Solitons and Vertex Operators in Affine Toda Field Theories

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Abstract

Affine Toda field theories are relativistic integrable theories in two dimensions. Each of them are associated to an affine Kac-Moody algebra $\mathfrak{g}^{(k)}$. When the coupling is chosen to be imaginary these theories possess soliton solutions. These nevertheless carry real energy and momentum. We obtain a vertex operator construction which determine the asymptotic behaviour of these solitons. This result gives some information about the topological charges of the solitons. Using this vertex operator it is also shown that a N-soliton solution at $t \to \pm\infty$ has the form of N isolated solitons. The overall lateral displacement of the soliton trajectories due to the scattering with several other solitons is determined and shown to be independent of the temporal order in which the collisions take place. This result constitutes a classical analogue of the Yang-Baxter equation. For the collision of two solitons, the time delay in the center-of-momentum frame is always negative and proportional to the logarithm of the function which comes from the normal ordering of two vertex operators. The soliton scattering matrix in semi-classical approximation is then derived from the time delay. Various results of twisted and non-simply laced theories are obtained from the simply laced ones by using folding procedures. In particular it is shown that the particles and solitons of the twisted theory are a subset of the unfolded theory and the criterion of "survival" is the same for both, particles and solitons.

Preface

This thesis is the result of the work carried out at Imperial College and University of Wales, Swansea, in the period between January 1992 and June 1995, under the supervision of Prof. D.I. Olive. No part of it has been previously submitted for a degree. Most of the original part of this thesis has been published in three papers[36, 37, 25].

I am particularly grateful to D.I. Olive for his guidance throughout this time. He has introduced me to the subject and it was a very valuable experience to collaborate with him. I would like to thank my collaborator A.Fring for so many discussions and also N. Dorey and T. Hollowood.

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DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree

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STATEMENT 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by giving explicit references. A bibliography is appended.

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Contents

Introduction

Two dimensional integrable field theories have been the subject of intense research in the last years. One class of these integrable theories are the conformal field theories which have the string theory as one of the original motivations. Conformal field theories have been also applied to the study of second order phase transitions of two dimensional materials. It has been pointed out by Zamolodchikov [63] that it is possible to add some special deformations to a conformal field theory which breaks conformal symmetry but preserves the integrabilty of the theory. One of the reason for studying two dimensional massive integrable theories is that the scattering-matrix is factorizable and it must satisfy Yang-Baxter equation and hence the old dream of obtaining an exact expression for S-matrices can be realized. Affine Toda field theories (ATFTs) provide a concrete example of Zamolodchikov concept of integrable deformation [30, 17].

If the coupling constant is take to be imaginary, ATFT present a degenerated vacua and soliton solutions interpolating between these vacua. In chapter 1 we shall discuss some general properties of solitons in $1+1$ dimensions. We also analyse the soliton solutions of sine-Gordon theory which is the simplest example of ATFTs with imaginary coupling.

In this thesis we shall use Lie algebraic methods to investigate various features of ATFTs and in particular their soliton solutions. Therefore we shall present in chapter 2 a very brief review of some of the principal points of finite and infinite dimensional Lie algebras which will be relevant in the following chapters.

As will be discussed in chapter 3, to each simple Kac-Moody algebra $\mathfrak{g}^{(k)}$ there corresponds an ATFT which equation of motion is obtained from a zero curvature condition. They are scalar theories with r species of particles where r denotes the rank of the Lie algebra associated to the Dynkin diagram obtained by removing the spot "0" of the Kac-Moody Dynkin diagram. The spins of the conserved charges are the exponents of $\mathfrak{g}^{(k)}$[47]. When $\mathfrak{g}^{(k)} = a_1^{(1)}$, we obtain sine/sinh-Gordon theory. For the untwisted theories, the particle masses are proportional to the components of the right Perron-Frobenius of the Cartan matrix [5, 21, 23]. By folding procedure the particle masses of ATFT based on a twisted $\mathfrak{g}^{(k)}$ is a subset of the masses of the unfolded theory. This has first been shown in a case by by case analysis[5]. Here we present a unified derivation for the criterion for "survival"[37]: in order to a particle associated to the vertex i of the unfolded Dynkin diagram to survive to the folding, it must satisfy the condition $\exp(-2\pi i\lambda_i \cdot \lambda_{\tau(0)}) = 1$ where τ is the inner automorphism used in the folding procedure. This turn out to be also the condition for survival of a soliton.

The solitons for ATFTs has been first constructed for the $a_n^{(1)}$ and $d_4^{(1)}$ theories[31] using τ-functions. It also has been shown that despite the complex nature of the energy and momentum, the soliton solutions have real energy and momentum, and positive masses. A number of authors[39, 3, 7] have obtained one soliton solutions for other $\mathfrak{g}^{(k)}$ using τ-functions. In chapter 4 we shall see a general approach for obtaining the multisoliton solutions exploiting the underlying Kac-Moody algebra[49, 50, 36, 37, 25]. Using this approach it has been extended the result that they have finite real energy and momentum for any untwisted [49] and twisted[37] algebras. Moreover it has been shown that the number of species of solitons is equal to the number of species of particles. In this formalism, the individual solitons are "created" by group elements obtained by exponentiating some generators $\hat{F}^i(\theta)$. Each exponential series terminates [50] with the highest non-vanishing power being expressed as a vertex operator for simply laced[36] and twisted algebras[37]. Here we extend this result for non-simply laced algebras. Using this vertex operator we showed explicitly that the soliton solutions correctly interpolated the degenerated vacua[36]. We also obtained a classical analogue of the crossing property of analytic S-matrix.

In chapter 5 we shall show that these vertex operators determine yet more detail of the asymptotic behaviour of the soliton solutions[25]: it is shown for any simply laced $\mathfrak{g}^{(k)}$ that a N-soliton solution at $t \to \pm\infty$ has the form of N isolated solitons and the only effect of the collisions is a lateral displacement in space-time or time-delay. For a two soliton solution this displacement is expressed in terms of the logarithms of the functions $X_{ij}(\theta_i - \theta_j)$ arising in the procedure of normal ordering of two vertex operators associated to solitons of species i and k. Furthermore it is shown that this functions take values restricted between 0 and 1, resulting that the time delay in the centre of momentum frame is always negative. The overall lateral displacement of a soliton trajectory due to the scattering with several other solitons is determined and shown to be independent of the temporal order in which the collisions take place and that the two-soliton time delay is the fundamental block as expected in an integrable theory. This result constitutes a classical analogue of the Yang-Baxter equation for the quantum scattering matrix featuring the factorisation property of n-particle S-matrix into two-particle S-matrix. Finally from the time delay we obtain the leading order of the soliton S-matrix in the semi-classical approximation.

In the conclusion we shall summarize our results and shall make a number comments indicating some open problems.

Chapter 1

Solitons in 2 dimensional relativistic field theories

1.1 Introduction

The concept of solitary waves was first introduced in August 1842 by J. Scott-Russell when he presented a report [54] at the British Association for Advancement of Science in which he describes the observation of a wave travelling in an narrow channel without change of shape or velocity. In 1895 Korteweg and de Vries[34] presented a non-linear equation, called nowadays as KdV equation, possessing solutions with the properties of solitary waves. A long time later it has been observed by computer simulation that when these solitary waves of the KdV equation are launched on a collision, they emerged from the collision having the same shapes and velocities with which they entered. Shortly before Perring and Skyrme[51] have observed the same properties for the solitary waves of sine-Gordon theory. It then has been coined the name soliton for the solitary waves with asymptotically preserves its shape upon collision with other solitary waves.

Based in these examples, let us give a more formal definition of solitary waves and solitons: consider a system of field equations with associated energy density $\epsilon(x,t)$. Its space integral is the conserved total energy E which is considered to be bounded from below. Without lost of generality the minimal value of the energy E can be set to zero. In this case we say that a solution of a field equation is localised when $\epsilon(x,t)$ at any time t is finite in some finite region in space and falls to zero sufficiently fast to give a finite energy E. Then we formally define a solitary wave as a localised solution of any non-linear equation of the form $f(x,t) = f(x - vt)$, where v is its velocity. Note that any static localised solution is a solitary wave with $v = 0$. Once a static solution is known, moving solutions are obtained by boosting.

Consider now a field theory with solitary wave solutions. Consider that exist another solution which in the far past consists of say N solitary waves with arbitrary initial positions and velocities and that this solution evolves to the far future giving once more N solitary waves. In this case this solution is called N-soliton solution and the solitary waves are called one-soliton solutions or just solitons. While all one-soliton solutions are solitary waves, the converse is not true. One of the attractive features of solitons is that they can be regarded as a classical theory of a particle with finite extension.

1.2 Solitons in 1+1 dimensions

We are interested in finding solitons solutions in relativistic scalar field theories in 1+1 dimensions[1]. We shall start by considering a single real scalar field theory whose dynamics is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{\beta^2}U(\beta\phi) \tag{1.2.1}$$

where $U(\beta\phi)$ is a function of ϕ but not of its derivatives, μ is a mass parameter and β is a dimensionless coupling constant. If we consider that $\phi = 0$ is a vacuum of the theory and that $U(\phi) \cong 1/2(\phi)^2$ for small ϕ, then the mass of the particle will be μ. One can then obtain the equation of motion

$$\partial_\mu\partial^\mu\phi = -\frac{\mu^2}{\beta^2}\frac{\partial U(\beta\phi)}{\partial\phi}. \tag{1.2.2}$$

The associated total energy functional is given by

$$E[\phi] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\partial_t\phi\right)^2 + \frac{1}{2}\left(\partial_x\phi\right)^2 + \frac{\mu^2}{\beta^2}U(\beta\phi)\right]. \tag{1.2.3}$$

In order to the energy be bounded from below, U must be bounded from below. We can always add a constant to U such that its minimum value is zero. Then the energy is minimised, and equal to zero, when $\phi(x,t)$ is a constant in space-time and equal to one of the zeros of U. These values of $\phi(x,t)$ are the vacua or "ground states" of the theory.

Note that β can be eliminated from (1.2.2) by performing the substitution $\phi \to \sigma := \beta\phi$. Let us consider static solutions for which the equation of motion reduces to

$$\partial_x^2\sigma = \mu^2\frac{\partial U(\sigma)}{\partial\sigma}. \tag{1.2.4}$$

[1]Throughout this thesis it will be adopted the conventions c $= 1 = \hbar$ and $ds^2 = dt^2 - dx^2$.

In order σ to be a solitary wave it must have localised energy and finite total energy. In view of (1.2.3) this implies that σ must approach one of the zeros of $U(\sigma)$, with are the vacua of the theory, as $x \to \pm\infty$.

From (1.2.4) one obtains that

$$(\partial_x \sigma)^2 - 2\mu^2 U(\sigma) = \text{const.} = 0 \tag{1.2.5}$$

where the constant vanishes due to the boundary condition $U(\sigma) \to 0$ and $\partial_x \sigma \to 0$ as $x \to \pm\infty$. Then one obtains the Bogomolny equation

$$\partial_x \sigma = \pm\sqrt{2\mu^2 U(\sigma)}. \tag{1.2.6}$$

Upon integration it gives

$$\int_{\sigma(x_0)}^{\sigma(x)} \frac{d\sigma}{\sqrt{2U(\sigma)}} = \pm\mu(x - x_0) \tag{1.2.7}$$

where x_0 is a integration constant which can be interpreted as the soliton coordinate.

The classical mass of a static solitary wave is equal to its energy. Therefore using (1.2.3) and (1.2.6) it follows that

$$\frac{M_{sol}}{\mu} = \frac{2\mu}{\beta^2}\int_{-\infty}^{\infty} dx U(\sigma) = \frac{1}{\beta^2}\left|\int_{\sigma(-\infty)}^{\sigma(\infty)} d\sigma\sqrt{2U(\sigma)}\right|. \tag{1.2.8}$$

We can observe that the soliton and particle mass ratio will increase as β decreases.

If we take the variable x as time and σ as the coordinate of a particle, then (1.2.4) can be interpreted as Newton's second law of a particle in a potential given by $-U(\sigma)$. Using this mechanical analogy it is easy to conclude that if $U(\sigma)$ has only one zero (i.e. the theory has one vacuum) then there are no non-trivial time-independent solutions of finite energy. On the other hand if $U(\sigma)$ has more than one zero, then there always exist static solitary waves. Such solutions must be monotonic, interpolating between one zero of $U(\phi)$ at $x \to -\infty$ to an adjacent zero of $U(\sigma)$ at $x \to \infty$.

Given a scalar theory in two dimensions we can always define a conserved current

$$j^\mu = \epsilon^{\mu\nu}\partial_\nu \phi. \tag{1.2.9}$$

The associated charge

$$Q_{top} = \int_{-\infty}^{\infty} dx j^0 = \phi(+\infty) - \phi(-\infty) \tag{1.2.10}$$

is called topological charge which, from what has been said previously, is non zero for the solitary waves.

Until the moment we did not need to give the explicit form of the potential. However in order to say that a solitary wave is a soliton we must specify $U(\phi)$. This will be done in the next section.

1.3 Sine-Gordon theory

The sine-Gordon Lagrangian is given by (1.2.1) with

$$U(\beta\phi) = 2\sin^2\frac{\beta\phi}{2}. \tag{1.3.11}$$

The Lagrangian has the discrete symmetries

$$\begin{aligned} \phi(x) &\rightarrow -\phi(x) \ , \\ \phi(x) &\rightarrow \phi(x) + \tfrac{2\pi}{\beta}n \ , \quad n \in \mathbb{Z} \ , \end{aligned} \tag{1.3.12}$$

and $U(\phi)$ has zeros at $\phi = 2\pi n/\beta$ with are the vacua of the theory with vanishing energy E. The associated equation of motion is

$$\partial^2\phi + \frac{\mu^2}{\beta}\sin\beta\phi = 0 \ . \tag{1.3.13}$$

From (1.2.8) we conclude that

$$\frac{M_{sol}}{\mu} = \frac{8}{\beta^2} \tag{1.3.14}$$

Before discussing about the solitons of this of this theory we shall make some few remarks about sine-Gordon. The first is that it is a classical integrable theory in the sense that it has an infinite number of conserved charges which are in involution. If we take $\beta \rightarrow i\beta$ in (1.3.13) we obtain sinh-Gordon theory which is also integrable but does not have a degenerated vacua and therefore does not possess solitons. Sinh-Gordon theory can be obtained by adding a deformation term to Liouville theory, which is conformal invariant. This can be seen in the following way: consider the equation of motion

$$\partial^2\phi + \frac{\mu^2}{\beta}\left(e^{\beta\phi} - \eta e^{-\beta\phi}\right) = 0 \ , \tag{1.3.15}$$

where η is a constant parameter. If we take $\eta = 0$ we obtain Liouville equation of motion. On the other hand if $\eta \neq 0$ we can perform a field redefinition $\bar{\phi}(x) = \phi(x) - (1/2\beta)\ln\eta$ and we obtain the sinh-Gordon equation of motion

$$\partial^2\bar{\phi} + \frac{\eta^{1/2}\mu^2}{\beta}\sinh\beta\bar{\phi} = 0 \ . \tag{1.3.16}$$

We can see that the particle mass is $\mu\eta^{1/4}$ with is singular as η approaches zero. We can then conclude that (1.3.15) describes a theory with a critical behaviour when $\eta = 0$. This example gives a Lagrangian realization of Zamolodchikov's ideia of obtaining integrable theories from CFT by adding some particular deformation[62].

Let us now analyse the soliton solutions of sine-Gordon. Putting the sine-Gordon potential in (1.2.7) and using the variable $\gamma = e^{-i\frac{\sigma}{2}}$, one can obtain the static solutions

$$e^{\frac{-i\beta\phi_\pm}{2}} = \frac{1+Qe^{\pm\mu x}}{1-Qe^{\pm\mu x}} \tag{1.3.17}$$

where Q is a complex constant of integration which must satisfy $Q^* = -Q$ since ϕ is real. This solution can be boosted by doing $x \to x\cosh\theta - t\sinh\theta$, resulting in

$$e^{\frac{-i\beta\phi_\pm}{2}} = \frac{1+Qe^{\pm\mu(x\cosh\theta - t\sinh\theta)}}{1-Qe^{\pm\mu(x\cosh\theta - t\sinh\theta)}} . \tag{1.3.18}$$

We can parametrize $Q = -ie^{\mp\mu x_0\cosh\theta}$ where x_0 is a real number and corresponds to the position of the soliton (i.e. the maximum of the energy density) when $t=0$. The ambiguity in sign corresponds to having a soliton or an anti-soliton since the two solutions run relatively backwards in space and therefore they have reversed spatial boundary conditions. Indeed, one can check that the topological charges (1.2.9) of these two solutions are

$$Q_{top}(\phi_\pm) = \pm\frac{2\pi}{\beta} . \tag{1.3.19}$$

In order to satisfy the reality condition of ϕ we can consider an alternative constant of integration $\bar{Q} = ie^{\mp\mu x_0\cosh\theta}$. In this case we shall have the static solution

$$e^{\frac{-i\beta\bar{\phi}_+}{2}} = \frac{1+ie^{\mu(x-x_0)}}{1-ie^{\mu(x-x_0)}} = e^{i\pi}e^{\frac{-i\beta\phi_-}{2}} . \tag{1.3.20}$$

Therefore for a given sign of x, the different choices of phases for the constant of integration will result on solutions with different topological charges.

Let us now analyse the two soliton solution. We shall show later that it can be written as

$$e^{-i\beta\frac{\phi}{2}} = \frac{1-Q_1W_1-Q_2W_2+Q_1Q_2W_1W_2X}{1+Q_1W_1+Q_2W_2+Q_1Q_2W_1W_2X} \tag{1.3.21}$$

where

$$W_i = e^{\mu(x\cosh\theta_i - t\sinh\theta_i)} \tag{1.3.22}$$

$$Q_i = e^{i\psi_i}e^{-\mu x_i^0\cosh\theta_i} \tag{1.3.23}$$

$$X = \tanh^2\left(\frac{\theta_1-\theta_2}{2}\right) . \tag{1.3.24}$$

where θ_i is the rapidity of soliton "i" and ψ_i is a phase. We shall consider that $\theta_1 > \theta_2$. Now let us track soliton 1. By tracking the faster soliton 1 we mean to remain in the vicinity of the soliton 1, which is near $x = v_1 t + x_1^0$. This implies that

$$W_1(\theta_1) = e^{\mu\cosh\theta_1(x-v_1t)}, \tag{1.3.25}$$

is hold fixed as time varies. On the other hand, for $x - v_1 t$ fixed, the time dependence of $W_2(\theta_2)$ is given by

$$W_2(\theta_2) \; = \; e^{\mu \cosh\theta_2 (x - v_2 t)} = \text{const} \;\; e^{\mu \cosh\theta_2 (v_1 - v_2) t} \quad . \tag{1.3.26}$$

Hence, in the past, $t \to -\infty$, $W_2(\theta_2)$ tends to 0 as $v_1 > v_2$. Thus, by (1.3.21), in the past

$$e^{-\frac{i\beta}{2}\phi} \;\to\; \frac{1 - Q_1 W_1}{1 + Q_1 W_1} \tag{1.3.27}$$

which we recognise as a single soliton solution 1 with velocity v_1. For comparison, let us track soliton 1 in the two soliton solution into the future, so $t \to \infty$, with W_1 fixed, so that W_2 tends to plus infinity. From (1.3.21) this yields

$$e^{-\frac{i\beta}{2}\phi} \;\to\; -\frac{1 - Q_1 W_1 X}{1 + Q_1 W_1 X} \tag{1.3.28}$$

Again we recognise a single soliton solution 1, with rapidity θ_1 and phase ψ_1. But now Q_1 has acquired a factor X which changes its modulus(but not the phase), and hence x_1^0. The effect is that

$$x_1^0 \to x_1^0 - \frac{\cosh\theta_1}{\mu} \ln X \quad . \tag{1.3.29}$$

Since $0 < X < 1$ this implies that soliton 1 suffer a displacement to the right. The same analyses can be done for the soliton 2. Once more the only effect of the collision will be a displacement, in this case to the left. Therefore if soliton 1 and soliton 2 have different phases ψ_i, i.e., one is a soliton and the other is an anti-soliton, then the solution describes a transmission with attractive force. However if the solitons are indistinguishable, i.e., with the same phases, then we can no longer say if the scattering describes a transmission or a reflection. In the last chapter we shall discuss in more detail this point. An important observation of the above result is that we have shown that the two soliton solution behaves asymptotically like two single soliton solutions. The same is true for an arbitrary N-soliton solution as will be seen in the last chapter. This proves that these solutions satisfy our definition of soliton.

We now want to obtain soliton solutions in other relativistic scalar field theories in 1 + 1 dimensions. Before doing this we shall review some points of Lie algebras.

Chapter 2

Lie Algebras

This chapter is intended to fix notations and conventions and to give a brief review of some of the principal points of finite and infinite dimensional Lie algebras that will be relevant for the subsequent chapters.

