariant is used in place of $A_{\nu;\mu} - A_{\mu;\nu}$. This last expression differs from covariant curl by a non-gauge invariant term $(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}) A_\alpha$. The curl $(D_\nu \Psi_\rho - D_\rho \Psi_\nu)$ respects the gauge invariance of L_{RS} in a covariant fashion as discussed below.

Now e^ℓ_μ, $\omega^{\ell m}_\mu$ and Ψ_μ are varied independently. Auxiliary fields must be introduced if we couple the theory to matter. We will consider them latter. The equations of motion are:

$$\delta\Psi_\mu: \qquad R^\mu \equiv \varepsilon^{\mu\nu\alpha\beta} \gamma_5 \gamma_\ell (e^\ell_\nu D_\alpha - \frac{1}{4} C^\ell_{\nu\alpha}) \Psi_\beta = 0$$

$$\delta\omega^{\ell m}_\mu: \qquad C^\ell_{\nu\alpha} \equiv D_\nu e^\ell_\alpha - D_\alpha e^\ell_\nu = \frac{i}{e} \bar{\Psi}_\nu \gamma^\ell \Psi_\alpha$$

$$\delta e^\ell_\mu: \qquad G^{\ell\mu} \equiv (R^{\ell\mu} - \frac{1}{2} e^{\ell\mu} R) = T^{\ell\mu}$$

where

$$T^{\ell\mu} \equiv \frac{i}{2} \cdot \frac{1}{e} \cdot \varepsilon^{\nu\mu\alpha\beta} \bar{\Psi}_\nu \gamma_5 \gamma^\ell D_\alpha \Psi_\beta$$

$$R_{\mu\ell} \equiv R_{\ell m\mu\nu}\ e^{\nu m}$$

The non-vanishing of $C^\ell_{\nu\alpha}$ indicates that we have torsion, since,

$$D_\nu e^\ell_\alpha = \Gamma^\beta_{\alpha\nu} e^\ell_\beta \qquad \text{if} \qquad e^\ell_{\mu;\lambda} = 0$$

For $C^\ell_{\nu\alpha} = 0$, the equation implies vanishing torsion

$$(\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\beta}_{\nu\alpha}) = 0$$

We may solve the 2nd. equation. Write

$$\omega_{\mu}^{\ell m} = \omega_{\mu}^{\ell m}(e) + K_{\mu}^{\ell m}$$

when $\omega_{\mu}^{\ell m}(e)$ is defined from

$$\partial_{\nu} e^{\mu\ell} + \{^{\mu}_{\alpha\nu}\}_g \; e^{\alpha\ell} + \omega_{\nu m}^{\ell}(e) \; e^{\mu m} = 0$$

or

$$\omega_{\nu}^{\ell m}(e) = \frac{1}{2}\left[e_{\mu}^{\ell} \, \partial_{\nu} \, e^{m\mu} - e_{\mu}^{m} \, \partial_{\nu} \, e^{\ell\mu} + \{^{\mu}_{\alpha\nu}\}_g (e_{\mu}^{\ell} \, e^{\alpha m} - e_{\mu}^{m} \, e^{\alpha\ell})\right]$$

Here $\{^{\mu}_{\alpha\nu}\}_g$ is to be expressed in terms of tetrad fields using $g^{\mu\nu} = e_{\ell}^{\mu} \, e^{\nu\ell}$ etc. We may show that

$$K_{\mu\ell m} = -\frac{i}{4} \, (\bar{\Psi}_{\ell} \, \gamma_{\mu} \, \Psi_m + \bar{\Psi}_{\mu} \, \gamma_{\ell} \, \Psi_m - \bar{\Psi}_{\mu} \, \gamma_m \, \Psi_{\ell}) = - K_{\mu m\ell}$$

If in addition we impose

$$e_{\mu}^{\ell}(\omega,\Gamma)_{;\lambda} = 0$$

we define a space-time connection $\Gamma^{\lambda}_{\mu\nu}$.

We observe that the manifest local gauge invariance of R.S.field is lost. Also the 1st. equation has a free index, so that the question of consistency arises. Does $D_{\mu}R^{\mu} = 0$ by virtue of the other equations of motion? Precisely this was the difficulty encountered in all previous attempts to couple higher spin fields to gauge theories such as electromagnetism or gravity.

