THE APPLICATIONS OF THE INVERSE SCATTERING PROBLEM IN GENERAL RELATIVITY

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I - INTEGRATION OF THE EINSTEIN EQUATIONS BY MEANS OF THE INVERSE SCATTERING PROBLEM TECHNIQUE AND CONSTRUCTION OF EXACT SOLITON SOLUTIONS

I.1 - INTRODUCTION

The purpose of the present paper is to describe a practical method (equivalent to the inverse scattering problem technique), allowing one to obtain explicity large classes of new exact solutions of the vacuum Einstein equations for the case when the metric tensor depends only on two variables, if simple particular solutions of the equations are known. Moreover, if developed further, the method allows one in principle to approach the problem of finding, in a certain sense, "all the solutions" of the equations of gravity for the two-dimensional case under consideration, and may lead to a solution of the corresponding Cauchy problem.

For definiteness we assume that the metric tensor depends on time and on one spacelike variable; this corresponds to wavelike and cosmological solutions of the gravitational equations. The case when both variables are spacelike (corresponding to stationary gravitational fields) will not be considered separately, since the corresponding solutions can also be obtained from the analysis given here by imposing certain boundary conditions and carrying out the required complex transformations. Moreover, we limit ourselves to that special (albeit quite widespread) case of two-dimensional metric where the interval has the form(*)

$$- ds^2 = f(-dt^2 + dz^2) + g_{ab}dx^a dx^b \tag{1.1}$$

Here the functions f and g_{ab} depend only on the variables t and z. For the coordinates we adopt the notation $(x^0, x^1, x^2, x^3) = (t,x,y,z)$. In this paper the Latin indices a, b, c, d take on the values 1 and 2 and refer to the variables x and y. We study this metric for the case of a vacuum gravitational field, when the Einstein equations reduce to the vanishing of the Ricci tensor.

A metric of this kind was first considered by Einstein and Rosen (1937) for a diagonal matrix g_{ab}, when the Einstein equations actually reduce to one linear equation in cylindrical coordinates. The inclusion of the off-diagonal component (**) g_{12} changes the situation radically, and converts the Einstein equations into a complicated essentially nonlinear problem. Equations for such a metric were first considered by Kompaneets (1958), who noted some of their general properties. In the past twenty years various authors, using different simplifying assumptions, have obtained a number of exact nontrivial solutions for a metric of the type (1.1) or its stationary analog (a large fraction of these results is listed in the review articles (Frolov, 1977; Belinskii and Khalatinikov, 1969), but no regular integration method has been found.

From the physical point of view the metric (1.1) and its stationary analog have many applications in gravitation theory. Suffice it to say that to this class belong the solutions for the Robinson-Bondi plane waves, cylindrical-wave solutions, homogeneous cosmological models

(*) We use a system of units where the speed of light is one. The four-dimensional metric is written in the form $-ds^2 = g_{ik}dx^i dx^k$, where g_{ik} has the signature (-+++).

(**) In the language of weak gravitational waves this corresponds to the appearance of a second independent polarization state of the wave. For a stationary analog of the metric (1.1) such a generalization means (under reasonable boundary conditions) that rotation has been included.

of Bianchi types I through VII, the Schwarzschild and Kerr solutions and their NUT-generalizations, Weyl's axially symmetric solution, etc. As applied to cosmology the metric (1.1) was discussed in a paper by Khalatnikov and one of the present authors (Belinskii and Khalatnikov, 1969), where it was shown that such a two-dimensional metric describes a general cosmological solution of the Einstein equations with a physical singularity on portions of the so-called "long eras". In the paper of Gowdy (1974) the metric (1.1) was used to find new vacuum solutions representing closed cosmological models. Recently there has been considerable interest in inhomogeneous cosmological models containing singularities having simultaneously a spacelike and a timelike character. Such models have recently been discussed on the basis of the metric form (1.1) in the paper of Tomita (1977). All this shows that, in spite of its relative simplicity, a metric of the type (1.1) encompasses a wide variety of physical cases, and that a method for integrating the corresponding Einstein equations could significantly move forward our understanding of various aspects of gravitation theory.

It turns out that this case can be successfully treated by means of the inverse scattering problem technique in its modified form (Zakharov and Shabat, 1978; Zakharov and Mikhailov, 1978). Moreover, Mikhailov and Zakharov (1978), have given a detailed exposition of this new method of integrating nonlinear differential equations, applied to a system which is quite close to the one to which the matrix $g_{ab}(t,z)$ is subject in the present paper. We explain the relation. The Einstein equations for the metric (1.1) are most conveniently investigated in light-cone coordinates ζ,η defined by the transformation

$$t = \zeta - \eta, \qquad z = \zeta + \eta. \tag{1.2}$$

In the sequel we shall always denote by g the two-dimensional matrix with elements g_{ab} (the two-dimensional block of the metric tensor (1.1)) and for the determinant we adopt the notation

$$\det g = \alpha^2. \tag{1.3}$$

The complete system of Einstein equations (in vacuum) for the metric (1.1)decomposes into two groups of equations (cf., e.g., Ref. 5). The first group determines the matrix g and can be written in the form of a single matrix equation:

$$(\alpha g_{,\zeta} g^{-1})_{,\eta} + (\alpha g_{,\eta} g^{-1})_{,\zeta} = 0 \ . \tag{1.4}$$

The second group expresses the metric coefficient f(t,x) by quadratures in terms of a given solution of Eq. (1.4) via the relations

$$\begin{aligned} (\ln f)_{\zeta} &= \frac{(\ln \alpha)_{\zeta\zeta}}{(\ln \alpha)_{\zeta}} + \frac{1}{4\alpha\alpha_{\zeta}} \operatorname{Sp} A^2, \\ (\ln f)_{\eta} &= \frac{(\ln \alpha)_{\eta\eta}}{(\ln \alpha)_{\eta}} + \frac{1}{4\alpha\alpha_{\eta}} \operatorname{Sp} B^2, \end{aligned} \tag{1.5}$$

where the matrices A and B (introduced for the convenience of the subsequente analysis) are defined as follows:

$$A = -\alpha g_{\zeta} g^{-1} \ ; \qquad B = \alpha g_{\eta} g^{-1} \ . \tag{1.6}$$

It is easy to establish (cf.Ref.5) that the integrability conditions for the equations (1.5) are automatically guaranteed if g is subject to Eqs. (1.3) and (1.4).

If one does not consider Eq. (1.5), the equation (1.4) has formally nontrivial solutions even if $\alpha \equiv 1$. That was the system of equations for a matrix g (in general complex and nonsymmetric) which was investigated in Ref. 9 where its integrability was proved and a procedure was described for the determination of the soliton solutions. Physically, such solutions are related to two-dimensional classical relativistic models of the theory of chiral fields. However, this case ($\alpha \equiv 1$) is not non-trivial when applied to a gravitational field described by the metric (1.1). It is easy to show (cf.Ref.5) that the presence of the additional field component f(t,x) related to the matrix g via the relations (1.5) leads for $\alpha \equiv 1$ only to the trivial solution, i.e., the Minkowski metric if one requires that the metric be real and have a physical signature.

In connection with this circumstance, the technique developed in Refs. 8,9, requires some generalization, since one cannot apply it literally to the problem considered here. As will be seen in the sequel, the general idea of the method remains the same: it is based on a study of the analytic structure of the eigenvalues of some operators (as functions of a complex spectral parameter λ), operators which can be associated according to a definite law to the system (1.3), (1.4) (the so called L-A pair). In particular, for solitonic solutions Eqs.(1.3) and (1.4) a fundamental role is played by the structure of the poles of the corresponding functions in the λ plane. For an α different from a constant the equations (1.3) and (1.4) require the introduction of generalized differential operators thus entering into the L-A pair, depend on the function $\alpha(\zeta,\eta)$, and contain differentiations also with respect to spectral parameter. For soliton solutions this leads to "floating" poles of the eigenfunctions, and instead of stationary poles λ_η = const (as was the case in Ref. 9) we shall have pole trajectories $\lambda_\eta(\zeta,\eta)$.

We try to develop our analysis in such a manner that the reading of this article should not require turning to all previous papers, if one is interested mainly in the results of the described method.

I.2 - THE INTEGRATION SCHEME

We now pass to a systematic investigation of Eqs. (1.3) and (1.4). Taking the trace of Eq. (1.4) with account of the condition (1.3) yields

$$\alpha_{\zeta\eta} = 0 \tag{2.1}$$

Thus, the square root of the determinant of the matrix g satisfies a wave equation (this result was already noted in Refs. 1,2) with a solution

$$\alpha = a(\zeta) + b(\eta) , \tag{2.2}$$

where $a(\zeta)$ and $b(\eta)$ are arbitrary functions. For the sequel we shall need a second independent solution of Eq. (2.1), which we denote by $\beta(\zeta,\eta)$ and choose in the form

$$\beta = a(\zeta) - b(\eta) . \tag{2.3}$$

It should be understood that the metric (1.1) admits in addition arbitrary coordinate transformations $z' = f_1(z+t)+f_2(z-t)$, $t' = f_1(z+t)-f_2(z-t)$ which do not affect the conformally flat form of the metric $f(-dt^2 + dz^2)$ in (1.1). By an appropriate choice of the functions f_1 and f_2 one can bring the functions $a(\zeta)$ and $b(\eta)$ in (2.2) to a prescribed form. If, for instance, the variable $\alpha(\zeta,\eta)$ is timelike (corresponding to solutions of cosmological type (Belinskii and Khalatnikov, 1970)) the coordinates can be chosen in such a manner that $\alpha = t$, $\beta = z$. It is however more convenient to carry through the analysis in a general form, without specifying the functions $a(\zeta)$ and $b(\eta)$ in advance, and turning to special cases as the necessity arises.

It is easy to see that Eq. (1.4) is equivalent to a system consisting of the relations (1.6) and two first-order matrix equations that define the matrices A and B. From Eqs. (1.6) and (1.4) follows the first obvious equation for A and B:

$$A_\eta - B_\zeta = 0 . \tag{2.4}$$

The second one is easily derived as an integrability condition for the relations (1.6) with respect to g. We obtain in this manner

$$A_\eta + B_\zeta + \alpha^{-1}\left[A,B\right] - \alpha_\eta \alpha^{-1} A - \alpha_\zeta \alpha^{-1} B = 0 \tag{2.5}$$

(here and in the sequel the square brackets denote the commutator).

The main step now consists in representing (2.4) and (2.5) in the form of compatibility conditions of a more general overdetermined system of matrix equations related to an eigenvalue-eigenfunction problem for some linear differential operators. Such a system will depend on a complex spectral parameter (which we denote by λ), and the solutions of the original equations for the matrices g, A, and B will be determined by the possible types of analytic structure of the eigenvalues in the λ plane. At present there does not exist a general algorithm for the determination of such systems, but for the concrete case of Eqs. (1.3) and (1.4) this can be done. For this purpose we introduce the following differential operators:

$$D_1 = \partial_\zeta - \frac{2\alpha_\zeta \lambda}{\lambda - \alpha}\partial_\lambda \quad , \quad D_2 = \partial_\eta + \frac{2\alpha_\eta \lambda}{\lambda + \alpha}\partial_\lambda \quad , \tag{2.6}$$

where the synbol ∂ with a subscript denotes partial differentiation with respect to the corresponding variable and λ is a complex parameter independent of the coordinates ζ and η. It is easy to verify that the commutator of the operators D_1 and D_2 vanishes exactly when α satisfies the wave equation. Thus, taking (2.1) into account we have

$$\left[D_1 \ , \ D_2\right] = 0 \ . \tag{2.7}$$

We now introduce the complex matrix function $\Psi(\lambda,\zeta,\eta)$ and consider the system of equations:

$$D_1\Psi = \frac{A}{\lambda-\alpha}\Psi \quad , \quad D_2\Psi = \frac{B}{\lambda+\alpha}\Psi \quad , \tag{2.8}$$

in which the matrices A and B do not depend on the parameter λ and are real (the same requirements are satisfied, of course, by the real function α). It then turns out that the compatibility conditions for the equations (2.8) coincide exactly with the equations (2.4)and (2.5). In order to see this it is necessary to operate with D_2 on the first of the equations (2.8) and with D_1 on the second one, and then subtract the results. On account of the commutativity of D_1 and D_2 we obtain zero in the left-hand side. In the right-hand side we get a rational function of λ which vanishes if and only if the conditions (2.4), (2.5) are satisfied. It is easy to see that a solution of the system (2.8) guarantees not only that the equations satisfied by the matrices A and B are true, but also yields a solution of the relations (1.6), i.e., directly the sought matrix $g(\zeta,\eta)$ that satisfies the original equations (1.3) and (1.4). The matrix $g(\zeta,\eta)$ is nothing else but the value of the matrix function $\Psi(\lambda,\zeta,\eta)$ at the point $\lambda = 0$:

$$g(\zeta,\eta) = \Psi(0, \ \zeta,\eta). \tag{2.9}$$

Indeed, in this case the equations (2.8) for $\lambda = 0$ (for solutions which are regular in the neighborhood of $\lambda = 0$) duplicate exactly the relations (1.6). The matrix $g(\zeta,\eta)$ must, of course, be real and symmetric. Below we shall formulate for the selection of the solutions of the equations (2.8) additional restrictions that guarantee this requirement.

The procedure of integration of the equations under consideration assumes the knowledge of at least one particular solution. Let $g_0(\zeta,\eta)$ be such a particular solution of the Einstein equations (1.3), (1.4) in terms of which by means of Eq. (1.6) one can determine the matrices $A_0(\zeta,\eta)$ and $B_0(\zeta,\eta)$, and with the help of (2.8) one can obtain the corresponding function $\Psi_0(\lambda,\zeta,\eta)$. We now make in the equations (2.8) the substitution

$$\Psi = \chi\Psi_0 \ . \tag{2.10}$$

Taking into account the fact that Ψ_0 satisfies the system (2.8), we obtain the following equations for the matrix $\chi(\lambda,\zeta,\eta)$:

$$D_1\chi = \frac{1}{\lambda-\alpha}\left(A\chi - \chi A_0\right) \ , \qquad D_2\chi = \frac{1}{\lambda+\alpha}\left(B\chi - \chi B_0\right) \ . \tag{2.11}$$

We now indicate additional conditions which need to be imposed on the matrix χ in order to assure the reality and symmetry of the matrix g. The first consists in requiring the reality of χ on the real axis of the λ plane (the matrix Ψ must also satisfy this conditions). This implies

$$\bar{\chi}(\bar{\lambda}) = \chi(\lambda), \qquad \bar{\Psi}(\bar{\lambda}) = \Psi(\lambda). \tag{2.12}$$

(Here and in the sequel the bar denotes complex conjugation. For the sake of brevity we often do not indicate the arguments ζ and η of the functions). The second condition is less trivial and is related to the following invariance property of the solutions of the system (2.11). Assume that the matrix $\chi(\lambda)$ satisfies the equations (2.11). Replacing in the argument λ by α^2/λ we form the new matrix $\chi'(\lambda)$:

$$\chi'(\lambda) = g\tilde{\chi}^{-1}(\alpha^2/\lambda)g_0^{-1}$$

(the tilde denotes transportation of a matrix). A direct verification convinces us that the new matrix $\chi'(\lambda)$ also satisfies the equations (2.12) if g is symmetric. We shall assume $\chi'(\lambda) = \chi(\lambda)$ which guarantees the symmetry of the matrix g. Thus, this condition takes the form

$$g = \chi(\alpha^2/\lambda)\ g_0\ \tilde{\chi}(\lambda). \tag{2.13}$$

Moreover, it is necessary to require that for $\lambda \to \infty$ the matrix $\chi(\lambda,\zeta,\eta)$ tend to the unit matrix

$$\chi(\infty) = I. \tag{2.14}$$

These relations imply

$$g = \chi(0)\ g_0, \tag{2.15}$$

a result which also follows from the conditions (2.9) - (2.10).

Thus, the problem now consists in solving the system (2.11) and in determining the matrix χ satisfying the supplementary conditions (2.12), (2.14). It is necessary to note the following important circunstance. The solution $g(\zeta,\eta)$ must also satisfy the requirement det $g = \alpha^2$. We assume that the function $\alpha(\zeta,\eta)$ is the same for the particular solution g_0 and for the generalized g (α is a given solution of the wave equation (2.1)), and that by definition the particular solution also satisfies the requirement det $g_0 = \alpha^2$. Therefore, as follows from (2.15) one must impose on the matrix χ yet another restriction: det $\chi(0) = 1$. It is more convenient not to worry about this condition during the calculations, and to use a simple renormalization of the final result in order to obtain the correct quantities. The latter will be called the physical quantities. It is easy to establish the legitmacy of this procedure from Eq. (1.4). If we had obtained a solution of that equation with det $g \neq \alpha^2$, the trace of (1.4) indicates that det g satisfies the equation

$$\left[\alpha(\ln \det g)_\zeta\right]_\eta + \left[\alpha(\ln \det g)_\eta\right]_\zeta = 0 \tag{2.16}$$

If one now forms the matrix g_{ph}:

$$g_{ph} = \alpha(\det g)^{-1/2}\ g, \tag{2.17}$$

it is easy to see that the latter again satisfies the equation (1.40) and moreover the condition det $g_{pg} = \alpha^2$. The matrices A and B are also subject to appropriate transformations:

$$\begin{aligned} A_{ph} &= A - \alpha\ \{\ln \left[\alpha\ (\det g)^{-1/2}\right]\}_\zeta\ I, \\ B_{ph} &= B + \alpha\ \{\ln \left[\alpha\ (\det g)^{-1/2}\right]\}_\eta\ I, \end{aligned} \tag{2.18}$$

where A and B are defined in terms of g according to (1.6) and A_{ph} and B_{ph} are defined by the same formulas but in terms of the matrix g_{ph}.

I.3 - CONSTRUCTION OF THE SOLITON SOLUTIONS

The solutions for the matrix $\chi(\lambda,\zeta,\eta)$are constructed by means of the method described in Refs. 8 and 9. In the general case the determination of χ reduces to solving the Riemann problem of analytic-function theory, which in turn reduces to the solution of a linear integral equation. We shall return to this in (I.6) and show there that the solution is determined by the analyticity properties of the matrix χ in the complex λ plane, and in general represents the sum of a soliton part and a nonsoliton part. In this section and in (I.4) and (I.5) we consider the purely solitonic solutions when the nonsoliton part is absent. This problem does not require the use of the Riemann problem (in fact it is a trivial special case of the Riemann problem) and can be explicitly solved to the end.

The existence of solutions of the soliton type is due to the presence in the λ plane of points of degeneracy (non-invertibility) of the matrix χ, i.e., points at which the determinant of χ vanishes in such a manner that the inverse matrix χ^{-1} has at these points simple poles. Thus, the purely solitonic solutions correspond to the case when χ^{-1} is representable by a rational matrix function of the parameter λ with a finite number of poles (we assume them to be simple) and which for $\lambda \to \infty$ tends to the unit matrix, as required by the condition (2.14). The matrix χ has the same properties, as can be easily seen from the supplementary condition (2.13). Indeed, (2.13) implies that if χ has n poles at the points $\lambda = \mu_k(\zeta,\eta)$ $(k = 1, \ldots, n)$ then χ^{-1} also has n poles at the points $\nu_k(\zeta,\eta)$ where $\nu_k = \alpha^2/\mu_k$. Moreover, it follows from (2.12) that the poles of the matrices χ and χ^{-1} are either on the real axis of the λ plane, or are paired: to each complex pole μ_k(or ν_k) corresponds the complex-conjugate pole $\bar{\mu}_k$(or $\bar{\nu}_k$). For uniformity in our calculation we shall assume that the poles of the matrix χ are complex and that among them there are no coinciding ones (the equations for the case when the poles are on the real axis can be obtained by taking an appropriate limit).

It follows that the matrix χ has the form:

$$\chi = I + \sum_{k=1}^{n} \left(\frac{R_k}{\lambda-\mu_k} + \frac{\bar{R}_k}{\lambda-\bar{\mu}_k}\right) ,$$

$$\chi^{-1} = I + \sum_{k=1}^{n} \left(\frac{S_k}{\lambda-\nu_k} + \frac{\bar{S}_k}{\lambda-\bar{\nu}_k}\right) , \tag{3.1}$$

where the matrices R_k and S_k (as well as the numerical functions μ_k and $\nu_k = \alpha^2/\mu_k$) no longer depend on λ. The matrices S_k can be expressed in terms of R_k by means of the obvious relation $\chi\chi^{-1} = I$. However, in the sequel we shall deal mainly with χ and the explicit expressions for S_k will not be needed.

It can be seen from (3.1) and (2.15) that the solution of the equations (1.4) for the matrix $g(\zeta,\eta)$ is

$$g(\zeta,\eta) = \left[I - \sum_{k=1} \left(\frac{R_k}{\mu_k} + \frac{\bar{R}_k}{\bar{\mu}_k}\right)\right] g_0. \tag{3.2}$$

We now determine the matrix R_k explicity. For this it is necessary to substitute (3.1) in (2.11) and to satisfy these equations at the poles $\lambda = \mu_k(\zeta,\eta)$. First of all it can be seen that these equations determine explicitly the dependence of the position of the poles on the coordinates ζ and η, i.e., the functions $\mu_k(\zeta,\eta)$. Indeed, the right-hand sides of (2.11) at the points $\lambda = \mu_k$ have only first-order poles, whereas the left-hand sides $D_1\chi$ and $D_2\chi$ have second order poles. The requirements that the coefficient of the powers $(\lambda - \mu_k)^{-2}$ vanish in the left-hand sides yields the following equations for the pole trajectories $\mu_k(\zeta,\eta)$:

$$\mu_{k,\zeta} = \frac{2\alpha_\zeta \mu_k}{\alpha - \mu_k} , \qquad \mu_{k,\eta} = \frac{2\alpha_\eta \mu_k}{\alpha + \mu_k} \tag{3.3}$$

These equations are invariant with respect to the substitution $\mu_k \to \alpha^2/\mu_k$, i.e., the function $\nu_k = \alpha^2/\mu_k$ also satisfies (3.3). The solutions of (3.3) are roots of the quadratic equation (in λ).

$$\alpha^2/\lambda + 2\beta + \lambda = 2\omega_k \tag{3.4}$$

where ω_k are arbitrary complex constants. It is easy to see that for each given ω_k Eq. (3.4) yields two solutions: a pole $\mu_k(\zeta,\eta)$ for the matrix χ and a pole $\nu_k = \alpha^2/\mu_k$ for the matrix χ^{-1}:

$$\mu_k = \omega_k - \beta - \left[(\omega_k - \beta)^2 - \alpha^2\right]^{1/2} \quad ,$$
$$\nu_k = \omega_k - \beta + \left[(\omega_k - \beta)^2 - \alpha^2\right]^{1/2} \tag{3.5}$$

Rewriting the equations (2.11) in the form

$$\frac{A}{\lambda-\alpha} = (D_1\chi)\chi^{-1} + \chi\,\frac{A_0}{\lambda-\alpha}\,\chi^{-1} \quad ,$$
$$\frac{B}{\lambda-\alpha} = (D_2\chi)\chi^{-1} + \chi\,\frac{B_0}{\lambda-\alpha}\,\chi^{-1} \quad , \tag{3.6}$$

we note that in order that they be satisfied at the poles $\lambda = \mu_k$ it is necessary that the residues at these poles vanish in the right-hand sides of (3.6), since the left-hand sides are holomorphic at the points $\lambda = \mu_k$. This requirement leads to the following equations for the matrices R_k:

$$R_{k,\zeta}\,\chi^{-1}(\mu_k) + R_k\,\frac{A_0}{\mu_k-\alpha}\,\chi^{-1}(\mu_k) = 0 \quad ,$$
$$R_{k,\eta}\,\chi^{-1}(\mu_k) + R_k\,\frac{B_0}{\mu_k-\alpha}\,\chi^{-1}(\mu_k) = 0 \quad , \tag{3.7}$$

where use has been made of the relation

$$R_k\,\chi^{-1}(\mu_k) = 0 \quad , \tag{3.8}$$

following from the identity $\chi\chi^{-1} = I$ (considered at the poles $\lambda = \mu_k$). It can be seen from (3.8) that R_k and $\chi^{-1}(\mu_k)$ are degenerate matrices for which the elements can be written in the form

$$(R_k)_{ab} = n_a^{(k)}\, m_b^{(k)} \;, \quad \left[\chi^{-1}(\mu_k)\right]_{ab} = q_a^{(k)}\, p_b^{(k)} \tag{3.9}$$

then (3.8) signifies that

$$m_a^{(k)}\, q_a^{(k)} = 0 \;. \tag{3.10}$$

Here and in the sequel summation will be understood over repeated vector and tensor indices a, b, c, d (they take the values 1, 2).

Substituting (3.9) into (3.7) we obtain the equations wich determine the evolution of the vectors $m_a^{(k)}$:

$$\left(m_{a,\zeta}^{(k)} + m_b^{(k)}\,\frac{(A_0)_{1a}}{\mu_k - \alpha}\right) q_a^{(k)} = 0 \quad ,$$
$$\left(m_{a,\eta}^{(k)} + m_b^{(k)}\,\frac{(B_0)_{1a}}{\mu_k - \alpha}\right) q_a^{(k)} = 0 \quad . \tag{3.11}$$

A solution of these equations is easily expressed in terms of a given particular solution Ψ_0 of the equations (2.8). Introducing the matrices

$$M_k = (\Psi_0^{-1})_{\lambda=\mu_k} = \Psi_0^{-1}(\mu_k, \zeta, \eta) \quad , \tag{3.12}$$

it is not hard to see that they satisfy the equations

$$M_{k,\zeta} + M_k \frac{A_0}{\mu_k-\alpha} = 0 \quad , \quad M_{k,\eta} + M_k \frac{B_0}{\mu_k-\alpha} = 0 \quad . \tag{3.13}$$

Thus, a solution of the equations (3.11) for the vectors $m_a^{(k)}$ will be (*):

$$m_a^{(k)} = m_{0b}^{(k)} (M_k)_{ba} \tag{3.14}$$

where the $m_{0b}^{(k)}$ are arbitrary complex constant vectors.

