SOLITONS OF MATTER IN GENERAL RELATIVITY

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1 - INTRODUCTION

The soliton concept has been widely used in particle physics to represent particles, even at a classical level, mainly due to the localized behaviour of soliton type of waves as well as its stability (1). It appears to us that solitons of matter in general relativity might be a good representation of very massive objects, e.g., galaxies. Also the collision of two galaxies shares some common features with the collision of two solitons, for instance the collision of two galaxies does not destroy the galaxies.

In general relativity, equations having soliton type of behaviour appear in the study of the vacuum Einstein equations for axially symmetric waves with two degrees of freedom (2, 3). Also the vacuum Einstein equations for stationary-axially-symmetric space-times have soliton (3,5) type of solution (Erst equations). Due to the close relation between vacuum Einstein equations for cylindrically symmetric space-times and the Einstein equations coupled with irrotational-perfect fluids with p = W equations of state for the same type of space-times, the concept of soliton also has appeared in this context (6). But, in this last case the mater does not have soliton behaviour, because the potencial that describes the matter obeys the usual linear wave equation in cylindrical coordinates.

The method used to solve the Einstein equations in the above mentioned cases are the inverse scattering method (2, 4) and the use of Bäcklund transformations (3, 5, 7).

In this paper we want to study the possibility of having matter propagating as solitary waves. In the next section we present a set of well known equations that have soliton type of solutions. In Section 3 we present a model of anisotropic fluid described by two perfect fluid components (8). In Section 4 we particularize the above mentioned model to yield matter evolution equations, general enough to include as particular cases all the equations exhibited in Section 2. In the next section, we further particularize the model to arrive to equations that present a clear soliton behaviour also these equations can be explicitly integrated. And, finally in Section 6, we present a discussion of the obtained results.

2 - EQUATIONS WITH SOLITARY WAVE TYPE OF SOLUTIONS

In general relativity we have two types of field variables: the field variables associated to the space-time, i.e., the metric $g_{\mu\nu}$, and second the field variables associated to the matter, e.g. for a fluid we have pressure, density, flux velocity, etc. In this paper we shall study a model of fluid whose velocity potencials obey evolution equations that are known to allow solitons as solutions.

Some well known scalar wave equations with solitary wave solutions are (1)

$$\phi_{tt} - \phi_{xx} = \begin{cases} \sin\phi & \text{sine-Gordon} \\ e^{\phi} & \text{Liouville} \end{cases} \tag{2.1}$$

A two-dimensional generalization of the sine-Gordon equations is (9)

$$\phi_{tt} - \phi_{xx} + \frac{1}{2}\sin 2\phi - \frac{\cos\phi}{\sin^3\phi}(\Psi_t^2 - \Psi_x^2) = 0 \tag{2.2a}$$

$$\Psi_{tt} - \Psi_{xx} - \frac{2}{\sin\phi\cos\phi}(\phi_t\Psi_t - \phi_x\Psi_x) = 0 \tag{2.2b}$$

here we have introduced the notation $\phi_x \equiv \partial\phi/\partial x$, $\phi_{tt} = \partial^2\phi/\partial t^2$, etc.

Another systems of equations are

$$(\Box - m^2)\,\psi = \begin{cases} -2g^2\,|\psi|^2\psi & \text{non-linear Klein-Gordon} \\ +2g^2\,|\psi|^2\psi & \text{Higgs,} \end{cases} \tag{2.3}$$

where $\Box \equiv \partial^2/\partial t^2 - \nabla^2$, also these two equations are attributed to the so-called ψ^4 field theory (1). And finally we have the equations

$$\phi_{tt} + \phi_t/t - \phi_{xx} - (\phi_t^2 - \phi_x^2)/\phi - \phi^3(\Psi_t^2 - \Psi_x^2) = 0 \tag{2.4a}$$

$$(t\,\phi^2\,\Psi_t)_t - (t\,\phi^2\,\Psi_x)_x = 0 \tag{2.4b}$$

This system of equations has been studied by many authors in the context of general relativity (10). It can be cast as

$$(t\,\gamma_t\,\gamma^{-1})_t - (t\,\gamma_x\,\gamma^{-1})_x = 0 \tag{2.5a}$$

with

$$\gamma \equiv t \begin{vmatrix} \phi & \phi\Psi \\ \phi\Psi & \Psi^2\phi + \phi^{-1} \end{vmatrix} \tag{2.5b}$$

the special representation (2.5a) is the starting point of the inverse scattering method used to solve these nonlinear equations.