What makes the consistency at all possible is that in $D_{\mu}R^{\mu}$ the commutation of two covariant derivatives reduce to the Einstein, rather than the full Riemann tensor, through the interplay of Dirac algebra and Ricci identities. We have

$$D_{\mu}R^{\mu} \equiv -\frac{1}{2} \, G^{\ell\mu} \, \gamma_{\ell} \, \Psi_{\mu} + \frac{1}{4} \, \epsilon^{\mu\nu\alpha\beta} \, C^{\ell}_{\mu\nu} \, \gamma_5 \, \gamma_{\ell} \, D_{\alpha} \, \Psi_{\beta}$$

But there is another identity derivable from Fierz rearrangements for anticommuting spinors

$$\frac{1}{2} \, T^{\ell\mu} \, \gamma_{\ell} \, \Psi_{\mu} - \frac{1}{4} \, \epsilon^{\mu\nu\alpha\beta} \, \bar{\Psi}_{\mu} \, \gamma^{\ell} \, \Psi_{\nu} \, \gamma_5 \, \gamma_{\ell} \, D_{\alpha} \, \Psi_{\beta} \equiv 0$$

Using this and the equations of motion one verifies the consistency

$$D_{\mu}R^{\mu} \cong 0.$$

The total variation of S_{SG} vanishes under the local s.s. gauge transformation:

$$\delta e_{\mu}^{\ell} = i \, \bar{\alpha} \, \gamma^{\ell} \, \Psi_{\mu}$$

$$\delta \Psi_{\mu} = 2 \, D_{\mu}\alpha$$

$$\delta\omega_{\mu\ell m} = B_{\mu\ell m} - \frac{1}{2} \, (e_{\mu m} \, B_{\nu\ell n} \, e^{\nu n} - e_{\mu\ell} \, B_{\nu m n} \, e^{\nu n})$$

where

$$B_{\ell}^{\lambda\mu} = i \, \bar{\alpha} \, \gamma_5 \, \gamma_{\ell} \, D_{\nu} \, \Psi_{\rho} \, \epsilon^{\lambda\mu\nu\rho}$$

The equivalent 2nd. order form is obtained by substituting $\omega = \omega(e) + K$ in the Lagrangian. The variation then is taken with $e^{\ell}_{\mu}, \Psi_{\mu}$ as independent fields. A contacts Seagull term arises

$$L_{con} \sim [\bar{\Psi}^{\ell} \gamma^{m} \Psi^{n} (\bar{\Psi}_{\ell} \gamma_{m} \Psi_{n} + 2 \bar{\Psi}_{m} \gamma_{\ell} \Psi_{n}) - 4 (\bar{\Psi}_{\ell} \gamma\cdot\Psi)^{2}]$$

The three invariances, coordinate, Lorentz and supersymmetry are not, however, independent. The commutator of two independent s.s. transformations characterized by $(\xi(x), \eta(x))$ yields a general coordinate transformation corresponding to the real displacement $i(\bar{\xi}\gamma^{\mu}\eta)$, together with field dependent Lorentz and s.s. transformations:

$$[\delta_S(\epsilon_1), \delta_X(\epsilon_2)] e^{\ell}_{\mu} = \delta_G(\xi^{\alpha}) e^{\ell}_{\mu} + \delta_L(\xi^{\alpha}\omega_{\alpha\ell m}) e^{\ell}_{\mu} + \delta_S(-\tfrac{1}{2}\xi^{\alpha}\Psi_{\alpha}) e^{\ell}_{\mu}$$

where

$$\xi^{\alpha} = 2i\, \bar{\epsilon}_2 \gamma^{\alpha} \epsilon_1(x)$$

and

$$[\delta_S(\epsilon_1), \delta_S(\epsilon_2)] \Psi_{\mu} = \text{as on } e^{\ell}_{\mu} + \bar{\epsilon}_1 \gamma^{\alpha} \epsilon_2 (\tfrac{1}{4} \gamma_{\alpha} g_{\mu\rho} + \tfrac{e}{2} \epsilon_{\mu\alpha\rho\tau} \gamma_5 \gamma_{\tau}) R^{\tau} +$$

$$+ \bar{\epsilon}_1 \sigma^{\rho\sigma} \epsilon_2 (\tfrac{1}{2} \sigma_{\rho\sigma} g_{\mu\tau} + g_{\mu\rho} g_{\sigma\tau} + \tfrac{e}{2} \epsilon_{\rho\sigma\tau\mu} \gamma_5) R^{\tau} \Omega + 0 + 0$$

(on-shell closure).

7.2 - SUPERGRAVITY AS THE GAUGE THEORY OF GRADED POINCARÉ ALGEBRA

Generators of SP_4 are

$$X_A = \{P^m, M^{\ell m}, Q_{\alpha}\}$$

and we indicate the corresponding parameters by $\{a^m, \lambda^{\ell m}, \bar{\epsilon}_{\alpha}\} = \eta^A$. When these parameters, in a globally s.s. theory, become x-dependent the Lagrangian is no more invariant and we need to introduce compensating gauge (vector) fields

$$\{e^{\ell}_{\mu}(x), \omega^{\ell m}_{\mu}(x), \Psi^{\alpha}_{\mu}(x)\} \equiv h^{A}_{\mu}(x)$$

to define a gauge covariant derivative D_{μ}. In the present case, ℓ, α are local tangent space indices while is the world vector index. We also define

$$\eta(x) = \eta^A(x) X_A = a_m P^m + \tfrac{1}{2} \lambda_{\ell m} M^{\ell m} + \bar{\epsilon} Q$$

which is Poincaré superalgebra valued scalar field, while

$$h_{\mu} = h^{A}_{\mu} X_A = e^{m}_{\mu} P_m + \tfrac{1}{2} \omega^{\ell m}_{\mu} M_{\ell m} + \bar{\Psi}_{\mu} Q$$

is super-algebra valued vector field for SP_4.