There remains the task of determining the vectors $n_a^{(k)}$ and thus the matrices R_k. This can be done by means of the supplementary condition (2.13) that must be satisfied by the matrix χ. Substituting (3.1) into (2.13) and considering the relation obtained in this manner at the poles of the matrix $\chi(\alpha^2/\lambda)$, i.e., at the points $\lambda = \nu_k = \alpha^2/\mu_k$, we reach the conclusion that the matrices R_k satisfy the following system consisting of n algebraic matrix equations:

$$R_k g_0 \left[I + \sum_{l=1}^{n} \left(\frac{\bar{R}_l}{\nu_k - \mu_l} + \frac{\bar{R}_l}{\nu_k - \bar{\mu}_l} \right) \right] = 0 \quad , \tag{3.15}$$

where $k = 1, \ldots, n$. Substituting the expression (3.9) for the matrices R_k we obtain a system of linear algebraic equations for the vectors $n_a^{(k)}$:

$$\sum_{l=1}^{n} \frac{m_b^{(l)} m_c^{(k)} (g_0)_{cb}}{\nu_k - \mu_l} n_a^{(l)} + \sum_{l=1}^{n} \frac{\bar{m}_b^{(l)} m_c^{(k)} (g_0)_{cb}}{\nu_k - \bar{\mu}_l} \bar{n}_a^{(l)} = - m_c^{(k)} (g_0)_{ca} \quad ,$$

$$\sum_{l=1}^{n} \frac{m_b^{(l)} \bar{m}_c^{(k)} (g_0)_{cb}}{\bar{\nu}_k - \mu_l} n_a^{(l)} + \sum_{l=1}^{n} \frac{\bar{m}_b^{(l)} \bar{m}_c^{(k)} (g_0)_{cb}}{\bar{\nu}_k - \bar{\mu}_l} \bar{n}_a^{(l)} = - \bar{m}_c^{(k)} (g_0)_{ca} \quad . \tag{3.16}$$

This completes the determination of the matrices R_k and from (3.2) one can now find a solution for the metric tensor $g(\zeta,\eta)$. We also note that from Eq. (3.6) one can obtain explicit expressions for the matrices A and B by equating the residues in the left-hand and right-hand sides of these equations at the poles $\lambda = \alpha$ and $\lambda = -\alpha$. As a result we obtain:

$$A = 2\alpha\alpha_\zeta \left(\sum_{l=1}^{n} \left[\frac{R_k}{(\alpha-\mu_k)^2} + \frac{\bar{R}_k}{(\alpha-\bar{\mu}_k)^2} \right] \right) \chi^{-1}(\alpha) + \chi(\alpha) A_0 \chi^{-1}(\alpha) \quad ,$$

$$B = 2\alpha\alpha_\eta \left(\sum_{l=1}^{n} \left[\frac{R_k}{(\alpha+\mu_k)^2} + \frac{\bar{R}_k}{(\alpha+\bar{\mu}_k)^2} \right] \right) \chi^{-1}(-\alpha) + \chi(-\alpha) B_0 \chi^{-1}(-\alpha) \quad . \tag{3.17}$$

Calculating the traces Tr A^2 and Tr B^2 we obtain from (1.5) the component $f(\zeta,\eta)$ of the matric tensor by quadratures. We note, however, that for those simplest solutions which we consider in the following sections the corresponding indefinite integrals encountered in the

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(*) In reality, in the solution (3.14) for the vectors $m_a^{(k)}$ there may also be arbitrary complex factors depending on the index k and the coordinates ζ,η . However, such factors reduce to an inessential renormalization of the vectors $m_a^{(k)}$ and disappear from the final expression for the matrices R_k; we therefore set them equal to one.

calculation of f can be evaluated explicitly and the solution can be expressed in terms of the particular solution g_0, f_0, as well as the quantities μ_k, $m_a^{(k)}$ in algebraic form.

I.4 - SIMPLE SOLITONS

In this section we consider soliton solutions for the simplest case: when the matrix χ has only one pole. If there is only one pole it can be situated only on the real λ axis (a complex pole has always a complex-conjugate partner).

All the results are easily obtained from the preceding general analysis. The position of the pole is determined by the equation $\lambda = \mu(\zeta,\eta)$, where μ is real and is expressed in terms of α and β according to Eq. (3.5):

$$\mu = \omega - \beta - \left[(\omega - \beta)^2 - \alpha^2\right]^{1/2} \quad ; \tag{4.1}$$

here ω is a real arbitrary constant. For μ to be real the functions α and β must satisfy the inequality

$$(\omega - \beta)^2 \geq \alpha^2 \quad , \tag{4.2}$$

the sense of which will become clear later. The matrix χ has the form

$$\chi = I + 2R/(\lambda - \mu) \quad , \quad R_{ab} = n_a m_b \quad , \tag{4.3}$$

where the vectors m_a and n_a are real. As follows from Eq. (3.12) and (3.14), the vector m_a is determined by the equations

$$m_a = m_{0b} M_{ba} \quad , \quad M = (\Psi_0^{-1})_{\lambda = \mu} \tag{4.4}$$

in which the arbitrary constant vector m_{0b} must be taken to be real and the matrix M will automatically be real on account of the conditions (2.12) and the reality of μ. The vector n_a is easily obtained from (3.16) (assuming that all the quantities in them are real and taking into account the fact that there is only one pole):

$$n_a = (\mu^2 - \alpha^2)\, m_b (g_0)_{ba} / 2\mu\, m_c m_d (g_0)_{cd} \quad . \tag{4.5}$$

Furthermore it is convenient to introduce the matrix P with the elements:

$$P_{ab} = m_c (g_0)_{ca}\, m_b / m_c m_d (g_0)_{cd} \quad . \tag{4.6}$$

From this definition it is clear that P has the properties

$$P^2 = P_1 \quad \det P = 0 \quad , T_r P = 1 \quad . \tag{4.7}$$

Now it is easy to express the matrices χ and χ^{-1} in terms of P:

$$\chi = I + \frac{\mu^2 - \alpha^2}{\mu(\lambda - \mu)} P, \qquad \chi^{-1} = I + \frac{\mu^2 - \alpha^2}{\alpha^2 - \lambda\mu} P \quad . \tag{4.8}$$

The equation (2.15) yields the matrix g:

$$g = (I - \frac{\mu^2 - \alpha^2}{\mu^2} P)\, g_0 \quad .$$

whence (taking account of $\det g_0 = \alpha^2$) it follows that

$$\det g = \alpha^1/\mu^2 \quad .$$

Thus, our solution does not satisfy the necessary condition det $g = \alpha^2$ and we must renormalize it, going over to the physical values in which we are interested, according to the procedure described at the end of I.2. We will denote (as in I.2) all physical quantities which yield the final result by the subscript "ph". In agreement with Eq. (2.17) we have $g_{ph} = \mu\alpha^{-1}g$ and obtain for the metric tensor g_{ph}

$$g_{ph} = (\frac{\mu}{\alpha} I - \frac{\mu^2 - \alpha^2}{\alpha\mu} P)\, g_0 \quad , \tag{4.9}$$

an expression which satisfies both original equations (1.3) and (1.4).

From (4.3) and (4.8) we determine the matrices χ and χ^{-1} at the points $\lambda = \pm\alpha$ and, substituting in Eq. (3.17), we determine the matrices A and B. We next use (2.18) to determine their physical values of A_{ph} and B_{ph} that satisfy Eqs. (2.4) and (2.5) and the relations (1.6)(with g replaced by g_{ph}):

$$\begin{aligned} A_{ph} &= \alpha_\zeta \frac{\alpha+\mu}{\alpha-\mu} (2P - I) + (I - \frac{\alpha+\mu}{\mu} P)\, A_0\, (I - \frac{\alpha+\mu}{\alpha} P) \quad , \\ B_{ph} &= \alpha_\eta \frac{\mu-\alpha}{\mu+\alpha} (2P - I) + (I - \frac{\mu-\alpha}{\mu} P)\, B_0\, (I - \frac{\alpha-\mu}{\alpha} P) \quad . \end{aligned} \tag{4.10}$$

We now calculate the traces Tr A_{ph}^2 and Tr B_{ph}^2 and substitute the results into the equations (1.5), thus obtaining the physical value f_{ph} of the metric component f. These rather lengthy calculations lead to a simple result: the indefinite integrals which occur in the calculation of f_{ph} in (1.5) turn out to be trivial and are easily calculated, and the final result is

$$f_{ph} = \frac{C\mu m_a m_b (g_0)_{ab}}{\left[\alpha(\omega - 2a)(\omega + 2b)\right]^{1/2}}\, f_0 \quad . \tag{4.11}$$

Here C is an arbitrary integration constant, a and b are the arbitrary functions from (2.2), (2.3) and f_0 is the particular solution for the component f corresponding to the particular solution g_0 (the function $f_0(\zeta,\eta)$ satisfies (1.5), where A and B are replaced by A_0 and B_0).

The equations (4.1), (4.4), (4.6), (4.9) and (4.11) give the final solution of the Einstein equations for the case of simple solitons. In order to obtain concrete solutions one must substitute into these equations some concrete particular solutions. In order to illustrate the method we consider the simplest case when the particular solution of the problem is the Kasner solution. It is easy to see that the equations (1.3) and (1.5) have the following exact solution:

$$g_0 = \begin{pmatrix} \alpha^{2s_1} & 0 \\ 0 & \alpha^{2s_2} \end{pmatrix} \quad , \quad f_0 = C_0\, \alpha_\zeta\, \alpha_\eta\, \alpha^{s_1^2 + s_2^2 - 1} \quad , \tag{4.12}$$

where C_0 is an arbitrary constant and s_1 and s_2 are constants satisfying the condition $s_1 + s_2 = 1$, so that they can be expressed in terms of one arbitrary constant parameter q:

$$s_1 = \frac{1}{2} + q \quad , \quad s_2 = \frac{1}{2} - q \quad . \tag{4.13}$$

We now obtain from Eq. (2.8) the corresponding particular solution for the matrix Ψ_0. One can choose for it the matrix

$$\Psi_0 = \begin{pmatrix} (\alpha^2 + 2\beta\lambda + \lambda^2)^{s_1} & 0 \\ 0 & (\alpha^2 + 2\beta\lambda + \lambda^2)^{s_2} \end{pmatrix} \tag{4.14}$$

Substituting (4.14) into (4.4) we obtain the vector m_a and then from (4.6), (4.9) and (4.11) we derive the explicit form of the solutions. We write out the final result for the special choice of coordinates when the arbitraty function $a(\zeta)$ and $b(\eta)$ have the forms:

$$a(\zeta) = \zeta + \omega/2 \quad , \qquad b(\eta) = - \eta - \omega/2 \quad . \tag{4.15}$$

This choice means that

$$\alpha = \zeta - \eta = t \quad , \qquad \beta = \zeta + \eta + \omega = z + \omega \tag{4.16}$$

(in these coordinates the solution (4.12), (4.13) takes on the usual Kasner form, and by means of a simple transformation of the time t it can be transformed to the standard synchronous form).

After simple calculations we obtain the final form of the metric

$$-ds^2 = \frac{C_1 t^{2q^2} ch(qr+C_2)}{(z^2-t^2)^{1/2}}(-dt^2+dz^2) + \frac{ch(s_1 r+C_2)}{ch(qr+C_2)} t^{2s_1} dx^2 + \frac{ch(s_2 r-C_2)}{ch(qr+C_2)} t^{2s_1} dy^2 - \frac{2sh(r/2)}{ch(qr+C_2)} t\, dxdy, \tag{4.17}$$

where C_1 and C_2 are arbitrary constants, and the function r is defined in the following manner:

$$e^r = 2\frac{z^2}{t^2} - 1 - 2\left[\frac{z^2}{t^2}\left(\frac{z^2}{t^2} - 1\right)\right]^{1/2}. \tag{4.18}$$

This is a solution of the cosmological type which cannot be called solitonic in the strict sense, since the velocity of the soliton here exceeds the speed of light. Indeed, let us consider, e.g., the field component g_{11} and determine the position of its extremum with respect to the spacelike variable z for various fixed instants of time t. It can be seen directly that for any t the extremum will correspond to the same constant value of the function $r = r_0$ = const. Then Eq. (2.18) shows that the world line of the extremum has the equation $z = t \cosh(r_0/2)$, and therefore the speed of this localized disturbance exceeds unity.

Thus, we are simply dealing with the time evolution of a given initial state of the field. The situation changes however if one sets $C_1 < 0$ in (4.17). Then the variable t becomes spacelike and z takes on the meaning of a time. Such a solution is already connected with cylindrical waves and t is the radial coordinate. If one takes the case when the $t = 0$ axis is free of singularities, i.e., if one chooses the Kasner indices in the form $s_1 = 0$, $s_2 = 1$ $(q = -1/2)$, then the extremum of the component g_{11} in the radial variable t also corresponds to the constant value $r = r_0 = 2C_2$, the world line of the extremum has the same equation as in the preceding case, but now the velocity of the disturbance is smaller than one. Such a solution describes a cylindrical solitary wave incident on the axis and reflected from it.

In both cases the solution (4.17), (4.18) makes sense only for $z^2 \geq t^2$. On the light cone $z^2 = t^2$ the function r vanishes and the matrix g coincides with the unperturbed particular solution g_0. The solution for g can also be defined in the region $z^2 < t^2$ using the following considerations, which have a general character and refer to all soliton solutions related to the real poles of the matrix $\chi(\lambda,\zeta,\eta)$. A real pole $\lambda = \mu$ is always given by the expression (4.1) with a real constant ω. If, moving along the coordinate plane, we go from the region (4.2) into a region where $(\omega - \beta)^2 < \alpha^2$, the quantity μ becomes complex and a continuation of the function g into this region will be the solution corresponding to the two-pole situation with $\lambda = \mu$ and $\lambda = \bar{\mu}$, where

$$\mu = \omega - \beta - i\left[\alpha^2 - (\omega - \beta)^2\right]^{1/2}$$

However, for such a function we have $|\mu|^2 = \alpha^2$ and the poles are situated on the circle $|\lambda|^2 = \alpha^2$. As will be shown in the next section, the matrix χ is identically equal to the unit matrix if its poles are situated on this circle. This implies that in the region

$(\omega - \beta)^2 < \alpha^2$ the solution g remains unperturbed and coincides identically with the particular solution g_0. The solution as a whole, while remaining itself continuous, suffers discontinuities of the first derivatives on the light cone $(\omega - \beta)^2 = \alpha^2$ (one can see from Eqs. (2.2), (2.3) that this equation yields a pair of straight lines ζ = const and η = const).This phenomenon requires, of course, additional investigation and appropriate interpretation. We note that such discontinuities do not occur in the solutions correspond to a χ matrix without poles on the real λ axis.

I.5 - TWO-SOLITON SOLUTIONS

In this section we consider the next-complicated case, when the matrix χ has a complex pole $\lambda = \mu$. On account of condition (2.12) it must also have the conjugate pole $\lambda = \bar{\mu}$; we thus deal with two poles. The matrix χ has the form

$$\chi = I + R/(\lambda - \mu) + \bar{R}/(\lambda - \bar{\mu}) \quad , \quad R_{ab} = n_a m_b \quad . \tag{5.1}$$

According to (3.14) the vector m_a is

$$m_a = m_{0b} M_{ba} \quad , \quad M = (\Psi_0^{-1})_{\lambda=\mu} \quad , \tag{5.2}$$

where m_{0b} is an arbitrary (now complex) vector. The matrix M is also complex. The vector n_a can be found from the equations (3.16), which are now two algebraic equations for n_a and $\bar{n}_a$ (as before, the index k takes on only one value). These equations have the following solution:

$$n_a = \frac{1}{\Delta}\left(\frac{m_c \bar{m}_d (g_0)_{cd}}{\nu - \bar{\mu}} \bar{m}_b (g_0)_{ba} - \frac{\bar{m}_c \bar{m}_d (g_0)_{cd}}{\bar{\nu} - \bar{\mu}} m_b (g_0)_{ba} \right) ,$$

$$\Delta = \frac{|m_a m_b (g_0)_{ab}|^2}{|\nu - \mu|^2} - \frac{|m_a \bar{m}_b (g_0)_{ab}|^2}{|\bar{\nu} - \mu|^2} \quad , \tag{5.3}$$

where $\nu = \alpha^2/\mu$. Substituting (5.3) and (5.2) into the expression $R_{ab} = n_a m_b$ we obtain the matrix R and from Eq. (3.2) we obtain the metric tensor g. We can now calculate the determinant of g and obtain

$$\det g = \alpha^6/\rho^4 \quad , \tag{5.4}$$

where ρ is the modulus of μ expressed in the form

$$\mu = \rho\, e^{i\phi} \quad . \tag{5.5}$$

Thus, the physical solution g_{ph} of Eqs. (1.3) and (1.4) will be

$$g_{ph} = \rho^2 \alpha^{-2} g \quad , \quad \det g_{ph} = \alpha^2 \quad . \tag{5.6}$$

The final expression for g_{ph} is:

$$(g_{ph})_{ab} = \frac{\rho^2}{\alpha^2} (g_0)_{ab} - \frac{\rho^2}{\alpha^2} \left(\frac{n_a m_c}{\mu} + \frac{\bar{n}_a \bar{m}_c}{\bar{\mu}} \right) (g_0)_{cb} \quad , \tag{5.7}$$

where the vectors n_a and m_a are defined by Eqs. (5.2) and (5.3). The function μ is defined as before as the solution of the quadratic equation, in which ω is now an arbitrary complex constant. Denoting

$$\omega = \omega_1 - i\omega_2 \quad , \tag{5.8}$$

we obtain for the modulus ρ and the phase Ψ from (3.4) the following system of equations:

$$\cos\phi = \frac{(2\omega_1 - 2\beta)\phi}{\alpha^2 + \rho^2} \quad , \quad \sin\Psi = \frac{2\omega_2 \rho}{\alpha^2 - \rho^2} \quad . \tag{5.9}$$

from which we can see that for $\omega_2 \neq 0$ the poles μ and $\bar{\mu}$ are either always inside the circle $|\lambda|^2 = \alpha^2 (\rho^2 < \alpha^2)$, or outside it $(\rho^2 > \alpha^2)$. For definiteness we shall consider that the poles are inside the circle and $\rho^2 \leq \alpha^2$. It can be seen from Eqs. (5.3) that as the poles tend to the circumference $\rho^2 - \alpha^2$ the quantity $1/\Delta$ tends to zero like $(\rho^2 - \alpha^2)^2$ and the vector n_a vanishes like $\rho^2 - \alpha^2$. It then follows from Eq. (5.7) that $g_{ph} \to g_0$. Thus, if the poles of the matrix are situated on the circle $\rho^2 = \alpha^2$ the solution g_{ph} remains unperturbed and coincides with the solution g_0.

Having obtained the solutions for g and g_{ph} we can now (just as in the previous case) determine the matrices A and B from (3.17) and their physical values A_{ph}, B_{ph} from (3.18). Substituting the quantities Tr A_{ph}^2 and Tr B_{ph}^2 into the equations (1.5) we obtain the metric component f_{ph} by quadratures.

In order to illustrate the results we take again for the particular solution g_0, Ψ_0, f_0 the Kasner solution (4.12)-(4.14) and consider only two special cases. The first is the isotropic case, when $s_1 = s_2 = 1/2$, and the second is flat space corresponding to $s_1 = 0$, $s_2 = 1$ $(q = -1/2)$.

If $s_1 = s_2 = 1/2$ we obtain the following solution for the metric:

$$-ds^2 = C_1 \alpha^{1/2} \sigma^{-1} Q(-dt^2 + dz^2) + \alpha Q^{-1} \{ \left[p_1^2 H - (1 - \sigma)^2 \cos 2\phi + 2p_1(1 - \sigma^2) \sin^2 \phi \right] dx^2 +$$

$$+ \left[p_1^2 H - (1 - \sigma)^2 \cos 2\phi - 2p_1(1 - \sigma^2) \sin^2 \phi \right] dy^2 - 2p_2(1 - \sigma^2) \sin 2\phi \, dx dy \} . \qquad (5.10)$$

Here we have introduced the notation:

$$Q = p_1^2 H - (1 - \sigma)^2 \quad , \quad H = 1 + \sigma^2 - 2\sigma \cos 2\phi \quad , \quad \sigma = \rho^2 \alpha^{-2} \qquad (5.11)$$

The quantities C_1, p_1, and p_2 are arbitrary constants restricted by the condition on p_1 and p_2

$$p_1^2 - p_2^2 = 1 \quad . \qquad (5.12)$$

The functions ρ and ϕ are determined from the equations (5.9) which involve two other arbitrary constants: ω_1 and ω_2 .

If one picks the coordinates in analogy with (4.15), i.e., in such a manner that $\alpha = t$ and $\omega_1 - \beta = z$, and if one analyzes the behavior of the field components g_{ab} as a function of the spacelike variable z at different times t, one can see that the solution (5.10)-(5.12) is of the two-soliton type and describes the interaction of two localized disturbances. For any fixed time t the matrix g will tend to the unperturbed solution $g_0 = \mathrm{diag}(t,t)$ at the infinities $z \to \pm \infty$. For all z we have $g_{11} > t$ and $g_{22} < t$. For sufficiently large values of $t (t >> \omega_2)$ each component g_{ab} has two extrema in the variable z, which are localized near the light cone $z^2 = t^2$. As t decreases these local disturbances start approaching one another, growing in amplitude. As $t \to 0$ (a singularity of cosmological character) both disturbances in the components g_{11} and g_{22} fuse into one concentrated near the origin $z = 0$, reaching at this stage some finite amplitude. The disturbances in the component g_{12} do not fuse as $t \to 0$, but approach each other to a finite minimal distance equal to $2\omega_2$.

By amplitudes we mean the absolute values of the extrema (with respecto to z) of the components of the matrix $(g - g_0) g_0^{-1}$. One can prove that as $t \to \infty$ the soliton amplitudes tend to zero, and as $t \to 0$ it is easy to calculate them from the asymptotic form of the matrix g corresponding to the solution (5.10) - (5.12). If $\alpha = t$ and $\omega_1 - \beta = z$, then as $t \to 0$ we get for g (here we have in mind every-where the matrix g_{ph}, i.e., the one that appears directly in the physical solution (5.10))

$$g = t \begin{pmatrix} \frac{z^2 + s^2\omega_2^2}{z^2 + \omega_2^2} & \frac{1 - s^2}{s} & \frac{z\omega_2}{z^2 + \omega_2^2} \\ \frac{1 - s^2}{s} & \frac{z\omega_2}{z^2 + \omega_2^2} & \frac{s^2 z^2 + \omega_2^2}{s^2(z^2+\omega_2^2)} \end{pmatrix} \tag{5.13}$$

where $s = (1 + p_1)/p_2$.

We now consider the case of solitons on a flat background, when $s_1 = 0$ and $s_2 = 1$ and when by means of a coordinate change the particular solution (4.12) can be reduced to the Minkowski metric. In this case the following choice of the functions $a(\zeta)$ and $b(\eta)$ turns out to be convenient:

$$2a = \omega_2 \operatorname{sh}(z + t) + \omega_1 \quad , \quad 2b = \omega_2 \operatorname{sh}(z - t) - \omega_1 \quad . \tag{5.14}$$

Whence, and from (2.2), we obtain:

$$\alpha = \omega_2 \operatorname{sh} z \operatorname{ch} t \quad , \quad \omega_1 - \beta = - \omega_2 \operatorname{ch} z \operatorname{sh} t \quad . \tag{5.15}$$

The equations (5.9) are simplest to solve for this choice of the functions α and β. For the modulus ρ and the phase Ψ we obtain:

$$\sin^2 \phi = \operatorname{ch}^{-2} t \quad , \quad \cos^2 \phi = \operatorname{th}^2 t \quad , \quad \rho^2 = \alpha^2 \operatorname{th}^2 (z/2) \quad . \tag{5.16}$$

The calculations lead to the following interesting result:

$$\begin{aligned} - ds^2 &= \omega (- dt^2 + dz^2) + \omega^{-1} (\gamma + a_1^2 \operatorname{sh}^2 z) dx^2 + \\ &+ \omega^{-1} \left[\gamma(2b_1 \operatorname{ch} z - a_1 \operatorname{sh}^2 z)^2 + \operatorname{sh}^2 z (r^2 + a_1^2 + b_1^2)^2\right] dy^2 - \\ &- 2\omega^{-1} \left[\gamma(2b_1 \operatorname{ch} z - a_1 \operatorname{sh}^2 z) + a_1 \operatorname{sh}^2 z (r^2 + a_1^2 + b_1^2)\right] dx\, dy \end{aligned} \tag{5.17}$$

where we have used the notations

$$\omega = r^2 + (b_1 - a_1 \operatorname{ch} z)^2 \quad , \quad \gamma = (a_1^2 - b_1^2 - m_1^2) \operatorname{ch}^2 t \ , \quad r = m_1 + (a_1^2 - b_1^2 - m_1^2)^{1/2} \operatorname{sh} t \ , \tag{5.18}$$

and the quantities a_1, b_1, and m_1 are arbitrary constants satisfying the requirement $a_1^2 \geq b_1^2 + m_1^2$. We note that the constant ω_2 is related to these variables by $\omega_2^2 = a_1^2 - b_1^2 - m_1^2$. This solution can be obtained from the known Kerr-NUT solution by means of a complex coordinate transformation:

$$\theta = iz \ , \ r = m_1 + (a_1^2 - b_1^2 - m_1^2)^{1/2} \operatorname{sh} t \ , \ \tau = x \ , \ \phi = y \quad , \tag{5.19}$$

where θ, r, ϕ, and τ are the Boyer-Lindquist coordinates. For $b_1 = 0$ we obtain the Kerr solution in these coordinates with angle parameter a_1 and the mass m_1. The metric (5.17) then corresponds to the case $a_1 \geq m_1$. This means that the Kerr solution can be obtained by means of the inverse scattering problem method discussed here, and also directly, by starting from the very outset not with the metric (1.1) but with its stationary anolog, and by choosing for the particular or "background" solution the flat space in spherical coordinates. Then the Kerr solution will represent a double stationary soliton.

In conclusion we note that in the derivation of the metrics considered above we have also used linear transformations of the coordinates x, y (with constant coefficients). These have allowed us to remove some inessential constants and to simplify the solutions.

I.6 - ON THE CONSTRUCTION OF SOLUTIONS IN GENERAL

Here we describe briefly a procedure of construction of solutions in the general case, when in addition to solitons there is also a nonsoliton part of the solution.

We define the numerical function $\omega(\lambda,\zeta,\eta)$ by means of the formula

$$\omega = \frac{1}{2}\,(\alpha^2/\lambda + 2\beta + \lambda) \quad . \tag{6.1}$$

It is easy to see that, taking (2.2) and (2.3) into account,

$$D_1\omega = 0 \quad , \quad D_2\omega = 0 \quad , \tag{6.2}$$

and consequently for an arbitrary matrix $\Pi(\omega)$ we also have $D_1\Pi(\omega) = 0$ and $D_2\Pi(\omega) = 0$.