2.1 Finite dimensional compact Lie algebras

Let $\mathfrak{g}$ be a finite dimensional compact Lie algebra of dimension d and rank r and let $\mathfrak{h}$ be a Cartan subalgebra(CSA) of $\mathfrak{g}$. A standard choice of basis for $\mathfrak{g}$ is to seek a maximal set of commuting hermitian generators H_i, $i = 1, 2, \ldots, r$, which form a basis for $\mathfrak{h}$, and $(d - r)$ step operators E_α which ad-diagonalise H_i. For most of the time it will use the Cartan-Weyl basis in which

$$\begin{aligned} \mathrm{Tr}(H_i, H_j) &= \delta_{i,j}, \\ \mathrm{Tr}(E_\alpha, E_\beta) &= \frac{2}{\alpha^2}\delta_{\alpha+\beta,0}, \\ \mathrm{Tr}(H_i, E_\alpha) &= 0, \end{aligned} \tag{2.1.1}$$

where the trace is taking in the adjoint representation. In this basis the commutation relations read

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, E_\alpha] &= \alpha^i E_\alpha, \\ [E_\alpha, E_\beta] &= \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root}, \\ \frac{2\alpha}{\alpha^2}\cdot H & \text{if } \beta = -\alpha, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{2.1.2}$$

where $N_{\alpha,\beta} = -N_{\beta,\alpha}$ are some non-vanishing constants whose exact form we shall not require. We shall also consider that

$$\begin{aligned} H_i^\dagger &= H_i, \\ E_\alpha^\dagger &= E_{-\alpha}. \end{aligned} \tag{2.1.3}$$

Let us denote the set of roots of $\mathfrak{g}$ by $\Phi(\mathfrak{g})$ and its set of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ by $\Pi(\mathfrak{g})$. In a simple root basis any root α is written as $\alpha = \sum_{i=1}^r k_i\alpha_i$ with k_i are either non-negative or non-positive integers and they are called positive and negative roots respectively. The integer $\mathrm{ht}(\alpha) := \sum_{i=1}^r k_i$ is called the height of α. We denote by ψ the root with maximal height and it is called highest root.

For each root α, $E_\alpha, , E_{-\alpha}, 2\alpha.H/\alpha^2$ form a su(2) subalgebra. Therefore the eigenvalues of $2\alpha.H/\alpha^2$ are integers. Its eigenvalues in the adjoint representation are $2\alpha.\beta/\alpha^2$, for all roots β, together with r zeros. Thus

$$\frac{2\alpha.\beta}{\alpha^2} \in \mathbb{Z} \qquad \text{for all roots } \alpha, \beta\ . \tag{2.1.4}$$

For each root α we can associate a Weyl reflection, σ_α , which is a linear transformation such that

$$\sigma_\alpha\beta = \beta - \frac{2\beta\cdot\alpha}{\alpha^2}\alpha \tag{2.1.5}$$

and which geometrically corresponds to a reflection of β in the plane perpendicular to α. Clearly $\sigma_\alpha^2 = 1$. One important property of the Weyl reflections is that if α and β are roots than $\sigma_\alpha\beta$ is also a root. Moreover, the set consisting of the identity operator, the Weyl reflections and all products of Weyl reflections form a group, called the Weyl group, and which will be denoted by $W(\mathfrak{g})$. As every element of $W(\mathfrak{g})$ maps the set of roots into itself, $W(\mathfrak{g})$ must be a subset of the set all permutations of the roots, which is a finite group. Therefore $W(\mathfrak{g})$ is also finite. It can be shown that any element of $W(\mathfrak{g})$ can be written as a product of simple root Weyl reflections σ_i, $i = 1, 2, \ldots, r$.

Sometimes it will be convenient to use the Chevalley basis in which,

$$H_{\alpha_i} := \frac{2\alpha_i}{\alpha_i^2}\cdot H\ \ . \tag{2.1.6}$$

Then the generators $H_{\alpha_i}, E_{\pm\alpha_j}$ satisfy the relations

$$\begin{aligned} \left[H_{\alpha_i}, H_{\alpha_j}\right] &= 0, \\ \left[H_{\alpha_i}, E_{\pm\alpha_j}\right] &= \pm K_{ji}E_{\pm\alpha_j}, \\ \left[E_{\alpha_i}, E_{-\alpha_j}\right] &= \delta_{i,j}H_{\alpha_i}. \end{aligned} \tag{2.1.7}$$

These commutators together with the Serre relations

$$(\mathrm{ad}\ E_{\pm\alpha_i})^{1-K_{ij}}\, E_{\pm\alpha_j} = 0 \tag{2.1.8}$$

are sufficient to describe the algebra $\mathfrak{g}$.

The information in the Cartan matrix

$$K_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}, \tag{2.1.9}$$

can be encoded into the Dynkin diagram which we shall denote by $\Delta(\mathfrak{g})$. In these diagrams each simple root is assigned a point and $K_{ij}K_{ji}$ lines are drawn between the points associated to α_i and α_j with an arrow pointing from α_i to α_j if $\alpha_i^2 > \alpha_j^2$. The set of roots $\alpha^{\mathrm{v}} = \frac{2\alpha}{\alpha^2}$ are called coroots. They form a root system and the associate algebra is called dual Lie algebra which is denoted by $\mathfrak{g}^{\mathrm{v}}$. In table 1 is given the list of all Dynkin diagrams associated to simple Lie algebras.

Let us now pass to the representation theory of $\mathfrak{g}$. Given a irreducible finite dimensional representation of $\mathfrak{g}$, we can take a basis of eigenvectors of H_i, $|\mu_j\rangle$, in which

$$H_i|\mu_j\rangle = \mu_j^i|\mu_j\rangle \tag{2.1.10}$$

and r-component vector μ is called weight vector. Recall that for any root α, $H_\alpha = 2\alpha.H/\alpha^2$ has only integer eigenvalues. Therefore,

$$\frac{2\alpha \cdot \mu}{\alpha^2} \quad \in \mathbb{Z} \ \text{ for all roots } \alpha. \tag{2.1.11}$$

This is the only condition on the allowed weights and it defines a lattice $\Lambda_W(\mathfrak{g})$ called weight lattice of $\mathfrak{g}$. Since the roots satisfy the same condition it implies that they are weights and by taking an integral combination of the roots it is generated the root lattice $\Lambda_R(\mathfrak{g})$, which is a sublattice of $\Lambda_W(\mathfrak{g})$.

The fundamental weights λ_i, defined by

$$\frac{2\alpha_i \cdot \lambda_j}{\alpha_j^2} = \delta_{ij} \quad , \qquad 1 \le i, j \le r, \tag{2.1.12}$$

can be take as a basis of $\Lambda_W(\mathfrak{g})$. Then any weight $\mu \in \Lambda_W(g)$ is of the form

$$\mu = \sum_{j=1}^{r} l_j \lambda_j \quad , \qquad l_j \in \mathbb{Z}. \tag{2.1.13}$$

Let us define

$$\rho^{\mathrm{v}} := \sum_{i=1}^{r} \lambda_i^{\mathrm{v}} \quad , \qquad \lambda_i^{\mathrm{v}} := \frac{2\lambda_i}{\alpha_i^2} \quad , \tag{2.1.14}$$

where λ_i^{v} are called the fundamental coweights. In a given finite dimensional representation, a weight λ is called the highest weight if $\rho^{\mathrm{v}} \cdot \lambda$ is the largest. It follows that the corresponding highest weight state $|\lambda\rangle$ will satisfy

$$E_\alpha|\lambda\rangle = 0 \ , \qquad \alpha > 0 \tag{2.1.15}$$

because otherwise there would exist a weight $\lambda + \alpha$ in the representation higher than λ. It can be shown that for each highest weight state $|\lambda\rangle$ there corresponds a finite dimensional irreducible representation and the weight state basis of this representation can be written as

$$\begin{aligned} |\mu\rangle &= E_{-\beta_1} \dots E_{-\beta_k}|\lambda\rangle \ , \\ H_i|\mu\rangle &= (\mu^i - \beta_1^i - \dots - \beta_k^i)\,|\lambda\rangle \ , \end{aligned} \tag{2.1.16}$$

with $\beta_1, \dots, \beta_k$ being positive roots.

The highest root ψ is the highest weight of the adjoint representation and the roots are its weights. It is easy to see that, for a root α, the scalar product $\rho^{\mathrm{v}} \cdot \alpha$ coincides with our previous definition of height of a root α.

2.2 Affine Lie algebras

2.2.1 Untwisted affine Lie algebras

Consider T^a, $1 \leq a \leq \dim \mathfrak{g}$, to be a basis of a finite dimensional compact Lie algebra $\mathfrak{g}$ satisfying the commutation relations

$$[T^a, T^b] = i f^{abc} T^c. \tag{2.2.17}$$

We introduce the generators

$$T_n^a := T^a \lambda^n \ , \qquad n \in \mathbb{Z} \tag{2.2.18}$$

where λ represents the unit circle in the complex plane $|\lambda| = 1$. These generators form an infinite dimensional algebra

$$[T_m^a, T_n^b] = i f^{abc} T_{m+n}^c \ . \tag{2.2.19}$$

It can be proven that this algebra admits a non-trivial central extension of the form

$$\begin{aligned} \left[T_m^a, T_n^b\right] &= i f^{abc} T_{m+n}^c + m \delta_{m,-n} \mathrm{Tr}\left(T^a, T^b\right) k \ , \\ [k, T_m^a] &= 0 \ , \end{aligned} \tag{2.2.20}$$

where k is the central element. This algebra is called untwisted affine Lie algebras or untwisted Kac-Moody algebras and we shall denote them by $\mathfrak{g}^{(1)}$. If the center k is zero the algebra is called loop algebra. Note that the elements T_0^a generate a subalgebra of $\mathfrak{g}^{(1)}$ isomorphic to $\mathfrak{g}$.

In the Cartan-Weyl basis, $\mathfrak{g}^{(1)}$ will read

$$\begin{aligned}
[H_m^i, H_n^j] &= m\delta_{i,j}\delta_{m,-n}k, \\
[H_m^i, E_n^\alpha] &= \alpha^i E_{m+n}^\alpha, \\
\left[E_m^\alpha, E_m^\beta\right] &= \begin{cases} N_{\alpha,\beta}E_{m+n}^{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root,} \\ \frac{2\alpha}{\alpha^2}\cdot H_{m+n} + m\delta_{m,-n}k & \text{if } \beta = -\alpha, \\ 0 & \text{otherwise,} \end{cases} \\
[k, E_m^\alpha] &= [k, H_m^i] = 0 \ .
\end{aligned} \tag{2.2.21}$$

Since $\lambda^* = \lambda^{-1}$ we shall have the hermiticity conditions

$$(H_m^i)^\dagger = H_{-m}^i \quad , \quad (E_m^\alpha)^\dagger = E_{-m}^{-\alpha} \quad , \quad (k)^\dagger = k \quad , \tag{2.2.22}$$

where the last condition can be obtained for instance from the first commutation relation in (2.2.21).

We could think to take as CSA the set k, H_0^i, $1 \leq i \leq r$, which form a $(r+1)$-dimensional Abelian subalgebra. However this Abelian subalgebra is not maximal since $[H_0^i, H_m^j] = 0 = [k, H_m^j]$. This problem can be solved by appending to $\mathfrak{g}^{(1)}$ an extra element d:

$$[d, T_n^a] = nT_n^a \ , \qquad [d, k] = 0 \ . \tag{2.2.23}$$

The element d provides a gradation to $\mathfrak{g}^{(1)}$ called homogeneous gradation. We can identify $d = \lambda\frac{d}{d\lambda} = -L_0$, where L_0 is an element of the Virasoro algebra. Instead of d, one can append to $\mathfrak{g}^{(1)}$ the element

$$d' := hd + T_0^3 \ , \tag{2.2.24}$$

where $T_0^3 = T^3 := \rho^{\mathrm{v}}.H$. The element d' provides the so-called principal gradation. We shall move between both gradations, using the one which is most convenient for our proposes.

Now we can take, H_0^i, $1 \leq i \leq r$, k, d as a basis of the CSA of $\mathfrak{g}^{(1)}$, which we shall denote by $\mathfrak{h}^{(1)}$. It will have as step operators, E_n^α corresponding to the root $(\alpha, 0, n)$ and H_n^i corresponding to $(0,0,n)$. A simple root system of $\mathfrak{g}^{(1)}$ will consist of the $(r+1)$ roots

$$a_i = (\alpha_i, 0, 0) \ , \ \ 1 \leq i \leq r \ , \qquad a_0 = (-\psi, 0, 1) \ . \tag{2.2.25}$$

Then any root of $\mathfrak{g}^{(1)}$ can be written as

$$a = \sum_{i=0}^{r} k_i a_i \tag{2.2.26}$$

where k_i are either non-negative or non-positive integers.

Analogously to finite dimensional Lie algebras, the Killing form on $\mathfrak{g}^{(1)}$ by definition must satisfy

$$\begin{aligned} \langle A, B \rangle &= \langle B, A \rangle \ , \\ \langle [A, BC] \rangle &= 0 \ . \end{aligned} \tag{2.2.27}$$

It is also considered that the generators T_0^a, which form an subalgebra isomorphic to $\mathfrak{g}$, have the same Killing form as the generators T^a of $\mathfrak{g}$, i.e.,

$$\left\langle T_0^a, T_0^b \right\rangle = \mathrm{Tr}(T^a, T^b) \ . \tag{2.2.28}$$

Then from the condition that the bilinear $\langle [d, T_m^a T_n^b] \rangle$, $\langle [T_m^a, kT_n^b] \rangle$, $\langle [T_m^a, T_0^b T_n^b] \rangle$ and $\langle [T_m^a, dT_n^b] \rangle$ must vanish it is obtained that

$$\begin{aligned} \left\langle T_m^a, T_n^b \right\rangle &= \delta_{m,-n} \mathrm{Tr}\left(T^a, T^b\right) \ , \\ \langle k, d \rangle &= 1 \ , \\ \langle T_m^a, k \rangle &= \langle T_m^a, d \rangle = \langle k, k \rangle = 0 \ , \\ \langle d, d \rangle &= \text{undetermined} \ . \end{aligned} \tag{2.2.29}$$

Since it is not possible to fix $\langle d, d \rangle$ we shall adopt the convention that $\langle d, d \rangle = 0$.

Like in finite dimensional Lie algebra, one can define a scalar product in the root space from the invariant bilinear form between elements of the CSA as follows: given two vectors in the root space $v = (v, v_k, v_d)$ and $w = (w, w_k, w_d)$ the inner product $v \cdot w$ is defined to be

$$\begin{aligned} v \cdot w = &\ \langle v_i H_0^i + v_k k + v_d d, \ \ w_i H_0^i + w_k k + w_d d \rangle \\ = &\ \textstyle\sum_{i=1}^{r} v_i w_i + v_k w_d + v_d w_k \ . \end{aligned} \tag{2.2.30}$$

This implies that for two roots $a = (\alpha, 0, n)$ and $a' = (\alpha', 0, n')$, $a \cdot a' = \alpha \cdot \alpha'$.

The extended Cartan matrix of $\mathfrak{g}^{(1)}$ is the matrix formed by the scalar products

$$K_{ij} = \frac{2a_i \cdot a_j}{a_j^2} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2} \ , \qquad 0 \le i, j \le r \ . \tag{2.2.31}$$

where $\alpha_0 := -\psi$ for the untwisted affine algebras. The associated Dynkin diagram, $\Delta\left(\mathfrak{g}^{(1)}\right)$, is obtained from $\Delta(\mathfrak{g})$ by the addition of a dot associated to $-\psi$.

Contrary to the finite dimensional case, the extended Cartan matrix is a singular matrix. If we consider that $\psi^2 = 2$ and that

$$\psi = \sum_{i=1}^{r} n_i \alpha_i = \sum_{i=1}^{r} m_i \alpha_i^{\vee} \ , \tag{2.2.32}$$

it then follows that

$$\sum_{i=0}^{r} n_i K_{ij} = 0 \quad \text{and} \quad \sum_{j=0}^{r} K_{ij} m_j = 0 \tag{2.2.33}$$

with $n_0 = m_0 = 1$. The numbers m_i for all $\mathfrak{g}^{(1)}$ are given in table 2.

A representation of $\mathfrak{g}^{(1)}$ in which the spectrum of d is bounded above is called a highest weight representation. These representations have particular interest because $-d = L_0$ is identified with the energy of a physical system whose spectrum must be bounded below. It can be proven that a irreducible weight representation can be built up from a highest weight state $|\mu, k, \nu\rangle$ which satisfies

$$\begin{array}{rcl} E_n^{\alpha}|\mu, k, \nu\rangle & = & 0 \quad \text{if either } n > 0 \text{ or } n = 0,\ \alpha > 0\ , \\ H_n^{i}|\mu, k, \nu\rangle & = & 0 \quad \text{if } n > 0 \end{array} \tag{2.2.34}$$

and $\hat{\mu} := (\mu, k, \nu)$ is the weight vector with respect to the CSA (H_0^i, k, d). It can also be proven that in order to $\hat{\mu}$ be a highest weight of $\mathfrak{g}^{(1)}$,

$$\frac{2\hat{\mu} \cdot a_j}{a_j^2} \in \mathbb{Z} \qquad \text{and non-negative} \tag{2.2.35}$$

for all simple roots of $\mathfrak{g}^{(1)}$.

The fundamental weights of $\mathfrak{g}^{(1)}$, Λ_i, $i = 0, 1, \ldots, r$ are defined by the relation

$$\frac{2\Lambda_i \cdot a_j}{a_j^2} = \delta_{ij}\ , \qquad 0 \leq i, j \leq r\ . \tag{2.2.36}$$

Then, it follows that

$$\begin{array}{rcl} \Lambda_i & = & (\lambda_i, m_i, 0)\ , \\ \Lambda_0 & = & (0, 1, 0)\ , \end{array} \tag{2.2.37}$$

where the last component of these weights are arbitrary and we have chosen it to vanishes. From (2.2.35) and (2.2.36) we obtain that a highest weight $\hat{\mu}$ of $\mathfrak{g}^{(1)}$ must have the form

$$\hat{\mu} = \sum_{i=0}^{r} l_i \Lambda_i \tag{2.2.38}$$

with l_i being non-negative integers. Moreover, from the scalar product $2\alpha_0 \cdot \hat{\mu}/\alpha_0^2$ we obtain that

$$x := \frac{2k}{\psi^2} = \sum_{i=0}^{r} l_i m_i \tag{2.2.39}$$

where x is called the level of the representation. However since we are using $\psi^2 = 2$, the level and the centre element coincide.

2.2.2 Twisted affine Lie algebras

Before to construct the twisted affine Lie algebras we need to mention some few results about automorphisms of Lie algebras which are linear transformations which preserve the commutators, i.e., $\tau([a,b]) = [\tau(a), \tau(b)]$. The set of these transformations form a group denoted by $\mathrm{Aut}(\mathfrak{g})$.

Let $\mathcal{G}$ be a connected Lie group with has $\mathfrak{g}$ as its corresponding Lie algebra. Then given a $T \in \mathfrak{g}$ and $G \in \mathcal{G}$, we can form a automorphism

$$T \to GTG^{-1}, \tag{2.2.40}$$

which is said to be inner. Then, the quotient group $\Gamma(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathcal{G}$ is formed by the automorphisms called outer.

Let us denote by $\mathrm{Aut}\Pi(\mathfrak{g})$, $\mathrm{Aut}\overline{\Pi}(\mathfrak{g})$ and $\mathrm{Aut}\Phi(\mathfrak{g})$ the groups of real r x r orthogonal matrices preserving respectively the simple root system $\Pi(\mathfrak{g})$, the extended simple root system $\overline{\Pi}(\mathfrak{g})$ formed by $\{\alpha_1, \alpha_2, \ldots, -\psi\}$ and the root system $\Phi(\mathfrak{g})$. Clearly,

$$\begin{array}{rl} \mathrm{Aut}\Pi(\mathfrak{g}) \subset \mathrm{Aut}\overline{\Pi}(\mathfrak{g}) \subset & \mathrm{Aut}\Phi(\mathfrak{g}) \\ W(\mathfrak{g}) \subset & \mathrm{Aut}\Phi(\mathfrak{g}) \end{array} \tag{2.2.41}$$

Since $\mathrm{Aut}\Pi(\mathfrak{g})$ and $\mathrm{Aut}\overline{\Pi}(\mathfrak{g})$ preserve scalar products, and thus Cartan matrices, we can conclude that

$$\begin{array}{rcl} \mathrm{Aut}\Pi(\mathfrak{g}) & \cong & \mathrm{Aut}\Delta(\mathfrak{g}) \ , \\ \mathrm{Aut}\overline{\Pi}(\mathfrak{g}) & \cong & \mathrm{Aut}\Delta(\mathfrak{g}^{(1)}) \ , \end{array} \tag{2.2.42}$$

where $\mathrm{Aut}\Delta(\mathfrak{g})$ and $\mathrm{Aut}\Delta(\mathfrak{g}^{(1)})$ are called diagram automorphism of the Dynkin diagrams $\Delta(\mathfrak{g})$ and $\Delta(\mathfrak{g}^{(1)})$. The group $\mathrm{Aut}\Delta(\mathfrak{g}^{(1)})$ consists of a semi-direct product of $\mathrm{Aut}\Delta(\mathfrak{g})$ and W_0. $\mathrm{Aut}\Delta(\mathfrak{g})$ is the subgroup of $\mathrm{Aut}\Delta(\mathfrak{g}^{(1)})$ in which the vertex 0 is fixed. W_0 is a subgroup of the Weyl group and its elements are label by the vertices symmetrically related to the vertex 0. It has been proven[46] that W_0 is isomorphic to $Z(\mathcal{G})$, the centre of the simply connected Lie group whose Lie algebra is $\mathfrak{g}$.