Let us remind the case of Yang-Mills theory in flat space-time:

$$A^{\mu} = A^{\mu}_{a} T^{a} \quad , \quad \omega = \omega_a(x) T^a \quad , \quad (T_a, T_b) = i f_{abc} T_c .$$

For the gauge variation we have

$$\delta(\text{gauge}) A^{\mu}_{a} = -\tfrac{1}{g} D^{\mu}_{ab} \omega_b = -\tfrac{1}{g} (D^{\mu}\omega)_a$$

or

$$\delta(\text{gauge}) A^{\mu} = -\tfrac{1}{g} \{\partial^{\mu}_{\omega} + ig [\omega, A^{\mu}]$$

Here

$$D^{\mu}_{ab} = (\delta_{ab}\, \partial^{\mu} - g\, f_{abc}\, A^{\mu}_{c}).$$

In the present case analogously we write

$$\delta(\text{gauge})\ h_{\mu} = \partial_{\mu}\, \eta + i\, [\eta, h_{\mu}] \quad ,$$

$$\delta(\text{gauge})\ h^{A}_{\mu} = (D_{\mu}\eta)^{A}$$

$$(D_{\mu}\eta)^{A} = \partial_{\mu}\eta^{A} + h^{B}_{\mu}\, \eta^{C}\, f^{A}_{CB}$$

where

$$[X_A, X_B] = f^{C}_{AB}\, X_C \equiv X_A X_B (-1)^{AB} X_B X_A$$

and

$$A = \ell,\ (\ell m),\ \alpha$$

are tangent space indices. The summation in D_{μ} is over all gauge connections, not only over the Lorentz connection. There is double counting, e.g., translations appear in the base manifold and in the tangent group.

We note

$$(D_{\mu}\eta)^{A} X_A = (\partial_{\mu}\eta^{A})\, X_A + i\, h^{B}_{\mu}\, \eta^{C}\, [X_C, X_B]$$

$$= (\partial_{\mu}\eta^{A})\, X_A + i\, h^{B}_{\mu}\, \eta^{C}\, (X_C X_B - (-1)^{BC} X_B X_C)$$

$$= \partial_{\mu}\, \eta + i\, h^{B}_{\mu}\, (\eta X_B - (-1)^{2BC} X_B\,)$$

$$= \partial_{\nu}\, \eta + i\, [\eta, h_{\mu}]$$

Thus

$$\delta(\text{gauge})\ h_{\mu} = (D_{\mu}\eta)^{A}\, X_A \equiv (\delta e^{m}_{\mu})\, P_m + \frac{1}{2}\, (\delta\omega^{\ell m}_{\mu})\, M_{\ell m} + (\delta\bar{\Psi})\, Q$$

From

$$[\eta, h_{\mu}] = [a.P + \frac{1}{2}\, \lambda.M + \bar{\varepsilon} Q,\ e^{\ell}_{\mu} P_{\ell} + \frac{1}{2}\, \omega^{\ell m}_{\mu}\, M_{\ell m} + \bar{\Psi}_{\mu}\, Q] \quad ,$$

$$(\delta e^{m}_{\mu})\, P_m = (\partial_{\mu} a^{m})\, P_m + \frac{i}{2}\, [a.P, \omega_{\mu}.M] + \frac{i}{2}\, [\lambda.M, e_{\mu}.P] + i\, [\bar{\varepsilon}Q,\ \bar{\Psi}_{\mu} Q] \quad ,$$

$$[a.P, \omega_{\mu}\, .M] = -\, i a^{n}\, \omega^{\ell m}_{\mu} (\eta_{\ell n} P_m - \eta_{mn} P_{\ell}) = -\, i\, \omega^{\ell m}_{\mu} (a_{\ell} P_m - a_m P_{\ell})$$

$$[\lambda.M, e^{\ell}_{\mu} P_{\ell}] = i\, \lambda^{\ell m} e^{n}_{\mu}\, (\eta_{\ell n} P_m - \eta_{mn} P_{\ell}) = i\, \lambda^{\ell m} (e_{\mu\ell} P_m - e_{\mu m} P_{\ell}) \quad ,$$