Of course, the equations presented here do not exhaust all the known equations that admit soliton solution. But all of them are second order differential equations and their general covariant generalization are straightforward.

We note that the system (2.4) is the only one that does not contain a "mass term". This fact, when dealing with the Einstein equations, is quite important, because almost always a mass term make the solving of the coupled Einstein equations very complicate, v.g., seldom one can find particular exact solutions. We shall return to this point later.

3 - A MODEL OF ANISOTROPIC FLUID

In this section we present the main formulas of a model of anisotropic fluid with two-perfect -fluid components recently studied by the author (8). The main reason to study this model is the appearence, in a very natural way, of two potentials that can be "forced" to obey soliton type of equations.

The stress-energy tensor for the anisotropic fluid is formed from the sum of two tensors, each of which is the energy-momentum tensor (EMT) of a perfect fluid, i.e.,

$$T^{\mu\nu}_{(u,v)} = t^{\mu\nu}(u) + t^{\mu\nu}(v) \tag{3.1}$$

$$t^{\mu\nu}(u) = (p + w)\,u^\mu u^\nu - p\,g^{\mu\nu} \tag{3.2}$$

$$t^{\mu\nu}(v) = (q + e)\,v^\mu v^\nu - q\,g^{\mu\nu} \tag{3.3}$$

with

$$u^\mu u_\mu = v_\mu v^\mu = 1 \quad , \; v^\mu \neq u^\mu \tag{3.4}$$

The vectors u^μ and v^μ are the velocities associated to each fluid, p and q are the pressures and w and e the fluids rest energy densities.

Making the transformations

$$u^\mu \rightarrow u^{*\mu} = \cos\alpha\, u^\mu + \left(\frac{q+e}{p+w}\right)^{1/2} \sin\alpha\, v^\mu \quad , \tag{3.5a}$$

$$v^\mu \rightarrow v^{*\mu} = -\left(\frac{p+w}{q+e}\right)^{1/2} \sin\alpha\, u^\mu + \cos\alpha\, v^\mu \quad , \tag{3.5b}$$

where

$$tg(2\,\alpha) = \frac{[(p+w)(q+e)]^{1/2}}{p+w-q-e}\; 2\, v^\mu u_\mu \quad , \tag{3.6}$$

we can cast the EMT (3.1) in the form

$$T^{\mu\nu} = \rho\, U^\mu\, U^\nu + S^{\mu\nu} \, , \tag{3.7}$$

$$S^{\mu\nu} = (\sigma - \pi)\, X^\mu X^\nu - \pi (\gamma^{\mu\nu} - U^\mu\, U^\nu) \; . \tag{3.8}$$

The quantities U^μ , X^μ, ρ, and π are the fluid flux velocity, the direction of anisotropy, the fluid rest energy density, the pressure along the anisotropy direction and the pressure on the perpendicular plane to X^μ respectively (8). These quantities are defined in terms of the two-fluid variables as

$$U^\mu = u^{*\mu}/(u^{*\alpha} u^*_{\ \alpha})^{1/2} \quad , \tag{3.9}$$

$$X^\mu = v^{*\mu}/(-v^{*\alpha}\, v^*_{\ \alpha})^{1/2} \quad , \tag{3.10}$$

$$\rho = T^{\mu\nu}\, U_\mu\, U_\nu = (p + w)\, u^{*\alpha}\, u^*_\alpha - (p + q) \quad , \tag{3.11}$$

$$\sigma = T^{\mu\nu}\, X_\mu\, X_\nu = p + q - (q + e)\, v^{*\alpha}\, v^*_\alpha \quad , \tag{3.12}$$

$$\pi = p + q \quad . \tag{3.13}$$

Note that

$$U^\mu\, U_\mu = -\, X^\mu\, X_\mu = 1 \quad , \tag{3.14}$$

$$X^\mu\, U_\mu = 0 \quad , \tag{3.15}$$

$$S^{\mu\nu}\, U_\nu = 0 \quad , \tag{3.16}$$

$$S^{\mu\nu}\, X_\nu = -\,\sigma\, X^\mu \tag{3.17}$$

Also we have

$$\rho = \frac{1}{2}\,(w - p + e - q) + \frac{1}{2}\,\{(p + w + q + e)^2 + 4(p+w)(q+e)\,[(u^\mu v_\mu)^2 - 1]\}^{1/2} \tag{3.18}$$

$$\sigma = -\,\frac{1}{2}\,(w-p+e-q) + \frac{1}{2}\,[(p+w-q-e)^2 + 4(p+w)(q+e)(u_\mu v^\mu)^2]^{1/2} \quad . \tag{3.19}$$

In general one need to add supplimentary conditions to close the model, this point was tested in some detail in the first paper of Ref. 8.