Given a $\tau \in$Aut Φ it can be lifted to an automorphism of $\mathfrak{g}$

$$\begin{array}{rcl} \psi_\tau(\alpha \cdot H) & = & \tau(\alpha) \cdot H, \\ \psi_\tau(E_\alpha) & = & \chi_\alpha E_{\tau(\alpha)}, \end{array} \tag{2.2.43}$$

for each $\alpha \in \Phi$. Here $\chi_\alpha = \pm 1$, the sign of χ_α being determined by the prescription that $\chi_\alpha = 1$ when α is a simple root, that

$$\chi_{\alpha+\beta} = \frac{N_{\tau(\alpha),\tau(\beta)}}{N_{\alpha,\beta}} \chi_\alpha \chi_\beta \tag{2.2.44}$$

and that $\chi_{-\alpha} = \chi_\alpha$.

Using this lift it can be proven that

$$\begin{aligned} \mathrm{Aut}\Phi(\mathfrak{g}) &\cong \mathrm{Aut}(\mathfrak{g},\mathfrak{h})/\mathcal{T} \ , \\ \mathrm{Aut}\Delta(\mathfrak{g}) &\cong \Gamma(\mathfrak{g}) \ , \end{aligned} \tag{2.2.45}$$

where $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ which preserves the CSA $\mathfrak{h}$ and $\mathcal{T}$ is the maximal torus of $\mathcal{G}$. Moreover $W(\mathfrak{g})$ is isomorphic to the subgroup formed by the inner automorphisms of $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$.

Let us consider a simple Lie algebra $\mathfrak{g}$ that possesses an outer automorphism. By what has being said previously the outer automorphism transformations can be constructed from symmetry transformations of $\mathrm{Aut}\Delta(\mathfrak{g})$, $\tau(\alpha_i) = \alpha_{\tau(i)}$. The only algebras with non-trivial $\mathrm{Aut}\Delta(\mathfrak{g})$ are a_r, d_r and e_6.

Suppose that an outer automorphism ψ_τ has order k, implying that ψ_τ has eigenvalues of the form $e^{2\pi i p/k}, 0 \geq p \geq k-1$. The elements of $\mathfrak{g}$ may be arranged in eigenvectors of ψ_τ. Let $\mathfrak{g}_p$ be the eigenspace corresponding to the eigenvalue $e^{2\pi i p/k}$, and let $T^{i,p}$ be an element of $\mathfrak{g}_p$, i. e.,

$$\psi_\tau(T^{i,p}) = e^{\frac{2\pi i p}{k}} T^{i,p}. \tag{2.2.46}$$

The subspace $\mathfrak{g}_0$ is a Lie algebra of $\mathfrak{g}$ because if $T^{i,0}, T^{j,0} \in \mathfrak{g}_0$ then $[T^{i,0}, T^{j,0}] \in \mathfrak{g}_0$. For $p \neq 0$, $\mathfrak{g}_p$ is the carrier space of a representation of $\mathfrak{g}_0$ because if $T^{i,0} \in \mathfrak{g}_0, T^{j,p} \in \mathfrak{g}_p$ then $[T^{i,0}, T^{j,p}] \in \mathfrak{g}_p$. So a representation R of $\mathfrak{g}_0$ may be defined by

$$[T^{i,0}, T^{j,p}] = \sum_{l=1}^{\dim\, \mathfrak{g}_p} R_{jl}(T^{i,0}) T^{l,p}, \qquad j = 1, 2, \ldots, \dim\, \mathfrak{g}_p \tag{2.2.47}$$

for all $T^{i,0} \in \mathfrak{g}_0$.

The algebra $\mathfrak{g}_0$ can be determined in the following way: let $\tau \in$Aut $\Delta(\mathfrak{g})$ and let $\langle i \rangle$ denote the set of vertices of $\Delta(\mathfrak{g})$ connected under the action of τ. τ is said to be direct if within each orbit $\langle i \rangle$ no two vertices are linked. For direct τ we define the generators

$$\begin{aligned} H_{\langle i \rangle} &:= \textstyle\sum_{i \in \langle i \rangle} H_{\alpha_i} \ , \\ E_{\pm\langle i \rangle} &:= \textstyle\sum_{i \in \langle i \rangle} E_{\pm\alpha_i} \ . \end{aligned} \tag{2.2.48}$$

These generators are in the fixed algebra $\mathfrak{g}_0$. Then using the fact that for a direct τ, $K_{\alpha\beta} = 0$ for $\alpha, \beta \in \langle i \rangle, \alpha \neq \beta$, we obtain that

$$\begin{aligned} \left[E_{\langle i \rangle}, E_{-\langle j \rangle}\right] &= \delta_{i,j} H_{\langle i \rangle}, \\ \left[H_{\langle i \rangle}, E_{\langle i \rangle}\right] &= 2E_{\langle i \rangle} \ , \end{aligned} \tag{2.2.49}$$

which means that $H_{\langle i\rangle}$ and $E_{\langle i\rangle}$ are correctly normalized. Then we obtain that

$$[H_{\langle j\rangle}, E_{\pm\langle i\rangle}] = K_{\langle i\rangle\langle j\rangle} E_{\langle i\rangle} \ , \tag{2.2.50}$$

where

$$K_{\langle i\rangle\langle j\rangle} = \sum_{j\in\langle j\rangle} K_{ij} = \frac{1}{|\langle i\rangle|} \sum_{i\in\langle i\rangle, j\in\langle j\rangle} K_{ij} \tag{2.2.51}$$

defines a Cartan matrix with $|\langle i\rangle|$ being the number of points in the orbit $\langle i\rangle$. It is possible to prove that set of elements $H_{\langle i\rangle}$ form a basis of the CSA of $\mathfrak{g}_0$, therefore $K_{\langle j\rangle\langle i\rangle}$ is the Cartan matrix of $\mathfrak{g}_0$ and the corresponding Dynkin diagram can be obtained from $\Delta(\mathfrak{g})$ by folding[46]. Then we obtain the following correspondence between $\mathfrak{g}$ and $\mathfrak{g}_0$:

$$a_{2l-1} \to c_l, \qquad d_{l+1} \to b_l, \qquad d_4 \to g_2, \qquad e_6 \to f_4 \ \ , \tag{2.2.52}$$

for $l \geq 2$. Note that all non-simply laced algebras can be obtained from simply laced ones as fixed subalgebras under a direct outer automorphism. The simple roots of $\mathfrak{g}_0$ will be given by

$$\alpha_{\langle i\rangle} = \frac{1}{|\langle i\rangle|} \sum_{i\in\langle i\rangle} \alpha_i \ \ \Rightarrow \ \ \alpha_{\langle i\rangle}^2 = \frac{1}{|\langle i\rangle|}\alpha_i^2 = \frac{2}{|\langle i\rangle|} \ , \tag{2.2.53}$$

$$\frac{2\alpha_{\langle i\rangle}}{\alpha_{\langle i\rangle}^2} = \sum_{i\in\langle i\rangle} \alpha_i \ . \tag{2.2.54}$$

Then it is direct to check that

$$\lambda^{\mathrm{v}}_{\langle i\rangle} = \sum_{i\in\langle i\rangle} \lambda_i \tag{2.2.55}$$

will have the correct inner products with the simple roots.

The only case in which a outer automorphism τ is not direct is when $\mathfrak{g} = a_{2l}$. In this case we must associate to the orbit $\langle l\rangle$ extra normalization factors,

$$\begin{aligned} H_{\langle i\rangle} &:= \ \ 2\textstyle\sum_{i\in\langle i\rangle} H_{\alpha_i} \\ E_{\pm\langle i\rangle} &:= \ \ \sqrt{2}\textstyle\sum_{i\in\langle i\rangle} E_{\pm\alpha_i} \ , \end{aligned} \tag{2.2.56}$$

in order to obtain the correct normalization (2.2.49). Then one obtain [46] that $\mathfrak{g}_0 = b_l$ for $l \geq 2$ and $\mathfrak{g}_0 = a_1$, for $l = 1$.

Now given an algebra $\mathfrak{g}$ with an outer automorphism of order k, the affine twisted Lie algebra $\mathfrak{g}^{(k)}$ is a subalgebra of $\mathfrak{g}^{(1)}$ formed by the generators

$$T^i_{mk+p} = \lambda^{mk+p} T^{i,p}, \qquad\qquad m \in \mathbb{Z} \tag{2.2.57}$$

where $T_p^i \in \mathfrak{g}_p$. Then the roots of $\mathfrak{g}^{(k)}$ will be of the form $(0, 0, mk+p)$ and $(\alpha, 0, mk+p)$ where α are roots of $\mathfrak{g}_0$ or non-zero weights of the representations $\mathfrak{g}_p$. Let $\alpha_i, i = 1, \ldots, l$ be the simple roots of $\mathfrak{g}_0$ and μ the highest weight of $\mathfrak{g}_1$. Then

$$\begin{aligned} a_i &= (\alpha_i, 0, 0) \qquad i = 1, \ldots, l, \\ a_0 &= (-\mu, 0, 1) \end{aligned} \tag{2.2.58}$$

form a simple root system of $\mathfrak{g}^{(k)}$ such that any root can be written as $a = \sum_{i=0}^{l} k_i \alpha_i$, which k_i are either non-negative or non-positive integers. The Cartan matrix is given by (2.2.31) with $\alpha_0 = -\mu$. The corresponding Dynkin diagrams are shown in table 3. The untwisted algebras could be seen as the affine algebras obtained by using a trivial outer automorphism and therefore in this case $\mathfrak{g}_0 = \mathfrak{g}$. One could think to replace the outer automorphism τ used in the above construction by an inner automorphism. However it can be proven [29] that the algebra obtained is isomorphic to $\mathfrak{g}^{(1)}$.

Note that the algebras $\mathfrak{g}_0$ originate two different affine algebras associated to Dynkin diagrams obtained by adding vertices associated to $(-\mu, 0, 1)$ or $(-\psi, 0, 1)$ to the Dynkin diagram $\Delta(\mathfrak{g}_0)$. ($\mathfrak{g}_0 = b_l$ give rise to $b_l^{(1)}, d_{l+1}^{(2)}$ and $\alpha_{2l}^{(2)}$.)In affine Toda theories this will correspond to alternative integrable deformations one can add to the conformal Toda theory associated to $\mathfrak{g}_0$.

There exist a different way of obtaining the twisted as well as the non-simply laced affine Lie algebras as sub-algebras of simply-laced ones which is by folding the extended Dynkin diagrams $\Delta(\mathfrak{g}^{(1)})$[46]. In order to understand this procedure we define the generators of $\mathfrak{g}^{(1)}$

$$\begin{aligned} \overline{H}_i &= H_0^{\alpha_i}, \qquad & \overline{H}_0 &= H_0^{-\psi} + k, \\ \overline{E}_i &= E_0^{\alpha_i}, \qquad & \overline{E}_0 &= E_1^{-\psi}, \end{aligned} \tag{2.2.59}$$

These generators satisfy commutation relations like the ones in (2.1.7) but with K_{ij} being the extended Cartan matrix associated to $\mathfrak{g}^{(1)}$. Like in finite dimensional Lie algebras, from the commutators between these generators together with the Serre relations determine the algebra. We can also lift a diagram automorphism of $\Delta(\mathfrak{g}^{(1)})$ can be to an automorphism of $\mathfrak{g}^{(1)}$,

$$\begin{aligned} \psi_\tau(\overline{H}_i) &= \overline{H}_{\tau(i)} \\ \psi_\tau(\overline{E}_i) &= \overline{E}_{\tau(i)} \end{aligned} \tag{2.2.60}$$

with $i = 0, 1, \ldots$. Then given a direct diagram automorphisms of $\Delta(\mathfrak{g}^{(1)})$ one can construct generators like (2.2.48),

$$\begin{aligned} \overline{H}_{\langle i \rangle} &:= \textstyle\sum_{i \in \langle i \rangle} \overline{H}_{\alpha_i} \,, \\ \overline{E}_{\pm\langle i \rangle} &:= \textstyle\sum_{i \in \langle i \rangle} \overline{E}_{\pm\alpha_i} \,. \end{aligned} \tag{2.2.61}$$

which belong to the fixed subalgebra and from their commutation relations one obtains a extended, folded, Cartan matrix

$$K_{\langle i\rangle\langle j\rangle} := \sum_{j\in\langle j\rangle} K_{ij} \tag{2.2.62}$$

from which is obtained the folded Dynkin diagram.

As we have said previously, $\mathrm{Aut}\Delta(\mathfrak{g}^{(1)})$ is a semidirect product of $\mathrm{Aut}\Delta(\mathfrak{g})$ and W_0. Folding with $\tau \in \mathrm{Aut}\Delta(\mathfrak{g})$ yields a non-simply laced affine Lie algebra. On the other hand using a $\tau \in W_0$ gives a twisted affine Lie algebra. This can be seen by direct inspection in tables 2 and 3 in which we conclude that for a given algebra the corresponding subalgebra associated to the folded Dynkin diagrams is the following:

$$c_l^{(1)} \subset a_{2l+1}^{(1)} \qquad b^{(1)} \subset d_{l+1}^{(1)} \qquad g_2^{(1)} \subset d_4^{(1)} \qquad f_4^{(1)} \subset e_6^{(1)} \tag{2.2.63}$$

$$a_{2l-1}^{(2)} \subset d_{2l}^{(1)} \qquad d_{l+1}^{(2)} \subset d_{l+2}^{(1)} \qquad e_6^{(2)} \subset e_7^{(1)} \qquad d_4^{(3)} \subset e_6^{(1)} \tag{2.2.64}$$

Therefore, for instance, the twisted affine algebra $d_4^{(3)}$ can be seen as a subalgebra of $e_6^{(1)}$ instead of $d_4^{(1)}$.

Since $K_{\tau(i)\tau(j)} = K_{ij}$ for $\tau \in \mathrm{Aut}(\mathfrak{g}^{(1)})$ it follows from (2.2.33) that

$$m_{\langle i\rangle} := m_i = m_{\tau(i)}. \tag{2.2.65}$$

Then from (2.2.62) we conclude that for direct diagram automorphisms

$$\sum_{\langle j\rangle} K_{\langle i\rangle\langle j\rangle} m_{\langle j\rangle} = 0 \ . \tag{2.2.66}$$

Finally from (2.2.53) and (2.2.65) we obtain that

$$\sum_{i=0} m_i \alpha_i^{\mathrm{v}} = \sum_{\langle i\rangle=\langle 0\rangle} m_{\langle i\rangle} \alpha_{\langle i\rangle}^{\mathrm{v}} = 0. \tag{2.2.67}$$

A similar procedure can be used to obtain $a_{2n}^{(2)}$ by using a non-direct reduction [46]. However for sake of simplicity we shall not consider this algebra in the following chapters.

Chapter 3

Particles in Affine Toda field theories

3.1 Introduction

An integrable field theory can be defined (naively) as a field theory with an infinite number of conserved charges in involution. Consequently any solution of such a system must be infinitely restricted. A soliton is precisely such a kind a solution, i.e., it is a localized wave which retains its shape even after collisions. Therefore these theories are a good staring point to look for soliton solutions.

Affine Toda field theories(ATFTs) are relativistic integrable theories in two dimensions. For each affine Lie algebra $\mathfrak{g}^{(k)}$ one can associated an ATFT. They are relativistic theories with r real scalars, where r is the rank of the Lie algebra associated to the Dynkin diagram obtained by removing the spot " 0 " of $\Delta(\mathfrak{g}^{(k)})$. Their Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi.\partial^\mu\phi - \frac{\mu^2}{\beta^2}\sum_{i=0}^{r}\frac{2m_i}{\alpha_i^2}e^{\beta\alpha_i.\phi} \tag{3.1.1}$$

where μ is a mass parameter and β is a dimensional constant. The coefficients were chosen such that $\phi = 0$ is a vacuum.

It is interesting to note that when $\mathfrak{g}^{(k)} = a_1^{(1)}$ we recover sinh-Gordon and sine-Gordon by taking β to be real or pure imaginary respectively. Therefore ATFT can be seen as a generalization of those theories. If the term $i = 0$ is omitted from the summation in (3.1.1) one obtains the Lagrangian density of conformal Toda field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi.\partial^\mu\phi - \frac{\mu^2}{\beta^2}\sum_{i=1}^{r}\frac{2m_i}{\alpha_i^2}e^{\beta\alpha_i.\phi} \tag{3.1.2}$$

These theories are conformal invariant since their improved energy momentum tensor is traceless. Then by adding to (3.1.2) a deformation term

$$\eta\frac{\mu^2}{\beta^2}\frac{2m_0}{\alpha_0^2}e^{\beta\alpha_0.\phi} \quad , \tag{3.1.3}$$

and performing the redefinition $\phi \to \phi + \frac{\rho^{\mathrm{v}}}{h\beta^2}\ln\eta$ we obtain once more the ATFT Lagrangian density (3.1.1). As we shall see later, ATFT is a integrable theory. Therefore, like in the case of sine/sinh-Gordon, ATFT is obtained from a conformal field theory by adding deformation which preserves integrability. Indeed, in [30, 17] it has been proved various connections between ATFT's and Zamolodchikov's integrable theories[62] at quantum level.

From (3.1.1) we obtain the equation of motion

$$\partial^2\phi + \frac{\mu^2}{\beta}\sum_{i=0}^{r}\alpha_i^{\mathrm{v}}m_i e^{\beta\alpha_i.\phi} = 0 \quad . \tag{3.1.4}$$

By using the folding procedure discussed in the last chapter we can obtain the twisted and the non-simply laced ATFT from the simply laced ones in the following way: consider the $(r+1)$ fields $\phi^i = \alpha_i.\phi$, $i = 0, 1, \ldots, r$, which, due to the condition (2.2.67), satisfy the constraint

$$\sum_{i=0}^{r}\frac{2m_i}{\alpha_i^2}\phi^i = 0 \ . \tag{3.1.5}$$

Then the equation of motion can be written

$$\partial^2\phi^i + \frac{\mu^2}{\beta}\sum_{j=0}K_{ij}m_j e^{\beta\phi^j} = 0 \ , \qquad i = 0, 1, \ldots, r, \tag{3.1.6}$$

subject to (3.1.5). This last form of the equation of motion and the constraint (3.1.5) are explicitly invariant under the transformation $\phi^i \to \phi^{\tau(i)}$ in which $K_{\tau(i)\tau(j)} = K_{ij}$ and $m_{\tau(i)} = m_i$. Therefore ATFT has a symmetry group isomorphic to $\mathrm{Aut}(\Delta(\mathfrak{g}^{(k)}))$. Using this symmetry, (2.2.62) and (2.2.65) we can fold the above equation and obtain

$$\partial^2\phi^{\langle i\rangle} + \frac{\mu^2}{\beta}\sum_{\langle j\rangle=\langle 0\rangle}K_{\langle i\rangle\langle j\rangle}m_{\langle j\rangle}e^{\beta\phi^{\langle j\rangle}} = 0 \ , \tag{3.1.7}$$

where

$$\phi^{\langle i\rangle} := \phi^i = \phi^{\tau(i)} \quad . \tag{3.1.8}$$

Therefore if $\phi^{\langle i\rangle}$ is a solution of the folded theory then it will also be a solution of the unfolded theory . In particular the particle mass spectrum of the folded theory automatically forms a subset of the unfolded mass spectrum.

Note that the substitution $\alpha_i \to 2\alpha_i/\alpha_i^2$ in (3.1.1) transforms the ATFT Lagrangian associated to $\Delta(\mathfrak{g}^{(k)})$ to the one associated to $\Delta(\mathfrak{g}^{(k)^{\vee}})$ which is obtained by reversing the arrows of the former. From tables 2 and 3 it can be seen that $(b_n^{(1)}, a_{2n-1}^{(2)})$, $(c_n^{(1)}, d_{n+1}^{(2)})$, $(g_2^{(1)}, d_4^{(3)})$ and $(f_4^{(1)}, e_6^{(2)})$ are the pairs of dual theories and $a_n^{(1)}$, $d_n^{(1)}$, $e_6^{(1)}$, $e_7^{(1)}$, $e_8^{(1)}$ and $a_{2n}^{(2)}$ are self dual theories.