$$[\bar{\varepsilon}Q,\ \bar{\Psi}_{\mu} Q] = -\, \bar{\varepsilon}\, \gamma^{\ell} \Psi_{\mu} P_{\ell}$$

it follows

$$\delta e^{\ell}_{\mu} = -\, i\, \bar{\varepsilon}\, \gamma^{\ell}\, \Psi_{\mu} + \partial_{\mu} a^{\ell} (\lambda^{m\ell}\, e_{\mu m} + \omega^{\ell m}_{\mu} a_m)$$

where $\delta \equiv \delta(\text{gauge})$. Note that e^{ℓ}_{μ} transforms as a gauge connection due to the presence of the term $\partial_{\mu}\, a^{\ell}$.

Consider the case of only local s.s. gauge transformations e.g. $a^{\ell} = 0$, $\lambda^{\ell m} = 0$, then

$$\delta e_\mu^\ell = - i \bar{\varepsilon} \gamma^\ell \Psi_\mu \qquad \text{"real"}$$

$$\delta \omega_\mu^{\ell m} = 0$$

and

$$\delta(\bar{\Psi}_\mu Q) = \partial_\mu \bar{\varepsilon} Q + \frac{i}{2} [\bar{\varepsilon} Q, \omega_\mu^{\ell m} M_{\ell m}] = (\partial_\mu \bar{\varepsilon}) Q + \frac{i}{2} \bar{\varepsilon}_\alpha \omega_\mu^{\ell m} i(\sigma_{\ell m})_{\alpha\beta} Q_\beta$$

or

$$\delta \bar{\Psi}_\mu^\alpha = \partial_\mu \bar{\varepsilon}_\alpha - \frac{1}{2} \omega_\mu^{\ell m} \bar{\varepsilon}_\beta (\sigma_{\ell m})_{\beta\alpha}$$

This implies

$$\delta \Psi_\mu^\alpha = \partial_\mu \varepsilon_\alpha + \frac{1}{2} \omega_\mu^{\ell m} (\sigma_{\ell m})_{\alpha\beta} \varepsilon_\beta$$

or

$$\delta \Psi_\mu = D_\mu \varepsilon$$

Here D_μ is the usual covariant derivative instead of the derivative D_μ.

Field strengths or curvatures which transform gauge covariantly may be defined by

$$[D_\mu(h), D_\nu(h)] - i R_{\mu\nu}(h)$$

where

$$D_\mu(h) = (\partial_\mu - i h_\mu).$$

We obtain

$$R_{\mu\nu}(h) = \partial_\mu h_\nu - \partial_\nu h_\mu - i [h_\mu, h_\nu]$$

Explicitly

$$R_{\mu\nu}^{\ell m}(M) = R_{\mu\nu}^{\ell m}(\omega) = R^{\ell m}{}_{\mu\nu}(\omega)$$

$$R_{\mu\nu}^\alpha(Q) = D_\nu \Psi_\mu^\alpha - D_\mu \Psi_\nu^\alpha$$

$$R_{\mu\nu}^\ell(P) = D_\nu e_\mu^\ell - D_\mu e_\nu^\ell + \frac{1}{2} \bar{\Psi}_\mu \gamma^\ell \Psi_\nu$$

We may obtain the second order formulation of N = 1 Supergravity if we impose the constraint $R_{\mu\nu}^\ell(P) = 0$. This leads to

$$\omega_\mu^{\ell m} = \omega_\mu^{\ell m}(e) + K_\mu^{\ell m} \quad .$$

A gauge covariant Lagrangian may be written as

$$L_{SG} = - \frac{1}{2} e R_{\mu\nu}^{\ell m}(M) e_m^\mu e_\ell^\nu - \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \bar{\Psi}_\mu \gamma_5 \gamma_\nu R_{\sigma\rho}(Q) = - \frac{1}{2} e R^{\ell m}{}_{\mu\nu}(\omega) e_m^\mu e_\ell^\nu - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\Psi}_\mu \gamma_5 \gamma_\nu D_\rho \Psi_\sigma$$

We may verify

$$\delta(\text{gauge}) L_{SG} = - \partial_\mu [\bar{\varepsilon} \gamma^\mu \sigma^{\rho\sigma} D_\rho \Psi_\sigma] \cong 0$$

Thus supersymmetry is not an internal symmetry but a space-time symmetry, like in the case of general coordinate transformations $\delta L = \partial_\mu (L \xi^\mu)$.

7.3 - AUXILIARY FIELDS IN N = 1 SUPERGRAVITY

A minimal set of auxiliary fields was found by Ferrara and van Nieuwenhuizen and Still and West, and a term calculus developed. The coupling to other matter supermultiplets is rather straightfound with the tensor calculus (see P.van Nieuwenhirizen, Cargèse Lactures,1978).