4 - THE MATTER EVOLUTION EQUATIONS

The equations of motion for the matter are obtained in the usual form, i.e., from the "conservation law",

$$T^{\mu\nu}{}_{;\nu} = 0 \; . \tag{4.1}$$

We shall "project" the equation (4.1) in the directions of u^μ, v^μ and in the directions that are perpendicular to both u^μ and v^μ . To perform this last projection we introduce the projection operator

$$H^\mu_\nu = \delta^\mu_\nu - \frac{1}{1-(u^\alpha v_\alpha)^2}\left[u^\mu u_\nu + v^\mu v_\nu - u^\alpha v_\alpha(u^\mu v_\nu + u_\nu v^\mu)\right] . \tag{4.2}$$

Some properties of this operator are

$$H^\mu_\nu \; v^\nu = H^\mu_\nu \, u^\nu = 0 \tag{4.3}$$

$$H^\mu_\alpha \; H^\alpha_\nu = H^\mu_\nu \; ; \; H_{\mu\nu} = H_{\nu\mu} \tag{4.4}$$

$$H^\alpha_\alpha = 2; \; \det |H^\mu_\nu| = 0 \; . \tag{4.5}$$

Transvecting (4.1) with u_μ , v_μ and H^α_μ we get, respectively

$$(p+w)_{,\nu}u^\nu + (p+w)u^\nu{}_{;\nu} + (q+e)_{,\nu}v^\nu u^\alpha v_\alpha + (q+e)v^\mu{}_{;\nu}v^\nu u_\mu + (q+e)u^\alpha v_\alpha v^\nu{}_{;\nu} - (q+p)_{,\nu}u^\nu = 0 \; , \tag{4.6}$$

$$(p+w)_{,\nu}u^\nu u^\alpha v_\alpha + (p+w)u^\mu{}_{;\nu}u^\nu v_\mu + (p+w)u^\alpha v_\alpha u^\nu{}_{;\nu} + (q+e)_{,\nu}v^\nu + (q+e)v^\nu{}_{;\nu} - (p+q)_{;\nu}v^\nu = 0 \; , \tag{4.7}$$

$$(p+w)u^\mu{}_{;\nu}u^\nu H^\alpha_\mu + (q+e)v^\mu{}_{;\nu}H^\alpha_\mu - (q+p)_{,\mu}H^\alpha_\mu = 0 \; . \tag{4.8}$$

We shall specialize the velocity of each fluid component in the following way

$$u_\mu = \phi_{,\mu}/(\phi_{,\alpha}\,\phi^{,\alpha})^{1/2} \; . \tag{4.9}$$

$$v_\mu = \Psi_{,\mu}/(\Psi_{,\alpha}\,\Psi^{,\alpha})^{1/2} \; . \tag{4.10}$$

In other words we impose to each fluid component the usual condition of irrototionality.

From (4.8) - (4.10) and (4.3) we get

$$\left[\frac{p+w}{\phi_{,\alpha}\phi^{,\alpha}}(\phi_{,\beta}\phi^{,\beta})_{,\mu} + \frac{q+e}{\Psi_{,\alpha}\Psi^{,\alpha}}(\Psi_{,\beta}\Psi^{,\beta})_{,\mu}\right]H^\mu_\alpha = 2(p+q)_{,\mu}\,H^\mu_\alpha \; . \tag{4.11}$$

This last equation will be satisfied identically if we choose

$$p + w = F\,(\phi,\Psi)\;\phi_{,\alpha}\;\phi^{,\alpha} \; , \tag{4.12}$$

$$q + e = H\,(\phi,\Psi)\;\Psi_{,\alpha}\;\Psi^{,\alpha} \; , \tag{4.13}$$

$$p + q = \frac{1}{2}\left[F\,\phi_{,\alpha}\phi^{,\alpha} + H\,\Psi_{,\alpha}\Psi^{,\alpha} - G(\phi,\Psi)\right] \; , \tag{4.14}$$

where F, H and G are arbitrary functionsof the scalar functions ϕ and Ψ .