We now consider in the complex λ plane the circle $|\lambda|^2 = \alpha^2$ and define on it the matrix function $G_0\,(\lambda,\zeta,\eta)$, which in general does not admit of analytic continuation off the circle, and depends only on the combination ω:

$$G_0 = G_0(\omega) \quad . \tag{6.3}$$

Putting $\lambda = \alpha e^{ir}$ on the circle, verify that the argument of G_0 is real and varies from $-\infty$ to $+\infty$. We require

$$G_0(\infty) = G_0(-\infty) = I. \tag{6.4}$$

Moreover, we shall assume that the matrix G_0 is real and symmetric:

$$\bar{G}_0(\lambda) = G_0(\lambda) \quad , \quad G_0 = \tilde{G}_0 \quad . \tag{6.5}$$

Let Ψ_0 be a particular solution of the equations (2.8). We define on the circle $|\lambda|^2=\alpha^2$ the new matrix function $G(\lambda,\zeta,\eta)$:

$$G(\lambda,\zeta,\eta) = \Psi_0\, G_0\, \Psi_0^{-1} \quad . \tag{6.6}$$

Since $D_{1,2}\, G_0(\omega) = 0$, we have the relations

$$D_1 G = \frac{1}{\lambda-\alpha}\,(A_0 G - G A_0) \quad , \quad D_2 G = \frac{1}{\lambda+\alpha}\,(B_0 G - G B_0) \tag{6.7}$$

One can now show that the determination of the matrix χ is closely related to finding the solution to the following problem (the Riemann problem) from analytic function theory. One is required to find the matrix function χ_1 holomorphic outside the circle $|\lambda|^2 = \alpha^2$, and the matrix function χ_2 holomorphic inside the circle, with the condition that the functions χ_1, χ_2 should satisfy on the circle the condition

$$\chi_1 = \chi_2\, G \quad . \tag{6.8}$$

Moreover, one can always require that the following normalization condition hold:

$$\chi_2(\infty) = I \quad . \tag{6.9}$$

If the matrices χ_1 and χ_2 are nonsingular in their domains of analyticity (i.e., their determinants do not have zeroes there), and have no poles, then the solution of the Riemann problem is unique. Acting on (6.8) with the operators D_1 and D_2 and making use of (6.7), it is easy to derive the relations

$$(D_1\chi_1 + \frac{1}{\lambda-\alpha}\chi_1 A_0)\chi_1^{-1} = (D_1\chi_2 + \frac{1}{\lambda-\alpha}\chi_2 A_0)\chi_2^{-1} \quad .$$

$$(D_2\chi_1 + \frac{1}{\lambda+\alpha}\chi_1 B_0)\chi_1^{-1} = (D_2\chi_2 + \frac{1}{\lambda+\alpha}\chi_2 B_0)\chi_2^{-1} \tag{6.10}$$

Each of these four expressions is defined (by the way they were derived)on the circle $|\lambda|^2=\alpha^2$, but the equations (6.10) also determine their analytic continuations into the whole complex λ plane. Since in their domains of analyticity the matrix χ_1, χ_2 are nonsingular and have no poles, the singularities exhibited by these expressions are obvious: the first two have a pole at $\lambda = \alpha$ the latter two have poles at $\lambda = -\alpha$. This implies that the quantities (6.10) have the form

$$(D_1\chi_1 + \frac{1}{\lambda-\alpha}\chi_1 A_0)\chi_1^{-1} = (D_1\chi_2 + \frac{1}{\lambda-\alpha}\chi_2 A_0)\chi_2^{-1} = \frac{1}{\lambda-\alpha} A \quad .$$

$$(D_2\chi_1 + \frac{1}{\lambda+\alpha}\chi_1 B_0)\chi_.^{-1} = (D_2\chi_2 + \frac{1}{\lambda+\alpha}\chi_2 B_0)\chi_2^{-1} = \frac{1}{\lambda+\alpha} B \; . \tag{6.11}$$

Where A and B are matrices which do not depend on λ. But the equations (6.11) now coincide with the equations (2.11), and since the system (6.11) is compatible, the matrices A and B satisfy the equations (2.4), (2.5). The matrix χ introduce before equals χ_2 (it is holomorphic at the point $\lambda = 0$ and tends to the unit matrix for $\lambda \to \infty$) and the matrix

$$g = \chi_2(0)\, g_0 \tag{6.12}$$

is the metric tensor satisfying the equations (1.4)

The matrices χ_1 and χ_2 must also satisfy some additional conditions similar to the conditions (2.12), (2.13) which follow from the symmetry and reality of the matrix G_0 and of the metric tensor g. These condition are now:

$$\bar{\chi}_{1,2}(\bar{\lambda}) = \chi_{1,2}(\lambda) \quad . \quad g = \chi_1(\alpha^2/\lambda)\, g_0 \bar{\chi}_2(\lambda) \quad . \tag{6.13}$$

Until now we have assumed that the matrix χ_1 and χ_2 are invertible in their domains of analyticity and have no poles there. The solution of this regular Riemann problem is reduced to a solution of a singular integral equation, as is well known. If one represents the inverse matrices χ_1^{-1} and χ_2^{-1} in the form

$$\chi_1^{-1} = I + \frac{1}{\pi i}\int_\Gamma \frac{\rho(z)}{\lambda - z + i0}\, dz \quad ,$$

$$\chi_2^{-1} = I + \frac{1}{\pi i}\int_\Gamma \frac{\rho(z)}{\lambda - z - i0}\, dz \quad , \tag{6.14}$$

where the contour Γ is the circle $|\lambda|^2 = \alpha^2$ and then substitutes these expressions into (6.8), one can see easily that the matrix function $\rho(z)$ satisfies the equations

$$\rho(z) + T(z,\zeta,\eta) \quad (I + \frac{1}{\pi i}\int_\Gamma \frac{\rho(z')}{z - z'} dz') = 0 \tag{6.15}$$

In this equations

$$T = (I - G)\,(I + G)^{-1} \tag{6.16}$$

is the Cayley transform of the matrix G; the points z and z' are situated on the circle of radius α and the integral is to be taken in the principal value sense.

A solution of equation (6.15) yields the purely non-soliton part of the solutions of the original equations (1.3), (1.4). In this case the meaning of the method consists in the fact that the equations (6.15) present considerably fewer difficulties than the original problem of integration of the equations (1.3), (1.4).

If the Riemann problem is not regular and the matrices χ_1 and χ_2 are degenerate (noninvertible) in their domains of analyticity, so that χ_1^{-1} and χ_2^{-1} have pole singularities there, the solutions will also involve solitons. The method exposed here also generalizes without difficulty to that case. In this case the right-hand sides of the expressions (6.14) for the matrices χ_1^{-1} and χ_2^{-1} will contain an additional term: the matrix $U(\lambda,\zeta,\eta)$ of the form

$$U = \sum_k \left(\frac{S_k}{\lambda - v_k} + \frac{\bar{S}_k}{\lambda - \bar{v}_k}\right) \tag{6.17}$$

which also enters as an additive term into the expression in parentheses in Eq. (6.15). In this case one has to add to the equation (6.15) a system of equations which determine the matrix $S_k(\zeta,\eta)$ (v_k are the same functions as in the purely solitonic case), but this system contains (linearly) also the contour integrals which occur in (6.14) considered as functions of λ at the poles $\lambda = v_k$. The derivation of these equations is simple and is based on the same method as used for the determination of the matrix R_k in the soliton case described above. The form of this complete system of equations will not be given here. We only indicate that the equations which determine pure soliton solutions follow from it in the special case when the matrix G is identically equal to the unit matrix. If G = I it follows from (6.16) and (6.15) that $T = 0$, $\rho = 0$.

We also note that the soliton of the general system of equations for $\beta \to \pm\infty$ tends to a purely solitonic one. Indeed, since the matrix G_0 is given on the circle $|\lambda|^2 = \alpha^2$, we may set in its argument ω $\lambda = \alpha\, e^{i\gamma}$. Then $\omega = \alpha\cos\gamma + \beta$ and for $\beta \to \pm\infty$ we obtain $\omega \to \pm\infty$, but on account of the condition (6.4) this implies $G_0 \to I$ and from (6.6) it follows that $G \to I$. But according to what was said above, for $G \to I$ the solution goes over into a solitonic one. A similar phenomenon occurs for $\alpha \to \pm\infty$ also.

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II - STATIONARY GRAVITATIONAL SOLITONS WITH AXIAL SYMMETRY

II.1 - INTRODUCTION

In a previous chapter which was based on the paper (1) we have shown that in the case in which the metric tensor depends on only two variables the gravitational equations form a system which is integrable by the method of the inverse scattering problem. The case was examined in which one of the variables is the time and the other is spacelike; this corresponds to cosmological and wave solutions of the equations of gravitation. It was pointed out that there is no difficulty in applying this method also to the case in which both the variables on which the metric tensor depends are spacelike, which corresponds to stationary gravitational fields. One possible interpretation of this case is that of a stationary gravitational field with axial symmetry. This class of solutions is important in the theory of gravitation, since it has a clear physical meaning. In this connection it is interesting to consider the case of axially symmetric stationary fields separately and to find the construction of the corresponding soliton solutions and their physical meaning. This is the purpose of the present paper. We shall also use this case as an example to carry to completion the procedure which we described earlier (1) for constructing exact soliton solutions, and deal with one important point which was left there incomplete. We shall explain the essence of the question, first introducing the metric and the corresponding Einstein equations.

Having in view the application to the case of stationary axially symmetric gravitational fields we write the metric in the form (*)

$$- ds^2 = f(d\rho^2 + dz^2) + g_{ab}\, dx^a\, dx^b \quad , \tag{1.1}$$

where the metric coefficients f and g_{ab} are functions of only two variables, ρ and z. We use for the coordinates the notation $(x^0, x^1, x^2, x^3) = (t, \phi, \rho, z)$. Throughout this paper the five Latin indices a, b, c, d, f run through the values 0 and 1 and correspond to the coordinates t and ϕ.

It is well known that in this case (by using the remaining freedom in the choice of the coordinates ρ and z) we can, without loss of generality, impose on the two-rowed matrix g (with components g_{ab}) the following supplementary condition:

$$\det g = - \rho^2 \quad . \tag{1.2}$$

It is now easy to show that the Einstein equations (in vacuum) for the metric (1.1), (1.2) separate into two groups. The first determines the matrix g and is of the form

$$(\rho g_\rho\, g^{-1})_\rho + (\rho g_z\, g^{-1})_z = 0 \quad . \tag{1.3}$$

The second group of equations determines the metric coefficient f for a given solution of Eq. (1.3) and can be written in the form

$$(\ln f)_\rho = - \rho^{-1} + (4\rho)^{-1}\, \mathrm{Sp}\, (U^2 - V^2) \quad , \tag{1.4}$$

$$(\ln f)_z = (2\rho)^{-1}\, \mathrm{Sp}\, (UV) \quad , \tag{1.5}$$

where the two-rowed matrices U and V are defined as follows:

$$U = \rho g_\rho\, g^{-1} \quad , \quad V = \rho g_z\, g^{-1} \quad . \tag{1.6}$$

It is easy to see that if instead of ρ and z we introduce the pair of complex variables $\zeta = (z + i\rho)$ and $\eta = (z - i\rho)$, then in the variables ζ and η the metric (1.1) and Eqs. (1.2)-(1.6) will be formally reduced to the same form as we studied previously (1). For this reason

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(*) A system of units is used in which the speed of light is equal to unity. The interval is written in the form $- ds^2 = g_{ik} dx_i dx_k$, where g_{ik} has the signature - + + + .

all of the formal side of the method for the case considered here can be obtained (*) from the results of our earlier paper (1). Of these results we shall present here only the basic points which are necessary for a complete exposition, and shall not go into the details of the proofs. The details can be found in Ref. 1.

Let us now turn to the point in the research which was not brought to completion in Ref. 1. As follows from what we have said, we can apply to the integration of Eqs. (1.2) - (1.6) the method given in Ref. 1., i.e., apply the method of the inverse scattering problem to the integration of the matrix equation (1.3) and thus get the major part of the metric coefficients g_{ab}. There then remains, however, the problem of calculating the metric coefficient f, which is given in quadratures by Eqs. (1.4) and (1.5).

In Ref. 1 it was shown by direct calculations that for the simple soliton solutions given there these quadratures can be performed completely (i.e., the integrals can be calculated explicitly), and the answer for the coefficient f can be expressed explicitly in terms of the appropriate partial or background solution of the problem and elementary functions, i.e., qualitatively in the same way as the metric components g_{ab}. This suggests that the same will be true also on the general case of an n-soliton solution. It turns out that this is indeed true, and the metric coefficient f in the general n soliton case, like the coefficients g_{ab}, can be calculated altogether explicitly. The analysis for this point is given in Sec. 3 of the present paper.

Finally, we point out that the question of the integrability of the equations of gravitation for the case considered has also been investigated by Maison (2), who proved the existence of an L - A pair for the Einstein equations, though in a somewhat different way from that followed in Ref. 1 and here (cf. Eqs. (2.1), (2.2)). Harrison (3) found the Bäcklund transformation for the Ernst equation corresponding to this problem.

II.2 - THE n-SOLITON SOLUTION FOR THE MATRIX g

Using the results of Ref. 1 (as explained in the Introduction), we can easily find the L - A pair for the matrix equation (1.3) in the variables ρ and z:

$$D_1\Psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\,\Psi \quad , \qquad D_2\Psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\,\Psi \quad , \tag{2.1}$$

where the commuting differential operators D_1 and D_2 are given by

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\,\partial_\lambda \quad , \qquad D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\,\partial_\lambda \tag{2.2}$$

and λ is a complex spectral parameter independent of the coordinates ρ and z. It is not hard to verify that the conditions of compatibility of the equations (2.1) for the matrix function $\Psi(\lambda,\rho,z)$ are identical with the original equations (1.3) and (1.6), if we rewrite them, and also the conditions for their compatibility, in terms of the matrices U and V, in the same way as this was done previously (1). The required matrix g is the value of the matrix $\Psi(\lambda,\rho,z)$ for $\lambda = 0$:

$$g(\rho,\, z) = \Psi(0,\, \rho,\, z) \quad . \tag{2.3}$$

The procedure for integrating the equations (2.1) preassumes the knowledge of some particular solution of the problem. Let g_0, U_0, V_0 be some particular solution of Eqs. (1.3) and (1.6), from which, with Eq. (2.1), the corresponding solution $\Psi_0(\lambda,\rho,z)$ has been found. We then seek the solution for Ψ in the form

(*) We may indicate that the formal transformations from the variables ζ,η,α,β and matrices A, B which we used previously to the variables ρ, z and matrices U, V of the present paper are the form $\zeta = (z + i\rho)/2$, $\eta = (z - i\rho)/2$, $\alpha = i\rho$, $\beta = z$, $A = -U - iV$, $B = -U + iV$.

$$\Psi = \chi \Psi_0 \tag{2.4}$$

and for $\chi(\lambda,\rho,z)$ we get from Eq. (2.1) the following equations:

$$D_1\chi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2} ,$$

$$D_2\chi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2} \tag{2.5}$$

Now (as before (1)) it can be shown that to assure that the matrix g is real and symmetric definite supplementary conditions have to be imposed on the solutions of Eq. (2.5). For the reality of g we have the requirements

$$\bar{\chi}(\bar{\lambda}) = \chi(\lambda) \quad , \quad \bar{\Psi}(\bar{\lambda}) = \Psi(\lambda) \tag{2.6}$$

(a bar denotes the complex conjugate), and for g to be symmetric we require

$$g = \chi(-\rho^2/\lambda)\; g_0\tilde{\chi}(\lambda) \tag{2.7}$$

(a tilde indicates transposition). Besides this, compatibility of Eqs. (2.7) with (2.3) requires

$$\chi(\infty) = I, \tag{2.8}$$

where I is the unit matrix (here, and often from now on, we omit the arguments ρ and z of functions for simplicity).

The soliton solutions for the matrix g correspond, as is well known, to the presence of pole singularities of the matrix $\chi(\lambda,\rho,z)$ in the complex plane of the spectral parameter λ. Let us consider the general case, in which the matrix χ has n such poles, which we assume to be simple. The matrix $\chi(\lambda,\rho,z)$ can then be represented in the form

$$\chi = I + \sum_{k=1}^{n}\frac{R_k}{\lambda-\mu_k} , \tag{2.9}$$

where the matrices R_k and the numerical functions μ_k now depend only on the variables ρ and z.

We note that in Ref. 1 an expression analogous to Eq. (2.9) was written in a form which obviously satisfies the condition (2.6) and which emphasizes the fact that complex poles (i.e. complex μ_k) of the matrix χ can exist only as conjugate pairs. Of course these requirements still hold here, but experience shows that writing χ in the form (2.9)considerably facilitates the calculations, which it is convenient to do by neglecting the conditions (2.6) and supposing (until the final form of the solutions is reached) that we have to do with n arbitrary complex poles $\lambda = \mu_k (k = 1, 2, \ldots, n)$. After the final form of the solution is obtained it is easy to assure that the matrix g is real by imposing definite supplementary conditions on the arbitrary constants that appear in the solution. This procedure is possible with an even number of complex poles in the sum (2.9), and is of course equivalent to introducing the complex poles at the very start as conjugate pairs. If, on the other hand, all of the μ_k in the sum (2.9) are real, then all of the matrices R_k will also be real and the matrix χ then satisfies Eq. (2.9) automatically.

Substitution of the expression (2.9) into Eq. (2.5) and the supplementary condition (2.7) completely determines the pole trajectories $\mu_k(\rho,z)$ and the matrices $R_k(\rho,z)$. The numerical functions μ_k are determined from the requirement that in the left sides of Eqs. (2.5) there are no poles of second order at the points $\lambda = \mu_k$. The result is that each function $\mu_k(\rho,z)$ (with each index $k = 1, 2, \ldots, n$) satisfies a pair of differential equations

$$\mu_{k,z} + 2\mu_k^2(\mu_k^2 + \rho^2)^{-1} = 0 \quad ,$$
$$\mu_{k,\rho} - 2\rho\mu_k\ (\mu_k^2 + \rho^2)^{-1} = 0 \quad , \qquad (2.10)$$

whose solutions are the roots of a quadratic algebraic equations

$$\mu_k^2 - 2(\omega_k - z)\ \mu_k - \rho^2 = 0 \quad , \qquad (2.11)$$

where ω_k are arbitrary constants (in general complex).

Accordingly, for each index k (i.e., for each pole) we have its own arbitrary constant ω_k, which determines two possible solutions for the trajectory of the pole $\mu_k(\rho,z)$:

$$\mu_k = \omega_k - z \pm \left[(w_k - z)^2 + \rho^2\right]^{1/2} \quad . \qquad (2.12)$$

The matrices R_k are degenerate, and their components can be written in the form

$$(R_k)_{ab} = n_a^{(k)}\ m_b^{(k)} \qquad (2.13)$$

The two component vectors $m_a^{(k)}$ are found directly from Eqs. (2.5) by requiring that they be satisfied at the poles $\lambda = \mu_k$, and the vectors $n_a^{(k)}$ are then determined from the condition (2.7). The vectors $m_a^{(k)}$ can be expressed in terms of the given partial solution fr the "wave" matrix $\Psi_0(\lambda,\rho,z)$ taken at the value μ_k for the argument λ. These vectors are of the following form:

$$m_a^{(k)} = m_{c0}^{(k)} \left[\Psi_0^{-1}\ (\mu_k,\rho,z)\right]_{ca} \qquad (2.14)$$

where Ψ_0^{-1} denotes the matrix inverse to Ψ_0. (Here and from now on summation is to be understood over repeated vector and tensor indices a, b, c, d, f, which run through the values 0 and 1. Summation over other indices occurs only when explicitly indicated). In Eq. (2.14) the $m_{c0}^{(k)}$ are arbitrary constants.

The vectors $n_a^{(k)}$ can then be determined from the following n-th order system of algebraic equations:

$$\sum_{\ell=1}^{n} \Gamma_{k\ell}\, n_a^{(\ell)} = \mu_k^{-1}\ m_c^{(k)}\ (g_0)_{ca} \quad , \quad k,\ell = 1,\ 2,\ \ldots,\ n \quad , \qquad (2.15)$$

where the matrix $\Gamma_{k\ell}$ is symmetric and its elements are

$$\Gamma_{k\ell} = m_c^{(k)}(g_0)_{cb}\ m_b^{(\ell)}\ (\rho^2 + \mu_k\mu_\ell)^{-1} \qquad (2.16)$$

(in these formulas $g_0(\rho,z)$ is a given particular solution of the original equations (1.3)).If we introduce the symmetric matrix $D_{k\ell}$ inverse to the matrix $\Gamma_{k\ell}$:

$$\sum_{p=1}^{n} D_{kp}\ \Gamma_{p\ell} = \delta_{k\ell} \quad , \qquad (2.17)$$

then we get from (2.15) for the vectors $n_a^{(k)}$

$$n_a^{(k)} = \sum_{\ell=1}^{n} D_{\ell k}\ \mu_\ell^{-1}\ N_a^{(\ell)} \quad , \qquad (2.18)$$

where

$$N_a^{(k)} = m_c^{(k)}\ (g_0)_{ca} \quad . \qquad (2.19)$$

According to Eqs. (2.3), (2.4), and (2.9) the required matrix g is

$$g = \Psi(0) = \chi(0)\, \Psi_0(0) = \chi(0)\, g_0 = (I - \sum_{k=1} R_k\, \mu_k^{-1})\, g_0 \qquad (2.20)$$

Now, using Eqs. (2.13), (2.18), and (2.19) we get the metric components g_{ab}:

$$g_{ab} = (g_0)_{ab} - \sum_{k,\ell=1}^{n} D_{k\ell}\mu_k^{-1}\, \mu_\ell^{-1}\, N_a^{(k)}\, N_b^{(\ell)} \ . \qquad (2.21)$$

With the expression (2.21) the matrix g is obviously symmetric. Let us now consider the question of its being real. If all of the functions $\mu_k(\rho,z)$ are real, the components g_{ab} are automatically real, if we take all of the arbitrary constants appearing in the solution to be real. In fact, the particular solution $\Psi_0(\lambda,\rho,z)$ is always taken to satisfy the second of the conditions (2.6), and consequently $\Psi_0(\lambda)$ is real on the real axis of the λ plane, i.e., at the points $\lambda = \mu_k$. It can now be seen from Eq. (2.14) that the arbitrary constants $m_{c0}^{(k)}$ that occur in the vectors $m_a^{(k)}$ must be taken real, and then the vectors $m_a^{(k)}$ will also be real. It then follows that all the other quantities from which the matrix g is constructed are real. We now suppose that there are also complex values among the functions $\mu_1, \mu_2, \ldots, \mu_n$. The conditions (2.6) then require that all the complex poles appear only as conjugate pairs; for each complex pole $\lambda = \mu$ its conjugate $\lambda = \bar{\mu}$ must also appear. Suppose there is such a pair of poles $\lambda = \mu_p$ and $\lambda = \mu_q$, with $\mu_q = \bar{\mu}_p$. To these poles there correspond vectors $m_a^{(p)}$ and $m_a^{(q)}$, which according to Eq. (2.14) are given by

$$m_a^{(p)} = m_{c0}^{(p)} \left[\Psi_0^{-1}\ (\mu_p, \rho, z)\right]_{ca} \ ,$$

$$m_a^{(q)} = m_{c0}^{(p)} \left[\Psi_0^{-1}\ (\mu_q, \rho, z)\right]_{ca} \ .$$

A simple analysis shows that the matrix g will be real if for each such pair of complex-conjugate poles the arbitrary constant $m_{c0}^{(p)}$ and $m_{c0}^{(q)}$ are taken conjugate to each other. This means that the vectors $m_a^{(p)}$ and $m_a^{(q)}$ corresponding to each pair of conjugate poles are also conjugate to each other ($m_a^{(q)} = \bar{m}_a^{(p)}$), since the function $\Psi_0(\lambda,\rho,z)$ satisfies the condition $\Psi_0(\bar{\lambda}) = \bar{\Psi}_0(\lambda)$. Accordingly, we can formulate the following rule that determines the choice of the arbitrary constants $m_{c0}^{(k)}$ in Eq. (2.14): to assure that the matrix g is real, it is necessary to choose the arbitrary $m_{c0}^{(k)}$ in Eq. (2.14) so that the vectors $m_a^{(k)}$ corresponding to real poles $\lambda = \mu_k$ are real and the vectors $m_a^{(p)}$ and $m_a^{(q)}$ corresponding to each pair of complex-conjugate poles $\lambda = \mu_p$ and $\lambda = \mu_q = \bar{\mu}_p$ are complex conjugate to each other.

Satisfying the requirements that g be real and symmetric is still not enough. It must not be forgotten that g must also satisfy the supplementary condition (1.2). We now calculate the determinant of the matrix g. The form (2.21) is not convenient for this calculation, and we use a different representation of our solution. We note that the process of perturbing the background solution g_0 and obtaining from it the n-soliton solution g, as described above, is formally equivalent to the introduction of the n solitons one at a time successively. The first step is to go from the background metric g_0 to the metric g_1 containing one soliton, corresponding to the presence in the matrix χ (which we at this stage call χ_1) only one pole $\lambda = \mu_1$.

This one-soliton solution is easily obtained from the results given above. The matrix $\chi_1(\lambda)$ and its inverse $\chi_1^{-1}(\lambda)$ can be written in the following form

$$\chi_1 = I + (\mu_1^2 + \rho^2)\mu_1^{-1}\ (\lambda - \mu_1)^{-1}\ P_1 \ ,$$
$$\chi_1^{-1} = I - (\mu_1^2 + \rho^2)(\rho^2 + \lambda\mu_1)^{-1}\ P_1 \ , \qquad (2.22)$$

where the matrix P_1 has the elements

$$(P_1)_{ab} = m_c^{(1)}\ (g_0)_{ca}\ m_b^{(1)}\ /\ m_d^{(1)}\ (g_0)_{df}\ m_f^{(1)} \qquad (2.23)$$

and accordingly has the following properties:

$$P_1^2 = P_1 \ , \quad Sp\ P_1 = 1 \ , \quad \det P_1 = 0 \ . \qquad (2.24)$$

The quantities μ_1 and $m_a^{(1)}$ are given by Eqs. (2.12) and (2.14) with k = 1. We now get for the matrix g_1:

$$g_1 = \chi_1(0)\ g_0 = \left[I - (\mu_1^2 + \rho^2)\ \mu_1^{-2}\ P_1\right]\ g_0 \tag{2.25}$$

It is not hard to calculate the determinant of g_1. Owing to the general relation

$$\det\ (I + F) = 1 + \mathrm{Sp}\ F + \det\ F$$

(which holds for an arbitrary two-rowed matrix F) and the properties (2.24) we get

$$\det\ \left[I - (\mu_1^2 + \rho^2)\ \mu_1^{-2}\ P_1\right] = -\ \rho^2\ \mu_1^{-2} \tag{2.26}$$

and consequently

$$\det\ g_1 = -\ \rho^2\ \mu_1^{-2}\ \det\ g_0 \quad . \tag{2.27}$$

We can now take the solution g_1 as a new particular or background solution and repeat the operation of adding a soliton to it, that corresponding to the pole $\lambda = \mu_2$. To do this we form the new background matrix function $\Psi_1 = \chi_1\ \Psi_0$, take its inverse Ψ_1^{-1} and calculate it at the point $\lambda = \mu_2$, and then find the corresponding vector $M_a^{(2)}$:

$$M_a^{(2)} = M_{c0}^{(2)}\ \left[\Psi_1^{-1}\ (\mu_2,\ \rho, z)\right]_{ca}$$

after which we construct the matrix P_2, in analogy with Eq. (2.23):

$$(P_2)_{ab} = M_c^{(2)}\ (g_1)_{ca}\ M_b^{(2)}\ /\ M_a^{(2)}\ (g_1)_{df}\ M_f^{(2)} \quad ,$$

which matrix has the same properties (2.24) as the matrix P_1.

When we now construct the matrix $\chi_2(\lambda)$ (this matrix is calculated from the same formulas (2.22), with the index 1 replaced with 2), we get the two-soliton solution g_2:

$$g_2 = \left[I - (\mu_2^2 + \rho^2)\ \mu_2^{-2}\ P_2\right]\left[I - (\mu_1^2 + \rho^2)\ \mu_1^{-2}\ P_1\right]\ g_0$$

Continuing this process, we get the n-soliton solution (2.21) in the form

$$g = \left(\prod_{k=1}^{n}\left[I - (\mu_k^2 + \rho^2)\ \mu_k^{-2}\ P_k\right]\right)\ g_0 \quad , \tag{2.28}$$

Where all of the matrices P_k satisfy the same conditions as the matrix P_1 does:

$$P_k^2 = P_k \quad , \quad \mathrm{Sp}\ P_k = 1 \quad , \quad \det\ P_k = 0 \quad . \tag{2.29}$$

Naturally the explicit form of the matrices P_k rapidly becomes cumbersome as k increases, and therefore this way of calculating solutions is less convenient than the one previously described. But the representation of the solution in the form (2.28) is useful for the study of some particular questions, and especially for calculating the determinant of the matrix g. The important thing for this is only that the matrices P_k have the properties (2.29), not their specific form. The contribution from each factor in Eq. (2.28) to the determinant of g can be calculated trivially, and the result is

$$\det\ g = (-1)^n\ \rho^{2n}\ \left(\prod_{k=1}^{n}\ \mu_k^{-2}\right)\ \det\ g_0 \quad . \tag{2.30}$$

If we take the particular solution g_0 as satisfying by definition the condition $\det\ g_0 = -\rho^2$, then it follows from Eq. (2.30) that the number of solitons n must always be even, since an odd number would change the sign of det g and violate the physical signature of the metric. Accordingly (in contrast with the case investigated earlier (1)) on a physical background all stationary axially symmetric solitons (even those which correspond to real poles $\lambda = \mu_k$), can appear only in pairs forming bound two-soliton states(*).