The appearence in the gange algebra of terms proportional to the field equations suggests that the addition of auxiliary fields is the transformation rules might restore off-shell closure. Another motivation for including auxiliary fields is to maintain equal number of degrees of freedom of fermionic and bosonic fields:

e_μ^ℓ 16 - 4 (general coord. transf. invariance)
- 6 (Local Lorentz gauge invariance)
= 6 degrees of freedom

Ψ_μ : 16 - 4 (Local s.s. gauge invariance)
= 12 degrees of freedom.

There is a mismatch of 6 degrees of freedom. The auxiliary fields found are

A'_m (axial) , S (scalar) , P (pseudo-scalar)

A_m is not a gauge field rather like Ψ, a local vector field. The Lagrangian takes the form

$$L = L^{(2)}(e,\omega) + L^{(3/2)}(e,\Psi,\omega) - \frac{e}{3}(S^2 + P^2 - A_m^2)$$

is invariant under s.s. gange transformation.

$$\delta e_\mu^m = \frac{K}{2}\bar{\varepsilon}\,\gamma^m\,\Psi_\mu$$

$$\delta\Psi_\mu = \frac{1}{K}(D_\mu + \frac{iK}{2}A_\mu\,\gamma_5)\,\varepsilon - \frac{1}{2}\gamma_\mu\,\eta\,\varepsilon$$

$$\delta S = \frac{1}{4}\bar{\varepsilon}\,\gamma\cdot R^{cov}$$

$$\delta P = -\frac{i}{4}\bar{\varepsilon}\,\gamma_5\,\gamma\cdot R^{cov}$$

$$\delta A_m = \frac{3i}{4}\bar{\varepsilon}\,\gamma_5\,(R_m^{cov} - \frac{1}{3}\gamma_m\,\gamma\cdot R^{cov})$$

where

$$\eta = -\frac{1}{3}(S - i\,\gamma_5 P - i\,\gamma^m A_m\,\gamma_5)$$

$$R^{\mu\ cov} = \varepsilon^{\mu\nu\rho\sigma}\,\gamma_5\,\gamma_\nu(D_\rho\,\Psi_\sigma - \frac{i}{2}A_\sigma\,\gamma_5\,\Psi_\rho + \frac{1}{2}\gamma_\sigma\,\eta\,\Psi_\rho)$$

Now

$$[\delta_S(\epsilon_1),\delta_S(\epsilon_2)]\,\phi = \{\delta_G(\xi^\alpha) + \delta_S(-\xi^\alpha\Psi_\alpha) + \delta_L[\xi^\alpha\tilde{\omega}_\alpha^{\ell m} + \frac{i}{e}\bar{\epsilon}_2\sigma^{\ell m}(S-i\gamma_5P)\epsilon_1]\}\,\phi$$

where

$$\tilde{\omega}_{\mu\ell m} = \omega_{\mu\ell m} - \frac{i}{3}\,\epsilon_{\mu\ell mn}\,A^n\quad ,$$

$$\xi^\mu_{(x)} = \frac{1}{2}\bar{\epsilon}_2\,\delta^\mu\,\epsilon_1$$

The structure "constants" now also depend on the auxiliary fields. The algebra closes off-shell. We have really field dependent "structure functions". This feature is not present in Yang-Mills theory or ordinary gravity.

8 - SUPERGRAVITY AS GEOMETRY OF SUPERSPACE

If one assumes the differential geometry of superspace to the super-Riemannian the connection with the space-time formulation of supergravity requires a rather delicate limiting procedure in superspace. The reason is that the field equations in Riemannian superspace do not admit as solution the flat superspace of ordinary global supersymmetry.

Wess and Zumino and Arulov, Volkov and Soroka introduced a (non-Riemannian) different differential geometry in superspace. The superspace of global s.s. comes out as a special case. We follow Wess-Zumino approach.

A general affine superspace is parametrized by coordinates $Z^M \equiv (X^m, \theta^\mu)$ where X^m are commuting space-time coordinates while θ^μ are anticommuting variables. The supervielbein matrix $E_M^A(Z)$, where $A = (a, \alpha)$ are tangent space indices and its inverse $E_A^M(Z)$ can be used to transform world (super) tensors into tangent space tensors and vice versa. The submatrices E_m^a and E_μ^α are bosonic, "even" elements, while E_m^α and E_μ^a consist of fermionic elements. We also introduce super-connections $\phi_{MA}{}^B(Z)$ w.r.t. (X,θ) dependent local tangent space transformations. We may define one-forms

$$E^A = dZ^M E_M{}^A$$

$$\phi_A{}^B = dZ^M \phi_{MA}{}^B$$

INDICES	TANGENT SPACE		CURVED SPACE	
Vector	a, b, ...	A, B, ...	m, n, ...	M, N, ...
Spinor	α, β, ... (α̲, β̲, ...)	A, B, ...	μ, ν, ... (μ̲, ν̲, ...)	M, N, ...
Internal	i, j, ... (α̲, β̲, ...)	A, B, ...	(μ̲, ν̲, ...)	M, N, ...