The evolution equations for ϕ and Ψ are obtained directly from (4.7) and (4.8), we find

$$F\phi,_{\alpha}\phi'^{\alpha}\,\Box\phi + H\phi'^{\beta}\phi,_{\beta}\,\Box\Psi + \frac{1}{2}F,_{\beta}\phi'^{\beta}\phi,_{\alpha}\phi'^{\alpha} - \frac{1}{2}H,_{\alpha}\phi'^{\alpha}\Psi,_{\beta}\Psi'^{\beta} + H,_{\alpha}\Psi'^{\alpha}\phi'^{\beta}\Psi,_{\beta} + \frac{1}{2}G,_{\alpha}\phi'^{\alpha} = 0\ , \quad (4.15)$$

$$H\Psi,_{\alpha}\Psi'^{\alpha}\,\Box\Psi + F\phi'^{\beta}\phi,_{\beta}\,\Box\phi + \frac{1}{2}H,_{\alpha}\Psi'^{\alpha}\Psi,_{\beta}\Psi'^{\beta} - \frac{1}{2}F,_{\alpha}\Psi'^{\alpha}\phi,_{\beta}\phi'^{\beta} + F,_{\alpha}\phi'^{\alpha}\phi,_{\beta}\Psi'^{\beta} + \frac{1}{2}G,_{\alpha}\Psi'^{\alpha} = 0\ , \quad (4.16)$$

where, now $\Box\phi \equiv \phi,_{\alpha}{}'^{\alpha}$, etc. And solving (4.15) - (4.16) for $\Box\phi$ and $\Box\Psi$ we get

$$F\,\Box\,\phi = \frac{1}{2}\,H_{\phi}\Psi,_{\alpha}\Psi'^{\alpha} - \frac{1}{2}\,F_{\phi}\phi'^{\alpha}\phi,_{\alpha} - F_{\Psi}\phi'^{\beta}\Psi,_{\beta} - \frac{1}{2}\,G_{\phi} \quad (4.17)$$

$$H\,\Box\,\Psi = \frac{1}{2}\,F_{\phi}\phi,_{\alpha}\phi'^{\alpha} - \frac{1}{2}\,H_{\Psi}\Psi'^{\alpha}\Psi,_{\alpha} - H_{\phi}\phi,_{\alpha}\Psi'^{\alpha} - \frac{1}{2}\,G_{\phi} \quad (4.18)$$

where we have introduced the notations, $H_{\phi} \equiv \partial H/\partial\phi$, $F_{\Psi} = \partial H/\partial\Psi$, etc.

From (4.12) - (4.14) we have that the EMT (3.1) can be written as

$$T_{\mu\nu} = F\phi,_{\mu}\phi,_{\nu} + H\Psi,_{\mu}\Psi,_{\nu} - \frac{1}{2}\,\delta\mu\nu(F\phi,_{\alpha}\phi'^{\alpha} + H\Psi,_{\beta}\Psi'^{\beta} - G). \quad (4.19)$$

It is interesting to point out that the evolution equations (4.17) and (4.18) as well as the EMT (4.19) can be obtained, in the usual way, from the Lagrangean density

$$L = \frac{1}{2}\sqrt{-g}(F\phi,_{\alpha}\phi'^{\alpha} + H\Psi,_{\alpha}\Psi'^{\alpha} - G) = \sqrt{-g}(p + q)\ . \quad (4.20)$$

In Ref. 8 we study a particular case of fluid in which the condition (4.1) was implemented by $t^{\mu\nu}(u);\nu = t^{\nu\beta}(v);\nu = 0$, ie.e. we had a kind of minimum coupling between the fluid components. Now with the specifications (4.12) - (4.14) we have a different kind of coupling, i.e., $t^{\mu\nu}(u);\nu = -\,t^{\mu\nu}(v);\nu \neq 0$, thus the fluids interact through a force density $j^{\mu} = t^{\mu\nu}{}_{;\nu}$ different from zero.

The particular choice of the physical variables given by (4.9), (4.10) and (4.12) - (4.14) tell us that the one fluid variables are related to the potencial ϕ and Ψ by

$$u^{*}_{\mu}(\phi,_{\beta}\phi'^{\beta})^{1/2} = \cos 2\ \phi,_{\mu} + (\tfrac{H}{F})^{1/2} \sin 2\ \Psi,_{\mu}\ , \quad (4.21a)$$

$$v^{*}_{\mu}(\Psi,_{\beta}\Psi'^{\beta})^{1/2} = -\ (\tfrac{F}{H})^{1/2} \sin 2\ \phi,_{\mu} + \cos 2\ \Psi,_{\mu}\ , \quad (4.21b)$$