We still have to construct an n-soliton solution g which not only satisfies Eq.(1.3) but also the supplementary condition (1.2). We shall call such a solution a physical one and denote it by $g^{(ph)}$. Constructing it is simple if we note that det g for any solution g of Eq. (1.3) satisfies the equation

$$\rho^{-1}[\rho(\ln\det g)_{,\rho}]_{,\rho} + (\ln\det g)_{,zz} = 0$$

Then it is easy to verify that the matrix

$$g^{(ph)} = -\rho\,(-\det g)^{-1/2}\,g \tag{2.31}$$

also satisfies Eq. (1.3), and also the condition $\det g^{(ph)} = -\rho^2$. Now supposing the number n of solitons is even and $\det g_0 = -\rho^2$, we get from Eqs. (2.30) and (2.31) the final expression for the metric tensor:

$$g^{(ph)} = -\rho^{-n}\,(\prod_{k=1}^{n}\mu_k)\,g \quad , \quad \det g^{(ph)} = -\rho^2 \tag{2.32}$$

where the matrix g is given by Eq. (2.21).

II.3 - CALCULATION OF THE METRIC COEFFICIENT f

It is also convenient to do the calculation of the coefficient f in two stages. First we calculate the value of f that follows from Eqs. (1.4) and (1.5) when we substitute in them the nonphysical solution g given by Eq. (2.21), which does not satisfy the condition $\det g = -\rho^2$, and then use a simple procedure to find the physical value of the coefficient, $f^{(ph)}$, which is obtained from these same Eqs.(1.4) and (1.5) when $g^{(ph)}$ is substituted in them instead of g.

To calculate f we must determine from Eqs.(2.5) the matrices U and V; this can be done by equating the left and right sides of these equations at the poles $\lambda = i\rho$ and $\lambda = -i\rho$ (cf. the analogous procedure in the previous paper (1)). Then calculating the traces $Sp(U^2 - V^2)$ and SP(UV) and substituting them in Eqs. (1.5) and (1.6) we find f by direct integration. It is a remarkable fact that this integration can actually be carried out. The key point in calculating the coefficient f corresponding to an n-soliton solution is to determine it for a one-soliton solution (which coefficient we denote as f_1), described by Eqs. (2.22) - (2.27). Having done the necessary calculations with the scheme indicated above (in analogy with the way this was done in Ref. 1), we get the following result for the one-soliton solution:

$$f_1 = C_1\, f_0\, \rho\mu_1^2\, (\mu_1^2 + \rho^2)^{-1}\, \Gamma_{11} \quad , \tag{3.1}$$

where C_1 is an arbitrary constant, f_0 is the particular(background) solution for the coefficient f, which corresponds to the solution g_0, and Γ_{11} is the single component of the matrix (2.16), which is all that exists in this case (k = 1 and ℓ = 1):

$$\Gamma_{11} = (\mu_1^2 + \rho^2)^{-1}\, m_c^{(1)}\, (g_0)_{cb}\, m_b^{(1)} \tag{3.2}$$

(the vector $m_a^{(1)}$ follows from Eq. (2.14) for k = 1).

The next step in the calculations is that, taking the solution g_1, f_1 as a new particular solution and repeating the operation just performed (as was explained in the foregoing section in connection with finding the matrix g_2), we get the coefficient f_2 that corresponds to the two-soliton solution with the poles $\lambda = \mu_1$ and $\lambda = \mu_2$. At this second step we already have to deal with only calculations of an algebraic nature, since the need for integration appears in the whole procedure only once, in the transition from the background solution g_0, f_0 to the solution g_1, f_1, which contains one soliton.

Omitting the details of the calculation, we give only the result:

$$f_2 = C_2\, f_2\rho^2\mu_1^2\mu_2^2\, (\mu_1^2 + \rho^2)^{-1}\, (\mu_2^2 + \rho^2)^{-1}\, (\Gamma_{11}\,\Gamma_{22} - \Gamma_{12}^2) \tag{3.3}$$

(*) Nevertheless we can obtain physical solutions with an odd number of solitons, but for this it is necessary to take a background solution with a nonphysical signature, for which $\det g_0 = \rho^2$.

Here C_2 is an arbitrary constant, f_0 is the same background solution as in Eq. (3.1), and Γ_{11}, Γ_{22}, and Γ_{12} are the components of the matrix (2.16). We now have three independent components of $\Gamma_{k\ell}$, since the indices k and ℓ can take two values, 1 and 2.

Equations (3.1) and (3.3) suggest that in the general n-soliton case the coefficient f is given by the expression

$$f_n = C_n\, f_0\, \rho^n \left(\prod_{k=1}^{n} \mu_k^2\right) \left[\prod_{k=1}^{n} (\mu_k^2 + \rho^2)\right]^{-1} \det \Gamma_{k\ell} \tag{3.4}$$

(where k, ℓ = 1, 2, ..., n). Since we see from Eqs. (3.1) and (3.3) that this formulat indeed holds for n = 1 and n = 2, we can prove that it holds in the general case by using the method of mathematical induction. This proof is given in the Appendix to the present paper, and shows that Eq. (3.4) is indeed correct in general.

Now we must determine the physical value $f_n^{(ph)}$ of the coefficient, i.e., the value that would be obtained from Eqs. (1.4) and (1.5) if we substituted in them the physical matrix $g^{(ph)}$ of Eq. (2.32) instead of g. From Eq. (2.31) we get the obvious relations

$$U^{(ph)} = \rho g_{,\rho}^{(ph)}\, g^{(ph)-1} = U + \left[1 - \frac{1}{2}\rho(\ln\det g)_{,\rho}\right] I \quad ,$$

$$V^{(ph)} = \rho g_{,z}^{(ph)}\, g^{(ph)-1} = V - \frac{1}{2}\rho(\ln\det g)_{,z} I \quad .$$

When we now substitute in Eqs. (1.4) and (1.5) the matrices $U^{(ph)}$ and $V^{(ph)}$ instead of U and V, we find that the physical coefficient $f_n^{(ph)}$ is given by the formula

$$f_n^{(ph)} = f_n\, \rho^{1/2}\, Q^{-1} \quad , \tag{3.5}$$

where f_n is the value of this coefficient which is given by Eq. (3.4) and the function Q is defined by the equations

$$(\ln Q)_{,z} = \frac{1}{2}\rho(\ln\det g)_{,\rho}\ (\ln\det g)_{,z} \quad ,$$

$$(\ln Q)_{,\rho} = \frac{1}{2}\rho\left[(\ln\det g)_{,\rho}^2 - (\ln\det g)_{,z}^2\right] \quad .$$

On substituting here the expression (2.30) for det g (with the condition det $g_0 = -\rho^2$)we find that these equations can be integrated easily, and the answer can be written in the form

$$Q^{-1} = \text{const}\ \rho^{-(n^2+2n+1)/2} \left(\prod_{k=1}^{n} \mu_k\right)^{n-1} \times \left[\prod_{k=1}^{n} (\mu_k^2+\rho^2)\right] \prod_{\substack{k,\ell=1\\ k>\ell}}^{n} (\mu_k - \mu_\ell)^2 \quad . \tag{3.6}$$

From this and Eqs. (3.4) and (3.5) we get the final expression for the physical value of the coefficient f:

$$f_n^{(ph)} = C_n^{(ph)}\, f_0\, \rho^{-n^2/2} \left(\prod_{k=1}^{n} \mu_k\right)^{n+1} \left[\prod_{\substack{k,\ell=1\\ k>\ell}}^{n} (\mu_k - \mu_1)^2\right]^{-1} \det \Gamma_{k\ell} \tag{3.7}$$

($C_n^{(ph)}$ is an arbitrary constant).

For clarity we point out that the product

$$\prod_{\substack{k,\ell=1\\ k>\ell}}^{n} (\mu_k - \mu_\ell)^2$$

is equal to 1 for n = 1, to $(\mu_2 - \mu_1)^2$ for n = 2, to $(\mu_3 - \mu_2)^2(\mu_3 - \mu_1)^2(\mu_2 - \mu_1)^2$ for n = 3, and so on. In deriving Eq. (3.7) we have assumed that no two of the quantities $\mu_1, \mu_2, \ldots, \mu_n$ are equal.

Accordingly, the final form of the n-soliton solution can be written in the form

$$- ds^2 = f_n^{(ph)} (d\rho^2 + dz^2) + g_{ab}^{(ph)} dx^a dx^b \quad , \tag{3.8}$$

where $f_n^{(ph)}$ is given by Eq. (3.7) and the matrix elements $g_{ab}^{(ph)}$ are determined by Eqs. (2.32) and (2.21).

II.4 - TWO-SOLITON SOLUTION ON A FLAT BACKGROUND

In this and the following sections we consider the application of the results presented above to the case in which the background metric g_0, f_0 is flat and given by the interval

$$- ds^2 = - dt^2 + \rho^2 d\phi^2 + d\rho^2 - dz^2 \quad . \tag{4.1}$$

That is, $f_0 = 1$ and $g_0 = \mathrm{diag}(-1, \rho^2)$ with the obvious property det $(g_0) = - \rho^2$. The matrix V_0 is equal to zero, and for the matrix U_0 we have $U_0 = \mathrm{diag}\ (0,2)$. From Eq. (2.1) we get the corresponding solution for $\Psi_0(\lambda,\rho,z)$:

$$\Psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & \rho^2-2z\lambda-\lambda^2 \end{bmatrix} \tag{4.2}$$

which satisfies the requirement $\Psi_0(0) = g_0$. From this and Eq. (2.14), using Eq. (2.11), we easily find the components of the vectors $m_a^{(k)}$:

$$m_0^{(k)} = C_0^{(k)} \quad , \quad m_1^{(k)} = C_1^{(k)} \mu_k^{-1} \quad , \tag{4.3}$$

where $C_0^{(k)}$ and $C_1^{(k)}$ are arbitrary constants.

Now from Eq. (2.16) we get the elements of the matrix Γ_{kl}:

$$\Gamma_{kl} = (-C_0^{(k)} C_0^{(\ell)} + C_1^{(k)} C_1^{(\ell)} \mu_k^{-1} \mu_\ell^{-1} \rho^2)(\rho^2 + \mu_k \mu_\ell)^{-1} \quad . \tag{4.4}$$

From Eq. (2.19) we get the components of the vectors $N_a^{(k)}$:

$$N_0^{(k)} = - C_0^{(k)} \quad , \quad N_1^{(k)} = C_1^{(k)} \mu_k^{-1} \rho^2 \quad . \tag{4.5}$$

Together with the expressions (2.12) for the functions μ_k we now have everything necessary for constructing n-soliton solutions on a flat-space background.

Let us now consider the simplest case of all. As was already stated at the end of Sec. 2, solitons on a physical background (with either complex or real poles) can appear only in pairs. Consequently, the simplest case will be a two-soliton solution, corresponding to two poles, $\lambda = \mu_1$ and $\lambda = \mu_2$. It is not hard to show by direct calculation that what we have here is just the Kerr-NUT solution. In our previous paper (1) it was already pointed out that a double stationary soliton on a flat background, corresponding to a pair of complex - conjugate poles, gives a Kerr-NUT solution with an "anomalously large" rotational moment(i.e.,a solution without horizons and with a base singularity). In fact, here we get precisely this situation for $\mu_2 = \bar{\mu}_1$. On the other hand, if both functions, μ_1 and μ_2, are real, the solution corresponds to the "normal" situation, with the singularity hidden from an outside observer by horizons.

These assertions can be verified by direct calculation of the metric. Let us represent the constant ω_1 and ω_2 that appear in the relations (2.11) and (2.12) in the form

$$\omega_1 = z_1 + \sigma \quad , \quad \omega_2 = z_1 - \sigma \quad , \tag{4.6}$$

where z_1 and θ are new arbitrary constants. We now introduce instead of ρ and z new coordinates γ and θ:

$$\rho = \left[(r - m)^2 - \sigma^2\right]^{1/2} \sin\theta \quad , \; z - z_1 = (r - m)\cos\theta \quad , \tag{4.7}$$

where m is an arbitrary constant whose meaning will be clear later. Then from Eq. (2.12)i is easy to express the quantities μ_1 and μ_2 in terms of the new variables r and θ. In this calculation we can choose the signs in the formula (2.12) either the same for μ_1 and μ_2, or else opposite. It is not hard to show that both cases lead to the same metric (to within linear transformations of the two coordinates t, ϕ in terms of each other, and a trivial conformal transformation, multiplication of the interval with a constant).

Let us consider first the case of like signs. If we choose the plus (*) sign in Eq. (2.12)for both values μ_1 and μ_2 , then substituting the expressions (4.6) and (4.7), we get

$$\mu_1 = 2(r - m - \sigma)\sin^2\frac{\theta}{2} \quad , \quad \mu_2 = 2(r - m - \sigma)\sin^2\frac{\theta}{2} \quad . \tag{4.8}$$

From this (using the expression (4.7) for ρ) and from Eq. (4.5) we find the components of the vectors $N_a^{(1)}$ and $N_a^{(2)}$, and from Eq. (4.4) we find the matrix $\Gamma_{k\ell}$ and its inverse $D_{k\ell}$ (in this case k,ℓ = 1, 2). After this we get from Eqs. (2.32) and (2.21) the components of the metric tensor $g_{ab}^{(ph)}$ and from Eq. (3.7) the metric coefficient $f_2^{(ph)}$. Substitution of these quantities in the interval (3.8) gives the final form of the solution, which can be reduced by simple linear transformations of the coordinates to the standard form of the Kerr-NUT solution in Boyer-Lindquist coordinates.

Omitting details, we point out that without loss of generality we can subject the arbitrary constants $C_0^{(k)}$ and $C_1^{(k)}$ that appear in the expressions (4.3) for the vectors $m_a^{(k)}$ to two conditions:

$$C_1^{(1)} C_0^{(2)} - C_0^{(1)} C_1^{(2)} = \sigma \quad , \quad C_1^{(1)} C_0^{(2)} + C_0^{(1)} C_1^{(2)} = - m \quad , \tag{4.9}$$

which are equivalent to the requirement that the variable r indeed be the Boyer-Lindquist radial coordinate. We then introduce two arbitrary constants a and b defined by

$$C_1^{(1)} C_1^{(2)} - C_0^{(1)} C_0^{(2)} = - b \quad , \quad C_1^{(1)} C_1^{(2)} + C_0^{(1)} C_0^{(2)} = a \quad . \tag{4.10}$$

From Eqs. (4.9) and (4.10) it follows that

$$\sigma^2 = m^2 - a^2 + b^2 \quad . \tag{4.11}$$

Now the metric (3.8) contains only the constants m, a, b and takes the form

$$- ds^2 = \omega\Delta^{-1} d_r{}^2 + \omega d\theta^2 - \omega^{-1}\{ (\Delta - a^2\sin^2\theta)\, d\tau^2 - 4\left[b\Delta\cos\theta - 4a\sin^2\theta(m\gamma + b^2)\right] d\tau d\phi +$$

$$+ \left[\Delta(a\sin^2\theta + 2b\cos\theta)^2 - \sin^2\theta(r^2 + b^2 + a^2)^2\right] d\phi^2 \} \quad , \tag{4.12}$$

where the variable τ is connected with t (the original coordinate $x^0 = t$) by the relation

$$\tau = t + 2a\phi \tag{4.13}$$

and the quantities ω and Δ are

$$\omega = \tau^2 + (b - a\cos\theta)^2 \quad , \quad \Delta = r^2 - 2mr + a^2 - b^2 \quad . \tag{4.14}$$

(*) We point out that the indicated choice of "signs" here has a precise meaning only for sufficiently large positive values of the variable r and real values of the constants ω_1 and ω_2. In the general case there is only a choice of one branch or the other of the solution of the quadratic equation (2.11).

It can be seen from this that the Kerr-NUT solution with horizons corresponds to real poles $\lambda = \mu_1$ and $\lambda = \mu_2$, since in this case the constant σ is real ($m^2 + b^2 > a^2$), and the constants ω_1 and ω_2 and the functions μ_1 and μ_2 are real along with σ. If the quantity σ is imaginary ($m^2 + b^2 < a^2$), then the constants ω_1 and ω_2 and the functions μ_1 and μ_2 are complex and conjugate to each other. This case corresponds to a solution without horizons. Furthermore the metric (4.12) and the constants m, a, b are of course still real, but the original constants $C_a^{(k)}$, as Eqs. (4.9) and (4.10) show, must be taken complex and related by $C_a^{(2)} = \bar{C}_a^{(1)}$, which we see from Eq. (4.3), means that also $m_a^{(2)} = \bar{m}_a^{(1)}$. This agrees with the rule for choosing real solutions with a complex-conjugate pair of poles that were formulated earlier in Sec. 2.

Let us now look at the second possibility for choosing the solutions of Eq. (2.11), the one that corresponds to using different signs in Eq. (2.12). Choosing the plus sign for μ_1 and the minus sign for μ_2, we get

$$\mu_1 = 2(r - m + \sigma)\sin^2\frac{\theta}{2} \quad , \quad \mu_2 = -2(r - m + \sigma)\cos^2\frac{\theta}{2} \quad . \tag{4.15}$$

Calculations like the foregoing ones show that in this case we again arrive at a Kerr - NUT metric, the only difference being that instead of the variables r, ϕ we will now have certain new coordinates r' and ϕ', conected with the original variables $x^0 = t$ and $x^1 = \phi$ by a linear transformation different from that in Eq. (4.13). The new relations are $\tau' = c_1 t + c_2\phi$, $\phi' = c_3\phi$, where the coefficients are real only if the constant σ is real (i.e., if μ_1 and μ_2 are real), and become complex when σ is imaginary. This means that for imaginary σ the matrix is complex in the original coordinates t, ϕ; this is quite natural, since in this case, as can be seen from Eq. (4.15), the poles $\lambda = \mu_1$ and $\lambda = \mu_2$ do not compose a complex-conjugate pair.

Besides this, the connection between the arbitrary constants $C_a^{(k)}$ and the parameters m, a, b are now different:

$$\begin{aligned} C_1^{(1)} C_1^{(2)} - C_0^{(1)} C_0^{(2)} &= \sigma \quad , \qquad C_1^{(1)} C_1^{(2)} - C_0^{(1)} C_0^{(2)} = -m \quad , \\ C_1^{(1)} C_0^{(2)} - C_0^{(1)} C_1^{(2)} &= a \quad , \qquad C_1^{(1)} C_0^{(2)} + C_0^{(1)} C_1^{(2)} = -b \quad . \end{aligned} \tag{4.16}$$

but the relation (4.11) between σ and the constants m, a, b is still valid.

In conclusions we point out that the only actual physical solution is that of Kerr, since the presence of the NUT parameter b makes the metric no longer asymptotically Euclidean and produces a number of nonphysical properties of the solution (the relevant analysis has been given by Misner (4)).

II.5 - THE n-SOLITON ON A FLAT BACKGROUND

In this section we consider some general properties of the n-soliton solution, confining ourselves to one of its possible types. We shall assume that on the background of a flat space with the metric (4.1) an even number n of solitons are introduced, corresponding to the poles $\lambda = \mu_1$, $\lambda = \mu_2$, ..., $\lambda = \mu_n$. We divide all of the functions μ_k ($k = 1, 2, \ldots, n$) into pairs and introduce the Greek index γ, which will number these pairs and takes only the odd values from 1 to $n - 1$: $\gamma = 1, 3, \ldots, n-1$. We thus have n/2 pairs of pole trajectories $(\mu_\gamma, \mu_{\gamma+1})$.

To understand the physical meaning of the solution it is helpful to examine first a special case which corresponds to a diagonal matrix g, i.e., to a static n-soliton field remaining after the rotation has been turned off. To obtain such a special case we set all of the arbitrary constants $C_0^{(k)}$ in Eq. (4.3) equal to zero, and then all the $m_0^{(k)}$ also equal to zero. It now follows from Eq. (2.15) that all the $n_0^{(k)} = 0$ and the matrices R_k as we can see from Eq. (2.15) take the form

$$R_k = \begin{pmatrix} 0 & 0 \\ 0 & n_1^{(k)} m_1^{(k)} \end{pmatrix}$$

This means that all the matrices P_k in the representation (2.28) of the solution take the form

$$P_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in accordance with the conditions (2.29). Then from Eqs. (2.28) and (2.32) we get the following solution for the diagonal case under consideration:

$$g_{00}^{(ph)} = \rho^{-n} \prod_{k=1}^{n} \mu_{k_1} \, g_{01}^{(ph)} = 0 \quad , \quad g_{11}^{(ph)} = - \rho^2/g_{00}^{(ph)} \quad . \tag{5.1}$$

The metric coefficient $f_n^{(ph)}$ can be found from Eq. (3.7); to do so we must calculate the determinant of the matrix $\Gamma_{k\ell}$ (with $C_0^{(k)} = 0$). It is simpler, however, to determine $f_n^{(ph)}$ directly from Eqs. (1.4) and (1.5), since the solution (5.1) is simple and easy to integrate. The result is

$$f_n^{(ph)} = \text{const}\ \rho^{(n^2+2n)/2} \left[\prod_{\substack{k,\ell=1 \\ k>\ell}}^{n} (\mu_k - \mu_\ell)^2\right] (\prod_{k=1}^{n} \mu_k)^{1-n} \left[\prod_{k=1}^{n} (\mu_k^2 + \rho^2)\right]^{-1} \tag{5.2}$$

We now determine from Eqs. (2.11) and (2.12) the functions μ_k, which we have arranged in the pairs $(\mu_\gamma , \mu_{\gamma+1})$. Confining our treatment to the case when the signs in Eq. (2.12)are chosen differently for the functions of each pair, we have

$$\begin{aligned} \mu_\gamma &= \omega_1 - z + \left[(\omega_1 - z)^2 + \rho^2\right]^{1/2} \quad , \\ \mu_{\gamma+1} &= \omega_{\gamma+1} - z - \left[(\omega_{\gamma+1} - z)^2 + \rho^2 \right]^{1/2} \end{aligned} \tag{5.3}$$

Instead of each pair of arbitrary constants ω_γ and $\omega_{\gamma+1}$, we introduce new constants z_γ and m_γ, setting

$$\omega_\gamma = z_\gamma - m_\gamma \quad , \qquad \omega_{\gamma+1} = z_\gamma + m_\gamma \quad . \tag{5.4}$$

If we now introduce n/2 pairs of functions $r_\gamma(\rho,z)$ and $\theta_\gamma(\rho,z)$(giving to each pair of poles its own "radial and angular coordinates") through the relations

$$\rho = \left[r_\gamma(r_\gamma - 2m_\gamma)\right]^{1/2} \sin\theta_\gamma \quad , \qquad z - z_\gamma = (\gamma_\gamma - m_\gamma)\cos\theta_\gamma \tag{5.5}$$

we get from Eq. (5.3)

$$\begin{aligned} \mu_\gamma &= 2(r_\gamma - 2m_\gamma) \sin^2 \frac{\theta_\gamma}{2} \quad , \\ \mu_{\gamma+1} &= - 2(r_\gamma - 2m_\gamma) \cos^2 \frac{\theta_\gamma}{2} \end{aligned} \tag{5.6}$$

Using these expressions for ρ and μ_k, we get from Eq. (5.1) the component $g_{00}^{(ph)}$ as the following product of n/2 factors:

$$g_{00}^{(ph)} = - (1 - 2m_1 r_1^{-1})(1 - 2m_2 r_2^{-1})\dots(1 - 2m_{n-1} r_{n-1}^{-1}) \quad . \tag{5.7}$$

For the case of the two-soliton solution Eq.(5.7) will have only one factor, the Schwarzschild expression for the coefficient $g_{00}^{(ph)}$. Calculating from Eq.(5.2) the coefficient $f_2^{(ph)}$ for this case and writing out the interval, we indeed get the standard expression for the Schwarzschild metric with radial radial coordinate r_1 and polar angle θ_1. This result also follows, of course, from the general form of the two-soliton Kerr-NUT solution, given in the preceding section (case (4.15), (4.16)) with $C_0^{(1)} = C_0^{(2)} = 0$.

To interpret the n-soliton static solution with the "potential" (5.7) we must choose a suitable radial variable. Any one of the functions $r_\gamma(\rho,z)$ could now serve as a radial coordinate, but it is most natural to define the radial variable in such a way that the dipole moment relative to it vanishes in the expansion at infinity of the Newtonian potential of the system in question. As is well known, the Newtonian potential here is $\Phi = 1 + g_{00}^{(ph)}$, and from Eq.(5.7) we have

$$\Phi = 1 - (1 - 2m_1 r_1^{-1})(1 - 2m_3 \ {}_3^{-1}) \ \ldots \ (1 - 2m_{n-1} r_{n-1}^{-1}) \ . \tag{5.8}$$

Let us try to define the "true" radial coordinate γ and polar angle θ by relations of the same form as Eq. (5.5):

$$\rho = \left[r(r - 2m)\right]^{1/2} \sin\theta \quad , \quad z - z_0 = (r - m)\cos\theta \quad , \tag{5.9}$$

but with new constants m and z_0, which are subject to definition. From Eqs.(5.9) and (5.5) we can find functions $r_\gamma(r,\theta)$ and $\theta_\gamma(r,\theta)$ and obtain their asymptotic expansions for $r \to \infty$ (in the first approximation we have for $r \to \infty$ simply $r_\gamma = r$ and $\theta_\gamma = \theta$). Substituting these expansions into Eq.(5.8), we find the expansion of the potential Φ, and from the condition that it must contain no dipole term we can determine the constants m and z_0. In this way we get

$$m = \sum_{\gamma=1}^{n-1} m_\gamma \quad , \quad z_0 = (\sum_{\gamma=1}^{n-1} m_\gamma z_\gamma)/\sum_{=1}^{n-1} m_\gamma \quad , \tag{5.10}$$

and then the expansion for Φ takes the form

$$\Phi = 2m\, r^{-1} + q(3\cos^2\theta - 1)\, r^{-3} + \ldots \quad , \tag{5.11}$$

where q is the quadrupole moment of the system. For the case of a four-soliton solution, for example, (where the index γ takes only the two values 1 and 3) we have

$$q = m_1 m_3 \left[(z_1 - z_3)^2 - m^2\right] (m_1 + m_3)^{-1} \quad .$$

These results show that the n-soliton static solution is a localized perturbation in an asymptotically flat space. For a sufficiently remote observer such a field can be regarded as an external field produced by n/2 localized axially symmetric structures, each of which has its own mass m and its center of mass lying on the axis of symmetry at the point with coordinate z. The equations (5.10) show that the total mass of all these n/2 objects (or pairs of solitons) is equal to the sum of their masses, and the coordinate z_0 of their common center of gravity is given by the usual expression of the mechanics of particles. All of the multipole moments of the field can also be expressed in definite ways in terms of the constants m_γ and z_γ.