Here we take dX^m to be odd and $d\theta^\mu$ to be even, the opposite of X^m and θ^μ; one is still working with Grassmann algebra. Similarly, E^a anticommute with each other and with E^α, while E^α commute with each other. Coordinate transformations in superspace mix X and θ but in such a way that the new X's are still even and the new θ's odd. Under an infinitesimal coordinate transformation specified by parameter $\xi^M(Z)$ we have for scalar V(Z)

$$\delta(\xi)\, V(Z) = \xi^M \frac{\partial V}{\partial Z^M} \equiv \xi^M \partial_M V$$

while for tensor $T_{LM}{}^N(Z)$:

$$\delta(\xi)\, T_{LM}{}^N = \xi^S \partial_S T_{LM}{}^N + U_L{}^{L'} T_{L'M}{}^N + U_M{}^{M'} T_{LM'}{}^N (-1)^{(m+m')(\ell+1)} - T_{LM}{}^{N'} U_{N'}{}^N$$

where

$$U_M{}^N = \partial_M \xi^N$$

and the small indices in exponent are 0 or 1 according as the corresponding capital index is bosonic or fermionic. We may also define densities D(Z) which transform as

$$\delta\theta = \partial_S(\xi^S \theta)\,(-1)^s \quad : \text{scalar density.}$$

The contractions are defined without extra sign if upper index is on the left. The extra factor $(-1)^s$ is due to contraction of type $A_S B^S$. Since a density changes by a sum of terms each of which is a derivative, the integral of a density over all of superspace is invariant under $\delta(\xi^M)$. Thus densities can be used as Lagrangians.

We note that

$$d = dX^m \frac{\partial}{\partial X^m} + d\theta^\mu \frac{\partial}{\partial \theta^\mu}$$

Satisfies

$$d^2 = 0$$

More details are given in 1980 Eric Lectures by B.Zumino.

Covariant derivative of tangent space vector u_A is written as

$$D_M u_A = \partial_M u_A - \phi_{MA}{}^B u_B$$

Imposing-

$$D_M (v^A u_A) = \partial_M (v^A u_A)$$

and using graded Leibnitz rule for derivative

$$D_M (v^A u_A) = (D_M v^A) u_A + (-1)^{ma} v^A D_M u_A$$

we derive

$$D_M u^A = \partial_M u^A + (-1)^{ma} u^B \phi_{MB}{}^A$$

For super-vielbeins

$$D_N E_A{}^M = \partial_N E_A{}^M - \phi_{NA}{}^B E_B{}^M$$

$$D_N E_M{}^A = \partial_N E_M{}^A + (-1)^{n(b+m)} E_M{}^B \phi_{NB}{}^A$$

since M index is "super-scalar" w.r.t. local tangent space rotations. However, the bosonic or fermionic nature of the index can not be ignored. We may define

$$D_A = E_A{}^M D_M$$

then

$$[D_C, D_D\} u_A = - R_{CDA}{}^B u_B - T_{CD}{}^B D_B u_A$$

They define super-curvature and supertorsion tensors. Here $[\,,\}$ indicates the graded **commutator.** **Applying** graded Jacobi identities to this relation we obtain Bianchi identities.(A 2-**component** notation is convenient to derive them).

$$\sum_{(ABC)} (D_A T_{BC}{}^D + T_{AB}{}^F T_{FC}{}^D - R_{ABC}{}^D) = 0$$

$$\sum_{(ABC)} (D_A R_{BCD}{}^F + T_{AB}{}^G R_{GCD}{}^F) = 0$$

where $\sum_{(ABC)}$ is cyclic sum:

$$\sum_{(ABC)} X_{ABC} = X_{ABC} + (-1)^{C(a+b)} X_{CAB} + (-1)^{a(b+c)} X_{BCA}$$

e.g. a permutation of two bosonic or one bosonic and one fermionic index gives rise to a change in sign, while for two fermionic indices there is no change of sign.

The geometric formulation, in an affine superspace requires general super-coordinate transformation invariance. We restrict these transformation so that even nature of X^m and odd nature of θ^μ is not changed. We restrict further the tangent space group, which is generally a general graded linear group, to be a local (X,θ) dependent Lorentz Group L. The superconnections then satisfy the restrictions:

$$\phi_M{}^{ab} = - \phi_M{}^{ba}$$

$$\phi_{M\alpha\beta} = \frac{1}{2} \phi_{Mab}(\sigma^{ab})_{\alpha\beta} \qquad ; \quad \sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]$$

$$\phi_{Ma\beta} = \phi_{M\alpha a} = 0$$

Thus $\phi_{MA}{}^B$ takes values in the algebra of the tangent space group L. No connections of type Γ^M_{NP} are introduced and no expressions of type $D_M T_N$ appear. Rather, only curls of type $(D_M T_N - D_N T_M)$ appear in the formulation; they are already tensorial.