When β is a pure imaginary and $\mathfrak{g}^{(k)}$ is untwisted, (3.1.1) has a discrete symmetry

$$\phi \to \phi + \frac{2\pi i}{\beta}\Lambda_W^{\vee}(\mathfrak{g}) \tag{3.1.9}$$

where $\Lambda_W^{\vee}(\mathfrak{g})$ is the coweight lattice of $\mathfrak{g}$ which has $\lambda_i^{\vee}$ as basis. Since $\phi = 0$ is a vacuum then by using this symmetry transformation we obtain that the constant solutions

$$\phi = \frac{2\pi i}{\beta}\Lambda_W^{\vee}(\mathfrak{g}) \tag{3.1.10}$$

form a degenerate vacua. From what has been said previously we can conclude that vacua for the twisted theories with imaginary coupling will be a subset of the vacua the corresponding unfolded theories.The existence of degenerate vacua indicates the possibility that ATFTs present solitons interpolating these vacua.

In the next sections we shall introduce an alternative basis for $\mathfrak{g}$ and $\mathfrak{g}^{(k)}$ which will be quite useful in the study of affine Toda field theories.

3.2 Alternative basis for Lie algebras

In order to construct this basis let us first define

$$T_3 := \rho^{\vee} \cdot H \quad \text{and} \quad T_{\pm} = \sum_{i=1}^{r} \sqrt{2c_i} E_{\alpha_i} \tag{3.2.11}$$

where $c_i := \sum_{j=1}^{r}(K^{-1})_{ij}$. These generators form the principal su(2) subalgebra of $\mathfrak{g}$,

$$[T_3, T_{\pm}] = \pm T_{\pm} \quad , \quad [T_+, T_-] = 2T_3. \tag{3.2.12}$$

Clearly,

$$[T_3, E_\alpha] = ht(\alpha)E_\alpha \quad \text{and} \quad [T_3, H_i] = 0. \tag{3.2.13}$$

Then we define the so-called principal element by

$$S := \exp\left(2\pi i T_3/h(\mathfrak{g})\right) \tag{3.2.14}$$

where $h(\mathfrak{g}) := \mathrm{ht}(\psi)+1$ is called the Coxeter number. By using the Baker-Campbell-Hausdorff formula $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \dots$, we obtain that

$$S E_\alpha S^{-1} = w^{\mathrm{ht}(\alpha)} E_\alpha \quad \text{and} \quad S H_i S^{-1} = H_i \ , \tag{3.2.15}$$

where $w := \exp(2\pi i/h)$. Let $\mathfrak{g}_n \subset \mathfrak{g}$ be the vector space formed by the generators $T^{(n)}$ satisfying $ST^{(n)}S^{-1} = w^n T^{(n)}$, $0 \leq n < h$. Clearly $[T^{(n)}, T^{(m)}] \in \mathfrak{g}_{[n+m]}$, where $[p] := p (\bmod h)$. Therefore S provides a $\mathbb{Z}_h$ grading of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{h-1}. \tag{3.2.16}$$

It is easy to conclude that $\mathfrak{g}_0 = \mathfrak{h}$ and

$$\begin{array}{rcl} \{E_{\alpha_1}, E_{\alpha_2}, \ldots, E_{\alpha_r}, E_{\alpha_0}\} & \in & g_1, \\ \{E_{-\alpha_1}, E_{-\alpha_2}, \ldots, E_{-\alpha_r}, E_{-\alpha_0}\} & \in & g_{h-1}, \end{array} \tag{3.2.17}$$

where $\alpha_0 := -\psi := -\sum_{i=1}^r m_i \alpha_i^{\text{v}}$. We then define the grade 1 and grade $(h-1)$ elements

$$\begin{array}{rcl} E_1 & = & \sum_{i=0}^r \sqrt{m_i}\; E_{\alpha_i} \;, \\ E_{h-1} & = & \sum_{i=0}^r \sqrt{m_i}\; E_{-\alpha_i} = E_1^\dagger \;, \end{array} \tag{3.2.18}$$

with $m_0 = 1$. Then, one has that

$$\left[E_1, E_1^\dagger\right] = \sum_{i=0}^r m_i \alpha_i^{\text{v}} \cdot H = 0. \tag{3.2.19}$$

Recall that a matrix M satisfying $[M, M^\dagger] = 0$ is called normal and is diagonalizable. Therefore we can conclude that E_1 is semi-simple, i.e., $\text{ad}(E_1)$ is diagonalizable. Now, it can be proven that any element contained in a CSA is necessarily semi-simple and conversely every semi-simple element may be embedded in a CSA. Therefore E_1 can be embedded in a CSA which we shall denote by $\mathfrak{h}'$. The CSA $\mathfrak{h}'$ is said to be in apposition to $\mathfrak{h}$ with respect to S [35]. We shall denote by h_i an orthonormal basis for $\mathfrak{h}'$ and e_α the corresponding step operators, with h_i and e_α satisfying (2.1.1), (2.1.2) and (2.1.3).

A semi-simple element T is said to be regular if the kernel of $\text{ad}(T)$ is a CSA. It can be shown ([35], corollary 6.4) that E_1 is a regular element. Now given an arbitrary element $x \cdot h \in \mathfrak{h}'$ we have that

$$0 = [x \cdot h, E_1] = \left[Sx \cdot hS^{-1}, wE_1\right] \tag{3.2.20}$$

which implies that $Sx \cdot hS^{-1} \in \mathfrak{h}'$. We can then define a map σ,

$$Sx \cdot hS^{-1} = \sigma(x) \cdot h. \tag{3.2.21}$$

Since $[x \cdot h, e_\alpha] = x \cdot \alpha\; e_\alpha$, it implies that

$$\left[x \cdot h, Se_\alpha S^{-1}\right] = S\left[\sigma^{-1}(x) \cdot h, e_\alpha\right] S^{-1} = x \cdot \sigma(\alpha)\; Se_\alpha S^{-1}. \tag{3.2.22}$$

So given any root α then $\sigma(\alpha)$ is also a root with $e_{\sigma\alpha} \sim Se_\alpha S^{-1}$. Because S provides an inner automorphism which preserves the CSA $\mathfrak{h}'$ it implies that σ is an element of the Weyl group of $\mathfrak{g}$. From the definition of S it is easy to see that S^h commutes with any element in $\mathfrak{g}$. Therefore $\sigma^h = 1$.

We have mentioned before that any element of $W(\mathfrak{g})$ can be written as a product of simple root Weyl reflections. In particular, an element formed by the product of the r simple root Weyl reflections is called a Coxeter element.These elements can be constructed by taking different choices of simple root basis and different ordering. However all Coxeter elements are conjugate to each other[6]. Therefore they share the same eigenvalues and order. It has been proven that the order of the Coxeter elements of a Lie algebra $\mathfrak{g}$ is equal to $h(\mathfrak{g})$. Therefore we can conclude that σ is a Coxeter element. From the fact that Coxeter elements acts on the root space as a real orthonormal rxr matrix, since they belong to $W(\mathfrak{g})$, and that they have order $h(\mathfrak{g})$ we can conclude that their eigenvalues must be of the form $e^{\frac{2\pi i\nu}{h}}$ where ν are integers lying between $0 \leq \nu < h$. The r numbers ν are called exponents of $\mathfrak{g}$, and they are shown in table 4.

Since the only algebras which have a degenerate exponent are d_{2n}, we shall abuse our notation and denote the by $q(\nu)$ the eigenvector of σ associated to the exponent ν, i.e.

$$\sigma q(\nu) = w^{\nu} q(\nu) \quad , \qquad w := e^{\frac{2\pi i}{h}} \quad . \tag{3.2.23}$$

Since $q(\mu)\cdot\sigma q(\nu) = \sigma^{-1}q(\mu)\cdot q(\nu)$ we can conclude that $q(\mu)\cdot q(h-\nu) \propto \delta_{\mu,\nu}$ and we shall fix the normalization such that

$$q(\mu)\cdot q(h-\nu) = h\delta_{\mu,\nu} \quad . \tag{3.2.24}$$

Given (3.2.23), we can also consider that

$$q(\nu)^* = q(h-\nu). \tag{3.2.25}$$

For the case of the degenerate exponent $\nu = 2n-1 = h/2$ of d_{2n}, we can choose two eigenvectors, $q_{(a)}(2n-1), a = 1,2$, such that $q_{(a)}(2n-1)\cdot q_{(b)}(2n-1) = h\delta_{a,b}$ and $q_{(a)}(2n-1)^* = q_{(a)}(2n-1)$.

Given the above properties of the vectors $q(\nu)$ we can conclude that the generators E_1 and E_{h-1} defined in (3.2.18) should be written in the basis orthonormal basis of $\mathfrak{h}'$ as

$$\begin{aligned} E_1 &= q(1)\cdot h, \\ E_{(h-1)} &= q(h-1)\cdot h. \end{aligned} \tag{3.2.26}$$

Since the elements E_1 and E_{h-1} exist for any Lie algebra, we can conclude the well known result that any Lie algebra has 1 and $(h-1)$ as exponents.

It is clear that the r elements

$$E_{\nu} := q(\nu)\cdot h \qquad \nu \text{ are exponents of } \mathfrak{g} \tag{3.2.27}$$

have grade ν under the adjoint action of S and form a basis for $\mathfrak{h}'$ since the vectors $q(\nu)$ form a basis in roots space.

Having found a basis of $\mathfrak{h}'$ with definite grade we want to determine the remaining basis elements with definite grade. First let us prove that σ split the roots into r orbits Ω_i of h elements[35]. Any algebra $\mathfrak{g}$ has hr roots. The action of σ on these roots splits them into s disjoint orbits, say, each at most with h elements, since $\sigma^h = 1$. Hence $hs \geq hr$. On the other hand, the quantities $\sum_{n=1}^{h} S^n e_\alpha S^{-n}$ associated to each orbit are linear independent and commute with S. So they lie in $\mathfrak{g}_0 = \mathfrak{h}$ and hence $s \leq r$. Therefore σ splits $\Phi(\mathfrak{g})$ in r orbits Ω_i with h roots.

Since S^h is in the centre, it is possible[23] to choose phases such that $S e_\alpha S^{-1} = e_{\sigma\alpha}$. Then we define the generators

$$F^i_\mu := \frac{1}{\sqrt{h}} \sum_{m=0}^{h-1} w^{-m\mu} e_{\sigma^m \gamma_i}, \qquad i = 1, \ldots, r, \qquad \mu = 0, 1, \ldots, h-1 \tag{3.2.28}$$

where γ_i are representative roots for the r orbits Ω_i, i.e. , γ_i lie in distinct orbits. It is not difficult to see that F^i_μ has degree μ. Moreover the generators F^i_μ are linearly independent. Therefore the generators E_ν, F^i_μ form a basis for $\mathfrak{g}$. From their definition we obtain that

$$\begin{aligned} \mathrm{Tr}(E_\mu, E_\nu^\dagger) &= q(\mu) \cdot q(\nu)^* = h\delta_{\mu,\nu} \ , \\ \mathrm{Tr}(F^i_\mu, (F^j_\nu)^\dagger) &= \tfrac{\delta_{i,j}}{h} \textstyle\sum_{m=0}^{h-1} w^{-m(\mu-\nu)} = \delta_{i,j}\delta_{\mu,\nu} \ , \\ \mathrm{Tr}(E_\mu, F^i_\nu) &= 0 \ . \end{aligned} \tag{3.2.29}$$

Before give the commutation relations of these generators we need to obtain more information about σ.

3.3 The Coxeter element $\sigma_-\sigma_+$.

We have mentioned before that the Coxeter elements form a conjugacy class in $W(\mathfrak{g})$. There exist however a particular choice of Coxeter element which is quite convenient. In order to define this Coxeter element we must first introduce the concept of bicolouration of a Dynkin diagram in which we associate signs "+" and "−" to the vertices of the Dynkin diagram in such a way that any two vertices linked together never have the same signs. Since every Dynkin diagram for simple Lie algebras does not contains loops, this partition is always possible and unique up to the ambiguity in the choice of the sign for each set. It separates the simple roots in such a way that K_{ij} is zero for every two different i and j belonging to the same set. We define

$$c(i) := \begin{cases} 1 & \text{if } i \in +, \\ -1 & \text{if } i \in -. \end{cases}$$

It is clear that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $c(i) = c(j)$ since K_{ij} vanishes. We then consider the special elements of $W(\mathfrak{g})$,

$$\sigma_+ := \prod_{i\in +}\sigma_i \quad \text{and} \quad \sigma_- := \prod_{i\in -}\sigma_i \ , \tag{3.3.30}$$

having the property $\sigma_\pm^2 = 1$. Then the product $\sigma_-\sigma_+$ is the particular Coxeter element mencioned previouly.

It is possible to choose a system of simple roots with respect to $\mathfrak{h}'$ so that the Coxeter element σ defined in the previous section can be written as

$$\sigma = \sigma_-\sigma_+. \tag{3.3.31}$$

Our next step will be to determine the eigenvectors $q(\nu)$ of $\sigma = \sigma_-\sigma_+$. In order to reach this aim we define

$$I_{ij} := 2\delta_{ij} - K_{ij} \tag{3.3.32}$$

which is called the incident matrix. Let $x_i^{(\nu)}$ be the left eigenvector of I_{ij} associated to the eigenvalue $\lambda(\nu)$, say. From the definition of $\sigma_\pm$ we obtain that

$$\begin{array}{rclcrcl} \sigma_{c(i)}\alpha_i &=& -\alpha_i, & \qquad & \sigma_{c(i)}\lambda_i &=& \lambda_i - \alpha_i, \\ \sigma_{-c(i)}\alpha_i &=& \alpha_i + I_{ij}\alpha_j, & & \sigma_{-c(i)}\lambda_i &=& \lambda_i, \\ (\sigma_+ + \sigma_-)\,\alpha_i &=& I_{ij}\alpha_j, & & & & \end{array} \tag{3.3.33}$$

where the notation $\sigma_{c(i)}$ is understood to mean either σ_+ or σ_-. Then the vector $X(\nu) := x_i^{(\nu)}\alpha_i$ will satisfy

$$(\sigma_+ + \sigma_-)\,X(\nu) = x_i^{(\nu)}I_{ij}\alpha_j = \lambda(\nu)X(\nu). \tag{3.3.34}$$

Since $\sigma_\pm^2 = 1$, then $\sigma^{-1} = \sigma_+\sigma_-$ and therefore

$$(\sigma_+ + \sigma_-)^2 = \left(\sigma + \sigma^{-1} + 2\right) \ . \tag{3.3.35}$$

From the last equation we can relate $\lambda(\nu)$ to the eigenvalues of σ and obtain that

$$\lambda(\nu) = 2\cos\frac{\pi\nu}{h}. \tag{3.3.36}$$

The vector $y_i^{(\nu)} := \alpha_i^2 x_i^{(\nu)}/2$ is the right-hand eigenvector of I_{ij} having $\lambda(\nu)$ as eigenvalue. Therefore

$$x_i^{(\nu)}I_{ij}y_j^{(\mu)} = \lambda(\nu)x_i^{(\nu)}y_j^{(\mu)} = \lambda(\mu)x_i^{(\nu)}y_j^{(\mu)}. \tag{3.3.37}$$

Except for $\mathfrak{g} = d_{2n}, \nu = 2n-1$, all the other eigenvalues $\lambda(\nu)$ are non-degenerate and therefore by chooing an appropiate normalization, we shall have that

$$x_i^{(\nu)}y_i^{(\mu)} = \delta_{\nu,\mu}. \tag{3.3.38}$$

In the degenerate case of d_{2n} we can always choose two vectors satisfying this condition.

Since $c(i)I_{ij} = -c(j)I_{ij}$, as $c(i) = -c(j)$ when $I_{ij} \neq 0$ and considering the above orthonormality condition, we can conclude that $x_i^{(h-\nu)} = \pm c(i)x_i^{(\nu)}$. We shall adopt the plus sign. This choice fix the bicolouration ambiguity if $\nu = h/2$ is an exponent. Let us define

$$X_\pm(\nu) := \frac{1}{2}\left[X(\nu) \pm X(h-\nu)\right] = \sum_{i\in\pm} x_i^{(\nu)}\alpha_i = \sum_{i\in\pm} y_i^{(\nu)}\alpha_i^{\mathrm{v}}. \tag{3.3.39}$$

which satisfies $\left(\sigma_+ + \sigma_-\right) X_\pm(\nu) = 2\cos\frac{\pi\nu}{h} X_\mp(\nu)$. From this relation and (3.3.33) we obtain directly that

$$\begin{aligned} \sigma_\pm X_\pm(\nu) &= -X_\pm(\nu)\ , \\ \sigma_\mp X_\pm(\nu) &= X_\pm(\nu) + 2\cos\tfrac{\pi\nu}{h} X_\mp(\nu). \end{aligned} \tag{3.3.40}$$

By taking scalar product of the last two equations by $X_\pm(\nu)$ and using the normalization (3.3.38), one can obtain that

$$\begin{aligned} X_+(\nu)\cdot X_+(\nu) &= X_-(\nu)\cdot X_-(\nu) = 1\ , \\ X_+(\nu)\cdot X_-(\nu) &= -\cos\tfrac{\pi\nu}{h}\ . \end{aligned} \tag{3.3.41}$$

Then one can check that

$$q(\nu) = \frac{\sqrt{h/2}}{\sin\frac{\pi\nu}{h}}\left(X_+(\nu) + e^{\frac{i\pi\nu}{h}} X_-(\nu)\right) \tag{3.3.42}$$

satisfy (3.2.23), (3.2.24) and (3.2.25). From this equation, using $\sigma v\cdot q(\nu) = v\cdot \sigma^{-1}q(\nu)$, we obtain that

$$\sigma^n\lambda_i\cdot q(\nu) = \frac{\sqrt{h/2}}{\sin\frac{\pi\nu}{h}} e^{\frac{2i\pi\nu}{h}\left(\frac{1-c(i)}{4}-n\right)} y_i^{(\nu)}\quad . \tag{3.3.43}$$

Let us define $\gamma_i := c(i)\alpha_i$, $i = 1, 2, \ldots, r$. Then, by using (3.3.43) and the identity [24]

$$\gamma_i = (1-\sigma^{-1})\sigma^{\frac{1-c(i)}{2}}\lambda_i\quad , \tag{3.3.44}$$

it results

$$\sigma^n\gamma_i\cdot q(\nu) = -i\sqrt{2h}\, e^{\frac{2i\pi\nu}{h}\left(\frac{1+c(i)}{4}-n\right)} y_i^{(\nu)}\quad . \tag{3.3.45}$$

Notice that I_{ij} is a matrix with strictly positive entries and it is irreducible since we are considering $\mathfrak{g}$ simple. So we can apply the Perron-Frobenius theorem (for a prove see for instance[27]) which says that for a given non-negative irreducible matrix M there exist a eigenvalue μ such that

i. $\mu \geq |\lambda|$ for all eigenvalues λ of M and μ has multiplicity 1,

ii. μ can be associated with strictly positive left and right eigenvectors.

We can conclude that for the incident matrix, $\mu = \cos\frac{\pi}{h}$ and therefore $x_i^{(1)}$ and $y_i^{(1)}$ can be taken strictly positive.

The Perron-Frobenius theorem is important in physics for example to prove the uniqueness of the ground state of quantum systems[27]. In affine Toda field theories we shall see that the mass (at the classical level) of particles and solitons are proportional $y_i^{(1)}$ and $x_i^{(1)}$ respectively. Therefore particle and soliton mass spectrum will be strictly positive with no massless particles or solitons.

From relation (3.3.45) we obtain some important results: by taking $\nu = 1$ and $n = 0$ and using the fact that $y_i^{(1)} > 0$, we obtain that it can never happen that $\gamma_i = \sigma^p \gamma_j$ for $i \neq j$, $p \in \mathbb{Z}$. This means that two different roots γ_i and γ_j can never be in the same orbit . Therefore we can take $\gamma_i,\ i = 1, 2, \ldots, r$ as representative of the r orbits.