$$\pi = \frac{1}{2}\left[F\phi,_{\alpha}\phi'^{\alpha} + H\Psi,_{\alpha}\Psi'^{\alpha} - G\right] \quad (4.22)$$

$$\delta = \frac{G}{2} + \frac{1}{2}\left[(F\phi,_{\alpha}\phi'^{\alpha} - H\Psi,_{\alpha}\Psi'^{\alpha})^{2} + \Psi FH(\phi,_{\alpha}\Psi'^{\alpha})^{2}\right]^{1/2} \quad (4.23)$$

$$\sigma = \rho - G\ . \quad (4.24)$$

The special case G = 0 will be of particular interest due to the mass character of G, as indicated by (4.20). Also, in this case, $\sigma = \rho$, i.e., we have a "stiff" equation of state along the anisotropy direction.

5 - SOLITONS OF MATTER

The Lagrangian density (4.20) is sufficiently general to yield any of the equations presented in Section 2. But, if we want to solve the Einsteins equations coupled to the EMT (4.19) we find different types of difficulties. The Einsteins equations coupled to the EMT (4.19) are equivalent to

$$R_{\mu\nu} = - (F\phi,_{\mu}\phi,_{\nu} + H\Psi,_{\mu}\Psi,_{\nu} - \frac{1}{2} g_{\mu\nu} G) \quad . \tag{5.1}$$

Let me study first one of the apparently simpler cases of (5.1), the particular case,

$$F = 1 \ , H = 0, \ G = \sin^2 \phi \quad , \tag{5.2}$$

for the plane symmetric metric,

$$ds^2 = 2\, e^{w(u,v)}\, dudv - e^{\mu(u,v)}\, (dx^2 + dy^2) \quad . \tag{5.3}$$

In this case the equations (5.1) reduce to

$$\mu_{++} + \frac{1}{2} \mu_+^2 - \mu_+ w_+ = - \phi_+^2 \tag{5.4a}$$

$$\mu_{--} + \frac{1}{2} \mu_-^2 - \mu_- w_- = - \phi_-^2 \tag{5.4b}$$

$$\mu_{+-} + w_{+-} + \frac{1}{2} \mu_+\mu_- = - \phi_+\phi_- + \frac{1}{2} e^w \sin^2 \phi \quad ,$$

$$\mu_{+-} + \mu_+\mu_- = \frac{1}{2}\ e^w \sin^2 \phi \quad , \tag{5.4c}$$

where, $\phi_+ \equiv \partial\phi/\partial u$, $\mu_- = \partial u/\partial v$, etc. The integrability condition (4.17) in this case gives

$$2\phi_{+-} + \mu_+\phi_- + \mu_-\phi_+ = - e^w \sin \phi \cos \phi \quad . \tag{5.5}$$

Taking $w = \mu = 0$ in (5.5) we see that this equation is equivalent to the sine-Gordon equation, fact that justifies our choice (5.2). The system of equations (5.4) - (5.5) are terrible non-linear and too coupled for the pourpose to find exact solutions, even a simple one. The origen of the difficulties is the term $G = \sin^2 \phi$, that at the same time is the basic term to have soliton solutions (11).

If we want to reproduce any of the equations (2.2), (2.3) or the Liouville equations, using this formalism we shall encounter the same type of difficulties as in the former case.

Let us take the following choice of the functions F, H and G,

$$F = \frac{1}{2}\, a\, \phi^{-2} \quad , \quad H = \frac{1}{a}\, a\, \phi^2 \quad , \quad G = 0 \quad , \tag{5.6}$$

where a is a constant. Then the Einstein equations (5.1) can be cast as

$$R_{\mu\nu} = - \frac{a}{2} (\phi^{-2}\ \phi,_{\mu}\phi,_{\nu} + \phi^2\ \Psi,_{\mu}\Psi,_{\nu}) \tag{5.7}$$

Now we shall study this field equations for the axially symmetric metric

$$d\Delta^2 = e^w\ (dt^2 - dv^2) - t(fd\theta + hdz)^2 - (t/f)\ dz^2 \quad , \tag{5.8}$$

where f, h and w are functions of t and v only. Note that letting $f \to 1$ and $h \to 0$ in(5.8) we get the metric (5.3), modulo a trivial change of variables. From (5.7), (5.8), (5.6), (4.17) and (4.18) we get

$$w_{00} - w_{11} - w_{0/t} + f^{-2}f_0^2 + f^2h_0^2 - t^{-2} = -a(\phi^{-2}\phi_0^2 + \phi^2\Psi_0^2) \quad , \tag{5.9a}$$