If we now suppose that in this system there appear "rotational motions either of the whole or of the individual elements" around the axis of symmetry, the resulting case will correspond to a nondiagonal metric with $g_{01}^{(ph)} \neq 0$. In the special case of a two-soliton system considered in the preceding section, this change corresponds to the change from the Schwarzchild solution to that of Kerr. Just as in that special case, we must also here assure that the solution with n solitons is asymptotically Euclidean. In the two-soliton case this made it necessary to set the NUT parameter equal to zero. This means that the off-diagonal component $g_{01}^{(ph)}$ of the metric must decrease like r^{-1} as $r \to \infty$ (in the Kerr-NUT solution $g_{01}^{(ph)} \to b\cos\theta + O(r^{-1})$ for $r\to\infty$). Then the coefficient of r^{-1} in $g_{01}^{(ph)}$ gives the total rotational moment of the system.

It is not hard to find the behavior of the components $g_{ab}^{(ph)}$ for $r \to \infty$ in the general case of an n-soliton metric. As in the two-soliton case we must introduce the notations (4.6) for each pair of constants ω_γ, $\omega_{\gamma+1}$ and for each pair of functions μ_γ, $\mu_{\gamma+1}$ we must introduce a pair of "coordinates" r_γ, θ_γ by the formulas (4.7). After this we get from Eq.(5.3) expressions for μ_γ and $\mu_{\gamma+1}$ of the form (4.15). At infinity all of the variables r_γ, θ_γ coincide, so that it is immaterial which pair we take as spherical coordinates r, θ if we are concerned only with the first terms of the expansion for $r \to \infty$.

Now from Eq. (4.3) we get the asymptotic form of the vectors $m_a^{(k)}$, and from Eqs.(4.4) and (2.15), that of the vectors $n_a^{(k)}$. From these it is now easy to find the behavior of the components $g_{ab}^{(ph)}$. The result shows that the asymptotic behavior of the metric coefficients $g_{ab}^{(ph)}$ for $r \to \infty$ is precisely the same as in the two-soliton case:

$$g_{00}^{(ph)} \to -1 , \quad g_{11}^{(ph)} \to r^2 \sin^2 \theta , \; g_{01}^{(ph)} \to b_1 \cos \theta + b_2 + O(r^{-1}) , \tag{5.12}$$

where b_1 and b_2 are constants which can be expressed in terms of $C_0^{(k)}$ and $C_1^{(k)}$. For the metric to be asymptotically Euclidean for $r \to \infty$ the parameter b_1 must be zero, which gives a supplementary condition connecting the constants $C_a^{(k)}$:

$$b_1(C_0^{(k)}, C_1^{(k)}) = 0 . \tag{5.13}$$

After this the constant b_2 can be eliminated from the asymptotic form of the metric coefficient $g_{01}^{(ph)}$ with a linear transformation of the form $t = t' = b_2$.

APPENDIX

We shall now prove the validity of Eq.(3.4). As was already stated in Sec. 3, we have only to show that is holds for the case m+ 1, on the assumption that it holds for the case m. We suppose that we have some solution g_n, f_n, Ψ_n of our problem, and on it as background we construct a solution g_{n+m}, f_{n+m}, Ψ_{n+m} by introducing m solitons corresponding to poles $\lambda = \mu_{n+1}$, $\lambda = \mu_{n+2}$, ..., $\lambda = \mu_{n+m}$. We assume that for such a "case m" Eq. (3.4) is true, and consequently the coefficient f is of the form

$$f_{n+m} = C_{n+m}\, f_n\, \rho^m \left(\prod_{k=1}^{m} \mu_{n+k}^2\right) \left[\prod_{k=1}^{m} (\mu_{n+k}^2 + \rho^2)\right]^{-1} D_{n+m} \quad . \tag{A.1}$$

where C_{n+m} is an arbitrary constant and D_{n+m} is the determinant of the matrix $\Gamma_{n+k,n+l}$ (relative to the indices k,l = 1, 2, ..., m).

$$\Gamma_{n+k,n+l} = m_a^{(n+k)} \, (g_n)_{ab} \, m_b^{(n+\ell)} \, (\rho^2 + \mu_{n+k}\, \mu_{n+\ell})^{-1} \quad . \tag{A.2}$$

Here and for what follows we have adopted the following conventions about indices: n and m are given constants; the letters k and ℓ are used to denote running indices which go through the values 1, 2, ..., m; and the Greek letters α, β denote indices (appearing later) that go through the m + 1 values 0,1,2,...,m.

As we have already said,

$$D_{n+m} = \det \Gamma_{n+k,n+\ell} \quad . \tag{A.3}$$

The vectors $m_a^{(n+k)}$ in Eq. (A.2) are constructed according to the rule (2.14):

$$m_a^{(n+k)} = m_{c0}^{(n+k)} \left[\Psi_n^{-1} \, (\mu_{n+k}, \rho, z)\right]_{ca} \tag{A.4}$$

Let us now consider that the solution g_n, f_n, Ψ_n was obtained from another solution g_{n-1}, f_{n-1}, Ψ_{n-1} by adding to it one soliton, corresponding to the pole $\lambda = \mu_n$. In this case according to Eqs.(2.22) and (2.25) we have

$$\Psi_n = \left[I + (\mu_n^2 + \rho^2)\ \mu_n^{-1}\ (\lambda - \mu_n)^{-1}\ P_n\right] \Psi_{n-1}$$

$$\Psi_n^{-1} = \Psi_{n-1}^{-1} \left[I - (\mu_n^2 + \rho^2)(\rho^2 + \lambda\mu_n)^{-1}\ P_n\right] \quad . \tag{A.5}$$

$$g_n = \Psi_n(0) = \left[I - (\mu_n^2 + \rho^2)\ \mu_n^{-2}\ P_n\right] g_{n-1} \quad . \tag{A.6}$$

The matrix P_n is constructed from Ψ_{n-1} and g_{n-1} according to the law (2.23):

$$P_n = \ell_c^{(n)}\ (g_{n-1})_{ca}\ \ell_b^{(n)}\ /\ \ell_f^{(n)}(g_{n-1})_{fd}\ l_d^{(n)} \quad . \tag{A.7}$$

where the vector ℓ_a is given by the expression

$$\ell_a^{(n)} = \ell_0^{(n)} \left[\Psi_{m-1}^{-1}\ (\mu_n, \rho,\ z)\right]_{ca} \quad . \tag{A.8}$$

Besides the vector $\ell_a^{(n)}$ we need the vectors $\ell_a^{(n+k)}$ $(k = 1, 2, \ldots, n)$, which are given by

$$\ell_a^{(n+k)} = m_{c0}^{(n+k)} \left[\Psi_{n-1}^{-1}(\mu_{n+k},\ \rho,\ z)\right]_{ca} \quad . \tag{A.9}$$

where $m_{c0}^{(n+k)}$ are the same arbitrary constants as appear in Eq. (A.4).

Now from Eq.(A.4), (A.5) and (A.7) we can obtain an expression for the vectors $m_a^{(n+k)}$ in terms of the vectors $\ell_a^{(n)}$ and $\ell_a^{(n+k)}$:

$$m_a^{(n+k)} = l_a^{(n+k)} - (E_{n,n})^{-1}\ E_{n,n+k}\ \ell_a^{(n)} \quad , \tag{A.10}$$

where we have introduced the matrix $E_{n+\alpha,n+\beta}(\alpha,\beta = 0,1,2,\ldots,m)$:

$$E_{n+\alpha,n+\beta} = l_c^{(n+\alpha)}\ (g_{n-1})_{cb}\ l_b^{(n+\beta)}\ (\rho^2 + \mu_{n+\alpha}\ \mu_{n+\beta})^{-1} \quad . \tag{A.11}$$

Then, substituting Eqs.(A.10) and (A.6) in (A.2), we find an expression for the matrix $\Gamma_{n+k,n+1}$ in terms of the matrix $E_{n+\alpha,n+\beta}$:

$$\Gamma_{n+k,n+1} = E_{n+k,n+1} - (E_{n,n})^{-1}\ E_{n,n+k}\ E_{n,n+1} \quad . \tag{A.12}$$

From Eq.(A.12) it follows that the determinants of the matrices $E_{n+\alpha,n+\beta}$ and $\Gamma_{n+k,n+1}$ are connected by the relation

$$\det E_{n+\alpha,n+\beta} = E_{n,n}\ \det \Gamma_{n+k,n+1} \quad . \tag{A.13}$$

Now from Eqs.(3.1) and (3.2) we get a connection between f_n and f_{n-1}:

$$f_n = C_n\ f_{n-1}\ E_{n,n}\ \rho\mu_n^2\ (\mu_n^2 + \rho^2)^{-1} \tag{A.14}$$

(C_n is an arbitrary constant). Substituting this expression in Eq. (A.1) and using Eqs. (A.3) and (A.13), we get

$$f_{n+m} = \text{const}\ .\ f_{n-1}\ \rho^{m+1}\ \left(\prod_{\alpha=0}^{m} \mu_{n+\alpha}^2\right) \left[\prod_{\beta=0}^{m} (\mu_{n+\beta}^2 + \rho^2)\right]^{-1} \det E_{n+\alpha,n+\beta} \quad . \tag{A.15}$$

This result, together with the expressions (A.8), (A.9), and (A.11) for the matrix $E_{n+\alpha,n+\beta}$ is nothing other than the formula (3.4) itself, except that it is for the "case m+1", with the solution g_{n+m}, f_{n+m}, Ψ_{n+m} being obtained from g_{n-1}, f_{n-1}, Ψ_{n-1} by adding m + 1 solitons to the latter. This analysis completes the proof that Eq. (3.4) is valid.

II.6 - REFERENCES

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III - ONE-SOLITON COSMOLOGICAL WAVES

III.1 - INTRODUCTION

The method of inverse solution of the scattering problem has been used by us (1) to describe a procedure for integrating the gravitational equations for the case of a metric tensor depending on only two variables. The metric we used was written in the form (*)

$$- ds^2 = f(- dt^2 + dz^2) + g_{ab}\, dx^a\, dx^b \quad , \tag{1.1}$$

where the functions f and g_{ab} depend on the coordinates t and z. Our notation for the coordinates is $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. The first Latin letters a and b always run through the values 1 and 2 and refer to the coordinates x and y. The Latin indices i and k, which occur later, refer to four-dimensional space and take the values 0, 1, 2, 3.

In the previous paper (1) we considered the Einstein equations corresponding to the interval (1.1) only in empty space. The application of a similar method to the integration of these equations in a space filled with matter is as yet an unsettled question. Meanwhile the solutions belonging to the class of metrics (1.1) include such fundamental exact solutions as the Friedmann cosmological models, for which the presence of matter is essential. It would certainly be interesting to construct new exact cosmological solutions describing the evolution of finite disturbances such as gravitational solitons, appearing against the background of a Friedmann space. For the reason we have noted, this cannot at present be done in general form.

There is, however, one special case in which the method already described (1) can still be applied even in a space with matter. This is the case of an ideal fluid with the "superrigid" equation of state $\varepsilon = p$, proposed by Zeld'ovich (2). The specific form of this equation of state will not play any decisive part in our work, since we shall deal with soliton perturbations of the gravitational field itself, not of the matter, which remains unperturbed in our solutions. From this point of view the matter serves only for the provision and maintenance of the Friedmann background solution, and it can be hoped that the qualitative picture of the behavior of gravitational solitons on this background will remain approximately the same for other equations of state. Besides this, exact solutions of the Einstein equations, analogous (in the sense that the behavior of the metric coefficients g_{ab} remains the same in them) to those obtained here for a space with matter, exist also in vacuum. The way they are found in the general case is described in Sec. 2, and the actual construction is given in Sec. 4.

In this paper we shall consider one-soliton solutions on the background of Friedmann models of all three types. Let us point out their main qualitative peculiarities. These solutions are inhomogeneous cosmological models, in which the distribution of the gravitational field at the initial time shows a clearly expressed maximum with respect to the spatial coordinates near some axis in three-dimensional space. During the expansion of the world this disturbance dies away, and after some finite interval of time it produces a gravitational wave moving away from the axis, with an amplitude decreasing with time. Accordingly, open models, during the

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(*) A system of units is used in which the speed of light and the gravitational constant are equal to unity. The interval is written in the form $- ds^2 = g_{ik}\, dx^i\, dx^k$, where g_{ik} has the signature $(-, +, +, +)$.

final stages of the infinite expansion, go over into Friedmann models. In the closed model, this process of homogenization (and also of isotropization) continues only up to the moment of maximum expansion. During the stage of contraction of the world the fractional perturbation of the gravitational field increases again, and at the final moment of the evolution it is again concentrated on an axis in three-dimensional space. In the open models this axis is topologically equivalent to an infinite straight line, and the soliton disturbance possesses cylindrical symmetry relative to it. In the closed space this assertion retains its meaning only locally, since the axes on which the soliton is concentrated at the initial and final times are circumferences of great circles of the thrre-dimensional spherical space of the Universe. Furthermore, the initial and final circles do not coincide and nowhere have any common points, being disposed normal to each other.

Another peculiarity of these solutions is the very possibility of treating them as perturbation of Freidmann models, since these solutions reduce, by a continuous limiting procedure with respect to an arbitrary constant parameter, to Friedmann metrics. This property is not completely trivial, since one-soliton solutions do not admit limiting reduction with respect to parameters taking them directly to the metric on whose background they are constructed by the method expounded in the previous paper (1). In the case studied here the one-soliton solution is close, not to the original background model, but to an exact copy of it, which can be obtained by a discrete symmetry transformation and can be regarded as a different specimen of the same solution on a different physical sheet. This is discussed in more detail in the Appendix. This interpretation means that after obtaining the final form of a one-soliton solution we forget about the method by which it was derived, and take as the background solution the one that is obtained by the appropriate passage to a limit.

The solutions obtained depend on two arbitrary constant parameters. Depending on the regions of variation of one of these parameters all solutions can be divided into two classes. Half of the solutions contain no singularities other than the usual cosmological singularities with respect to the time, which are already contained in the background solution itself. This fact, together with the existence of the limiting transition with respect to the parameters to the background Friedmann models makes this set of solutions extremely satisfactory from the physical point of view; they describe perturbations of the Friedmann models which are finite (but with an infinitesimal case) and every-where regular. The other half of the solutions, in addition to the background cosmological singularities, have discontinuities of the energy density of the matter and of the first derivatives of the metric coefficients on the light cone. The existence of such discontinuities in one-soliton solutions was already pointed out in Ref. 1. We emphasize that everything we have said about limiting transitions to background solution, and about what is to be taken as being a background solution, relates only to the first set of regular solutions. We shall not consider the case of the discontinuous solutions in this paper.

Solutions with the indicated properties describe one possible mechanism for the production of gravitational waves of cosmological origin. Their sources are inhomogeneities of the gravitational field near the initial cosmological singularity and the dynamics of these inhomogeneities during the further expansion of space. In the course of time the inhomogeneities disappear (at least in open models), but they leave behing a trace in the form of decaying waves which still exist for some time in the universe at later stages of its evolution . This entire process, however - the appearance of an inhomogeneity, its prewave stage, and its final product, a gravitational wave, makes up a single whole, the evolution of a gravitational soliton. In the present case we are dealing with solitons that have cylindrical symmetry, and we cannot call them localized disturbances in the ordinary sense of the word. Nevertheless, the existence of this example allows us to suppose that analogous phenomena can occur with three-dimensional perturbations against the background of uniform cosmological models.

III.2 - THE GRAVITATIONAL EQUATIONS AND THE FRIEDMANN BACKGROUND MODELS

If the matter filling space is an ideal fluid with the equation of state $\varepsilon = p$, its energy-momentum tensor is

$$T_{ik} = 2\varepsilon u_i u_k + \varepsilon g_{ik} \quad , \quad u^i u_i = -1 \quad , \tag{2.1}$$

and the Einstein equations take the form

$$R_{ik} = 2\varepsilon u_i u_k \quad . \tag{2.2}$$

Since for the metric form (1.1) the components R_0a and R_3a of the Ricci tensor are identically equal to zero, it follows from Eq. (2.2) that the velocity components u_a must also be equal to zero. It can be seen from this that the main part of the Einstein equations, which determines the matrix components g_{ab}, has the form $R_{ab} = 0$, and is thus the same as in vacuum. For this reason the method developed in Ref. 1 can still be applied in the present case.

It is not hard to show that with the use of gravitational hydrodynamics we can, without limiting the generality of our solution, express the matter field in terms of a single scalar function ϕ, which we call the fluid potential:

$$\varepsilon = p = -\frac{1}{2} \phi_{,k} \phi^{,k} \quad , \quad u_i = (2\varepsilon)^{-1/2} \phi_i \quad . \tag{2.3}$$

The Einstein equations and the equations of hydrodynamics now become the following system:

$$R_{ik} = \phi_{,i} \phi_{,k} \quad , \quad \phi_{;k}^{;k} = 0 \quad . \tag{2.4}$$

The possibility of this representation of an ideal liquid with the equation of state $\varepsilon = p$ was noted in Ref. 3. The fact that the components u_a of the velocity are zero means that the potential is a function of only two variables, t and z. Denoting differentiation with respect to t with a dot, and that with respect to z with a prime, we get from Eqs. (2.3) and (1.1):

$$\varepsilon = p = (2f)^{-1} (\dot\phi^2 - \phi'^2) \quad , \tag{2.5}$$

$$u_0 = (2\varepsilon)^{-1/2} \dot\phi \quad , \quad u_3 = (2\varepsilon)^{-1/2} \phi' \quad , \quad u_a = 0$$

As in our earlier paper (1), we shall denote by g a two-rowed matrix g_{ab}, and for its determinant and derivatives we introduce the notations

$$\det g = \alpha^2 \quad , \quad A = -\alpha g_{\zeta} g^{-1} \quad , \quad B = \alpha g_{\eta} g^{-1} \quad , \tag{2.7}$$

where the comma indicates ordinary differentiation and instead of t and z we have introduced the light variables ζ and η :

$$t = \zeta - \eta \quad , \quad z = \zeta + \eta \quad . \tag{2.8}$$

If we now write the metric coefficient F as a product

$$f = f_v F \quad , \tag{2.9}$$

it is easy to show that the equations (2.4) can be divided into four groups. The first and second of them exactly repeat the Einstein equations in vacuum for the metric

$$- ds^2 = f_v (- dt^2 + dz^2) + g_{ab} dx^a dx^b \quad . \tag{2.10}$$

These equations can be written (cf.Ref.1) in the form

$$(\alpha g_{\zeta} g^{-1})_{\eta} + (\alpha g_{\eta} g^{-1})_{\alpha} = 0 \quad , \tag{2.11}$$

$$(\ln f_v)_{\zeta} = (\ln \alpha)_{\zeta\zeta}/(\ln \alpha)_{\zeta} + (\mathrm{Sp}\, A^2)/4\alpha\alpha_{\zeta} \quad , \tag{2.12}$$

$$(\ln f_v)_{\eta} = (\ln \alpha)_{\eta\eta}/(\ln \alpha)_{\eta} + (\mathrm{Sp}\, B^2)/4\alpha\alpha_{\eta} \quad . \tag{2.13}$$

The third group is just a wave equation for the potential ϕ:

$$(\alpha\phi_\zeta)_\eta + (\alpha\phi_\eta)_\zeta = 0 \tag{2.14}$$

and the fourth group determines the factor F which corrects for the matter

$$(\ln F)_\zeta = \phi_\zeta^2/(\ln \alpha)_\zeta \quad , \qquad (\ln F)_\eta = \phi_\eta^2/(\ln \alpha)_\eta \tag{2.15}$$

It follows from Eqs. (2.10) and (2.7) that the function α satisfies the usual wave equation as before:

$$\alpha_{\eta\zeta} = 0 \quad . \tag{2.16}$$

With this condition the equations (2.15) are automatically compatible if ϕ satisfies Eq. (2.14).

Accordingly, we see that to solve the problem we must first integrate Eqs. (2.11)-(2.15), thus constructing some exact solution of the Einstein equations with the metric (2.10). This part of the problem has already been studied in Ref. 1. After this we must determine the fluid potential ϕ from Eq. (2.14) and with it find from Eq. (2.15) the coefficient F. Substituting this in Eq. (2.9), we get the desired metric (1.1), and the potential ϕ determines the energy density and the components of the velocity of the matter in accordance with the relations (2.5) and (2.6).

In the framework of the metric (1.1) we must now determine the Friedmann solutions. The standard forms for these, when four-dimensional spherical coordinates are used, contain a dependence on two space coordinates, while the interval (1.1) assumes a dependence on only one space variable. However, there exists a transformation of the three-dimensional coordinates which allows us to reduce the Friedmann solution to the form (1.1). This transformation (found for a different reason) is given in Appendix D of Ref. 4, and here we need only a special case of the result. The element of length in three-dimensional space in the closed model is given by the expression

$$d\ell^2 = a^2 \, (d\chi^2 + \sin^2 \chi \sin^2 \theta \, d\phi^2 + \sin^2 \chi \, d\theta^2) \quad , \tag{2.17}$$

where the variables χ, θ, ϕ range over the limits $0 \le \chi \le \pi$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. The transformation

$$\sin z = \sin \chi \sin \theta , \quad \cos z \sin y = \sin \chi \cos \theta , \quad x = \phi \tag{2.18}$$

reduces (2.17) to the following form:

$$d\ell^2 = a^2 \, (dz^2 + \sin^2 z dx^2 + \cos^2 z dy^2) \quad , \tag{2.19}$$

in which the ranges of variation of the coordinates are $0 \le z \le \pi/2$, $0 \le x \le 2\pi$, $-\pi \le y \le \pi$. The three-dimensional line element of the open space is described by the expression

$$d\ell^2 = a^2 \, (d\chi^2 + sh^2 \chi \sin^2 \theta \, d\phi^2 + sh^2 \chi \, d\phi^2) \tag{2.20}$$

with the following ranges for the coordinates: $0 \le \chi \le \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. The transformation analogous to Eqs. (2.18) is

$$sh \, z = sh \, \chi \sin \theta \quad , \quad ch \, z \, sh \, y = sh \, \chi \cos \theta \, , \, x = \phi \quad , \tag{2.21}$$

and this reduces Eq. (2.20) to the form

$$d\ell^2 = a^2 \, (dz^2 + sh^2 \, z dx^2 + ch^2 \, z dy^2) \quad , \tag{2.22}$$

where the coordinates vary in the range $0 \le z < \infty$, $0 \le x \le 2\pi$, $-\infty < y < +\infty$. We choose the three-dimensional length element of the planar space in the form

$$d\ell^2 = a^2 \, (dz^2 + z^2 \, dx^2 + dy^2) \tag{2.23}$$

and assume that in this expression $0 \le z \le \infty$, $0 \le x \le 2\pi$, $-\infty \le y \le +\infty$, so that the variables z, x, y form an ordinary cylindrical coordinate system.

With the use of Eqs. (2.23), (2.22) and (2.19) it is now easy to establish the form of the Friedmann solutions in the framework of the metric (1.1). For the flat model we have

$$- ds^2 = t(- dt^2 + dz^2 + z^2 dx^2 + dy^2) \quad , \quad \phi = (\tfrac{3}{2})^{1/2} \ln t, \quad \varepsilon = \tfrac{3}{4} t^3 \quad , t \ge 0 \quad .(2.24)$$

For the open model

$$-ds^2 = a_0^2 \mathrm{sh}\, 2t(-dt^2+dz^2+\mathrm{sh}^2 z dx^2+\mathrm{ch}^2 z dy^2) \ , \ \phi = (\tfrac{3}{2})^{1/2} \ln \mathrm{th}\, t, \ \varepsilon = 3a_0^{-2}\mathrm{sh}^{-3} 2t, \quad t \ge 0 \ . \quad (2.25)$$

And, finally, for the closed model we have

$$-ds^2 = a_0^2 \sin 2t(-dt^2+dz^2+\sin^2 z dx^2+\cos^2 z dy^2) \ , \ \phi = (\tfrac{3}{2})^{1/2} \ln \mathrm{tg}\, t, \ \varepsilon = 3a_0^{-2}\sin^{-3} 2t, \ 0 \le t \le \pi/2. (2.26)$$

In the last two solutions a_0 is an arbitrary constant. For simplicity the analogous constant in Eq. (2.24) has been given a fixed value.

To obtain the soliton solutions with the models (2.24) - (2.26) as backgrounds, it is necessary, in accordance with the procedure described in Ref. 1, that we now determine the wave matrix $\Psi(\lambda, t, z)$ corresponding to these metrics; after this, the construction of the solutions reduces to mere algebraic operations. It turns out that for all three models the LA equations can be integrated rather simply and the matrix Ψ can be expressed in terms of elementary functions. The details are given in the Appendix, and in what follows we give only the final forms of the resulting expressions, so that if the reader is not interested in the way they are found there is no need to refer to the Appendix or to the previous paper (1).

In concluding this section we recall that in accordance with the discussion in the Introduction we are considering only the solutions that are associated with a perturbation of the gravitational field. The matter potential ϕ remains unperturbed in our models, although there is no difficulty in obtaining, by applying precisely the same technique to Eqs. (2.14)-(2.15), exact solutions containing along with the gravitational fields also the soliton fields ϕ.

III.3 - SOLITON SOLUTIONS ON BACKGROUND OF FRIEDMANN FIELDS

The one-soliton solution on the background of the flat model (2.24) is

$$\begin{aligned} - ds^2 = {} & \ell^2 s^{-2}\, t \left[s^2t^2 + (t^2 + \mu)^2\right]\left[\ell^2t^2 + (t^2 + \mu)^2\right]^{-1} (- dt^2 + dz^2) + \\ & + t\left[s^2t^2 + (t^2 + \mu)^2\right]^{-1}\{\left[s^2t^2z^2 + \ell^2 (t^2 + \mu)^2 + qz^2(t^2 + \mu) - q^2\mu\right] dx^2 + \\ & + \left[s^2t^2 + (t^2 + \mu)^2 - q(t^2 + \mu)\right] dy^2 + 2qs\, \mu\, dx\, dy\} \ . \end{aligned} \quad (3.1)$$

Here the quantities s, ℓ, and q are arbitrary constants, related to each other by the equation

$$q = s^2 - l^2 \quad (3.2)$$

The quantity μ is a function of the coordinates and is given by the expression

$$\mu = - \tfrac{1}{2} \ (l^2 + t^2 + z^2) + \tfrac{1}{2} \ \left[(\ell^2 + t^2 + z^2)^2 - 4\, t^2\, z^2\right]^{1/2} \quad . \quad (3.3)$$

Here the second term contains the arithmetic value of the root (*).

(*) We impose this condition only for definiteness. The opposite sign of the square root in Eq. (3.3) leads to the same physical results. The same is true for the function μ in the expressions (3.25) (see further discussion).

We note that the determinant of the matrix g found from Eq. (3.1) is of the same form as in the background solution (2.24): $\det g = \alpha^2$, where $\alpha = tz$. The fluid potential for this solution also retains the unperturbed form

$$\phi = (\tfrac{3}{2})^{1/2} \ln t \quad , \tag{3.4}$$

so that the matter is stationary ($u_3 = 0$). The energy density can be found easily from Eq. (2.5):

$$\varepsilon = \tfrac{3}{4}\, s^2 \ell^{-2} t^{-3} \left[\ell^2 t^2 + (t^2 + \mu)^2\right] \left[s^2 t^2 + (t^2 + \mu)^2\right]^{-1} \quad . \tag{3.5}$$

The deviation of this value from the background value is due only to the perturbation of the metric (the metric coefficient f) and not to a perturbation of the matter field as such.