We shall denote by $\gamma_{\bar{\imath}}$ the orbit which contains $(-\gamma_i)$. Since $\gamma_{\bar{\imath}}$ and $(-\gamma_i)$ lie in the same orbit, there exist some integer power of σ which connects one to the other, i.e. $\gamma_{\bar{\imath}} = \sigma^l(-\gamma_i)$. Using once more the fact that $y_i^{(1)}$ is strictly positive, it follows that[24]

$$\gamma_{\bar{\imath}} = \sigma^{\frac{c(i)-c(\bar{\imath})}{4}+\frac{h}{2}}(-\gamma_i) \quad . \tag{3.3.46}$$

Moreover, since the power of σ must be an integer we obtain that

$$c(\bar{\imath}) = (-1)^h c(i) \quad . \tag{3.3.47}$$

Let us now come back to our discussion about the alternative basis for $\mathfrak{g}$. From the definitions of E_ν and F_μ^i , using (3.2.25) and (3.3.46) we obtain the hermitian conjugation properties

$$E_\nu^\dagger = E_{h-\nu} \ , \qquad (F_\mu^i)^\dagger = w^{\left(\frac{c(i)-c(\bar{\imath})}{4}+\frac{h}{2}\right)\mu} F_{h-\mu}^{\bar{\imath}} \quad . \tag{3.3.48}$$

The commutation relations of E_ν and F_μ^i will be

$$\begin{aligned} [E_\mu, E_\nu] &= 0 \ , \\ [E_\mu, F_\nu^i] &= \gamma_i \cdot q(\mu) F_{[\mu+\nu]}^i \ , \end{aligned} \tag{3.3.49}$$

where $[n] := n(\mathrm{mod} h)$. These commutations are obtained directly from the definition of E_ν and F_μ^i. The commutator $\left[F_\mu^i, F_\nu^j\right]$ can be obtained by calculating the Killing forms $\mathrm{Tr}\left([F_\mu^i, F_\nu^j], E_\rho\right)$ and $\mathrm{Tr}\left([F_\mu^i, F_\nu^j], F_\rho^k\right)$. However it will not be necessary for our future proposes.

3.4 Alternative basis for untwisted affine Lie algebras

We have seem that the set of generators E_μ and F^i_ν defined in (3.2.27),(3.2.28) , provides a basis for $\mathfrak{g}$. Therefore, we can use them to construct the generators

$$E^m_\mu := \lambda^m E_\mu \ , \qquad F^{i,n}_\nu := \lambda^n F^i_\nu \ , \tag{3.4.50}$$

which form a basis for the untwisted loop algebra. From (2.2.20) and (3.3.49), considering $k = 0$, we obtain that these basis elements satisfy the following commutation relations:

$$\begin{aligned} \left[E^m_\mu, E^n_\nu\right] &= 0 \ , \\ \left[E^m_\mu, F^{i,n}_\nu\right] &= \gamma_i \cdot q(\mu) F^{i,m+n}_{[\mu+\nu]} \ , \\ \left[d, E^m_\mu\right] &= m E^m_\mu \ , \qquad \left[d, F^{i,m}_\mu\right] = m F^{i,m}_\mu \ . \end{aligned} \tag{3.4.51}$$

Note that these basis elements have a definite homogeneous grade. Alternatively one can define the generators

$$\begin{aligned} E^{(p)}_{\mu+mh} &= \lambda^{\frac{\mu}{h}} E^m_\mu = \lambda^{\frac{\mu}{h}+m} E_\mu \\ F^{i(p)}_{\nu+nh} &= \lambda^{\frac{\nu}{h}} F^{i,n}_\nu = \lambda^{\frac{\nu}{h}+n} F^i_\nu . \end{aligned} \tag{3.4.52}$$

Their commutation relations can be obtained directly from (3.4.51). In this basis a generator $X^{(p)}(\lambda)$ will satisfy

$$S X^{(p)}(\lambda) S^{-1} = X^{(p)}(e^{2\pi i}\lambda) \ , \tag{3.4.53}$$

where S is the principal element, (3.2.14). Finally we define the generators

$$\begin{aligned} \hat{E}_{\mu+mh} &= \lambda^{-\frac{T_3}{h}} \ E^{(p)}_{\mu+mh} \ \lambda^{\frac{T_3}{h}} \ , \\ \hat{F}^i_{\nu+nh} &= \lambda^{-\frac{T_3}{h}} \ F^{i(p)}_{\nu+nh} \ \lambda^{\frac{T_3}{h}} \ , \end{aligned} \tag{3.4.54}$$

which will satisfy

$$\begin{aligned} \left[\hat{E}_{\mu+mh}, \hat{E}_{\nu+nh}\right] &= 0 \ , \\ \left[\hat{E}_{\mu+mh}, \hat{F}^i_{\nu+nh}\right] &= \gamma_i \cdot q(\mu) \hat{F}^i_{[\mu+\nu]+(m+n)h} \ , \\ \left[d', \hat{E}_{\mu+mh}\right] &= (\mu+mh)\hat{E}_{\mu+mh} \ , \qquad \left[d', \hat{F}^i_{\mu+mh}\right] = (\mu+mh)\hat{F}^{i,m}_{\mu+mh} \ . \end{aligned} \tag{3.4.55}$$

Therefore these new basis elements have a definite principal grade. The numbers $\mu + mh$ are the exponents of $\mathfrak{g}^{(1)}$. For notational simplicity we shall write $M = [M] + mh$ where it is understood that $[M]$ lies between 0 and $h-1$.

In order to the generator E_μ to have grade μ under the action of S, it must have the general form

$$E_\mu = \sum_{ht(a)=\mu} \sqrt{a_\alpha} E_\alpha + \sum_{ht(-\beta)=\mu-h} \sqrt{b_\beta} E_{-\beta} \quad , \tag{3.4.56}$$

Then from (3.4.54) we can conclude that

$$\hat{E}_{\mu+mh} = \sum_{ht(a)=\mu} \sqrt{a_\alpha} E^m_\alpha + \sum_{ht(-\beta)=\mu-h} \sqrt{b_\beta} E^{m+1}_{-\beta} \quad , \tag{3.4.57}$$

In particular from (3.2.19) and (2.2.59) we obtain that

$$\hat{E}_1 = \sum_{i=1}^{r} \sqrt{m_i} E^0_{\alpha_i} + E^1_{\alpha_0} = \sum_{i=0}^{r} \sqrt{m_i} \;\; \overline{E}_i \quad . \tag{3.4.58}$$

Therefore the generators $\hat{E}_M$ with $M > 0$ are a combination of positive-step operators. This implies that for any highest-weight state $|\Lambda\rangle$

$$\hat{E}_M |\Lambda\rangle = 0 \quad \text{if } M > 0 \quad . \tag{3.4.59}$$

Using (3.2.29) and (3.4.54) we obtain that

$$\begin{aligned} \langle \hat{E}_M, \hat{E}_N \rangle &= h\delta_{M+N,0} \; , \\ \langle \hat{E}_M, \hat{F}^i_N \rangle &= 0 \; , \end{aligned} \tag{3.4.60}$$

In order to pass from the loop algebra to the full Kac-Moody algebra we can obtain the coefficient of k which will appear in one commutator by using the fact that $< d, k >= 1$, $d' = hd + T^3_0$ and (3.4.60):

$$\begin{aligned} \langle d, [\hat{E}_M, \hat{E}_N] \rangle &= \langle [\tfrac{d'}{h}, \hat{E}_M], \hat{E}_N \rangle = \delta_{M+N,0} M \; , \\ \langle d, [\hat{E}_M, \hat{F}^i_N] \rangle &= - \tfrac{1}{h} \langle T^3_0, [\hat{E}_M, \hat{F}^i_N] \rangle = -\delta_{M+N,0} \tfrac{\gamma_i \cdot q([M])}{h} \langle T^3_0, \hat{F}^i_0 \rangle \; . \end{aligned} \tag{3.4.61}$$

From the first Killing form we can conclude that

$$[\hat{E}_M, \hat{E}_N] = \delta_{M+N,0} M k \quad . \tag{3.4.62}$$

The second Killing form gives the coefficient of k which appears in $[\hat{E}_M, \hat{F}^i_N]$. However this term can be absorbed by redefining $\hat{F}^i_0 \to \hat{F}^i_0 - \langle T^3_0, \hat{F}^i_0 \rangle k/h$.

3.5 Alternative basis for non–simply laced and twisted affine Lie algebras.

We have seem in the end of the last chapter that the non-simply laced and twisted affine Lie algebras can be obtained as subalgebras of the simply laced ones by folding using a $\tau \in$Aut$(\Delta(\mathfrak{g}^{(1)}))$ and direct. In this section we shall obtain these subalgebras in terms of the generators $\hat{E}_M$ and $\hat{F}^i_N$ of the unfolded algebras.

From (2.2.59) we can conclude that ψ_τ leaves invariant the subspaces of $\mathfrak{g}^{(1)}$ with principal grade 0,1 and -1. As these subspaces generate all $\mathfrak{g}^{(1)}$ it follows that ψ_τ respects the principal grade. Furthermore by the definition of $\hat{E}_1$, (3.4.58), and the fact that $m_i = m_{\tau(i)}$ we can conclude that

$$\psi_\tau(\hat{E}_1) = \hat{E}_1 \quad , \qquad \psi_\tau \in \mathrm{Aut}(\Delta(\mathfrak{g}^{(1)}) \tag{3.5.63}$$

As ψ_τ respect the principal grade, $\psi(\hat{E}_M)$ must belong to grade M. Then using (3.5.63), the commutation relation $[[\hat{E}_M, \hat{E}_1], \hat{E}_1^\dagger]$ and the fact that $\gamma_i.q(1)$ is proportional to $y^{(i)}(1)$ and therefore never vanishes, we can conclude that

$$\psi_\tau(\hat{E}_M) = \eta(\tau, M)\hat{E}_M \tag{3.5.64}$$

where $\eta(\tau, M)^k = 1$, k being the order of τ. For the case $\mathfrak{g}^{(1)} = d^{(1)}_{2n}$, $M = (2n-1)$ mod(h), the phase $\eta(\tau, M)$ must be replaced by a two dimensional matrix.

Similarly we can conclude that

$$\psi_\tau(\hat{F}^i_N) = \sum_j d_{ij}\hat{F}^j_N \tag{3.5.65}$$

where $d_{ij} \neq 0$ if and only if $\gamma_j.q(1) = \gamma_i.q(1)$.

As have been mentioned before the non-simply laced are obtained by folding a simply laced $\mathfrak{g}^{(1)}$ with a $\tau \in \mathrm{Aut}\Delta(\mathfrak{g})$ and direct. One can note looking at table 4 that the exponents of the non-simply laced algebras are a subset of the exponents of the corresponding unfolded algebra. It has been proven[50] that for this subset of exponents

$$\psi_\tau(\hat{E}_M) = \hat{E}_M \tag{3.5.66}$$

They also have concluded that

$$\psi_\tau(\hat{F}^i_N) = \hat{F}^{\tau(i)}_N \quad \text{for } \tau \in \mathrm{Aug}(\Delta(\mathfrak{g}^{(1)})). \tag{3.5.67}$$

Therefore the generators of $\mathfrak{g}^{(1)}$ which belong to the non-simply laced subalgebra will be

$$\begin{cases} \hat{E}_M & \text{if M is a exponent of the non-simply laced subalgebra,} \\ \hat{F}^{\langle i\rangle}_N := \sum_{i\in\langle i\rangle}\hat{F}^i_N & N \in \mathbb{Z} \end{cases} \tag{3.5.68}$$

When $\hat{E}_M$ satisfies (3.5.66), by using (3.4.56) and (3.5.67) we obtain that

$$\gamma_{\tau(i)} \cdot q([M]) = \gamma_i \cdot q([M]). \tag{3.5.69}$$

Then from (2.2.53) and the fact that all the roots in the orbit $\langle i \rangle$ have same color, it follows that the generators (3.5.68) which form an basis for the non-simply laced Kac-Moody algebra will satisfy the commutation relation

$$\left[\hat{E}_M, \hat{F}_N^{\langle i \rangle}\right] = \gamma_{\langle i \rangle} \cdot q([M]) \hat{F}_{M+N}^{\langle i \rangle} \quad . \tag{3.5.70}$$

The twisted algebras are obtained from a simply laced $\mathfrak{g}^{(1)}$ by using a $\tau \in W_0$ and direct. As we shall show under this automorphism

$$\psi_\tau(\hat{F}_N^i) = e^{-2\pi i \lambda_{\tau(0)} \cdot \lambda_i} \hat{F}_N^i. \tag{3.5.71}$$

Moreover we shall see that the generators of $\mathfrak{g}^{(1)}$ which will generate the twisted subalgebra are [37]

$$\begin{cases} \hat{E}_M : & \text{if M is an exponent of the twisted algebra } \mathfrak{g}^{(k)} \quad , \\ \hat{F}_N^i : & N \in \mathbb{Z} \text{ with } e^{-2\pi i \lambda_{\tau(0)} \cdot \lambda_i} = 1. \end{cases} \tag{3.5.72}$$

where the exponents of the twisted algebra $\mathfrak{g}^{(k)}$ are the exponents of $\mathfrak{g}^{(1)}$ different from h/k, $2h/k$, $\ldots$, $(k-1)/k$ mod(h), k being the order of τ and h the Coxeter number of $\mathfrak{g}^{(1)}$.

The proof that (3.5.72) form a basis for the folded twisted Kac-Moody algebra is quite involved. Our first aim will be determine the fixed subalgebra $\mathfrak{g}_0$ of a finite dimensional Lie algebra $\mathfrak{g}$ under a $\tau \in W_0(\mathfrak{g})$. We shall first consider a more general case in which τ is not necessarily direct and $\mathfrak{g}$ is not necessarily simply-laced. Suppose that $\tau^k = 1$. Because $W_0(\mathfrak{g}) \subset W(\mathfrak{g})$, τ can be lifted to an inner automorphism ψ_τ of $\mathfrak{g}$:

$$\psi_\tau(a) = T a T^{-1}, \qquad a \in \mathfrak{g}, T \in \mathcal{G}. \tag{3.5.73}$$

From the definition of E_1 we obtain that

$$\psi_\tau(E_1) = E_1. \tag{3.5.74}$$

As E_1 is regular the above result implies that T lies in $\mathcal{H}'$, the maximal torus of $\mathcal{G}$ whose Lie algebra is the Cartan subalgebra in apposition $\mathfrak{h}'$. Moreover, since $\tau^k = 1$, it follows that $T^k \in Z(\mathcal{G})$, the centre of $\mathcal{G}$. We can therefore conclude that

$$T = e^{-2\pi i Y \cdot h}, \qquad kY \in \Lambda_W(\mathfrak{g}^{\mathrm{v}}) \tag{3.5.75}$$

where Y is a vector that we shall now determine.

Taking the scalar products with the simple roots of $\mathfrak{g}$ enables us to deduce[50]

$$\tau(\rho^{\mathrm{v}}) - \rho^{\mathrm{v}} = -h \lambda^{\mathrm{v}}_{\tau(0)} \tag{3.5.76}$$

which implies that

$$z(\tau) := TST^{-1}S^{-1} = e^{-2\pi i \lambda^{\vee}_{\tau(0)} \cdot H} = e^{-2\pi i \lambda^{\vee}_{\tau(0)} \cdot h} \tag{3.5.77}$$

where in the last equality was used the fact that $\mathfrak{h}'$ is conjugate to $\mathfrak{h}$, i. e.,

$$h_i = PH_iP^{-1}, \qquad i = 1, \ldots, r, \qquad P \in \mathcal{G}, \tag{3.5.78}$$

and that the group element $z(\tau)$ lies in $Z(\mathcal{G})$ and therefore commutes with P. On the other hand, using (3.2.21), (3.5.75) and (3.5.77) we find that

$$z(\tau) = e^{-2\pi i (1-\sigma) Y \cdot h}. \tag{3.5.79}$$

Comparing expressions (3.5.77) and (3.5.79) we have that

$$(1-\sigma)Y = \lambda^{\vee}_{\tau(0)} + \Lambda_R(\mathfrak{g}^{\vee}). \tag{3.5.80}$$

where the element of the coroot lattice $\Lambda_R(\mathfrak{g}^{\vee})$ is undetermined.

As $(1-\sigma)$ is non-singular since 1 is not eigenvalue of σ, it has a unique inverse. Furthermore it maps the coweight lattice of $\mathfrak{g}$ into this coroot lattice due to the identity (3.3.44). Therefore,

$$T = e^{-2\pi i (1-\sigma)^{-1} \lambda^{\vee}_{\tau(0)} \cdot h} \tag{3.5.81}$$

up to an element of $Z(\mathcal{G})$ dependent on the undetermined element of the coweight lattice of $\mathfrak{g}$. This undetermined factor is innocuous as it has no effect on the adjoint action of T, (3.5.73). We shall take advantage of this ambiguity by replacing $\lambda^{\vee}_{\tau(0)}$ in (3.5.80) by $w\left(\lambda^{\vee}_{\tau(0)}\right)$ where $w \in W(\mathfrak{g})$ is a Weyl group element. Then, instead of (3.5.81),

$$T = e^{-2\pi i (1-\sigma)^{-1} w \lambda^{\vee}_{\tau(0)} \cdot h} \tag{3.5.82}$$

We shall now add the assumption that $\mathfrak{g}$ is simply laced and that $\tau \in W_0(\mathfrak{g})$ is direct. The latter condition excludes the element of order 4 in $W_0\ (d_{2n+1}) \cong Z_4$.

Let us define

$$Y' := w^{-1}Y = (1-\sigma')^{-1}\lambda_{\tau(0)} \tag{3.5.83}$$

where

$$\sigma' = w^{-1}\sigma w \tag{3.5.84}$$

is equally a Coxeter element of $W(\mathfrak{g})$ since it is conjugate to σ. There always exist a Coxeter element σ' such that

$$\lambda_{\tau^p(0)} = \left(1 + \sigma' + \sigma'^2 + \cdots + \sigma'^{p-1}\right)\lambda_{\tau(0)}\ , \qquad p = 1, \ldots, k. \tag{3.5.85}$$

Formally $\lambda_{\tau^k(0)} = \lambda_0 = 0$ and this is guaranteed by (3.5.85) if

$$(1 - \sigma'^k)\lambda_{\tau(0)} = 0. \tag{3.5.86}$$

Unfortunately the verification of (3.5.86) and (3.5.87) has to be done on a case by case basis and is therefore relegated to the appendix A. It follows from (3.5.85) that

$$\sigma'\lambda_{\tau^p(0)} = \lambda_{\tau^{p+1}(0)} - \lambda_{\tau(0)}. \tag{3.5.87}$$

Summing from values of p running from 1 to $(k-1)$ yields

$$Y' = \frac{1}{k}\left(\lambda_{\tau(0)} + \lambda_{\tau^2(0)} + \cdots + \lambda_{\tau^{(k-1)}(0)}\right). \tag{3.5.88}$$

Moreover, if we define

$$q'\left(\frac{ph}{k}\right) = \frac{1}{k}\sum_{m=1}^{k-1} e^{\frac{-2\pi i pm}{k}}\lambda_{\tau^m(0)} \tag{3.5.89}$$

we likewise find

$$\sigma' q'\left(\frac{ph}{k}\right) = e^{\frac{2\pi i p}{k}} q'\left(\frac{ph}{k}\right). \tag{3.5.90}$$

Then we can write

$$Y' = -\sum_{p=1}^{k-1} q'\left(\frac{ph}{k}\right). \tag{3.5.91}$$

Now let us define $\mathcal{G}_0$, the fixed subgroup of $\mathcal{G}$ with respect to ψ_τ,

$$\mathcal{G}_0 = \{a \in \mathcal{G}; \quad TaT^{-1} = a\}. \tag{3.5.92}$$

The structure of $\mathcal{G}_0$ can be determined by considering the isomorphic group conjugate to $\mathcal{G}_0$ within $\mathcal{G}$

$$\mathcal{G}_0' := W^{-1}\mathcal{G}_0 W \tag{3.5.93}$$

whose elements commutes with

$$T' := W^{-1}TW = e^{-2\pi i Y'.h} \tag{3.5.94}$$

The generators of $\mathfrak{g}_0'$ comprise the complete CSA in apposition of $\mathfrak{g}$, and the step operators $e_{\alpha'}$ for roots α' with respect to their CSA,

$$e_{\alpha'} = W^{-1}e_\alpha W \sim e_{w^{-1}\alpha}, \tag{3.5.95}$$

which satisfy

$$e^{-2\pi i Y'.w^{-1}\alpha} = 1. \tag{3.5.96}$$

But by (3.5.88) and the fact that the weights $\lambda_{\tau^p(0)}$ are minimal so that for any root β, $\lambda_{\tau^p(0)}.\beta = 0$ or ± 1, we have

$$|Y'.\beta| \leq 1 - \frac{1}{k} < 1. \tag{3.5.97}$$

Hence the only solutions to condition (3.5.96) satisfy is

$$Y'.w^{-1}\alpha = Y.\alpha = 0. \tag{3.5.98}$$

This implies that it is orthogonal to each of the minimal fundamental weights $\lambda_{\tau(0)}, \ldots, \lambda_{\tau^{k-1}}(0)$. This, in turn, means that

$$\mathfrak{g}_0 = \mathfrak{g}_0^0 \oplus \underbrace{u(1) \oplus \ldots \oplus u(1)}_{(k-1)\ \text{terms}} \tag{3.5.99}$$

where $\mathfrak{g}_0^0$ is semisimple. As $\mathfrak{g}_0$ has same rank as $\mathfrak{g}$, since $\mathfrak{h}' \subset \mathfrak{g}_0$,

$$\text{rank}(\mathfrak{g}_0^0) = \text{rank}(\mathfrak{g}) + 1 - k. \tag{3.5.100}$$

Furthermore, the semisimple factor $\mathfrak{g}_0^0$ has a Dynkin diagram, $\Delta(\mathfrak{g}_0^0)$, obtained by deleting from $\Delta(\mathfrak{g}^{(1)})$ the k vertices $0, \tau(0), \ldots, \tau^{(k-1)}(0)$. Thus, referring to Table 2, we see that

$$\begin{array}{ll} \Delta(d_{2l}^{(1)}) \to \Delta(a_{2l-1}) \text{ deleting } (0, 2l-1), & \Delta(d_{l+2}^{(1)}) \to \Delta(d_{l+1}) \text{ deleting } (0,1), \\ \Delta(e_7^{(1)}) \to \Delta(e_6) \text{ deleting } (0,6), & \Delta(e_6^{(1)}) \to \Delta(d_4) \text{ deleting } (0,1,5). \end{array} \tag{3.5.101}$$

This results agree with (2.2.64). In particular, we see that $\mathfrak{g}_0^0$ is actually simple in all cases except when deleting vertices $0, n, 2n, \ldots (m-1)n$ from $\Delta(a_{mn-1}^{(1)})$ yields m copies of $\Delta(a_{n-1})$.