Consider a linear transformation in the tangent space

$$\delta v^A = v^B X_B{}^A$$

then

$$\delta u_A = - X_A{}^B u_B$$

For example,

$$\delta(\delta^A_B) = \delta^C{}_B X_C{}^A - X_B{}^C \delta_C{}^A = X_B{}^A - X_B{}^A = 0$$

Also

$$\delta(v^A u_A) = (\delta v^A) u_A + v^A (\delta u_A) = v^B X_B{}^A u_A - v^A X_A{}^B u_B = 0$$

For Lorentz group L in tangent space:

$$X_a{}^b = L_a{}^b \qquad\qquad L_{ab} = - L_{ba}$$

$$X_\alpha{}^\beta = \frac{1}{2} L_a{}^b (\sigma_b{}^a)_\alpha{}^\beta \qquad\qquad L \equiv \left(\begin{array}{c|c} L_a{}^b & 0 \\ \hline 0 & X_\alpha{}^\beta \end{array}\right)$$

$$X_a{}^\alpha = X_\alpha{}^a = 0$$

Note that $X_\alpha{}^\beta$ describes the same Lorentz transformation as $L_a{}^b$ when applied to spinors and

$$\sigma_b{}^a = \frac{1}{4} [\gamma_b, \gamma^a]$$

Considered as a matrix in the last two indices, the curvature belongs to the algebra of the tangent space group. Thus like $\phi_{MA}{}^B$ we have

$$R_{AB,ab} = - R_{AB,ba}$$

$$R_{AB,\alpha\beta} = \frac{1}{2} R_{AB,ab} (\sigma^{ab})_{\alpha\beta}$$

$$R_{AB,a\alpha} = R_{AB,\alpha a} = 0$$

The final restriction on the geometry consists in imposing constraints on the supertorsion. Tangent group restriction along with these restrict the number of component fields in the theory.

They cannot be too stringent so as to allow only rigid superspace. Wess and Zumino impose

$$T_{\alpha\beta}{}^{c} = 2i\ (\gamma^{c})_{\alpha\beta}$$

$$T_{ab}{}^{c} = T_{\alpha\beta}{}^{\gamma} = T_{\alpha b}{}^{c} = 0$$

having $T_{\alpha b}{}^{\gamma}$ and $T_{ab}{}^{\gamma}$ undetermined. The last one is related to R.S. field strength $(\partial_a \psi_b^{\gamma} - \partial_b \psi_a^{\gamma})$. The first one is suggested from rigid supersymmetry (*).

(*) We found for the covariant derivatives

$$D_A: \quad \begin{aligned} D_a &= i\,P_a = \partial_a \\ D_\alpha &= i\,C_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - \frac{1}{2}(\gamma^a\theta)_\alpha\,\partial_a \end{aligned}$$

In the "curved" superspace (rigid supersymmetry)

$$D_M = (\partial_m, \partial_\mu) \equiv \left(\frac{\partial}{\partial X^m}, \frac{\partial}{\sigma\theta_\mu}\right)$$

and

$$D_A = E_A{}^M D_M = E_A{}^m\,\partial_m + E_A^\mu\,\partial_\mu$$

so that

$$D_a = E_a{}^m\,\partial_m + E_a{}^\mu\,\partial_\mu$$

$$D_\alpha = E_\alpha{}^m\,\partial_m + E_\alpha{}^\mu\,\partial_\mu$$

$$\therefore\ E_a{}^m = \delta_a^m \quad , \quad E_a^\mu = 0$$

$$E_\alpha^\mu\frac{\partial}{\partial\theta_\mu} = i\,C_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} = i\,C_{\alpha\beta}\,\delta_\beta^\mu\frac{\partial}{\partial\theta_\mu}$$

$$\therefore\ E_\alpha^\mu = i\,C_{\alpha\beta}\,\delta_\beta^\mu$$

$$E_\alpha^m\,\partial_m = -\frac{1}{2}(\gamma^a\theta)_\alpha\,\partial_a$$

Now for any vector field A_a we have

$$A_a = E_a{}^M A_M = E_a{}^m A_m + E_a{}^\mu A_\mu = E_a{}^m A_m$$

Making use of

$$A^a E_a{}^m + A^\alpha E_\alpha{}^m = A^m$$

or

$$A^a \delta_a{}^m + A^\alpha E_\alpha{}^m = A^m$$

we find $A^\alpha E_\alpha{}^m = 0$. Hence

$$(\gamma^a\theta)_\alpha\,\partial_a = (\gamma^a\theta)_\alpha\,E_a{}^m\,\partial_m = (\gamma^m\theta)_\alpha\,\partial_m$$

or

$$E_\alpha{}^m = -\frac{1}{2}(\gamma^m\theta)_\alpha$$

$$E_A{}^M\ (\text{rigid}) = \begin{pmatrix} \delta_a^m & 0 \\ -\frac{1}{2}(\gamma^m\theta)_\alpha & iC_{\alpha\beta}\delta_\beta^\mu \end{pmatrix}$$