From these formulas we see that if we let the parameter q go to zero ($s^2 \to \ell^2$) the solution goes over into the background, Eq. (2.24). We now determine the field of the soliton as the precise deviation of the metric from its background value. This field can be described with a symmetrical perturbation matrix H, which is constructed according to exactly the same rule as in the infinitesimal case:

$$H_{11} = (g_{11} - g_{11}^{(0)})\,(g_{11}^{(0)})^{-1} \quad , \qquad H_{22} = (g_{22} - g_{22}^{(0)})\,(g_{22}^{(0)})^{-1} \quad , \tag{3.6}$$

$$H_{12} = H_{21} = g_{12}(g_{11}^{(0)}\, g_{22}^{(0)})^{-1/2} \quad , \tag{3.7}$$

where the quantities with superscript zero relate to the background solution (2.24).

Besides the matrix H, the soliton is also characterized by the perturbation of the metric coefficient f. It is more convenient, however, to consider instead of this the perturbation of an equivalent quantity, the energy density ε, for which we write

$$E = (\varepsilon - \varepsilon_0)\ \varepsilon_0^{-1} \quad . \tag{3.8}$$

From Eqs. (3.1), (3.5), and (2.24) we get:

$$H = q\left[s^2 t^2 + (t^2 + \mu)^2\right]^{-1} \begin{pmatrix} t^2 + \mu - q\mu z^{-2} & s\mu z^{-1} \\ s\mu z^{-1} & -t^2 - \mu \end{pmatrix} , \tag{3.9}$$

$$E = q\ell^{-2}\,(t^2 + \mu)^2 \left[s^2 t^2 + (t^2 + \mu)^2\right]^{-1} \quad . \tag{3.10}$$

Let us examine the behavior of these quantities near the moment $t = 0$ of the initial cosmological singularity. It is easy to show that the first nonvanishing terms of the matrix H for $t \to 0$ (and arbitrary values of z) are given by the expression

$$H = qs^{-2}\,(\ell^2 + z^2)^{-1} \begin{pmatrix} s^2 & -sz \\ -sz & -\ell^2 \end{pmatrix} \tag{3.11}$$

and for the quantity E the first nonvanishing term is

$$E = q^2 s^{-2} t^2\,(\ell^2 + z^2)^{-2} \quad . \tag{3.12}$$

It can be seen from Eq. (3.11) that the field of the perturbation H is concentrated, during the first few moments of the evolution, near the axis $z = 0$ of the axial symmetry, in a cylindrical volume with the characteristic radius $z \sim 1$. The components H_{11} and H_{22} have extrema with respect to the variable z right on the axis $z = 0$, and H_{12} has extrema at distance $z = \ell$ from the axis. The perturbation of the energy density (i.e., the metric coefficient f) is proportional to t^2 in the first nonvanishing approximation, and is already of the next order of smallness as compared with the main terms of the expansion of the matrix H. Nevertheless, as can be seen from Eq. (3.12), the distribution of the quantity E with respect to z also localized on the axis z = 0 with the characteristic width z ~ 1.

For simplicity we will suppose that the constants s and l are of the same order of magnitude. Then there is a single characteristic length ℓ in the solution, and the asymptotic expressions (3.11) and (3.12) are the first terms of the expansion of the solution in powers of t/ℓ in the region where $t \leq \ell$. As in the time t increases we come to the region $t >> \ell$, in which all the components of the matrix H go to zero for $t \to \infty$. However, the laws of this dying away are different for points located near the light cone z = t and far from it. If $t >> \ell$, and $z < t$, then we get from Eqs. (3.3) and (3.9) the following assymptotic expression for H:

$$H = q\,(t^2 - z^2)^{-1} \begin{pmatrix} 1 & -sz(t^2-z^2)^{-1} \\ -sz(t^2-z^2)^{-1} & -1 \end{pmatrix} , \qquad (3.13)$$

from which it can be seen that both near z = 0 and also at any other fixed point in space the perturbation field falls off for t according to the law $H_{11} \sim H_{22} \sim \ell^2 t^2$, $H_{12} \sim \ell^4 t^4$. On the light cone the expressions (3.13) diverge, but this is due only to the fact that they cease to be applicable when we get into the strip $t - z \lesssim 1$ adjacent to the light line z = t. The behavior of the matrix H inside this strip can be estimated by determining its asymptotic behavior on the cone z = t itself for $t >> \ell$. The main term for this is easily found from Eqs. (3.3) and (3.9):

$$H = q\,(s^2 + 1^2)^{-1}\, t^{-1} \begin{pmatrix} 1 & -s \\ -s & -1 \end{pmatrix} \qquad (3.14)$$

Thus we see that for any given time $t >> \ell$ the amplitude of the perturbation H at points of the light cone is of the order of ℓt^{-1} and is very large in comparison with its values at other points of space (where the components of H are of orders $1^2 t^{-2}$ and $1^4 t^{-4}$). This means that the initial perturbation H, which for $t \to 0$ was concentrated near the axis $z \to 0$ with characteristic dimension z ~ 1, while decreasing with time produces in the later stages a gravitational wave moving out from the axis with the speed of light. The amplitude of this wave also decreases with time, and the field distribution in it, concentrated on the light cone, has the same characteristic width $\delta z \sim \ell$ as the initial cosmological perturbation of the metric. It must be remembered, however, that these assertions, as always, have only an approximate meaning. Actually the quantities H and E contain, besides the wave part, perturbations relating to the background geometry, and it is hard to give an exact meaning to each of these effects by itself. This fact is well illustrated in an analysis of the relative perturbation E. At the moment when the evolution begins this quantity is vanishingly small and is given by the expression (3.12). For $t >> \ell$, in the region $z > t$ the approximation for E is

$$E = q\ell^2\, s^{-2}\, t^2\, (t^2 - z^2)^{-2} \quad . \qquad (3.15)$$

At points of the light cone z = t we have for $t >> \ell$

$$E = q\,(s^2 + \ell^2)^{-1} + O\,(\ell t^{-1}) \qquad (3.16)$$

and finally, in the region $t > z$ with $t >> \ell$ we get

$$E = q\ell^{-2} - q\ell^{-2}\, s^2\, t^2\, (t^2 - z^2)^{-2} \quad . \qquad (3.17)$$

It can be seen from these expressions that the fractional perturbation of the metric coefficients f, and along with it the energy density ε, remain small only at the initial moment of the evolution and at the points of space where $z >> t$, i.e., in regions not yet reached by the gravitational wave. In regions $t > z$, through which the wave was already passed, there remains a final decreasing perturbation $E = q\ell^2 \sim 1$, which reduces a change of the constant parameters of the background Friedmann solution.

Accordingly, in the final stages of the expansion for $t \to \infty$ we have instead of Eq. (2.24) the following asymptotic behavior:

$$- ds^2 = {}^2 s^{-2}(- dt^2 + dz^2) + \ z^2dx^2 + dy^2), \qquad (3.18)$$

$$\varepsilon = \frac{3}{4}\ s^2\ 1^{-2}\ t^{-3} \quad . \qquad (3.19)$$

This phenomenon illustrates the interaction between the wave and background parts of the solution. It may be possible to speak here of an exchange of energy between the gravitational wave and the background, but this would require that one give some satisfactory definition of these concepts.

Finally, we must discuss the physical meaning of the arbitrary constants contained in the solutions (3.1) - (3.5). The foregoing analysis has shown that the constant ℓ is the characteristic width of the initial distribution of the soliton field. After this we can associate the constant s with the amplitudes of this distribution. A different, and not less clear, physical meaning of the constant s can be obtained if we examine in more detail the development in time of the profile of the component H_{22} of the perturbation. For $t \to 0$ the shape of this profile follows from Eq. (3.11).

Let us now determine the extrema of H_{22} with respect to the variable z at an arbitrary time t, by considering the equation $\partial H_{22}/\partial z = 0$. It is easy to show that this equation has two solutions: one of them is $z = 0$, independently of the time t (which corresponds to the smooth behavior of H_{22} on the axis of symmetry), and the second solution gives the following world line: $z^2 = s^{-1}(st + \ell^2)(t - s)$ (we assume $s > 0$; otherwise the formula must be written with s replaced with $|s|$). It follows that there is a second extremum on the profile of H_{22}, but it appears only after a finite time interval $t = s$ after the beginning of the evolution. Up to the time $t = s$ the distribution of H_{22} with respect to z has a smooth nature as in Eq. (3.11). After the time $t = s$ the world line of the second extremum(*) moves out toward increasing values of the coordinate z, and for $t \to \infty$ it asymptotically approaches the light line $z = t - q/2s$. This means that the time $t = s$ marks the beginning of the wave stage of the evolution of the soliton, i.e., the generation of the gravitational wave. Thus the variable s has the meaning of the delay time, or the time of embryonic development of the wave. The pattern of the behavior of the component H_{22} is shown in Fig. 1.

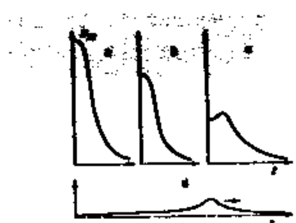

FIGURE 1 - Behavior in time of the profile of the absolute value of the perturbation H_{22} in the flat model. The sequence of plots corresponds to increasing values of the time t: (a) distribution of the perturbation at the time the evolution starts, $t \to 0$; (b) the initial perturbation begins to die away; (c) profile near the critical time $t - s$, beginning of production of the wave; (d) the wave recedes to infinity with the speed of light, its amplitude decreasing as $|q/2\ st|$.

(*) Throughout its entire extent this world line remains spacelike and corresponds to the phase velocity of propagation of the wave. The physical velocity of the wave is equal to the speed of light for large times, and in other regions it is not uniquely definable. The values of the quantity H_{22} at the extremal points $z^2 = s^{-1}(st + \ell^2)(t - s)$ are given by the simple expression $H_{22} = - q/2\ st$, from which it is apparent that as t increases this extreme value decays in the same way as the field H on the light cone $z = t$, Eq. (3.14).

Let us now pass on to the one-soliton solutions with the open and closed Friedmann models as backgrounds. These metrics can be written in the following unified form:

$$- ds^2 = a_0^2 \, rs^{-2} \, k^{-1} \sin 2\,kt \; QL^{-1} \; (- dt^2 + dz^2) +$$

$$+ (2k)^{-1} \sin 2\,ktQ^{-1} \Big[(2k^{-2}a_0^2 L \sin^2 kz + \sigma\mu \cos^2\gamma + \sigma s^{-2} \, \mu R \sin^2 \gamma) \, dx^2 +$$

$$+ s^2 \, r^{-1} \, (2a_0^2 \, L \cos^2 kz - \sigma k^2 \, \mu \sin^2 \gamma - \sigma k^2 s^{-2} \, \mu R \cos^2 \gamma) \, dy^2 \; +$$

$$+ \sigma k^2 \, s^{-1} \, \mu(R - s^2) \cos \; 2\gamma \; dx \, dy\Big] \quad , \qquad (3.20)$$

$$Q = s^2 \sin^2 kt + R \cos^2 kt \quad , \quad L = r \sin^2 kt + R \cos^2 kt \quad . \qquad (3.21)$$

The fluid potential is of the form

$$\phi = (\tfrac{3}{2})^{1/2} \ln \, (k^{-1} \, \mathrm{tg} \, kt) \qquad (3.22)$$

and from Eq. (2.5) we get the energy density

$$\epsilon = 3a_0^{-2} \, s^2 \, r^{-1} \, k^3 \, LQ^{-1} \sin^{-3} 2\,kt \quad . \qquad (3.23)$$

In these formulas a_0, s, k, σ, r, and γ are arbitrary constants, connected by two relations:

$$r = - k^{-2} \, \mathrm{tg}^2 \, 2\gamma \quad , \quad \sigma = s^2 - r \quad . \qquad (3.24)$$

The quantities μ and R are functions of the coordinates given by

$$\mu = a_0^2 (2k^2)^{-1} \{\cos 2\gamma + \cos 2kt \cos 2kz - \Big[(\cos 2\gamma + \cos 2kt \cos 2kz)^2 - \sin^2 2kt \sin^2 2kz\Big]^{1/2}\} \qquad (3.25)$$

$$R = k^6 a_0^{-} \cos^{-2} 2\gamma \Big[a_0^2 k^{-4} \mathrm{tg}^2 kt (\cos 2\gamma - \cos 2kz) - \mu k^{-2} \cos^{-2} kt\Big]^2 \quad , \qquad (3.26)$$

where the square bracket taken to the power in $\frac{1}{2}$ in Eq. (3.25) means the arithmetic root.

The solution depends on two essentially new constant parameters, γ and s. The constant a_0 is of the same nature as in the background models (2.25) and (2.26), and the constant k (if $k \neq 0$) can be eliminated by a transformation of the constants and a scale transformation $(kt, kz) \to (t, z)$. The choice of the constant k determines the type of the model. For real values of k (in this case we can take $k = 1$) the solution describes the evolution of a soliton on the background of the closed Friedmann model. For imaginary k (here we can set $k = i$) we get the analogous solution on the background of the open model, and the case $k = 0$ reduces to the soliton perturbation of the flat model, which we have already discussed. It is not hard to carry out the passage to the limit $k \to 0$, by setting $a_0^2 = \frac{1}{2}$ and renaming the constants in the following way:

$$s = 2s' , \quad \sigma = 4q, \quad \cos 2\gamma = -1 -2 \, k^2 \ell^2 \quad ,$$

where the constants s', l, and q are to be regarded as independent of k. In this case we get from Eqs. (3.24) - (3.26) in the limit $k \to 0$:

$$r = 4\ell^2, \quad q = s'^2 - l^2, \quad R = 4k^2 \, (t^2 + \mu)^2 \quad ,$$

and for the function μ we get the formula (3.3), if we consider $k^2 < 0$ (for $k^2 > 0$ we get a result analogous to Eq. (3.3), but with the minus sign for the square root). It can now be verified that the limit of the metric (3.20) for $k \to 0$ exists and can be reduced by a simple transformation to the form (3.1), in which s' will appear instead of s.

It is easy to see that the solution (3.20) - (3.26) goes over into the background solutions (2.24) - (2.26) through taking the limit with respect to the parameter σ ($\sigma \to 0$). As has already been pointed out, we shall consider here only those regions of variation of the arbitrary constants in which our solutions have no additional singularities beyong the initial cosmological singularities that are already present in the background models. The solution (3.20) - (3.26) in fact has this property if the constant r is positive:

$$r = - k^{-2}\, tg^2\, 2\gamma > 0 \quad . \tag{3.27}$$

This condition means that for the closed model (real k) we must choose a purely imaginary γ, and for the open model vice versa: imaginary k, but real γ. The constant s is always real, and consequently, with the condition $r > 0$ the parameter $\sigma = s^2 - r$ can in fact go to zero. It is easy to see that for $\sigma = 0$ the metric gives

$$- ds^2 = a_0^2\, k^{-1} \sin 2kt\, (-dt^2 + dz^2 + k^{-2} \sin^2 kz\, dx^2 + \cos^2 kz\, dy^2) \quad , \tag{3.28}$$

and it follows from Eq. (3.23) that

$$\varepsilon = 3a_0^{-2}\, k^3 \sin^{-3} 2kt \quad . \tag{3.29}$$

For $k = 0$ ($a_0^2 = 1/2$), $k = i$, and $k = 1$ the form (3.28) becomes identical with the respective metrics (2.24), (2.25), and (2.26). There is similar agreement for the potential ϕ and the energy density ε.

We note also that the determinant of the matrix g, i.e., of the two-row block g_{ab}, has the following simple form:

$$\det g = \alpha^2 \quad , \quad \alpha = a_0^2\, (2k^2)^{-1} \sin 2kt \sin 2kz \quad , \tag{3.30}$$

and, as can be seen from Eq. (3.28), remains the same as in the unperturbed background metrics.

For the case of the open model the solution (3.20) - (3.26) describes approximately the same pattern of evolution of the soliton as is found in the flat model. With the closed model, on the other hand, there are naturally qualitative differences because there are no infinite values for either the time or the space coordinates. For this reason we confine ourselves here to closed model only. In all further formulas the parameter k is regarded as real, and the parameter γ, as imaginary. In the closed space the evolution of the model occupies a finite time interval from the moment $kt = 0$ (the big bang) the time of collapse of the Universe, $kt = \pi/2$. It is not hard to show that near the initial instant $kt \to 0$ the asymptotic form of the solution (3.20) - (3.26) is

$$\begin{aligned}
g_{11} &= a_0^2 k^{-3} \sin 2kt \sin^2 kz \left[1 + \sigma s^{-2} \sin^2\gamma \sin^2 kz (\cos^2 kz - \sin^2 \gamma)^{-1}\right] \quad , \\
g_{22} &= a_0^2 k^{-1} \sin 2kt \cos^2 kz \left[1 - \sigma r^{-1} \sin^2\gamma \sin^2 kz (\cos^2 kz - \sin^2 \gamma)^{-1}\right] \quad , \\
g_{12} &= - a_0^2\, \sigma (4ks)^{-1} \cos 2\gamma \sin 2kt \sin^2 2kz (\cos 2\gamma + \cos 2kz)^{-1} \quad , \\
f &= a_0^2\, k^{-1} \sin 2kt \quad , \quad \varepsilon = 3k^3\, a_0^{-2} \sin^{-3} 2kt \quad .
\end{aligned} \tag{3.31}$$

and near the finite cosmological singularity $kt \to \pi/2$ we get for these same quantities:

$$g_{11} = a_0^2\, k^{-3}\sin 2kt\, \sin^2 kz\left[1 + \sigma s^{-2}\sin^2\gamma\, \cos^2 kz(\sin^2 kz - \sin^2\gamma)^{-1}\right] ,$$

$$g_{22} = a_0^2\, k^{-1}\sin 2kt\, \cos^2 kz\left[1 - \sigma r^{-1}\sin^2\gamma\, \cos^2 kz(\sin^2 kz - \sin^2\gamma)^{-1}\right] ,$$

$$g_{12} = a_0^2\, \sigma(4ks)^{-1}\cos 2\gamma\, \sin 2kt\, \sin^2 2kz\, (\cos 2\gamma - \cos 2kz)^{-1} , \qquad (3.32)$$

$$f = rs^{-2}\, a_0^2\, k^{-1}\sin 2kt \quad , \qquad \epsilon = 3s^2 r^{-1} k^3 a_0^{-2}\sin^{-3} 2kt \quad .$$

In Eq. (3.31) one must take $\sin 2kt \simeq 2kt$, and analogously in (3.32) $\sin 2kt \simeq \pi - 2kt$.

The field of the soliton for the solution (3.20) - (3.26) is determined as before by the perturbation matrix H and the fractional change E of the energy density These components are given by the same formulas (3.6) - (3.8), in which the quantities g_{ab} and ϵ must be taken to mean the expressions shown in Eq. (3.20) and (3.23), and the corresponding quantities with the index zero refer to the background solution (3.28), (3.29). Setting

$$k = 1 \quad , \quad \sin\gamma = i\Delta \quad , \qquad s = 2p\Delta(1 + \Delta^2)^{1/2}(1 + 2\Delta^2)^{-1} \quad , \qquad (3.33)$$

where Δ and p are new arbitrary constants (and Δ is already real), we get from Eqs.(3.24) and (3.31) the asymptotic values of the perturbation fields H and E for $t \to 0$:

$$H_{11} = (1 - p^2)p^{-2}\Delta^2\sin^2 z(\cos^2 z+\Delta^2)^{-1} \quad , \quad H_{22} = (p^2-1)\Delta^2\sin^2 z(\cos^2 z+\Delta^2)^{-1} \quad ,$$

$$H_{12} = (1 - y^2)p^{-1}\Delta(1+\Delta^2)^{1/2}\sin z\, \cos z(\cos^2 z+\Delta^2)^{-1} \quad , \quad E = 0 \quad , \qquad (3.34)$$

and from Eqs. (3.24) and (3.32) we get their asymptotic forms for $t \to \pi/2$:

$$H_{11} = (1-p^2)p^{-2}\Delta^2\cos^2 z(\sin^2 z+\Delta^2)^{-1} \quad , \quad H_{22} = (p^2-1)\Delta^2\cos^2 z(\sin^2 z+\Delta^2)^{-1} \quad ,$$

$$H_{12} = (p^2-1)p^{-1}\Delta(1+\Delta^2)^{1/2}\sin z\, \cos z(\sin^2 z+\Delta^2)^{-1} \quad , \quad E = p^2 - 1 \quad . \qquad (3.35)$$

These formulas show clearly the distribution of the perturbation H in the initial and final moments of the evolution. Near the initial instant the absolute values of the components H_{11} and H_{22} are largest at $z = \pi/2$ and equal to zero for $z = 0$. The absolute value of H_{12} has its maximum in the region $\pi/4 < z < \pi/2$. With the passage of time the maxima in the distributions of the quantities H_{11} and H_{22} are shifted in space, and at the finite time $t = \pi/2$ they are at $z = 0$, while H_{11} and H_{22} go to zero at the former position of the maxima, $z = \pi/2$. The extremum of the component H_{12} also shifts during the cycle of evolution, through a finite distance in the direction of smaller values of z, and for $t \to \pi/2$ it is in the range $0<z<\pi/4$. Figure 2 and 3 show the initial and final profiles of the perturbations H_{22} and H_{12} as functions of the value of the parameter Δ, which determines the widths of the corresponding distributions (for definiteness we consider $p > 1$, $\Delta > 0$ and use a fixed value of the parameter p).

In a closed space ($0 \leq z \leq \pi/2$) we can speak of localization of the perturbations only for a sufficiently small value of Δ. If $\Delta \ll 1$, then we can see from Eqs. (3.34) and (3.35) and the figures that the field of the soliton at the beginning and at the end of the evolution is concentrated near $z = \pi/2$ and $z = 0$, respectively, in narrow ranges of width $\delta z \sim \Delta$, which are much smaller than the number $\pi/2$, i.e., than the linear extent of the Universe in the

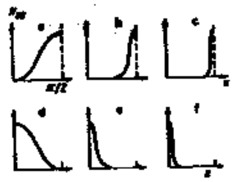

FIGURE 2 - Profiles of the initial and final distributions of the perturbation component H_{22} in closed models. Curves a, b and c correspond to the beginning of the evolution, $t \to 0$, and d, e, and f, to the final time $t \to \pi/2$; a and d correspond to very large values of the parameter Δ, for which the width of the soliton is comparable with the size of the Universe; b, d and c, e show the change of shape of the initial and final distributions as Δ is made smaller. The value of H_{22} at the maximum is p^2-1 throughout.

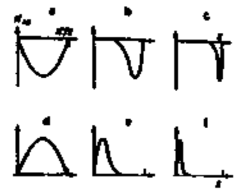

FIGURE 3 - Profiles of the initial and final distributions for the component H_{12}. The upper row shows profiles near the time $t=0$, and the lower row shows them near the final moment $t=\pi/2$. Curves a and d are for very large values of the parameter Δ, and b, e and c, f show the change of shape of the distributions as Δ is made smaller. The respective extreme values of H_{12} on the upper and lower diagrams are $(1-p^2)/2p$ and $(p^2-1)/2p$, and the coordinate values at which they occur are given by the equations $\cos 2z_0 = -(1+2\Delta^2)^{-1}$ and $\cos 2z_0 = (1+2\Delta^2)^{-1}$, respectively.

coordinate z. With this condition the picture of the evolution of the soliton in the stage of expansion of space is partially similar to what happens in the flat model; during a short time interval after the beginning ($t \lesssim \Delta$) the perturbation H in the region around $z = \pi/2$ will die away, without changing the general shape and width of its profiles. Near the points with $z = 0$, on the other hand, the perturbation H begins to grow. After a critical time $t \sim \Delta$ this process will continue, but along with it a gravitational wave appears from the region near $z = \pi/2$ and is propagated toward $z = 0$ with the speed of light. At the time of maximum expansion, $t = \pi/4$, it has passed through a "quarter of the Universe" and reaches the region with $z = \pi/4$.

With further increase of the time from $t = \pi/4$ to $t = \pi/2$ the perturbation H becomes concentrated in the region at $z = 0$, and after a time $t \sim \pi/2 - \Delta$ it absorbs the wave which has arrived there. The final distribution of the field of the soliton is given by Eqs. (3.35) and again has a small width $\delta z \sim \Delta$. It can be shown that the distribution of the field in the gravitational wave itself is similarly small in width during the entire time of its propagation from the region $z = \pi/2$ to the region at $z = 0$. The process is shown schematically in Fig. 4.

For large values of the parameter Δ both the initial and the final distributions of the field of perturbations has a width of the order of the size of the whole Universe (corresponding to the profiles shown in Fig. 2, a and d). For any observer to study the profile of the soliton will require a time of the order of the entire cycle of evolution of the Universe, and the usual interpretation of a soliton as a single localized disturbance can be applied in this case only in a conventional sense.

As for the perturbation E of the energy density, in the approximation considered here it is zero at the beginning of the evolution, and Eq. (3.35) shows that at the concluding stage of collapse it becomes constant in space, producing a change of the parameters of the background Friedmann model. Here we again encounter the same phenomenon as was described in the analysis of the perturbations on the background of the flat model.

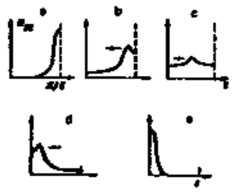

FIGURE 4 - Schematic representation of the evolution of a soliton in the closed model. The sequence a-e corresponds to variation of the time from $t = 0$ to $t = \pi/2$. The picture corresponds to rather small values of the parameter Δ.

In conclusion we note that the geometrical loci of the points $z = \pi/2$ and $z = 0$ are circles in the closed three - dimensional space of the Universe, and are great circles on this hypersphere. It can be seen from Eqs. (2.18) and (2.19) that in standard four-dimensional spherical coordinates the equation of the circle $z = \pi/2$ is $\chi = \pi/2$, $\theta = \pi/2$ and it may be arbitrarily called the equator. The equation of the circle $z = 0$ is $\theta = 0$ and $\theta = \pi$, and it can be called the polar axis. These circles have no points in common. The equator and polar axis so defined are completely equivalent and can be interchanged by a suitable transformation of the four-dimensional coordinates. These closed curves are the equivalent of what gas an infinite axis of cylindrical symmetry of the soliton in the open models.

III.4 - VACUUM SOLUTIONS

It was shown in Sec. 2 that for any solution of the form (1.1) in a space with matter described by a potential ϕ there is a corresponding solution of the gravitational equations in vacuum, of the form (2.10). By using Eq. (2.9) this solution can be written in the form:

$$- ds^2 = fF^{-1} (-dt^2 + dz^2) + g_{ab}\, dx^a\, dx^b \quad , \tag{4.1}$$

where the functions f and g_{ab} are precisely the same as in the solution with matter, and the coefficient F is determined by the equations (2.15). It is easy to find this coefficient for the solutions (3.20) - (3.26) by substituting in Eq. (2.15) the expressions for the potential ϕ(Eq. (3.22)) and the function α Eq.(3.30)). A simple integration gives

$$F = F_0 (\sin 2kt)^{1/2} (\sin 4k\zeta)^{-1/2} (\sin 4k\eta)^{-1/2} \quad , \tag{4.2}$$

where ζ and η are the light variables (2.8) and F_0 is an arbitrary constant. Substituting this result along with the metric coefficients f and g_{ab} of Eq. (3.20) in the expression (4.1), we get the desired vacuum solution.