$$- w_{00} + w_{11} - w_{0/t} + f^{-2}f_1^2 + \qquad f^2h_1^2 = -a(\phi^{-2}\phi_1^2 + \phi^2\Psi_1^2) \ , \tag{5.9b}$$

$$- w_{1/t} + f^{-2}f_0f_1 + f^2h_0h_1 = -a(\phi^{-2}\phi_0\phi_1 + \phi^2\Psi_0\Psi_1) \quad ,$$

$$f_{00} + f_{0/t} - f_{11} - (f_0^2 - f_1^2)/f - f^3\ (h_0^2 - h_1^2) = 0 \quad , \tag{5.10a}$$

$$(tf^2h_0)_0 - (tf^2h_1)_1 = 0 \quad , \tag{5.10b}$$

$$\phi_{00} + \phi_{0/t} - \phi_{11} - (\phi_0^2 - \phi_1^2)/\phi - \phi^3(\Psi_0^2 - \Psi_1^2) \tag{5.11a}$$

$$(t\phi^2\Psi_0)_0 - (t\phi^2\Psi_1)_1 = 0 \quad . \tag{5.11b}$$

where we have made use of the notation $\Psi_0 \equiv \partial\Psi/\partial t$, $f_1 \equiv \partial f/\partial t$, etc. Solving the equations (5.9) for w we get

$$w = -\frac{1}{2} \; h \; t + \Sigma \quad , \tag{5.12}$$

$$\Sigma\left[h,f,\phi,\Psi\right] = \Omega\left[f,h\right] + a\Omega\left[\phi,\Psi\right] \quad , \tag{5.13}$$

$$\Omega\left[f,h\right] \equiv \int \frac{t}{2} \{\left[f^{-2}(f_0^2 + f_1^2) + f^2 \; (h_0^2 + h_1^2)\right]dt + 2\left[f^{-2}f_0 f_1 + f^2 h_0 h_1\right]dr\} \tag{5.14}$$

The integravility conditions for $\Omega\left[\phi,\Psi\right]$ and $\Omega\left[f,h\right]$ are the equations (5.11) and (5.10). But equations (5.11) and (5.10) are completely equivalent and they are equal to (2.4). Thus any set of particular solutions to (2.4) or better to (2.5) will give us solutions to (5.7)with an w given by (5.12).

In the present model we can have propagation of the metric coefficients f and h, as well as propagation of the matter variables ϕ and Ψ as solitary waves. The metric and the matter solitary waves do not need to be equal e.g., we can have the metric coefficients propagating as a one-soliton and the matter as an n-soliton.

The general formulas (4.22) - (4.24) for this particular case reduce to

$$\pi = \frac{a}{4} \; \sqrt{t} \; e^{-\Sigma} \left[(\phi_0^2 - \phi_1^2)\phi^{-2} + (\Psi_0^2 - \Psi_1^2)\phi^2\right] \tag{5.15}$$

$$\sigma = \rho = \frac{a}{4} \; \sqrt{t} \; e^{-\Sigma} \{\left[\phi^{-2}(\phi_0^2 - \phi_1^2) - \phi^2 \; (\Psi_0^2 - \Psi_1^2)\right]^2 + 4 \; (\phi_0\Psi_0 - \phi_1\Psi_1)^2\}^{1/2} \tag{5.16}$$

These relations tell us that if we have a localized ϕ and Ψ we can have a localized $\rho = \sigma$ and π, i.e., a model of a massive body.

6 - DISCUSSION

The general formalism studied in Section 4 was particularized in Section 5 in such a way that we found soliton equations for the potencial ϕ and Ψ. We belive that other possible particularizations of F, H and G will also yield soliton equations. The case G = 0 is particularly interesting as stressed along the paper, we see that in this case the Einstein equations (5.1) for the axially symmetric metric (5.8) can be formally integrated. Particular F and H can made the system of equations (4.17) and (4.18) to admit solitary waves as solutions, they can be found by discovering if the resulting system of equations have a Bäcklund transformation or a Lax pair.This point is under active consideration by the author using Clairin method (12).

Some physical aspects of the two fluid model, as well as the "choice" (4.12) - (4.14) need to be better understood in order to apply the model to more realistic situations. We are also working along this line.

The Lagrangean density (4.20) is interesting by itself, we have that many models of field theory are particular cases of (4.20), e.g., the Higgs equations presented in Section 2. Thus, all those models have a fluid interpretation given by (4.21) - (4.24) .

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