Calculating $[D_A, D_B\}$ in the present case leads to vanishing curvature and only a non-vanishing supertorsion $T_{\alpha\beta}^{c}$.

Working out the Bianchi identities and using the constraints, all components of the super-curvature and supertorsion can be expressed in terms of 3 superfields $G_{\alpha\dot{\beta}}$, $W_{\alpha\beta\gamma}$, R and their conjugates (2-component formalism used). $G_{\alpha\dot{\beta}}$ is hermitian and $W_{\alpha\beta\gamma}$ totally symmetric. We also find differential relations:

$$D^{\alpha} W_{\alpha\beta\gamma} = D_{\beta}{}^{\dot{\epsilon}} G_{\gamma\dot{\epsilon}} + D_{\gamma}{}^{\dot{\epsilon}} G_{\delta\dot{\epsilon}} \quad , \qquad D^{\alpha} G_{\alpha\dot{\beta}} = D_{\dot{\beta}} R^{*}$$

$$D_{\dot{\epsilon}} W_{\alpha\beta\gamma} = D_{\dot{\epsilon}} R = 0 \quad , \qquad \text{where } D_{\beta\dot{\epsilon}} = {}^{a} D_{a}.$$

$G_{\alpha\dot{\beta}}(X,\theta)$ in its expansion contains at $\theta\bar{\theta}$ level a tensor which contains the Einstein tensor, at $\theta\theta\bar{\theta}$ level we find a spinor which is the Rarita-Schwinger operator (left hand side of the R.S. equation). $R(X,\theta)$ contains the scalar curvature tensor at $\theta\theta$ level. $W_{\alpha\beta\gamma}$ contains the R.S. field strength at $\theta = 0$ level and Weyl conformal spinor at θ level.

The auxiliary fields are also contained in the superspace formalism. They may be obtained by solving constraints on torsion. More details are given in Zumino's lectures at Cargèse (1978) and Erice (1980).

REFERENCES (*)

Chapter 1:

1 - See for example: Spinor Space Representation of Lorentz Group, P.Srivastava, Rev. Bras. Fis., 4, 507 (1974).

2 - F.A.Berezin, The Method of Second Quantization, Academic Press, 1966.

Chapter 2:

3 - J.Wess and B. Zumino, Nucl.Phys., B70, 39(1974).

4 - B.Zumino in Cargèse Lectures 1978, Recent Developments in Gravitation, Editors M.Levy and S.Deser (Plenum Press, 1979).

5 - J.Wess, Lecures given at Boulder School 1979; Lectures given at VIII GIFT Inter. Seminar, Lecture Notes in Physics 77 (Springer-Verlag).

6 - P.Fayet and S.Ferrara, Phys.Rep. 32C, 249 (1977) - contains a long list of references.

7 - A.Salam and J.Strathdee, Phys.Rev., D11, 1521 (1975); Forschr.Phys., 26, 57(1978) - contains a long list of references.

Chapter 3 and 4:

8 - A.Salam and J.Strathdee, See ref. 7.

9 - B.Zumino, ref. 4.

10- P.van Nieuwenhirizen, Supergravity, Phys.Rep.68(4),181(1981) - contains a long list of references.

Chapter 5:

11- See for example ref. 10.

Chapter 6:

12- Proc.II Escola de Cosmologia e Gravitação, Vols. I, II, III, Editor M.Novello, C.B.P.F., Rio de Janeiro, 1980.

13- E.Lord, Tensor Relativity and Cosmology, Tata Mc Graw Hill, 1976. See also P. Srivastava, C.B.P.F. - N.F. 009/82 and ref. 10.

Chapter 7 and 8:

14- D.Z.Freedman, P.van Nieuwenhuizen and S.Ferrara, Phys.Rev.D13,3214 (1976);D14,912(1976). See also ref.10.

15- S.Deser and B.Zumino, Phys.Lett., 62B, 335(1976).

16- See also Cargèse Lectures, ref. 4.

(*) Due to lack of time in preparing these notes references are mainly given for the sources frequently used by the author. The complete list of references with reference to the original literature may be found in these references.