We note here that an interesting qualitative study of closed vacuum cosmological models with metrics of the type (1.1) has been given by Gowdy (5).

III.5 - APPENDIX

We shall here describe briefly the method for deriving the solutions presented in Sec. 3. As shown in the previous paper (1), the main step in finding them is the determination of the matrix functions $\Psi(\lambda,\zeta,\eta)$ corresponding to the background metrics (3.28). Such a function satisfies the equations

$$D_1\Psi = (\lambda - \alpha)^{-1} A\Psi \quad , \quad D_2\Psi = (\lambda + \alpha)^{-1} B\Psi \quad , \tag{A.1}$$

where the operators D_1 and D_2 are given by

$$D_1 = \partial_\zeta - 2(\lambda - \alpha)^{-1}\alpha_\zeta \lambda\partial_\lambda \quad , \quad D_2 = \partial_\eta + 2(\lambda + \alpha)^{-1}\alpha_\eta \lambda\partial_\lambda \quad . \tag{A.2}$$

Here λ is a complex spectral parameter, ζ and η are the variables (2.8), α^2 is the determinant of the matrix g of the background solution, which is of the form (3.30), and the matrices A and B are defined from Eq. (2.7) with the same matrix g, which, as shown in Eq. (3.28), is

$$g = \mathrm{diag}(a_0^2 k^{-3} \sin 2kt \sin^2 kz,\ a_0^2 k^{-1} \sin 2kt \cos^2 kz) \tag{A.3}$$

(the commas in Eq. (A.2) and the letter ∂ denote ordinary differentiation).

Integration of Eqs. (A.1) and (A.2) with $k \neq 0$ leads to the following diagonal matrix Ψ:

$$\Psi_{11} = \left[a_0^4 k^{-6} \sin^2 2kt \sin^2 kz - a_0^2 k^{-4} \lambda \cos 2kt - k^{-2}(\alpha^2 + 2\beta\lambda + \lambda^2)\right]^{1/2} ,$$

$$\Psi_{22} = (\alpha^2 + 2\beta\lambda + \lambda^2)\,\Psi_{11}^{-1} \quad , \quad \Psi_{12} = 0 \quad , \qquad \text{(A.4)}$$

where

$$\alpha = a_0^2 (2k^2)^{-1} \sin 2kt \sin 2kz \quad , \quad \beta = - a_0^2 (2k^2)^{-1} \cos 2kt \cos 2kz \quad . \qquad \text{(A.5)}$$

The limit $k \to 0$ cannot be taken directly in Eqs. (A.4) and (A.5), as in the case of the flat model, but it is easy to find a solution for which it is possible. The point is that the matrix Ψ and the function β (the second solution of the wave equation satisfied by α)are not uniquely determined. The function β is determined up to an arbitrary additive constant, and the matrix Ψ, up to multiplication from the fight by an arbitrary matrix of the argument $= 1/2\ (\alpha^2\lambda^{-1} + 2\beta + \lambda)$: using this freedom, we can reconstruct the solution (A.4), (A.5) so that it has a limit for $k = 0$. We have not done this, however, and in constructing the solutions (3.20) - (3.26) for $k \neq 0$ we have used just the formulas (A.4), (A.5)(the intecated transformation for $k = 0$ would not change anything in the solution except to redefine the constants). The matrix Ψ for the flat model can be found either by the method indicated or by direct integration of Eqs. (A.1) and (A.2). The result, which we have used in constructing the solution (3.1), can be written in the form

$$\Psi_{11} = \left[(z^2+\lambda)(\alpha^2 + 2\beta\lambda + \lambda^2)\right]^{1/2} \quad , \quad \Psi_{22} = (\alpha^2 + 2\beta\lambda + \lambda^2)\Psi_{11}^{-1} \quad .$$

$$\Psi_{12} = 0 \quad , \qquad \alpha = tz \quad , \qquad \beta = 1/2\ (t^2 + z^2) \quad . \qquad \text{(A.6)}$$

The further operations that lead to the solution are merely algebraic and are explained in Ref. 1. We shall not repeat them here, but we point out the following important features. Starting from the background metric (3.28) and the Ψ function (A.4) - (A.6), we arrive at solutions in which the natures of the variables t and z are in a certain sense reversed. Whereas in the background solutions (3.28) the matrix g has an isotropic cosmological singularity with respect to t and fictitious coordinate singularities with respect to z, in the one-soliton solutions an isotropic physical singularity appears with respect to the space variable z, and fictious ones with respect to t. When we try to take the limit with respect to a parameter to obtain the background metric we get instead of the metric coefficients g_{ab} from Eq. (3.28) the same functions except that t and z are interchanged. Accordingly, to recover the cosmological character of the model, one must interchange the coordinates t and z and at the same time choose the correct sign of the metric coefficient f (so that the variable t will actually be timelike). This must be done first in the vacuum solution, and then one can turn to the solution with matter.

The final sequence of operations is:

1 - With the metric (3.28) and the potential (3.22) we determine the vacuum background solution (the coefficient F for changing from f to f_v is given by Eq. (4.2));

2 - We apply the one-soliton perturbation to the vacuum solution;

3 - In the result so found we make the interchange $t \to z$, $z \to t$ and choose the correct sign of f_v;

4 - we again return to the solution with matter with the same potential (3.22).

The transition coefficient F remains unchanged, since the function α is not changed by the interchange $t \to z$, $z \to t$, and Eq. (2.15) is also unchanged when there is no change of the potential ϕ. But the solution of these equations, i.e., the coefficient F itself, does not have this symmetry, and $F(T,z) \neq F(z,t)$, which is important in this sequence of transformations. After these operations we obtain a solution which can be reduced to the form (3.20) by a certain linear transformation (with constant coefficients) of the variables x and y.

Analogous operations in the case of the flat model give the metric (3.1). As was pointed out in Sec. 3, a transition from (3.20) to (3.1) in the limit $k \to 0$ exists, although it does not exist in explicit form between Eqs. (A.4), (A.5) and (A.6). The linear transformation of the coordinates x, y is made from considerations of convenience of the final result; only after this transformation do we get the metric (3.20), in which: (a) there is a transition in the limit with respect to a parameter to the form (3.28); (b) the coefficient g_{11} goes to zero at $z = 0$, and (c) for the closed model the coefficient g_{22} goes to zero for $kz = \pi/2$. For the flat model the analogous transformation serves to satisfy conditions (a) and (b) and to make the behavior of the metric for $z \to \infty$ the same as in the background solution (2.24).

In the investigation of the properties of the solution (3.20) - (3.26) it is necessary to use certain identities connecting the functions μ (Eq. (3.25))and R (Eq. (3.26)). We give them here. The function μ is a solution of the quadratic equation

$$\mu^2 - (a_0^2 k^{-2} \cos 2\gamma - 2\beta)\mu + \alpha^2 = 0 \quad , \tag{A.7}$$

where α and β are given by Eq. (A.5). Besides this, the following two identities hold:

$$\begin{aligned} r\alpha^2\mu^{-1}\mathrm{tg}^2kt+\mu R &= a_0^2k^{-2}(\cos 2\gamma + \cos 2kz)\, L \quad , \\ r\mu + \alpha^2\mu^{-1} R\, \mathrm{ctg}^2\, kt &= a_0^2\, k^{-2}\, (\cos 2\gamma - \cos 2kz)\, L \quad , \end{aligned} \tag{A.8}$$

from which one further relation can easily be derived:

$$r\mu\, (\cos^2\gamma - \sin^2 kz) + \mu R(\sin^2\gamma - \sin^2 kz) = a_0^2(2k^2)^{-1}\sin^2 2kz(r\, \sin^2 kt - R\, \cos^2 kt) \ . \tag{A.9}$$

The quantity L in Eq. (A.8) is determined by Eq. (3.21).

III.6 - REFERENCES

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IV - TWO-SOLITON WAVES IN ANISOTROPIC COSMOLOGY

IV.1 - INTRODUCTION

In the present chapter (which are based on paper (1)) we investigate a new exact wavelike solution of gravitational-field equations in the framework of the inverse-scattering problem technique.

This solution is a natural extension of a particular solution found by us and it describes the propagation of two plane waves and their mutual interaction in an anisotropic cosmological model.

The exact two-soliton solution for the metric tensor g_{ph} is obtained in sect. 2 according to the integration scheme described in paper (1) and it is constructed by the use of the same $\Psi_0(\lambda,\eta,\zeta)$matrix function solved for in that article and related to a g_0 Kasner solution as the background field.

The degree of deviation from the background metric is introduced as a soliton field defined in sec. 3.

The behaviour of these field components is considered in detail in sec. 4 mainly in the interesting region and for a characteristic class of cosmological models, i.e., for Kasner indices s_1 and s_2 both positive. The general case for s_1 larger than unity needs further investigation and will be discussed elsewhere.

In the limit $z \to \pm \infty$ one finds that the soliton field has not enough time to propagate and, therefore, the perturbation approaches zero.

In remote time, as $t \to \infty$, the disturbance would decay and would disappear.

In the limit $t \to 0$, near the cosmological singularity, the deviation is described by two planelike positive disturbances approaching each other at the origin $z = 0$ and growing in amplitude. For a class of cosmological models, when $(3 - \sqrt{5})/4 > s_2 > 0$, the soliton field approaches a finite minimal distance; for another class $1/2 > s_2 > (3 - \sqrt{5})/4$, the two solitons fuse into one soliton concentrated near the origin, in a time $t_i \geq 0$; in the first case the maxima propagate with a velocity less than unity, in the second with a velocity that exceeds unity.

In the same limit $t \to 0$, the g_{22} perturbation is described by two negative disturbances approaching each other and fusing into a concentrated soliton in a time $t_i \geq 0$.The two-soliton field for the mixed component consists of a negative and a positive disturbance whose separation approaches a finite minimal distance $\pm z_1$ at $t = 0$.

Some interesting remarks on the behaviour of the initial time t_i for different cosmological models are given.

Finally the soliton fields are studied along the light-cone $t^2 = z^2$, $t \to \infty$, where the perturbation is shown to be maximal. Therefore, one may verify that the asymptotic propagation velocity approaches the velocity of light.

This soliton wave solution on a homogeneous Kasner background may also be seen as a non-trivial example of an inhomogeneous anisotropic cosmological model. We hope that this kind of investigation may lead to the understanding of some open cosmological problems.

IV.2 - THE BACKGROUND FIELD AND THE GENERAL SOLUTION

Let us consider the simplest homogeneous cosmological Bianchi type-I model in vacuum,i.e. the Kasner solution, written in the following standard form:

$$- ds^2 = - d\tau^2 + \tau^{2p_3} dz^2 + \tau^{2p_2} dy^2 + \tau^{2p_1} dx^2 \quad , \tag{2.1}$$

where the constants p_1, p_2, p_3 satisfy the two relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \quad .$$

It is obvious that, after a time transformation, we can express our metric in the following form:

$$- ds^2 = f_0 (- dt^2 + dz^2) + g^0_{ab} dx_a dx_b \quad ,$$

i.e.

$$- ds^2 = t^{s_1^2+s_2^2-1} (- dt^2 + dz^2) + t^{2s_2} dy^2 + t^{2s_1} dx^2 \quad , \tag{2.2}$$

where the constants s_1 and s_2 are related by only one constraint: $s_1 + s_2 = 1$.

It is easy to show the relations with the previous parameters:

$$p_1 = \frac{1 - s_2}{s_2^2 - s_2 + 1} \quad , \quad p_2 = \frac{s_2}{s_2^2 - s_2 + 1} \quad , \quad p_3 = \frac{s_2\ (s_2 - 1)}{s_2^2 - s_2 + 1} \quad . \qquad (2.1a)$$

We will use this metric (2.2) in order to construct on this background our exact two-soliton solutions.

This solution is derived from a technique that has been explained in paper (1).

Two particular cases ($s_1 = s_2 = 1/2$ and $s_1 = 1$, $s_2 = 0$) have been previously (1) considered; here we will extend our investigation to the general case, i.e. when s_1 and s_2 are arbitrary constants, but positive.

The calculation leads us to the following formula for the physical (1) metric tensor $g_{ab}^{(ph)}$ and the function f_{ph} for (a, b = 1, 2):

$$g_{11}^{ph} = t^{2s_1}\{\frac{\rho^2}{t^2} + \frac{(t^2-\rho^2)^2}{D}\left[\sin^2\delta - \frac{t^2}{\rho^2}\sin^2(\phi+\delta) - (\frac{t}{\rho})^{4s_1}L_0^2\sin^2\phi\right] -$$

$$- 2\rho^2\ \frac{t^2 - \rho^2}{D}\left[\frac{t^2}{\rho^2} + L_0^2\ (\frac{t}{\rho})^{4s_1}\right]\sin^2\phi\ \} \qquad (2.3)$$

$$g_{22}^{ph} = t^{2s_2}\{\ \frac{\rho^2}{t^2} + \frac{(t^2 - \rho^2)^2}{D}\left[\sin^2\delta - \frac{t^2}{\rho^2}\sin^2(\phi - \delta) - (\frac{t}{\rho})^{4s_2}\ L_0^{-2}\sin^2\phi\right] -$$

$$- 2\rho^2\ \frac{t^2 - \rho^2}{D}\left[\frac{t^2}{\rho^2} + L_0^{-2}(\frac{t}{\rho})^{4s_2}\right]\sin^2\phi\ \} \quad , \qquad (2.4)$$

$$g_{12}^{ph} = -\frac{t^2}{\rho}\ \frac{t^2 - \rho^2}{D}\sin\phi\{\ L_0\ (\frac{t}{\rho})^{2s_1}\left[t^2\sin(\phi - \delta) + \rho^2\sin(\phi + \delta)\right] +$$

$$+ L_0^{-1}\ (\frac{t}{\rho})^{2s_2}\left[t^2\sin(\phi + \delta) + \rho^2\sin(\phi - \delta)\right]\ \} \ , \qquad (2.5)$$

$$f_{ph} = -4\ \frac{w_i^2}{\sin^2\delta_0}\ \frac{\rho^4}{t^4}\ \frac{Dt^{2s_1^2+s_2^2-1}}{(t^2-\rho^2)^2(t^4+\rho^4-2t^2\rho^2\cos 2\phi)\sin^2\phi} \quad , \qquad (2.6a)$$

$$D \equiv -\ t^2\ \frac{\sin^2\delta}{\rho^2}\ (t^2 - \rho^2)^2 + \rho^2\sin^2\phi\left[L_0(\frac{t}{\rho})^{2s_1} + L_0^{-1}\ (\frac{t}{\rho})^{2s_2}\right]^2\ \} \qquad (2.6b)$$

where the arbitrary parameters are only three real constants: δ_0, w_i, L_0.

We defined $\delta = (s_1 - s_2)\phi + \delta_0$.

The variables $\phi = \phi(t,z)$ and $\rho = \rho(t,z)$ are related to the complex quantity $\mu = \rho\exp(i\phi)$ that is defined (1) as follows:

$$\mu^2 - 2(z - iw^2)\ \mu + t^2 = 0 \quad . \qquad (2.7)$$

In the following we will assume that the real constant w_i is positive without loss of generality, as well as L_0 and δ_0.

From (2.7) one obtains

$$\cos\phi = \frac{2z\rho}{t^2+\rho^2} \quad , \tag{2.8}$$

$$\sin\phi = \frac{2w_i\ \rho}{t^2-\rho^2} \quad . \tag{2.9}$$

From (2.7) and its conjugate equation one obtains an implicit definition of the real function $\rho = \rho(t,x)$ and them from (2.8) and (2.9) a definition of $\phi = \phi(t,x)$.

IV.3 - THE SOLITON FIELDS

Let us introduce the following soliton fields:

$$H_{aa} \equiv \frac{g_{aa}^{ph} - g_{aa}^{0}}{g_{aa}^{0}} \quad , \quad H_{12} \equiv \frac{g_{12}^{ph}}{\sqrt{g_{11}^{0}g_{22}^{0}}} \tag{3.1}$$

where g_{ab}^{0} is the background metric (2.2).

These fields indicate a level of deviation from the original metric. Let us also introduce the perturbation function

$$\Delta f \equiv \frac{f_{ph} - f_0}{f_0} \tag{3.2}$$

where f_0 is the background function (2.2), $f_0 = t^{s_1^2+s_2^2-1}$ and f_{ph} is defined in (2.6a).

Then the exact formula for the perturbation function is

$$\Delta f = -\frac{w_i^2}{\sin^2\delta_0}\left(\frac{\rho}{t}\right)^4 \frac{4Dt^2}{(t^2-\rho^2)^2(t^4+\rho^4-2t^2\rho^2\cos 2\phi)\sin^2\phi} - 1 \quad . \tag{3.3}$$

From (2.3) - (2.5) and (3.1) we found the general formulae for the perturbation fields H_{ab}

$$H_{11} = \frac{t^2}{\rho^2}\frac{(t^2-\rho^2)}{D}\left[\sin^2\delta - \sin^2(\phi+\delta)\right] - \frac{(t^2-\rho^2)\sin^2\phi}{D}\left[L_0^2\left(\frac{t}{\rho}\right)^{4s_1}t^2 - L_0^{-2}\left(\frac{t}{\rho}\right)^{4s_2}\rho^2\right] \quad , \tag{3.4}$$

$$H_{22} = \frac{t^2}{\rho^2}\frac{(t^2-\rho^2)}{D}\left[\sin^2\delta - \sin^2(\phi-\delta)\right] - \frac{(t^2-\rho^2)\sin^2\phi}{D}\left[L_0^{-2}\left(\frac{t}{\rho}\right)^{4s_2}t^2 - L_0^{2}\left(\frac{t}{\rho}\right)^{4s_1}\rho^2\right] \quad , \tag{3.5}$$

$$H_{12} = -\frac{t}{\rho}\,\frac{t^2-\rho^2}{D}\sin\phi\,\left\{L_0\left(\frac{t}{\rho}\right)^{2s_1}\left[t^2\sin(\phi-\delta)+\rho^2\sin(\phi+\delta)\right] + L_0^{-1}\left(\frac{t}{\rho}\right)^{2s_2}\left[t^2\sin(\phi+\delta)+\rho^2\sin(\phi-\delta)\right]\right\} . \tag{3.6}$$

Due to the complexity of the general behaviour of the solitons we will first consider those cases in which the s_1 and s_2 parameters are both positive and, due to the symmetry between components 11 and 22, we will discuss in detail the cases $0 < s_2 < s_1$ and we will easily extend the results to the cases $s_2 > s_1 > 0$.

The most general case ($s_2 < 0$, $s > 1$) will be discussed elsewhere.

IV.4 - THE ASYMPTOTIC BEHAVIOUR OF THE SOLITON FIELD IN THE REGION $s_1 > s_2 > 0$

Case $z \to \pm\infty$.

Let us first consider the asymptotic behaviour in regions far from the light-cone: for the limits $|z| \gg t$ and $|z| \gg 2w_i$, i.e. for $z \to \pm\infty$.

From (2.7) - (2.9) one finds

$$\rho = \frac{t^2}{2|z|} \quad , \quad \sin\phi = \frac{w_i}{|z|} \quad , \quad \cos\phi = \pm 1 \quad . \tag{4.1}$$

Then, introducing these results in formulae (3.4) - (3.6), one obtains

$$H_{11} = \left(\frac{t}{2w_i}\right)^{-2\Delta s}\left[\left(\frac{w_i}{z}\right)^{4s_2}\frac{1}{\sin^2\delta_0}\,L_0^2\right] + \left[\frac{2w_i}{z}\mathrm{ctg}\delta_0 + \frac{w_i^2}{z^2}(\mathrm{ctg}^2\delta_0 - 1)\right] \quad , \tag{4.2}$$

$$H_{22} = \left(\frac{t}{2w_i}\right)^{2\Delta s}\left[\left(\frac{w_i}{z}\right)^{4s_1}\frac{1}{\sin^2\delta_0}\,L_0^{-2}\right] - \left[\frac{2w_i}{z}\,\mathrm{ctg}\delta_0 + \frac{w_i^2}{z^2}(\mathrm{ctg}^2\delta_0 - 1)\right] \quad , \tag{4.3}$$

$$H_{12} = \frac{1}{\sin\delta_0}\left[-\left(\frac{t}{2w_i}\right)^{-\Delta s} L_0\left(\frac{w_i}{z}\right)^{2s_2} + \left(\frac{t}{2w_i}\right)^{\Delta s} L_0^{-1}\left(\frac{w_i}{z}\right)^{2s_1}\right] \quad . \tag{4.4}$$

where $\Delta s = s_1 - s_2 > 0$.

In order to avoid "bad behaviour" of this asymptotic for vanishing values of $\sin\delta_0$ and in order to preserve the z negative-positive symmetry, we assume $\delta_0 = \pi s_2$ in the following.

From (4.2) - (4.4) one verifies that all the H_{ab} fields are vanishing as z goes to infinity.

This result confirms the smooth connection with the background metric i.e. $H_{ab} = 0$.

From (3.3) one obtains in the limit $z \to \infty$

$$\Delta f = 0 \quad . \tag{4.5}$$

Therefore, also the f_{ph} and f_0 are smoothly connected as z goes to inifinity.

In particular, the first nonvanishing term of Δf is proportional to t^2/z^2.

Case $t \to \infty$.

Let us now consider the behaviour of the soliton fields in a finite region of space a long time after the singularity: $t \gg |z|$ and $t \gg 2w_i$, i.e. $t \to \infty$. From relations (2.7) - (2.9) one obtains

$$\rho = t - w_i \quad , \quad \sin\phi = 1' \quad , \quad \cos\phi = \frac{z}{t} \tag{4.6}$$

Substituting these results in formulae (3.4) - (3.6) one obtains

$$H_{11} = \frac{2w_i}{t} \; \frac{L_0^2 - 1}{L_0^2 + 1} \quad , \tag{4.7}$$

$$H_{22} = - H_{11} \quad , \tag{4.8}$$

$$H_{12} = \frac{4zw_i}{t^2} \; \Delta s (L_0 + L_0^{-1})^{-1} \quad . \tag{4.9}$$

In order to see the z-dependence of H_{11} or H_{22}, one should take into account the expansion of (3.4), (3.5) up to the fourth power in z. Clearly all the soliton fields are vanishing in the limit $t \to \infty$ and this result confirms the "good behaviour" of such solutions.

Finally we consider the perturbation function Δf in the present limit:

$$\Delta f = \frac{(L_0 + L_0^{-1})^2}{4 \sin^2 \delta_0} - 1 > 0 \tag{4.10}$$

(in this paper $\delta_0 = \pi s_2$). This positive variation of the function f_{ph} indicates an energetic exchange between solitons and background due to the wave propagation. The same effect was described in (1).

Case $t \to 0$.

Let us now consider the behaviour near the singularity when $t \ll |z|$ and $t \ll 2w_i$ i.e., when $t \to 0$.

In this case one may consider the global z-dependence of the soliton fields H_{ab}.

From relations (2.7) - (2.9) one obtains in the limit

$$\rho = \frac{t^2}{2\sqrt{w_i^2 + z^2}} \quad , \quad \sin \phi = \frac{w_i}{\sqrt{w_i^2 + z^2}} \quad , \quad \cos \phi = \frac{z}{\sqrt{w_i^2 + z^2}} \quad . \tag{4.11}$$

From these limits and formulae (3.4) - (3.6) one finds

$$H_{11} = (\frac{t}{2w_i})^{-2\Delta s} \left[(\frac{w_i^2}{w_i^2+z^2})^{2s_2} L_0^2 \frac{1}{\sin^2\delta} \right] + \left[\frac{2w_i z}{w_i^2+z^2} \operatorname{ctg}\delta + \frac{w_i^2}{w_i^2+z^2} (\operatorname{ctg}^2\delta - 1) \right] \quad , \tag{4.12}$$

$$H_{22} = (\frac{t}{2w_i})^{2\Delta s} \left[(\frac{w_i^2}{w_i^2+z^2})^{2s_1} L_0^{-2} \frac{1}{\sin^2\delta} \right] + \left[- \frac{2w_i z}{w_i^2+z^2} \operatorname{ctg}\delta + \frac{w_i^2}{w_i^2+z^2} (\operatorname{ctg}^2\delta - 1) \right] \quad , \tag{4.13}$$

$$H_{12} = (\frac{t}{2w_i})^{-\Delta s} \left[- (\frac{w_i^2}{w_i^2+z^2})^{s_2} L_0 \frac{\sin(\delta-\phi)}{\sin^2 \delta} \right] + (\frac{t}{2w_i})^{\Delta s} \left[(\frac{w_i^2}{w_i^2+z^2})^{s_2} L_0^{-1} \frac{\sin(\delta+\phi)}{\sin^2 \delta} \right] \quad . \tag{4.14}$$

$$\Delta f = \frac{\sin^2 \delta - \sin^2 \delta_0}{\sin^2 \delta_0} \tag{4.15}$$

One can see that the behaviour of the soliton fields is dominated in H_{11} and H_{12} by the first time-dependent term. These quantities diverge as time approaches zero. The reason of this fictitious singularity (not to be confused with the true cosmological singularity) is related to the choice of the metric form. Indeed, if one introduces a co-ordinate transformation of the form $x'_1 = (x + y)/2$, $x'_2 = (x - y)/2$, one can write a new expression in place of (2.2):

$$-ds^2 = -t^{s_1^2+s_2^2-1}(dt^2-dz^2)+(t^{2s_1}+t^{2s_2})(dx_1^2+dx_2^2)+2(t^{2s_1}-t^{2s_2})\,dx_1\,dx_2 \quad . \tag{4.16}$$

Then one will find for this new background g'^{0}_{ab} a physical solution $g'^{(ph)}_{ab}$ related to $g^{(ph)}_{ab}$ in formulae (2.3) - (2.5):

$$g'^{(ph)}_{11} = g^{(ph)}_{11} + g^{(ph)}_{22} + 2g^{(ph)}_{12} \quad , \tag{4.17}$$

$$g'^{(ph)}_{22} = g^{(ph)}_{11} + g^{(ph)}_{22} - 2g^{(ph)}_{12} \quad , \tag{4.18}$$

$$g'^{(ph)}_{12} = g^{(ph)}_{11} - g^{(ph)}_{22} \quad . \tag{4.19}$$

In particular in the $t \to 0$ asymptotic the soliton field will avoid the problem that we found in (4.12) due to divergent behaviour of H_{11}.

From definition (3.1) and (4.17) - (4.18) one obtains

$$H'_{11} = f_1(z) + g_1(z) + 2h_1(z) \quad , \tag{4.20}$$

$$H'_{22} = f_1(z) + g_1(z) - 2h_1(z) \quad . \tag{4.21}$$

For the mixed components of the soliton field one needs to redefine the formula as follows:

$$H'_{12} = \frac{g'^{(ph)}_{12} - g'^{0}_{12}}{g'^{0}_{12}} = g_1(z) - f_1(z) \quad , \tag{4.22}$$

where $f_1(z)$ is the first term in square bracket in (4.12), $g_1(z)$ is the second term in square bracket in (4.13) and $h_1(z)$ is the first term in square bracket in (4.14). Using the angle ϕ in place of z, one can rewrite these quantities in a more compact form:

$$f_1(z) = L_0^2 \frac{\sin^{4s_2}\phi}{\sin^2\delta} \quad , \tag{4.23}$$

$$g_1(z) = \frac{\sin^2(\phi-\delta)}{\sin^2\delta} - 1 \quad , \tag{4.24}$$

$$h_1(z) = -L_0 \frac{\sin^{2s_2}\phi \cdot \sin(\delta-\phi)}{\sin^2\delta} \quad . \tag{4.25}$$

(it is easy to verify that, in order to have a determinant $\det g'_{ab} = t^2$, the relation $f_1(g_1 + 1) = h_1^2$ should be and is indeed satisfied).