Notice also that it follows from (3.5.98) that if e_α is a generator of $\mathfrak{g}_0^0$

$$w^{-1}\alpha.q'\left(\frac{ph}{k}\right) = \alpha.q\left(\frac{ph}{k}\right) = 0, \qquad p = 1, 2, \ldots, k-1. \tag{3.5.102}$$

Thus the roots of $\mathfrak{g}_0^0$ are all perpendicular to the $(k-1)$ dimensional subspace spanned by the $(k-1)$ eigenvectors of σ, (the Coxeter element induced by S) corresponding to the $(k-1)$ exponents $h/k, 2h/k, \ldots, (k-1)h/k$. This was first observed by Dorey[14] for the particular case of $\mathfrak{g} = e_6$. Therefore we can conclude that the $(k-1)$ dimensional Abelian algebra that appears in (3.5.99) is formed by the generators

$$q\left(\frac{h}{k}\right).h \ , \ q\left(\frac{2h}{k}\right).h \ , \ldots , \ q\left(\frac{(k-1)h}{k}\right).h. \tag{3.5.103}$$

We shall now consider the action of S on $\mathfrak{g}_0^0$. From (3.5.77), using the fact that $z(\tau)$ lies in $Z(\mathcal{G})$, it follows that for any pair of integers m and n

$$T^m S^n = z(\tau)^{mn} S^n T^m. \tag{3.5.104}$$

As $T^k \in Z(\mathcal{G})$, it implies that $z(\tau)^k = 1$, and therefore from (3.5.77) we see that S^k commutes with T. Hence S^k lies in $\mathcal{G}_0$ while S itself does not when $k \geq 2$. Nevertheless

$$SaS^{-1} \in \mathcal{G}_0 \qquad \text{for all} \qquad a \in \mathcal{G}_0 \tag{3.5.105}$$

as T commutes with SaS^{-1} by virtue of (3.5.77) and (3.5.92). Thus conjugation by S produces an outer automorphism of $\mathcal{G}_0$, with the property that its k^{th} is inner since $S^k \in \mathcal{G}_0$.

Because $q\left(\frac{ph(\mathfrak{g})}{k}\right)$ are eigenvectors of σ, S acts on the Abelian terms in (3.5.99) diagonally:

$$Sq\left(\frac{ph}{k}\right).hS^{-1} = \sigma q\left(\frac{ph}{k}\right).h = e^{\frac{2\pi i p}{k}} q\left(\frac{ph}{k}\right).h. \tag{3.5.106}$$

As these eigenvectors are orthogonal to all the roots of $\mathfrak{g}_0^0$, (3.5.101), S acts directly on $\mathbf{g}_\tau^{\mathbf{0}}$. So if $e_\alpha \in \mathfrak{g}_0^0$, than,

$$Se_\alpha S^{-1} = e_{\sigma(\alpha)} \in \mathfrak{g}_0^0. \tag{3.5.107}$$

In fact, the roots of $\mathfrak{g}_0^0$ fall into complete orbits of $h(\mathfrak{g})$ elements under σ. This is because if one element of an orbit is orthogonal to the eigenvectors of σ mentioned above then so are all the other roots in the orbit. Therefore

$$F_\mu^i := \frac{1}{\sqrt{h}} \sum_{m=0}^{h-1} w^{-m\mu} e_{\sigma^m \gamma_i} \in g_0^0 \quad \text{if} \quad \gamma_i.q\left(\frac{ph}{k}\right) = 0; \quad p = 1, \ldots, (k-1) \tag{3.5.108}$$

for any value $\mu = 1, 2, \ldots, h(\mathfrak{g})$. If there are l such orbits then $\mathfrak{g}_0^0$ has $lh(\mathfrak{g})$ roots. On the other hand we know that $\mathfrak{g}_0^0$ has $r(\mathfrak{g}_0^0)h(\mathfrak{g}_0^0)$ roots, where $h(\mathfrak{g})$ and $h(\mathfrak{g}_0^0)$ are the Coxeter numbers of $\mathfrak{g}$ and $\mathfrak{g}_0^0$ respectively. Thus

$$lh(\mathfrak{g}) = r(\mathfrak{g}_0^0)h(\mathfrak{g}_0^0) = (r(\mathfrak{g}) + 1 - k)h(\mathfrak{g}_0^0) \tag{3.5.109}$$

This makes it clear that the action of σ on the roots of $\mathfrak{g}_0^0$ differs from the action of its own Coxeter element. This different action is that of the twisted Coxeter element. The element σ^k is inner as far as $\mathcal{G}_0^0$ is concerned, since $S^k \in \mathcal{G}_0^0$, even though σ is outer. Thus σ, being an automorphism of $\mathfrak{g}_0^0$ root system is composed of an element of $\mathrm{Aut}\Delta(\mathfrak{g}_0^0)$ of order k times an element of $W(\mathfrak{g}_0^0)$. It therefore has the same properties as the twisted Coxeter element of $\mathfrak{g}_0^0$ described in [57]. In particular the twisted element of $\mathfrak{g}_0^0$ has the same order as the Coxeter element of $\mathfrak{g}$, namely $h(\mathfrak{g})$. By extension of the usual definition of an exponent, one can define the twisted exponents of $\mathfrak{g}_0^0$ as being those powers of exp $2\pi i/h(\mathfrak{g})$ which occur as eigenvalues of σ applied to the Cartan subalgebra in apposition of $\mathfrak{g}_0^0$. By our construction, the twisted exponents are given by the set of exponents of $\mathfrak{g}$ less the $(k-1)$ exponents $h(\mathfrak{g})/k, 2h(\mathfrak{g})/k, \ldots, (k-1)h(\mathfrak{g})/k$ associated with the Abelian subalgebra of $\mathfrak{g}_0$, (3.5.102).

Using (3.3.44), (3.5.80) and $Y.(\sigma-1)\lambda_i = (1-\sigma)Y.\sigma\lambda_i$ we obtain that[50]

$$e^{-2\pi i Y.\gamma_i} = e^{-2\pi i \lambda^{\vee}_{\tau(0)}.\lambda_i} = e^{-2\pi i \lambda_{\tau(0)}.\lambda_i} \tag{3.5.110}$$

where in the last equality we have used the fact that $\lambda^{\vee}_{\tau(0)} = \lambda_{\tau(0)}$ since the root associated to the vertex symmetrically related to vertex 0 has the same length as ψ, i. e., $\sqrt{2}$. Therefore we can conclude that

$$\begin{aligned} \psi_\tau\left(e_{\gamma_i}\right) &= e^{-2\pi i \lambda_i\cdot\lambda_{\tau(0)}} e_{\gamma_i} \\ \psi_\tau\left(F^i_\nu\right) &= e^{-2\pi i \lambda_i\cdot\lambda_{\tau(0)}} F^i_\nu \end{aligned} \tag{3.5.111}$$

In order to e_{γ_i}, or equivalently F^i_ν, to be in $\mathfrak{g}_0^0$, λ_i must satisfy the condition

$$e^{-2\pi i \lambda_i\cdot\lambda_{\tau(0)}} = 1 \quad . \tag{3.5.112}$$

It is easy to evaluate this condition. Using the numbering of table 1, the results are

$$\begin{aligned} e_6\ (\tau(0)=1):&\quad i=3,6 \\ e_7\ (\tau(0)=6):&\quad i=1,2,3,6 \\ d_{l+2}\ (\tau(0)=1):&\quad i=1,2,\ldots,l \\ d_{2l}\ (\tau(0)=2l):&\quad i=2,4,\ldots,2l-2,2l \ \text{ for } l \text{ even} \\ &\quad i=2,4,\ldots,2l-2,2l-1 \ \text{ for } l \text{ odd} \end{aligned} \tag{3.5.113}$$

We can therefore conclude the generators of $\mathfrak{g}$ which belong to $\mathfrak{g}_0^0$ are

$$\begin{aligned} E_\mu:&\qquad \mu \ \text{ is a twisted exponent of } \mathfrak{g}_0^0, \\ F^i_\nu:&\qquad \nu=1,\ldots,h(\mathfrak{g})-1 \quad \text{such that } e^{-2\pi i\lambda_i.\lambda_{\tau(0)}}=1. \end{aligned} \tag{3.5.114}$$

Given this result one can obtain that (3.5.72) is the subset of generators of the Kac-Moody algebra $\mathfrak{g}^{(1)}$ which is will belong to the Kac-Moody twisted subalgebra $\mathfrak{g}_0^{0\,(k)}$.

3.6 Classical Integrability of Affine Toda field theories.

The classical integrability of ATFT relies on the fact that its equation of motion can be written as a zero curvature condition. From this condition one can construct an infinite number of local conserved charges in involution. We shall sketch the argument for proving the existence of these charges. A more detailed and rigourous proof can be seem in [47, 60, 21].

Let $\Phi = \phi_i H_0^i$ where H_0^i is in the CSA of $\mathfrak{g}^{(k)}$ with the central term $k = 0$. And let us define the gauge fields [1]

$$\begin{aligned} A_+ &= \tfrac{1}{2}\beta\partial_+\Phi + \mu e^{\frac{1}{2}\beta\Phi} E_1^{(p)} e^{-\frac{1}{2}\beta\Phi} \\ A_- &= -\tfrac{1}{2}\beta\partial_-\Phi + \mu e^{-\frac{1}{2}\beta\Phi} E_1^{(p)^\dagger} e^{\frac{1}{2}\beta\Phi} \end{aligned} \qquad (3.6.115)$$

For simplicity we shall re-scale $\lambda^{1/h} \to \lambda$. In order to guarantee that under a Lorentz transformation $A_\pm \to e^{\pm\theta} A_\pm$, the spectral parameter must transform as $\lambda \to e^\theta \lambda$. From the condition that F_{-+} vanishes implies that Φ must satisfy

$$\beta\partial_+\partial_-\Phi + \mu^2 \left[e^{\beta\Phi}\lambda E_1 e^{-\beta\Phi}, \lambda^{-1} E_1^\dagger \right] = 0 \ . \qquad (3.6.116)$$

Using the definition of E_1, one can conclude that (3.6.116) gives the ATFT equation of motion.

Performing the gauge transformation $A_\pm^{\gtrless} = e^{-\frac{\beta\Phi}{2}} A_\pm e^{\frac{\beta\Phi}{2}} + \beta\partial_\pm\Phi$ we obtain

$$\begin{aligned} A_+^{\gtrless} &= \beta\partial_+\Phi + \mu\lambda E_1 \\ A_-^{\gtrless} &= -\mu e^{-\beta\Phi}\lambda^{-1} E_1^\dagger e^{\beta\Phi} \ . \end{aligned} \qquad (3.6.117)$$

Note the gauge fields $A_\pm^{\gtrless}$(or $A_\pm$) satisfy condition (3.4.53)

$$S A_\pm^{\gtrless}(\lambda) S^{-1} = A_\pm^{\gtrless}(e^{\frac{2\pi i}{h}}\lambda) \qquad (3.6.118)$$

where S is the principal element, (3.2.15). Assuming that at $x \to \pm\infty$, ϕ goes to one of the vacua (i.e., $\phi(x = \pm\infty) = 0$ for β real or $\phi(x = \pm\infty) = \frac{2\pi i}{\beta}\Lambda(\mathfrak{g}^\vee)$ for β pure imaginary), it implies that $A_\pm^{\gtrless}$ will satisfy the boundary conditions

$$\begin{aligned} A_\pm^{\gtrless}(x^+ = +\infty) &= A_\pm^{\gtrless}(x^+ = -\infty) \ , \\ A_\pm^{\gtrless}(x^- = +\infty) &= A_\pm^{\gtrless}(x^- = -\infty) \ . \end{aligned} \qquad (3.6.119)$$

Now consider a gauge transformation $a_\pm = \omega A_\pm^{\gtrless}\omega^{-1} + \omega\partial_\pm\omega^{-1}$ such that ω has the form

$$\omega = \exp\left(\lambda^{-1}\omega_1 + \lambda^{-2}\omega_2 + \ldots\right) \ , \qquad (3.6.120)$$

$$S\omega_k S^{-1} = e^{\frac{-2\pi i k}{h}}\omega_k \ . \qquad (3.6.121)$$

Then $a_\pm$ also satisfy (3.6.118). We also assume that $a_\pm$ satisfy the boundary conditions (3.6.119). It is possible to prove [47] that one can choose the quantities ω_k in such a way that

$$[a_+, E_1] = 0 \ . \qquad (3.6.122)$$

[1] We define the light-cone coordinates as $\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$

As E_1 is a regular element this implies that a_+ must take values in the CSA $\mathfrak{h}'$ which has the elements E_μ as basis. Moreover, since a_+ satisfies (3.6.118) it implies that it must have the form

$$a_+ = \lambda E_1 + \sum_\mu \sum_{m=0}^{\infty} \lambda^{-\mu-mh} E_\mu^\dagger I_{\mu+mh}(x) \quad . \tag{3.6.123}$$

We can write $a_- = a_-^h + a_-^e$, where a_-^h is spanned by $E_\mu \in \mathfrak{h}'$ and a_-^e by the e_α. Then, from the zero curvature condition one obtains that

$$\partial_+ a_-^e + \left[a_+, a_-^e\right] = 0 \tag{3.6.124}$$

$$\partial_- a_+ - \partial_+ a_-^h = 0 \tag{3.6.125}$$

Expanding a_-^e in descending powers of λ one can deduce from (3.6.123) and (3.6.124) that a_-^e must vanishes.

The relation (3.6.125) gives a conservation equation with $j^\mu = \epsilon^{\mu\nu} a_\nu$. Therefore the integral of a_+ over x^+ is conserved since

$$\partial_- \int_{-\infty}^{+\infty} dx^+ a_+ = -\int_{-\infty}^{+\infty} dx^+ \partial_+ a_+ = 0 \quad , \tag{3.6.126}$$

using the boundary conditions of a_+. As λ is arbitrary, there are infinitely many conserved charges

$$Q_{\mu+mh} = \int_{-\infty}^{\infty} \mathrm{d}x^+ I_{\mu+mh}(x) \quad . \tag{3.6.127}$$

Using the fact that under a Lorentz transformation $\lambda \to e^\theta \lambda$, we can obtain that $Q_M \to e^{M\theta} Q_M$. Therefore Q_M has spin M.

If instead of gauge transforming by $e^{-\frac{\beta\Phi}{2}}$ in (3.6.117), we gauge transform by $e^{\frac{\beta\Phi}{2}}$, then we obtain

$$\begin{array}{rcl} A_+^< & = & \mu e^{\beta\Phi} \lambda^{-1} E_1 e^{-\beta\Phi} \quad , \\ A_-^< & = & -\partial_+ \Phi - \mu \lambda^{-1} E_1^\dagger \quad . \end{array} \tag{3.6.128}$$

Then we can repeat the previous steps with $E_1^\dagger$ replacing E_1 and λ^{-1} replacing λ. This will result in conserved charges with opposite spins from the previous ones. Therefore we can conclude that the spins of the conserved charges of ATFT associated to $\mathfrak{g}^{(k)}$ are equal to the exponents of $\mathfrak{g}^{(k)}$. It has been proven [47] that the charges of ATFT are in involution.

3.7 Particle mass spectrum and couplings

Let us now analyse the mass and three point coupling of the particles of ATFT. The equation (3.6.116) can be linearised to

$$\partial_+ \partial_- \Phi + \mu^2 \left[\left[\Phi, E_1^{(p)}\right], E_1^{(p)\dagger}\right] + O(\Phi^2) = 0 \quad . \tag{3.7.129}$$

As $\Phi = 0$ is a vacuum of the theory the second term gives the poles of the propagators at tree level which correspond to the masses of the quantum particles at the tree level. As we have mentioned before, the generators F_0^i form a basis for the Cartan subalgebra. Therefore, instead of using the generators H_0^i, we can expand Φ in terms of $F_0^{i(p)}$, i.e., $\Phi = \phi_i F_0^{i(p)}$. Then from

$$\begin{aligned} \left[\left[F_0^{i(p)}, E_1^{(p)}\right], E_1^{(p)\dagger}\right] &= |q(1).\gamma^i|^2 F_0^{i(p)} \\ &= 2hy_i^{(1)^2} F_0^{i(p)} \end{aligned} \tag{3.7.130}$$

it follows that the mass associated to the particle of species i is [5, 21, 23]

$$m_i = \mu\sqrt{2h}y_i^{(1)} \ . \tag{3.7.131}$$

Remember that $y_i^{(1)}$ are the components of the right Perron-Frobeniush eigenvector that are strictly positive. It follows that all the particles will have mass greater than zero. The masses for the particles of ATFT based on the twisted algebras correspond to the set of generators F_0^i which "survive" to the folding. This set is given by (3.5.113) and confirms the result of [5] in which they used a case by case analysis.

The equation of motion (3.2.18) can be obtained from the Lagrangian density

$$\mathcal{L} = \left\langle \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - e^{\beta\Phi}E_1^{(p)}e^{-\beta\Phi}E_1^{(p)\dagger} \right\rangle \ . \tag{3.7.132}$$

It is also not difficult to prove that this Lagrangian density is equal to (3.1.1).

Expanding it to the third order in Φ we obtain the term defining the three point coupling C_{ijk},

$$\frac{\beta^3}{3!}C_{ijk}\phi_i\phi_j\phi_k = \frac{\beta^3}{3!}\left\langle \left[\left[F_0^{i(p)}, E_1^{(p)}\right], \left[F_0^{j(p)}, E_1^{(p)\dagger}\right]\right] F_0^{k(p)} \right\rangle \phi_i\phi_j\phi_k \ . \tag{3.7.133}$$

The above Killing form has been calculated in [23] and it has been proven that C_{ijk} is not zero if and only if there exist three roots in the orbits Ω_i, Ω_j and Ω_k which adds to zero. This result confirmed Dorey's fusing rule which has been conjectured in [13]. From the classical mass spectrum and the three point coupling using the S-matrix axioms[16, 63] it has been possible to conjecture an exact S-matrix for the particles of ATFT associated to simpy laced algebras[5, 12, 13, 24]

Chapter 4

Solitons in Affine Toda field theories

4.1 Introduction

In this chapter we shall analyse the soliton solutions of ATFT. Let us then consider (3.6.115), but now

$$\Phi = \phi . H_0 + \phi_k K + \phi_d d' \tag{4.1.1}$$

For simplicity we shall consider that $\mathfrak{g}^{(k)}$ to be untwisted. It is convenient to perform the constant gauge transformation $A_\pm \to \lambda^{T_0^3/h} A_\pm \lambda^{-T_0^3/h}$ which, by using (3.4.54), will result in

$$\begin{aligned} A_+ &= \tfrac{1}{2}\beta\partial_+\Phi + \mu e^{\frac{1}{2}\beta\Phi}\hat{E}_1 e^{-\frac{1}{2}\beta\Phi} \\ A_- &= -\tfrac{1}{2}\beta\partial_-\Phi + \mu e^{-\frac{1}{2}\beta\Phi}\hat{E}_{-1}^{\dagger} e^{\frac{1}{2}\beta\Phi} \end{aligned} \tag{4.1.2}$$

The field strength will transformation as $F_{-+} \to \lambda^{T_0^3/h} F_{-+} \lambda^{-T_0^3/h}$ resulting in

$$\partial_+\partial_-\Phi + \frac{\mu^2}{\beta}\left[e^{\beta\Phi}\hat{E}_1 e^{-\beta\Phi}, \hat{E}_{-1}\right] = 0 \quad . \tag{4.1.3}$$

This is the equation of motion of conformal Toda affine theory [4, 1]. Writing this equation in terms of the components fields we obtain that

$$\begin{aligned} \partial_+\partial_-\phi + \frac{\mu^2}{\beta} e^{\beta\phi_d} \sum_{i=0}^{r} m_i \alpha_i^{\vee} e^{\beta\alpha_i.\phi} &= 0 \quad , \\ \partial_+\partial_-\phi_k + \frac{\mu^2}{\beta} e^{\beta\phi_d} e^{\beta\alpha_0.\phi} &= 0 \quad , \\ \partial_+\partial_-\phi_d &= 0 \quad . \end{aligned} \tag{4.1.4}$$

From these equations one can note that ϕ_k is an auxiliary field and ϕ_d is a free field. If one choose to put $\phi_d = 0$ then ϕ will satisfy the ATFT equation of motion(3.1.4).