Let us describe schematically these solutions (Fig. 1 (a) - (c)):

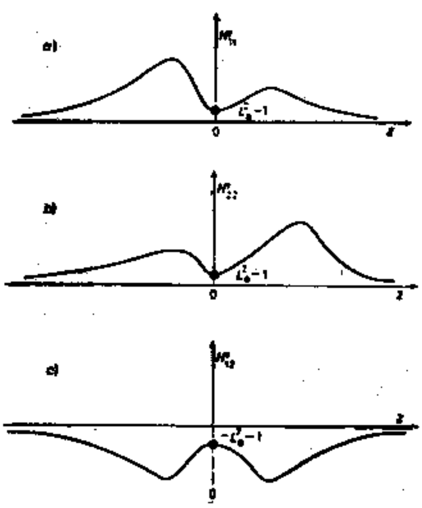

FIGURE 1 - Schematical behaviour of the soliton fields $H'_{11}, H'_{22}, H'_{12}$ as functions of the co-ordinate z at the time t=0 in (a),(b),(c), respectively. In this primed system, all of the soliton fields have finite values and are described by formulae (4.20) - (4.22).

From inspection of formulae (4.12) - (4.14) and (4.20) - (4.22), one can see that $H'_{11}(-z) = H'_{22}(+z)$ and $H'_{12}(z) = H'_{12}(-z)$.

Now that the nonsingular behaviour of the solution in a proper co-ordinate system is proved, it is convenient to reconsider in detail our previous dynamic solutions (4.12) - (4.14); we will assume an arbitrarily small but finite time parameter in those formulae.

By inspection one sees that in the limit $z \to \pm\infty$ the power dependence in z is obviously the same as the one obtained in the limit $z \to \pm\infty$ in formulae (4.2) - (4.4).

The field H_{11} is composed of two terms: the first (dominant because it is proportional to $t^{-2\Delta s}$) is $f_1(z)$; the second term is the second one in square bracket in formula (4.12); let us indicate it by $i_1(z)$. Then $H_{11} = (t/2w_o)^{-2\Delta s} \cdot f_1(z) + i_1(z)$.

The analyses of the two functions and of their first derivatives lead us to the following conclusions.

Case (a) - Two soliton fields, well separated in space, are created and move away from each other symmetrically.

Case (b) - A double-soliton field, i.e. an overlapping of two solitons, will appear and will "split" into two independent ones that will move away one from the other.

Case (a) $\frac{3-\sqrt{5}}{4} > s_2 > 0$.

By formulae (4.11) and (4.12) and their first derivatives in z one obtains the typical scenario shown in figure 2.

The symmetry in positive and negative values of z is preserved by our previous request $\delta_0 = \pi s_2$. The maximum co-ordinates, $z = \pm z_i$, are solutions of the equation $dH_{11}/dz = 0$; from (4.12) one obtains

$$-\frac{2\sin^2\phi}{w_i\sin^2\delta}\mathrm{ctg}\,\delta\left\{\left(\frac{t}{2w_i}\right)^{-2\Delta s}L_0^2\sin^{4s_2}\phi\cdot\left[2s_2\frac{\mathrm{ctg}\,\phi}{\mathrm{ctg}\,\delta}-\Delta s\right]+\left[\frac{s_1\sin2(\phi+\delta)}{\mathrm{ctg}\,\delta}-\Delta s\sin^2(\phi+\delta)\right]\right\}=0. \quad (4.26a)$$

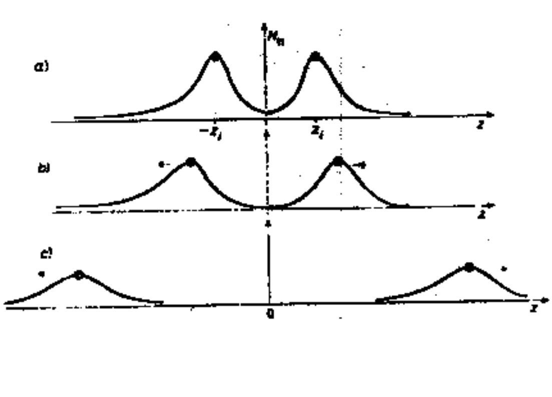

FIGURE 2 - Evolutionary behaviour near the singularity (t = 0) for the soliton field $H_{11}(z)$ defined by formula (4.12), for those cosmological backgrounds in which $(3-\sqrt{5})/4 > s_2 > 0$ under the assumption of plane symmetry ($H_{11}(z) = H_{11}(-z)$, i.e. $\delta_0 = \pi s_2$. (a) t = 0, (b) $t = 2w_i$, (c) $t > 2w_i$.

In particular, for small enough time the first term in square bracket will be dominant and, therefore (once we took into account the trivial solution z = 0, ctgδ=0), eq. (4.26a) reduces to

$$\frac{\mathrm{tg}\,\delta}{\mathrm{tg}\,\phi} = \frac{\Delta s}{2s_2} \quad . \qquad (4.26b)$$

From (4.26b) one may understand also the existence of two characteristic solutions i.e. cases (a) and (b). Indeed, when $1/2 > s_2 > (3-\sqrt{5})/4$, there is only one solution at z = 0 of (4.26a) and no solution of (4.26b), while, if $0 < s_2 < (3-\sqrt{5})/4$, there are two distinct solutions at $z = \pm z_i$ in (4.20b).

When $\delta_0 \neq \pi s_2$, one should adapt formulae (4.12) - (4.14) in order to avoid the fictitious singularity when $\sin\delta = 0$ for some z. If $0 < \delta_0 < 2\pi s_2$, there is no need of any correction and from (4.12) and (4.26) one obtains two asymmetric perturbations schematically shown in figure 3.

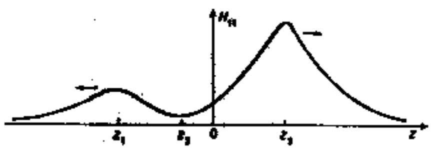

FIGURE 3 - As in fig. 2, under the less restrictive conditions $\delta_0 \neq \pi s_2$; t = 0.

The two peaks move immediately due to the fact that both terms (in (4.26a)) in square brackets have the same sign near the origin.

Case (b) $\frac{1}{2} \geq s_2 \geq \frac{3 - \sqrt{5}}{4}$.

In this case the soliton field H_{11} will be, at the very beginning, an overlapping of the two disturbances. Taking the first derivative and putting it equal to zero, one may find an initial time, t_i, when the maxima at the origin split into two distinct peaks. When $t \to t_i \ll 1$ and for $z \to 0$, one finds (4.26a)

As in the previous case z_1, z_2, z_3 are defined by eq. (4.26b). These situations are more complicated and less easy to analyse. Therefore, we will again assume $\delta_0 = \pi s_2$ in what follows.

Then it is possible to describe the world-line of the peak from the first derivative in z of the soliton field H_{11} in (4.12). If $(3 - \sqrt{5})/4 \geq s_2 > 0$, then the world-lines of the peak are shown in fig. 4.

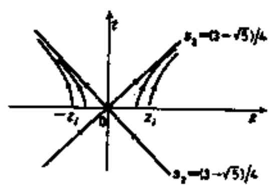

FIGURE 4 - The world-lines of the maximum values of the soliton field H_{11} for the case $(3 - \sqrt{5})/4 \geq s_2 > 0$, $\delta_0 = \pi s_2$; this soliton field is depicted in fig. 2.

$$\frac{t}{2w_i} = \left[L_0 \; \frac{6s_2 - 4s_2^2 - 1}{4(1 - s_2)^2}\right]^{1/2\Delta s} \tag{4.27}$$

Clearly, when the right-hand side in (4.27) exceeds unity, the approximation is no longer valid.

Apparently this is always the case as $\Delta s \to 0$, i.e. when $s_2 \to \frac{1}{2}$.

In reality the situation is more complicated: if, for example, $L_0 = 1$, then there is a finite limit of (4.27) as $\Delta s \to 0$. Indeed

$$\lim_{\Delta s \to 0} \frac{t}{2w_i} = \lim_{\Delta s \to 0} \left[\frac{1 - \Delta s - \Delta s^2}{(1 + \Delta s)^2}\right]^{1/2\Delta s} = \exp\left[-\frac{3}{2}\right] \quad . \tag{4.28}$$

From (4.28) it is clear that, if $L_0 < 1$, then $t_i/2w_i$ is always less than unity and (4.27) always holds.

We can describe $t/2w_i$ for the three possible situations $L_0 > 1$, $L_0 < 1$ and $L_0 = 1$. (Note that, when $\Delta s = 0$, $L_0 = 1$, eqs. (2.3) - (2.6) give the trivial solution $g^0_{ab} = g_0^{(ph)}$).

Figure 5 describes the cases $L_0 > 1$.

Figure 6 describes the cases $L_0 < 1$.

In general one may describe the dynamical behaviour of the soliton field H_{11} as shown in Fig. 7.

The important point to stress from formula (4.27) and from diagram 6 is that t_i is not a monotonic function of s_2 if $L_0 < 1$.

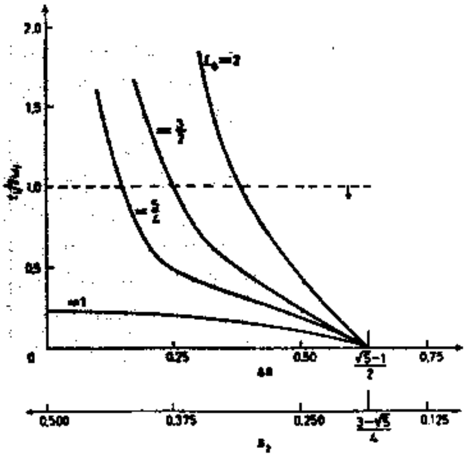

FIGURE 5 - The dimensionless initial time $t_i/2w_i$ (at which the two solitons of H_{11} decouple, i.e., the time when the original soliton "splits" into two distinct solitons) as a function of the cosmological parameter Δs (or equivalently of s_2), for various values of the arbitrary constant L_0: $L_0 \geq 1$. See formula (4.30) as well as fig. 7.

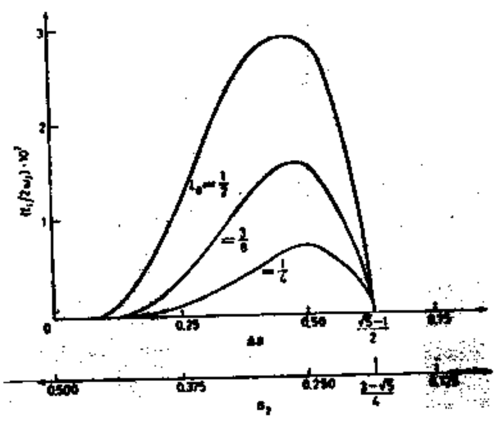

FIGURE 6 - As in fig. 5, for the case $L_0 < 1$.

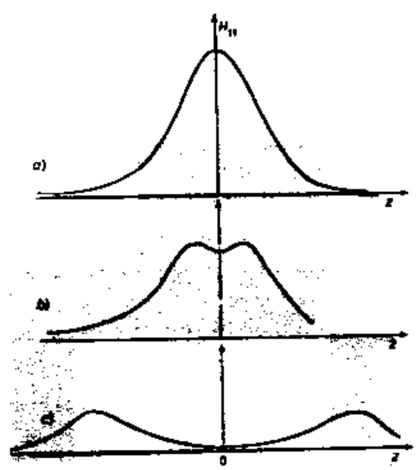

FIGURE 7 - As in fig. 2 when $\frac{1}{2} > s_2 > (3 - \sqrt{5})/4$.

(a) $t_i > t > 0$.

(b) $t \gtrsim t_i$

(c) $t > 2w_i$.

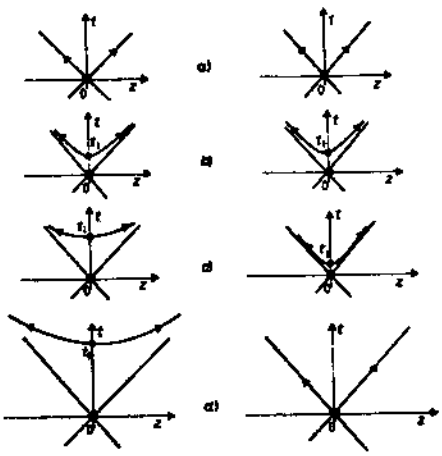

FIGURE 8 - The world-lines of the maximum values of the soliton field H_{11} for cases $\frac{1}{2} > s_2 > (3 - \sqrt{5})/4$, $\delta_0 = \pi s_2$, described in fig. 7. Subcases $L_0 \geq 1$ (l.h.s.) and $L_0 < 1$ (r.h.s) are related to the values $t_i/2w$, analysed in fig. 5 and fig. 6 and described by formula (4.30). (a) $s_2 = (3 - \sqrt{5})/4$, (b) $\frac{1}{2} > s_2 > (3 - \sqrt{5})/4$, (c) $s_2 \lesssim \frac{1}{2}$, (d) $s_2 = \frac{1}{2}$.

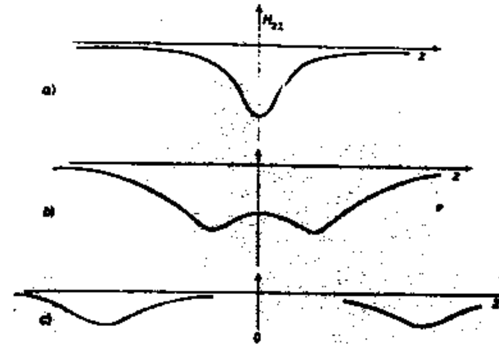

FIGURE 9 - Evolutionary behaviour near the singularity ($t = 0$) for the soliton field $H_{22}(z)$ defined by formula (4.13) for the entire range of values $\frac{1}{2} > s_2 > 0$ under the assumption of plane symmetry ($\delta_0 = \pi s_2$). (a) $t_i > t > 0$, (b) $t \gtrless t_i$, (c) $t > 2w_i$.

Indeed the world-line of the peaks of the solitons may be described in two situations: $L_0 > 1$ or $L_0 < 1$ (fig. 8).

Now we can consider the soliton field H_{22}.

From eqs. (4.11) and (4.13) one obtains the general behaviour for H_{22} shown in fig. 9.

Unlike the situation for H_{11}, no complications like case (a) arise for H_{22}, when s_2 is in the range $\frac{1}{2} > s_2 > 0$.

In analogy to H_{11}, for $(1 + \sqrt{5})/4 > s_2 > \frac{1}{2}$ (i.e. $(3 - \sqrt{5})/4 < s_1 < \frac{1}{2}$) H_{22} behaves as H_{11} by an interchange of s_1 with s_2.

The world-line of the minima of the two solitons may be obtained by requiring $dH_{22}/dz = 0$, i.e. (*)

$$-2w_i^{-1}\frac{\sin^2\phi}{\sin^2\delta}\mathrm{ctg}\delta\{(\frac{t}{2w_i})^{2\Delta s}L_0^{-2}\sin^{4s_1}\phi\left[2s_1\frac{\mathrm{ctg}\phi}{\mathrm{ctg}\delta}-\Delta s\right]+\left[s_2\frac{\sin 2(\phi-\delta)}{\mathrm{ctg}\delta}-\Delta s\,\sin^2(\phi-\delta)\right]\}\,. \qquad (4.29)$$

(*)Note that (4.29) and (4.26a) are equivalent after the exchange $s_2 \leftrightarrow s_1$, $\Delta s \leftrightarrow -\Delta s$, $\delta \leftrightarrow -\delta$.

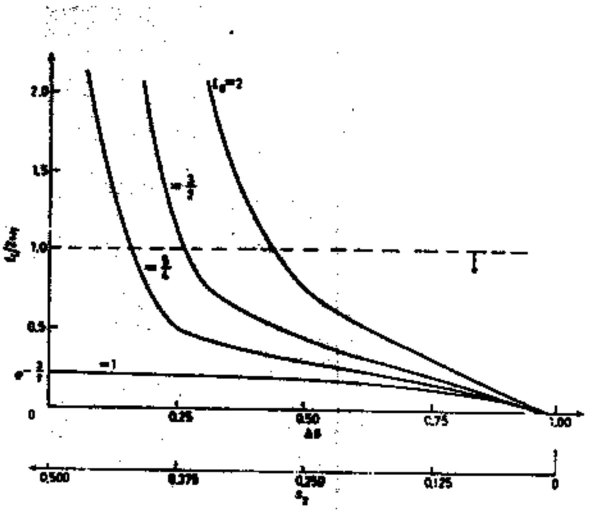

FIGURE 10A - The dimensionless initial time $t_1/2w_i$, at which the two solitons of H_{22} decouple, i.e. when the original soliton "splits" into two distinct solitons, as a function of the cosmological parameter Δs(or equivalently of s_2), for various values of the arbitrary constant L_0: $L_0 \geq 1$.

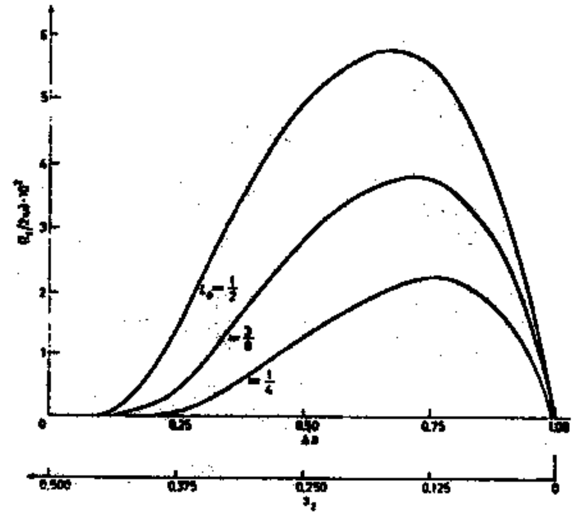

FIGURE 10B - As in fig. 10a when L_0 < .

Here again the question arises of how long a time after t = 0 do the two negative solitons "split"; solving (4.29) in the limit $z \to 0$, one finds (*)

$$\frac{t}{2w_i} = \left[L_0 \, \frac{4s_2^2}{-4s_2^2 + 2s_2 + 1}\right]^{1/2\Delta s} \tag{4.30}$$

This function may be described for the two cases: $L_0 > 1$ in fig. 10a., $L_0 < 1$ in fig. 10b.

One should remember that in fig. 10a, when $t_1/2w_i$ exceeds unity, formula (4.30) is no longer valid, because it violates a previous assumption ($t_1 \ll 2w_i$).

In analogy with fig. 5, 6 the world-lines of the two negative peaks are similar to those described in fig. 8 ($L_0 > 1$, $L_0 < 1$), when the range of s_2 is extended from $\frac{1}{2} > s_2 > (3 - \sqrt{5})/4$ to $\frac{1}{2} > s_2 > 0$.

Note that the limiting cases $s_2 = \frac{1}{2}$, when $L_0 \leq 1$, are the same for t_i in (4.27) and t_i in (4.30).

Finally let us consider the mixed component of the soliton field in formula (4.14), H_{12}.

Then one can see that the first term in square brackets is multiplied by the dominant term $(t/2w_i)^{-\Delta s}$ and it will be perturbed by the second term, proportional to $(t/2w_i)^{\Delta s}$, as time passes. The combined effect can be described dynamically (fig. 11).

As in previous cases one has to investigate the first derivative of H_{12} and, by requiring the result to be zero, one may obtain the zeros of the maxima and minima; from formula (4.14) one finds

(*)One obtains the same result from eq.(4.27) by the obvious following interchange: $s_2 \to s_1$, $\Delta s \to -\Delta s$.

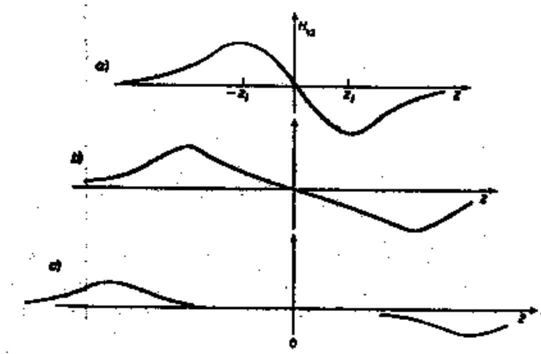

FIGURE 11 - Evolutionary behaviour near the singularity (t = 0) for the mixed-component soliton field $H_{12}(z)$ defined by formula (4.14) for the entire range of values of s_2. (a) t = 0, (b) $t = 2w_i$, (c) $t > 2w_i$.

$$- \frac{\sin^2\phi}{w_i \sin^2\delta} ctg\delta \{ - (\frac{t}{2w_i})^{-\Delta s} L_0 \sin^{2s_2}\phi \cdot \sin(\delta-\phi) \cdot 2 \left[s_2 \frac{ctg\phi}{ctg\delta} - s_2 \frac{ctg(\delta-\phi)}{ctg\delta} - \Delta s \right] +$$

$$+ (\frac{t}{2w_i})^{\Delta s} L_0^{-1} \sin^{2s_1}\phi \, \sin(\delta+\phi) \cdot 2 \left[s_1 \frac{ctg\phi}{ctg\delta} + s_1 \frac{ctg(\delta+\phi)}{ctg\delta} - \Delta s \right] \} = 0 \quad . \tag{4.31}$$

For $t << 2w_i$, i.e. when the first term is dominant, then (4.31) is equivalent to

$$ctg\delta \left[s_2 \frac{ctg\phi}{ctg\delta} - s_2 \frac{ctg(\delta-\phi)}{ctg\delta} - \Delta s \right] = 0 \quad . \tag{4.32}$$

The zeros of the term in square bracket are the zeros $\pm z_i$.

Contrary to previous cases (H_{11}, H_{22}) the disturbances described by H_{12} are created at a finite distance and propagate (*) immediately after t > 0. In fig. 12 a typical world-line is shown of the maxima and minima.

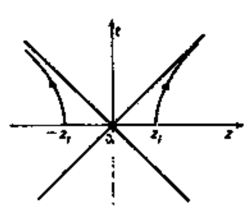

FIGURE 12 - Typical world-lines of the maximum and minimum values of the mixed component of the soliton field H_{12} in Fig.11.

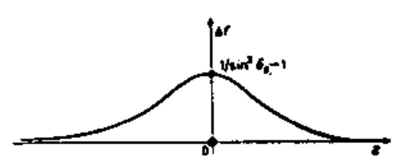

FIGURE 13 - Spatial behaviour of the function Δf defined in (4.15) near the singularity t = 0.

(*) With a velocity less then unity.

From the asymptotic formula (4.15) for the deviation function Δf we found in (4.15)

$$\Delta f = \frac{\sin^2 \delta - \sin^2 \delta_0}{\sin^2 \delta_0} > 0 \quad .$$

In the particular case when $\Delta s = 0$, i.e. $\delta_0 = \pi/2$, $\Delta f = 0$.

We can describe Δf as function of space as shown in fig. 13.

We can summarize our analysis of the metric near the singularity as follows:

- the field H_{11} describes two summetric positive disturbances $H_{11}(z) = H_{11}(-z)$ leaving each other; in case (a), $(3 - \sqrt{5})/4 > s_2 > 0$, the two solitons appear separated from the beginning; in case (b), $\frac{1}{2} > s_2 > (3 - \sqrt{5})/4$, the two solitons overlap near the singularity and decouple after an initial time t_1 described in fig. 5 and fig. 6;
- the field H_{22} describes two negative disturbances that appear overlapped and decouple after an initial t_1 described in fig. 10a and 10b;
- the field H_{12} finally describes two disturbances, one positive and one negative, at a finite distance from the origin defined by eq. (4.32). In this case $H_{12}(z) = - H_{12}(-z)$.

The two peaks move immediately away as shown in fig. 11 and 12.

The perturbation Δf is static and is described in fig. 13.

Case $t = |z|$, $t \to \infty$.

We investigate now the asymptotic behaviour of the soliton field along the light-cone.

From relations (2.7) - (2.9) we obtain

$$\rho = t - \sqrt{w_i t} \quad , \quad \sin \phi = \sqrt{\frac{w_i}{t}} \quad , \quad \cos \phi = 1 \quad . \tag{4.33}$$

Then from the exact solution (3.4) - (3.6) we obtain the following asymptotic expression for the soliton fields:

$$H_{11} = \sqrt{\frac{w_i}{t}} \left[\frac{4 \sin 2\delta_0 + 2(L_0^2 - L_0^{-2})}{4 \sin^2 \delta_0 + (L_0 + L_0^{-1})^2} \right] + \frac{w_i}{t} \left[\frac{4 \cos 2 \delta_0}{4 \sin^2 \delta_0 + (L_0 + L_0^{-1})^2} \right] \quad , \tag{4.34}$$

$$H_{22} = \sqrt{\frac{w_i}{t}} \left[\frac{-4 \sin 2\delta_0 + 2(L_0^2 - L_0^{-2})}{4 \sin^2 \delta_0 + (L_0 + L_0^{-1})^2} \right] - \frac{w_i}{t} \left[\frac{4 \cos 2 \delta_0}{4 \sin^2 \delta_0 + (L_0 + L_0^{-1})^2} \right] \quad , \tag{4.35}$$

$$H_{1=} = - 4 \sqrt{\frac{w_i}{t}} \; \frac{L_0 - L_0^{-1}}{4 \sin^2 \delta_0 + (L_0 + L_0^{-1})^2} \quad . \tag{4.36}$$

One sees immediately that in this limit the typical power behaviour for the field is

$$H_{ab} \sim \frac{1}{\sqrt{t}} \quad ,$$

while, if one reconsiders the previous limit ($t \to 0$, $z \to \pm \infty$, $t \to \infty$) and requires $t = |z|$,then

$$H_{ab} \sim \frac{1}{t} \quad .$$

This means that the largest perturbation lies on the light-cone and, therefore asymptotically, the perturbation propagates at the velocity of light.

In conclusion of this analysis one may calculate from (4.33) and the definition of Δf in(3.3) the perturbation in this limit:

$$\Delta f = \frac{(L_0 + L_0^{-1})^2}{4 \sin^2 \theta_0} > 1 \quad . \tag{4.37}$$

This perturbation is larger than in other cases (*), i.e. for Δf found in eq. (4.15) $(t \to 0)$, eq. (4.10) $(t \to \infty)$, eq. (4.5) $(z \to \infty)$.

Therefore, it proves that the perturbation is localized on the light-cone in agreement with the arguments just expressed for the soliton field H_{ab}.

Therefore, asymptotically the soliton waves propagate at the velocity of light.

IV.5 - REFERENCES

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3 - V.A.Belinskii and I.M.Khalatnikov, Sov. Phys. JETP, 30, 1174, 1970.
4 - V.A.Belinskii and D.Fargion,"The inverse-scattering problem technique in gravitation theory"(in press).

(*) If one indicates by Δf_4 the perturbation in light-cone in the limit (4.37), Δf_3 in the $t \to 0$ limit (4.15), Δf_2 in $t \to \infty$ limit (4.10) and Δf_1 in the $z \to \infty$ limit in (4.5), then one obtains the disequality $\Delta f_4 > \Delta f_2 > \Delta f_3 > \Delta f_1$.