The general solution of (4.1.3) has been obtained in [43, 4] for $\mathfrak{g}^{(k)}=a_1^{(1)}$ using the Leznov-Saveliev [38] method and later extended to the other untwisted cases in [2, 49]. This solution can be written as

$$e^{-\beta\Lambda.\varphi}=e^{-\beta\Lambda.\varphi^0}\langle\Lambda|U(x^+)V(x^-)|\Lambda\rangle \tag{4.1.5}$$

where Λ is an arbitrary highest weight of $\mathfrak{g}^{(k)}$, $\varphi^0:=(\phi^0,\phi_k^0,\phi_d^0)$ satisfies a free field equation, $\varphi:=(\phi,\phi_k,\phi_d)$ and $U(x^+),V(x^-)$ satisfy

$$\begin{aligned}\partial_+U&=-\mu e^{\beta\phi^{0,+}.H_0}\hat{E}_1e^{-\beta\phi^{0,+}}U\ ,\\ \partial_-V&=-\mu Ve^{-\beta\phi^{0,-}.H_0}\hat{E}_{-1}e^{\beta\phi^{0,-}}\ ,\end{aligned} \tag{4.1.6}$$

where $\phi^{0,\pm}$ are the left and right moving parts of ϕ^0.

Taking $\Lambda=\Lambda_i$, $i=0,1,\ldots r$, we obtain that

$$\begin{aligned}e^{-\beta(\lambda_i.\phi+m_i\phi_k)}&=e^{-\beta(\lambda_i.\phi^0+m_i\phi_k^0)}\langle\Lambda_i|VU|\Lambda_i\rangle\qquad i=1,2,\ldots,r\ ,\\ e^{-\beta\phi_k}&=e^{-\beta\phi_k^0}\langle\Lambda_0|VU|\Lambda_0\rangle\ .\end{aligned} \tag{4.1.7}$$

Therefore it follows that

$$e^{-\beta\lambda_i.\phi}=e^{-\beta\lambda_i.\phi^0}\frac{\langle\Lambda_i|VU|\Lambda_i\rangle}{\langle\Lambda_0|VU|\Lambda_0\rangle^{m_i}}\qquad i=1,2,\ldots,r\ . \tag{4.1.8}$$

Until now we have kept β arbitrary. Now we shall consider that β is pure imaginary, in which case there exist a degenerated vacua. We shall also consider that $\phi^{0,\pm}$ is constant and taking values in one of the vacua $(2\pi i/\beta)\Lambda_W(\mathfrak{g}^{\text{v}})$[49]. Then from (4.1.6) we obtain that

$$\begin{aligned}U&=e^{-\mu\hat{E}_1x^+}g_1(0)\ ,\\ V&=g_2(0)\,e^{-\mu\hat{E}_{-1}x^-}\ ,\end{aligned} \tag{4.1.9}$$

where $g_1(0)$ and $g_2(0)$ are group valued integration constants. Substituting these results in (4.1.8) will give

$$e^{-\beta\lambda_i.\phi}=\frac{\langle\Lambda_i|e^{-\mu\hat{E}_1x^+}g(0)e^{-\mu\hat{E}_{-1}x^-}|\Lambda_i\rangle}{\langle\Lambda_0|e^{-\mu\hat{E}_1x^+}g(0)e^{-\mu\hat{E}_{-1}x^-}|\Lambda_0\rangle^{m_i}}\qquad i=1,2,\ldots,r\ , \tag{4.1.10}$$

where $g(0):=g_1(0)g_2(0)$ and the constant phase $\exp(-\beta\lambda_i.\phi^0)$ has been absorbed by a redefinition of ϕ. We shall show that (4.1.10) satisfies the definition for soliton solutions given in chapter one. However we don't have a proof that it gives all soliton solutions from the general solution of ATFT.

Consider that $g(0)$ is generated by $\hat{E}_M$. Those $\hat{E}_M$ with positive (negative) M annihilate the highest-weight states on the right (left) leaving a function of the level. As the level of the numerator and denominator is the same, this function cancels being therefore a trivial solution. In [49] it is argued that the non-trivial solutions are produced by generators which diagonalise the ad-action of $\hat{E}_M$. They are given by

$$\hat{F}^i(z) = a_i \sum_{N=-\infty}^{+\infty} z^{-N} \hat{F}^i_N \qquad i = 1, 2, \ldots, r \quad , \tag{4.1.11}$$

where z is a complex parameter and a_i is a normalization constant that we shall fix later. It follows that

$$\left[\hat{E}_M, \hat{F}^i(z)\right] = \gamma_i \cdot q([M]) z^M \hat{F}^i(z) \quad . \tag{4.1.12}$$

Therefore the generators $\hat{F}^i(z), i = 1, 2, \ldots, r$, are indeed eigenvectors of the adjoint action of $\hat{E}_M$.

Then the Kac-Moody group element $g(0)$ can be parametrized as follows[49]:

$$g(0) = \prod_{p=1}^{n} g_p(0) \tag{4.1.13}$$

where

$$g_p(0) := e^{Q_p \hat{F}^{i_p}(z_p)} \tag{4.1.14}$$

where Q_p are some complex numbers which physical interpretation we shall analyse later. By using (4.1.12) it results that

$$g(t) := e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} = \prod_{p=1}^{n} g_p(t) \tag{4.1.15}$$

where

$$g_p(t) = \exp\left(Q_p W_{i_p} \hat{F}^{i_p} z_p\right) \tag{4.1.16}$$

and

$$W_{i_p} = \exp\left\{-\mu \left[\gamma_{i_p} \cdot q(1) z_p x^+ - \gamma_{i_p} \cdot q(1)^* z_p^{-1} x^-\right]\right\} \tag{4.1.17}$$

Since the ATFT equation of motion is Lorentz invariant, a Lorentz boost $x^\pm \to e^{\pm\theta} x^\pm$ applied to any solution must produce another solution. From expression (4.1.17) we can see that this can be achieved by replacing

$$z_p \to e^{-\theta} z_p \qquad , \quad p = 1, 2, \ldots, r \tag{4.1.18}$$

Let us consider (4.1.15) with $n = 1$, which corresponds to one-soliton solution, and require that it can be put at rest , i.e., that the coefficient of $t = (x^+ + x^-)/\sqrt{2}$ vanishes. This happens if

$$z = \pm \sqrt{\frac{\gamma_i \cdot q(1)^*}{\gamma_i \cdot q(1)}} = \pm i e^{-\pi\left(\frac{1+c(i)}{2h}\right)} \tag{4.1.19}$$

The ambiguity in sign corresponds to have a soliton or an anti-soliton since the two solutions run relatively backwards in space. Let us take the plus sign. Then, using (4.1.18) we can conclude that boost this static solution to an arbitrary rapidity θ is equivalent to take

$$z = ie^{-\theta}\sqrt{\frac{\gamma_i \cdot q(1)^*}{\gamma_i \cdot q(1)}} = ie^{-\theta-\pi\left(\frac{1+c(i)}{2h}\right)} \tag{4.1.20}$$

Using this parametrization of z in (4.1.17) gives

$$W_{i(m)}(\theta_{(m)}) = e^{\mu_{i(m)}\left(x\cosh\theta_m - t\sinh\theta_{(m)}\right)}. \tag{4.1.21}$$

where $\mu_{i_{(m)}}$ is the mass of the particle of species "$i_{(m)}$", (3.7.131). Like in the sine-Gordon case, the complex number Q_m can be parametrised as

$$Q_m = e^{i\psi_m}e^{-\mu_{i(m)}x_m^0\cosh\theta_m} \tag{4.1.22}$$

where x_m^0 relates to the space coordinate of the m'th soliton at $t = 0$. The phase ψ_m relates to the topological quantum number defined as the difference between the values the affine Toda field takes at large distances. Certain discrete values are forbidden by the requirement that the solution should not develop singularities as x varies over space.

Note that from (4.1.10) we obtain that

$$\phi = -\frac{1}{\beta}\sum_{j=0}^{r}\alpha_j^{\vee}\ln\tau_j \tag{4.1.23}$$

where

$$\tau_j := \langle\Lambda_j|g(t)|\Lambda_j\rangle \tag{4.1.24}$$

Written in this form it becomes clear that τ_j are the tau-functions which appear in the Hirota method used in [31, 39, 3, 7] to construct soliton solutions.

It has been first proven by Hollowood in [31] for the $a_r^{(1)}$ theories that although the soliton solutions are not real they nevertheless have real energy and momentum. This result has been extended for other untwisted algebras in [49]. They have obtained that the momentum components for a multisoliton solution have the form

$$P^{\pm} = \sum_p \frac{M_{i_p}}{\sqrt{2}}e^{\pm\theta_p} \tag{4.1.25}$$

where

$$M_i = \frac{2h}{|\beta^2|}\frac{2\mu_i}{\gamma_i^2} \tag{4.1.26}$$

is the soliton mass and μ_i is the particle mass (3.7.131) of species "i". As the particle masses are proportional to the components of the right Perron-Frobenius vector $y_i^{(1)}$

we can conclude that the soliton masses are proportional to the components of the left Perron-Frobenius vector $x_i^{(1)}$. Moreover we can observe that

$$\frac{M_j}{\mu_j}(\mathfrak{g},\beta) = \frac{\mu_j}{M_j}(\mathfrak{g}^{\mathrm{v}}, 2h/\beta)\ , \tag{4.1.27}$$

which bears a resemblance to the electro-magnetic duality conjectures in four dimensions [28].

The result (4.1.25) indicates that (4.1.15) describes a combination of individual one-soliton solutions. Moreover as the energy $E = (P^+ + P^-)/\sqrt{2}$ is finite, it indicates this solution must approach the vacua of the theory as $x \to \pm\infty$ as a soliton solution must do. Indeed, we shall prove this result explicitly in the next sections.

The other conserved charges for a single solitons for ATFT based on untwisted $\mathfrak{g}^{(1)}$ has been calculated in [22]. It has been shown that, like the soliton masses, all the other charges for the soliton solutions are real and proportional to the left eigenvectors $x_i^{(\mu)}$. This gives more evidence for the existence for a unitary theory inside ATFT with imaginary coupling. It also indicates a duality between the solitons of ATFT based on $\mathfrak{g}^{(1)}$ and the particles of ATFT based on $\mathfrak{g}^{\mathrm{v}(1)}$ since their charges are proportional at least at the classical level.

4.2 Principal vertex operator construction for simply laced Kac-Moody algebras.

Let us consider a level one irreducible representation which has Λ_j as highest-weight. For these representations when $\mathfrak{g}^{(1)}$ is simply-laced there exist a vertex operator construction [26, 55, 50] for $F^i(z)$,

$$\hat{F}^i(z) = F_{ji}\, Y_-^i\, Y_+^i \tag{4.2.28}$$

where

$$Y_\pm^i = \exp\left(\sum_{M>0} \frac{\gamma_i . q(\mp[M]) z^{\mp M}}{\mp M} \hat{E}_{\pm M}\right)\quad , \tag{4.2.29}$$

$$F_{ji} = \langle \Lambda_j | \hat{F}^i(z) | \Lambda_j \rangle\quad . \tag{4.2.30}$$

Using (4.3.59) it is easy to see that (4.2.28) has the correct commutation relation with $\hat{E}_N$ for level one representations. It has also the correct expectation value in the highest-weight state $|\Lambda_j\rangle$ by virtue of (3.4.59).

The expectation values F_{ji} can be determined in the following way: all the level 1 fundamental weights Λ_j correspond to vertices $\Delta(\mathfrak{g}^{(1)})$ related to the vertices "0" by diagram automorphisms of $\Delta(\mathfrak{g}^{(1)})$, therefore there exist a $\tau \in W_0$ such that $\Lambda_j = \Lambda_{\tau(0)}$. It follows that

$$\begin{aligned} \langle \Lambda_0 | \hat{F}^i(z) | \Lambda_0 \rangle &= \langle \Lambda_{\tau(0)} | \tau(\hat{F}^i(z)) | \Lambda_{\tau(0)} \rangle \\ &= e^{-2\pi i \lambda_i . \lambda_{\tau(0)}} \langle \Lambda_{\tau(0)} | \hat{F}^i(z) | \Lambda_{\tau(0)} \rangle \end{aligned} \tag{4.2.31}$$

where we have used the fact that

$$\tau(\hat{F}^i(z)) = e^{-2\pi i \lambda_i . \lambda_{\tau(0)}} \hat{F}^i(z) \tag{4.2.32}$$

which is a consequence of (3.5.71). We can fix $a_i = \langle \Lambda_0 | \hat{F}^i(z) | \Lambda_0 \rangle^{-1}$ in (4.1.11). This will imply that for any level one representation with Λ_j as highest weight,

$$F_{ji} = e^{-2\pi i \lambda_i . \lambda_j} \tag{4.2.33}$$

with $\lambda_0 := 0$. Using the normal ordering procedure in which $\hat{E}_M$ with positive suffices are moved to the right of those with negative one obtains that [50]

$$\hat{F}^i(z_i)\hat{F}^k(z_k) = X_{ik}(z_i, z_k) : \hat{F}^i(z_i)\hat{F}^k(z_k) : \tag{4.2.34}$$

where

$$: \hat{F}^i(z_i)\hat{F}^k(z_k) := e^{-2i\pi(\lambda_i + \lambda_k)\cdot\lambda_j} Y_-^i Y_-^k Y_+^i Y_+^k \tag{4.2.35}$$

and

$$X_{ik}(z_i, z_k) = \exp\left(- \sum_{N>0} \frac{\gamma_i \cdot q([N])^* \gamma_k \cdot q([N])}{N} \left(\frac{z_k}{z_i}\right)^N \right) \tag{4.2.36}$$

and with the sum converging when $|z_i| > |z_k|$. The function $X_{ik}(z_i, z_k)$ can be put in the form [50]

$$X_{ik}(z_i, z_k) = \prod_{p=1}^{h} \left(z_i - \omega^{-p} z_k \right)^{\sigma^p(\gamma_i)\cdot\gamma_k} \tag{4.2.37}$$

where $\omega = \exp(2\pi i/h)$. The exponents, $\sigma^p(\gamma_i)\cdot\gamma_k$, being scalar products of roots of a simply laced Lie algebra can take the values 0, ± 1, ± 2. Thus $X_{ij}(z_i, z_j)$ can be analytically extended to a meromorphic function of the complex variables z_i and z_k. Note that

$$X_{ii}(z, z) = 0 \quad , \tag{4.2.38}$$

as the factor of $p = h$ vanishes and the other factors are regular. This implies the nilpotency property

$$\hat{F}^i(z)\hat{F}^i(z) = 0 \quad . \tag{4.2.39}$$

Until now we have considered $\hat{F}^i(z)$ in fundamental representations of level 1. Now let us consider another fundamental representation with weight Λ_k of level m_k

greater than 1. The corresponding level one highest weight state $|\Lambda_k\rangle$ can be represented as a state occurring in the m_k-fold tensor product with weights $\Lambda_{j(1)}, ..., \Lambda_{j(m_k)}$ which must satisfy

$$e^{-2\pi i \lambda_i \cdot \lambda_k} = e^{-2\pi i \lambda_i \cdot \sum_{p=1}^{m_k} \lambda_{j(p)}}, \tag{4.2.40}$$

but the detailed construction of the state will, fortunately, be immaterial for the results that follow. So

$$\hat{F}^i(\theta) = \hat{F}^i(\theta)_{(1)} + \hat{F}^i(\theta)_{(2)} + \cdots + \hat{F}^i(\theta)_{(m_k)}, \tag{4.2.41}$$

and similarly for the Heisenberg subalgebra,

$$\hat{E}_M = \hat{E}_{M(1)} + \hat{E}_{M(2)} + \cdots + \hat{E}_{M(m_k)} \quad . \tag{4.2.42}$$

In this notation the bracketed subscript denotes which of the spaces it is upon which the operator acts non-trivially, i.e. ,

$$T_{(m)} = \underbrace{1 \otimes 1 \otimes \ldots 1 \otimes T}_{m} \otimes 1 \otimes \ldots \tag{4.2.43}$$

Clearly,

$$T_{(m)} T_{(n)} = T_{(n)} T_{(m)} \qquad \text{for } m \neq n \quad . \tag{4.2.44}$$

Thus the terms in (4.2.41) mutually commute and each have vanishing square as each is given by the vertex operator construction (4.2.28) since they are in a level one representation. As a simple example consider the case $m_k = 2$. In this case

$$\begin{aligned} \hat{F}^i(z)^2 &= 2\hat{F}^i(z)_{(1)} \hat{F}^i(z)_{(2)} \quad , \\ \hat{F}^i(z)^3 &= 0 \quad . \end{aligned} \tag{4.2.45}$$

For an arbitrary level m_k we shall have

$$\begin{aligned} \frac{(\hat{F}^i(z))^{m_k}}{m_k!} &= \hat{F}^i(z)_{(1)} \hat{F}^i(z)_{(2)} \ldots \hat{F}^i(z)_{(m_k)} \\ &= e^{-2\pi i \lambda_i \cdot \lambda_k} Y^i_{-(1)} \ldots Y^i_{-(m_k)} Y^i_{+(1)} \ldots Y^i_{+(m_k)} \\ (\hat{F}^i(z))^{m_k+1} &= 0 \quad . \end{aligned} \tag{4.2.46}$$

Let us once more consider the case when $m_k = 2$. Using (4.2.44) and (4.2.42) we obtain that

$$\begin{aligned} Y^i_{\pm(1)} Y^i_{\pm(2)} &= e^{\left(\sum_{M>0} \frac{\gamma_i \cdot q(\mp[M]) z^{\mp M}}{\mp M} \hat{E}_{\pm M(1)}\right)} e^{\left(\sum_{M>0} \frac{\gamma_i \cdot q(\mp[M]) z^{\mp M}}{\mp M} \hat{E}_{\pm M(2)}\right)} \\ &= e^{\left(\sum_{M>0} \frac{\gamma_i \cdot q(\mp[M]) z^{\mp M}}{\mp M} \left(\hat{E}_{\pm M(1)} + \hat{E}_{\pm M(2)}\right)\right)} \\ &= e^{\left(\sum_{M>0} \frac{\gamma_i \cdot q(\mp[M]) z^{\mp M}}{\mp M} \hat{E}_{\pm M}\right)} = Y^i_{\pm} \end{aligned} \tag{4.2.47}$$

Then it is easy to conclude that for an arbitrary level m_k

$$\frac{(\hat{F}^i(z))^{m_k}}{m_k!} = e^{-2\pi i\lambda_i\cdot\lambda_k} Y_-^i Y_+^i \tag{4.2.48}$$

where

$$Y_\pm^i = \exp\left(\sum_{M>0} \frac{\gamma_i.q(\mp[M])z^{\mp M}}{\mp M}\hat{E}_{\pm M}\right) \quad . \tag{4.2.49}$$

This is a new vertex operator construction[36]. Of course (4.2.28) is now seen to be the special case of (4.2.48) when $m_k = 1$.

The operator product of two vertex operators is

$$\frac{(\hat{F}^i(z))^{m_k}}{m_k!}\frac{(\hat{F}^{i'}(z'))^{m_k}}{m_k!} = [X_{i,i'}(z,z')]^{m_k} : \frac{(\hat{F}^i(z))^{m_k}}{m_k!}\frac{(\hat{F}^{i'}(z'))^{m_k}}{m_k!} : \tag{4.2.50}$$

This result can be extended to an arbitrary product resulting

$$\frac{(\hat{F}^{i(1)}(z_{(1)}))^{m_k}}{m_k!}\cdots\frac{(\hat{F}^{i(n)}(z_{(n)}))^{m_k}}{m_k!} = \prod_{1\le p,q\le n}[X_{i(p),i(q)}(z_{(p)},z_{(q)})]^{m_k} : \frac{(\hat{F}^{i(1)}(z_{(1)}))^{m_k}}{m_k!}\cdots\frac{(\hat{F}^{i(n)}(z_{(n)}))^{m_k}}{m_k!} : \tag{4.2.51}$$

4.3 Principal vertex operator construction for non-simply laced and twisted Kac-Moody algebras.

Let us extend our vertex operator construction of the previous section for twisted Kac-Moody algebras. Using (4.2.32) we can conclude that in order to $\hat{F}^i(z)$ belong to the folded subalgebra $\mathfrak{g}_0^{0(k)}$, λ_i must satisfy the condition $e^{-2i\pi\lambda_i\cdot\lambda_{\tau(0)}} = 1$. They clearly diagonalise the ad-action of $\hat{E}_M \in \mathfrak{g}_0^{0(k)}$. Let us now consider how the fundamental highest weight representations of $\mathfrak{g}_0^{0(k)}$ follows from those of $\mathfrak{g}^{(1)}$ via folding. What happens is that the inequivalent fundamental representations of $\mathfrak{g}^{(1)}$ with highest weights $\Lambda_j, \Lambda_{\tau(j)}, \ldots$, become identified as a single fundamental representation of $\mathfrak{g}_0^{0(k)}$ whose highest weight is denoted by $\Lambda_{\langle j\rangle}$. This is because (2.2.61) have the same action on each of these states and they generate $\mathfrak{g}_0^{0(k)}$. Moreover $m_j = m_{\tau(j)}$ the level of $\Lambda_{\langle j\rangle}$ is the same as the level of $\Lambda_j, \Lambda_{\tau(j)}, \ldots$. Therefore we can take

$$|\Lambda_{\langle j\rangle}\rangle = |\Lambda_j\rangle \quad \text{or} \quad |\Lambda_{\tau(j)}\rangle \quad \text{or} \quad \ldots \tag{4.3.52}$$

As $\mathfrak{g}^{(1)}$ is simply laced,

$$\hat{F}^i(z)^{m_j+1} = 0 \tag{4.3.53}$$

in a $\mathfrak{g}^{(1)}$ irrep with highest weight Λ_j. This therefore remains true in the $\Lambda_{\langle j\rangle}$ irrep of $\mathfrak{g}_0^{0(k)}$ for the surviving $\hat{F}^i(z)$. In order to establish that the vertex operator construction (4.2.48) for the surviving $\hat{F}^i(z)$ makes sense in $\mathfrak{g}_0^{0(k)}$ as well as $\mathfrak{g}^{(1)}$ we need to check two things.

First consider the sum in the exponential. In $\mathfrak{g}^{(1)}$ the sum includes all the exponents of $\mathfrak{g}^{(1)}$ but in $\mathfrak{g}_0^{0(k)}$ the sum should be restricted to the twisted exponents of $\mathfrak{g}_0^{0(k)}$. This restriction is automatically guaranteed as when $\hat{F}^i(z)$ lies in $\mathfrak{g}_0^{0(k)}$, γ_i satisfies condition (3.5.102) . Thus the surviving $\hat{F}^i(z) \in \mathfrak{g}_0^{0(k)}$ are obtained by exponentiating the twisted principal Heisenberg subalgebra ones.

The second point concerns the phase factor in (4.2.28). We have to show that this phase is unaltered if Λ_j is replaced by $\Lambda_{\tau(j)}$ so that it is independent of the representative Λ_j or $\Lambda_{\tau(j)}$ chosen for $\Lambda_{<j>}$. Consider the expectation value

$$\langle\Lambda_{\tau(j)}| \frac{\left(\hat{F}^i(z)\right)^{m_j}}{m_j!} |\Lambda_{\tau(j)}\rangle = e^{-2\pi i\lambda_i.\lambda_{\tau(j)}} \quad . \tag{4.3.54}$$

Lifting τ to the outer automorphism ψ_τ of $\mathfrak{g}^{(1)}$ expression (4.3.54) equals

$$\langle\Lambda_j| \frac{\left(\psi_\tau^{-1}\left(\hat{F}^i(z)\right)\right)^{m_j}}{m_j!} |\Lambda_j\rangle \, . \tag{4.3.55}$$

But $\psi_\tau^{-1}\left(\hat{F}^i(z)\right) = \hat{F}^i(z)$ precisely when $\hat{F}^i(z) \in \mathfrak{g}_0^{0(k)}$. Thus this expression equals expression (4.3.54) with $\tau(j)$ replaced by j. It follows that the vertex operator construction (4.2.48) holds for the $F^i(z) \in \mathfrak{g}_0^{0(k)}$ with the sum in the exponentials $Y_\pm^i$ running over the twisted exponents.

Let us now consider that $\mathfrak{g}^{(1)}$ is non-simply laced. We shall see that also in this case the highest non-vanishing power of $\hat{F}^i(z)$ is a vertex operator. Once more we shall use the fact that each of such algebras can be obtained from a simply laced one, $\mathfrak{g}^{(1)}$, as a grade zero subalgebra $\mathfrak{g}_0^{(1)}$ under an element of $\tau \in \mathrm{Aut}\Delta(\mathfrak{g})$ lifted to ψ_τ.

It has been proven in [50] that

$$\psi_\tau\left(\hat{F}^i(z)\right) = \hat{F}^{\tau(i)}(z) \quad . \tag{4.3.56}$$

Therefore the generators

$$\hat{F}^{\langle i\rangle}(z) := \sum_{i\in\langle i\rangle} \hat{F}^i(z) \tag{4.3.57}$$

are in $\mathfrak{g}_0^{(1)}$. It follows from (3.5.70) that they are eigenvectors under the adjoint action of $\hat{E}_M \in \mathfrak{g}_0^{(1)}$.

By the same arguments of the twisted algebras we can conclude that the fundamental representations of $\mathfrak{g}^{(1)}$ with highest weights $\Lambda_j, \Lambda_{\tau(j)}, \ldots$ become identified as a single representation of $\mathfrak{g}_0^{(1)}$, denoted by $\Lambda_{\langle j\rangle}$.

Since $X_{ik}(z,z) = X_{ki}(z,z)$ it follows that

$$\left[\hat{F}^i(z), \hat{F}^k(z)\right] = 0 \quad . \tag{4.3.58}$$

Then using (4.3.57) and the fact that $\hat{F}^i(z)^{m_{\langle j\rangle}+1} = 0$ in the irrep $\Lambda_{\langle j\rangle}$ we obtain that

$$\left(\hat{F}^{\langle i\rangle}(z)\right)^{p_i m_j} = (p_i m_j)! \prod_{i\in\langle i\rangle} \frac{\left(\hat{F}^i(z)\right)^{m_j}}{m_j!}, \tag{4.3.59}$$

$$\left(\hat{F}^{\langle i\rangle}(z)\right)^{p_i m_j+1} = 0, \tag{4.3.60}$$

where $p_i := |\langle i\rangle|$ is the number of roots in the orbit $\langle i\rangle$. Now each factor in the product (4.3.59) can be written as a vertex operator. Then performing a normal ordering (4.2.50) and using (2.2.54) and (2.2.55) it will result that

$$V^{\langle i\rangle}(z) := \frac{\left(\hat{F}^{\langle i\rangle}(z)\right)^{p_i m_j}}{(p_i m_j)!} = \left(a_{\langle i\rangle}\right)^{m_j} e^{-2\pi i \lambda^{\mathrm{v}}_{\langle i\rangle}\cdot\lambda_j}\, Y_-^{\langle i\rangle} Y_+^{\langle i\rangle} \tag{4.3.61}$$

where

$$Y_\pm^{\langle i\rangle} = \exp\left(\sum_{M>0} \frac{\gamma^{\mathrm{v}}_{\langle i\rangle}\cdot q(\mp[M]) z^{\mp M}}{\mp M} \hat{E}_{\pm M}\right) \quad , \tag{4.3.62}$$

$$a_{\langle i\rangle} = \prod_{0\le p<q\le p_i} X_{\tau^p_{(i)}\tau^q_{(j)}}(z,z) \quad , \tag{4.3.63}$$

The sum in (4.3.62) runs over the exponents of $\mathfrak{g}^{(1)}$. However in order to the vertex operator construction to make sense in the non-simply laced algebra $\mathfrak{g}_0^{(1)}$ the sum must be on its exponents. Let us see that this happens indeed.

For the cases in which τ is of order two and M is not a degenerated exponent, the relation (3.5.64) implies that if E_M does not belong to $\mathfrak{g}_0^{(1)}$, then

$$\psi_\tau(\hat{E}_M) = -\hat{E}_M \tag{4.3.64}$$

Therefore using (3.5.67) and (3.4.56) we can conclude that

$$\gamma_i \cdot q([M]) = -\gamma_{\tau(i)} \cdot q([M]) \tag{4.3.65}$$

This implies that $\gamma_i^{\mathrm{v}} \cdot q([M])$ will vanish if $\hat{E}_M$ does not belong to $\mathfrak{g}_0^{(1)}$. The same result can be obtained by explicit computation for the order three automorphism of $d_4^{(1)}$ and for the degenerated exponents of $d_{2n}^{(1)}$. Therefore (4.3.61) makes sense in the fixed non-simply laced subalgebra $\mathfrak{g}_0^{(1)}$ since the sum is restricted to its exponents.

Therefore we have proven that for any Kac-Moody algebra, the highest non-vanishing power of $\hat{F}^i(z)$ has a vertex operator construction given by (4.3.61).(with $p_i = 1$ and $a_i = 1$ for the simply laced or twisted cases.)

The normal ordering of two vertex operators is

$$V^i(\theta)V^{i'}(\theta') = [X_{i,i'}(z,z')]^{m_k} : V^i(\theta)V^{i'}(\theta') : \tag{4.3.66}$$

but now

$$X_{ik}(z_i, z_k) = \prod_{p=1}^{h} \left(z_i - \omega^{-p} z_k\right)^{\sigma^p(\gamma_i^\vee)\cdot\gamma_k^\vee} \quad . \tag{4.3.67}$$

Of course this expression of X_{ik} coincides with the previous one when $\mathfrak{g}^{(k)}$ is not non-simply laced.

4.4 Asymptotic behaviour of solitons solutions

Now we are in position to show that in the limits $x \to \pm\infty$ the soliton solutions tend to a vacua.

Using relation (3.4.59) we obtain that

$$< \Lambda_k|V^i(z)|\Lambda_k > = a_k e^{-2\pi i \lambda_i^\vee\cdot\lambda_k}, \tag{4.4.68}$$

a surprisingly simple result, holding for all $k = 0, 1, 2, \ldots r$ and $i = 1, 2, \ldots r$. Note that (4.4.68) is independent of z (or θ). This is a special case of a more general result: the expectation value of any power of $\hat{F}^i(\theta)$ is independent of θ. From the definitions of d' and $\hat{F}^i(\theta)$ we obtain that

$$[d', \hat{F}^i(\theta)] = \frac{d}{d\theta}\hat{F}^i(\theta). \tag{4.4.69}$$

Hence

$$[d', (\hat{F}^i(\theta))^a] = \frac{d}{d\theta}(\hat{F}^i(\theta))^a. \tag{4.4.70}$$

The expectation value of this vanishes as d' has equal, real eigenvalues to each side. Thus $< \Lambda_k|(\hat{F}^i(\theta))^a|\Lambda_k >$ is independent of the rapidity θ and so is Lorentz invariant.

Using (4.3.61), the solution for the single soliton created by $g(t) = exp\left\{W_i Q \hat{F}^i(\theta)\right\}$ has the form

$$e^{-\beta\lambda_k\cdot\Phi} = \frac{1 + \cdots + a_i^{m_k}\, e^{-2\pi i\lambda_i^\vee\cdot\lambda_k}(QW_i)^{m_k p_i}}{[1 + \ldots + a_i\ (QW_i)^{p_i}]^{m_k}} \tag{4.4.71}$$

where the dots indicates intermediate powers of QW_i. These intermediate powers, whose coefficients we have not calculated, do not affect the asymptotic limit $x \to \pm\infty$, which are equivalent to $W_i \to \infty$ or $W_i \to 0$. Thus

$$e^{-\beta\lambda_k\cdot\Phi} = \begin{cases} e^{-2\pi i\lambda_i^\vee\cdot\lambda_k} & x \to \infty \\ 1 & x \to -\infty \end{cases} \tag{4.4.72}$$

In particular, this shows that, asymptotically, Φ does approach one of the "degenerate vacua", so satisfying the boundary conditions expected of a soliton solution. We can also conclude that the topological charge will satisfy

$$Q_{top} = \phi(+\infty) - \phi(-\infty) = -\frac{2\pi i}{\beta}\left(\lambda_i^{\rm v} + \Lambda_R(\mathfrak{g}^{\rm v})\right) \qquad (4.4.73)$$

Using the normal ordering (4.3.66) we obtain that the two soliton solution created by $\hat{F}^i(\theta)$ and $\hat{F}^{i'}(\theta')$ takes the form

$$e^{-\beta\lambda_k\cdot\Phi} = \frac{1 + \cdots + (a_i a_{i'})^{m_k} e^{-2\pi i(\lambda_i^{\rm v}+\lambda_{i'}^{\rm v})\cdot\lambda_k}\left((QW_i)^{p_i}(Q'W_{i'})^{p_{i'}}\right)^{m_k} X_{i,i'}(z,z')^{m_k}}{\left[1 + \ldots + a_i a_{i'}\left((QW_i)^{p_i}(Q'W_{i'})^{p_{i'}}\right) X_{i,i'}(z,z')\right]^{m_k}} \qquad (4.4.74)$$

with asymptotic limits $\exp[-2\pi i(\lambda_i^{\rm v} + \lambda_{i'}^{\rm v})\cdot\lambda_k]$ and 1. Again this confirms that the solution interpolates degenerate vacua. This argument is readily extended to more solitons.

As mentioned in the begin of the previous chapter, the solutions of non-simply laced or twisted ATFT are the subset of the solutions of simply laced ATFT which satisfies (3.1.8). From (4.1.24) we obtain that

$$\phi_i = -\frac{1}{\beta}\sum_{j=0}^{r} K_{ij}\ln\tau_j \qquad i = 0, 1, 2, \ldots \qquad (4.4.75)$$

Therefore in order to $\phi_i = \phi_{\tau(i)}$ the τ_j function must satisfy the symmetry condition $\tau_j = \tau_{\tau(j)}$. This happens whenever the group element satisfies $\tau(g(0)) = g(0)$, i. e., it is generated by the folded algebra. This implies that $\hat{F}^i(\theta)$ satisfies (3.5.112) for the twisted algebras or (4.3.57) for the non-simply laced algebras. In this last case we see that the single soliton of species $\langle i\rangle$ of the folded theory will arise from folding a special $|\langle i\rangle|$-soliton solution of the unfolded theory.

Let us now obtain the soliton masses for the "folded" theory [37]. For a symmetric configuration (not necessarily a solution) for the unfolded theory, $\phi_i = \phi_{\tau(i)}$, the potential term which appears in the Lagrangian and in the energy-momentum tensor will satisfy,

$$\begin{aligned} V(\phi) &= \frac{\mu^2}{\beta^2}\sum_{i=0}\frac{2m_i}{\alpha_i^2}e^{\beta\phi_i} \\ &= \frac{\mu^2}{\beta^2}\sum_{\langle i\rangle=0}\frac{2m_{\langle i\rangle}}{\alpha_{\langle i\rangle}^2}e^{\beta\phi_{\langle i\rangle}} \end{aligned}$$

The same result applies to the kinetic terms $\partial_\mu\phi.\partial^\mu\phi$ and $\partial_\mu\phi.\partial_\nu\phi$ in the Lagrangian and energy-momentum tensor. Therefore we can conclude that a symmetric configuration for the unfolded theory yields a configuration of the folded theory such that the respective Lagrangian and energy-momentum tensor assume identical values.

In particular, if we have a symmetric multisoliton solution of the unfolded theory, it will yield a multisoliton solution of the folded theory with the same energy and momentum. Nevertheless we cannot conclude that the solitons have the same masses in the folded and unfolded theory as soliton number is not conserved necessarily preserved in the folding as we now see.

In the case that the folded theory is twisted, a multisoliton solution of species $n(1), n(2), \ldots, n(k)$ satisfying $\exp\{-2\pi i \lambda_{n(i)}.\lambda_{\tau(0)}\} = 1$ is symmetric and survives as a soliton with the same interpretation. Its energy and momentum is given by

$$\sqrt{2}P^{\pm} = \sum_{i=1}^{k} M_{n(i)} e^{\pm\theta_{n(i)}} \; .$$

and the folded soliton solution possesses the same mass as the unfolded one.

On the other hand, if the folded theory is untwisted but non simply laced, the symmetric solution could involve an exponential of (4.3.57),

$$e^{Q\hat{F}^{\langle i\rangle}(z)} = \prod_{i\in\langle i\rangle} e^{Q\hat{F}^{i}(z)}.$$

The left hand side would create a single soliton of the folded theory but the right hand side would create a superposition of $|\langle i\rangle|$ solitons of the unfolded theory all with the same coordinate and rapidity. We have used the fact that the pieces in (4.3.57) mutually commute since they have same rapidity.

When we equate the consequent contributions to the energy and momentum tensor of the soliton we have

$$\sqrt{2}P^{\pm} = M_{\langle i\rangle} e^{\pm\theta} = \sum_{i\in\langle i\rangle} M_i e^{\pm\theta}.$$

Hence we conclude that the mass of the soliton of species $|\langle i\rangle|$,

$$M_{\langle i\rangle} = |\langle i\rangle| M_i$$

This confirms the result (4.1.26) for non-simply laced algebras.

4.5 The Crossing Relations for Solitons

In this section we shall prove the following relation between group elements

$$e^{Q\hat{F}^{\bar{\imath}}(\theta)} = e^{-2\pi i \lambda_{\bar{\imath}}\cdot\lambda} Q^{x} Y_{-}^{\bar{\imath}}(\theta) e^{\frac{1}{Q}\hat{F}^{i}(\theta+i\pi)} Y_{+}^{\bar{\imath}}(\theta) \tag{4.5.76}$$

This relates the analytic continuation in rapidity, $\theta \to \theta + i\pi$, of the group element $e^{\frac{1}{Q}\hat{F}^{i}(\theta)}$ creating a soliton of species i, to the group element $e^{Q\hat{F}^{\bar{\imath}}(\theta)}$ creating an antisoliton of species $\bar{\imath}$. The remaining factors $Y^{\bar{\imath}}(\theta), Z^{\bar{\imath}}(\theta)$ can be thought of as "crossing matrices". The reason for saying this is that in a local quantum field theory, the S-matrix describing the scattering of particles obeys a "crossing property", [16]. In two dimensions, the S-matrix for the elastic scattering of an antiparticle on a target is obtained by the analytic continuation $\theta \to i\pi - \theta$ of the S-matrix for the elastic scattering of the particle on the same target. If the particle has internal degrees of freedom, then an additional crossing matrix multiplies the S-matrix. One difference is that the crossing property just described, holds in the quantum theory as the S-matrix elements are probability amplitudes. The solitons are the only sort of particles possible in a classical theory such as we consider. Thus, (4.5.76) is actually the classical analogue of the crossing property and is remarkable because it is formulated purely in terms of the underlying Kac-Moody group. What is attractive about this is that, hitherto, the crossing property of the affine Toda S-matrix has been incorporated in a rather ad-hoc way. Relation (4.5.76) suggests the possibility of a more systematic understanding.

In order to prove (4.5.76) we shall take $\mathfrak{g}^{(1)}$ to be untwisted and simply laced. Consider a representation of level x with highest weight Λ, constructed again as occurring in the decomposition of tensor product of x fundamental representations of level 1. Consider a power a of $\hat{F}^i(\theta)$ less then x:

$$\frac{(\hat{F}^i(\theta))^a}{a!} = \hat{F}^i(\theta)_{(1)}\hat{F}^i(\theta)_{(2)}\dots\hat{F}^i(\theta)_{(a)} + \dots \tag{4.5.77}$$

where the dots indicates extra terms of a similar structure obtained by assigning the "a" quantities $\hat{F}^i(\theta)$ separately to the x different species. There are consequently $\begin{pmatrix} x \\ a \end{pmatrix}$ terms on the right hand of (4.5.77) in all. Similarly, if we consider the power $x-a$

$$\frac{(\hat{F}^i(\theta))^{x-a}}{(x-a)!} = \hat{F}^i(\theta)_{(a+1)}\hat{F}^i(\theta)_{(a+2)}\dots\hat{F}^i(\theta)_{(x)} + \dots, \tag{4.5.78}$$

The first terms on the right hand side of (4.5.77) and (4.5.78) multiplied give $(\hat{F}^i(\theta))^x/x!$ and are thus naturally complementary. There is a similar pairing between any term on the right hand side of equation (4.5.77) and one on the right hand side of (4.5.78). Thus the respective number of terms $\begin{pmatrix} x \\ a \end{pmatrix}$ and $\begin{pmatrix} x \\ x-a \end{pmatrix}$ must be equal as indeed they are. We shall now relate the first two terms exhibited above so that a similar relation will follow for all the $\begin{pmatrix} x \\ a \end{pmatrix}$ complementary pairs

$$
\begin{aligned}
&\hat{F}^i_{(1)}(\theta)\hat{F}^i_{(2)}(\theta)\cdots\hat{F}^i_{(a)}(\theta) = \\
&\quad = e^{-2\pi i\lambda_i\cdot\left(\sum_{p=1}^{a}\lambda_{j(p)}\right)} Y^i_-(\theta)_{(1)}\dots Y^i_-(\theta)_{(a)} Y^i_+(\theta)_{(1)}\dots Y^i_+(\theta)_{(a)} = \qquad (4.5.79)\\
&\quad = e^{-2\pi i\lambda_i\lambda} Y^i_-(\theta)\left\{e^{2\pi i\lambda_i\sum_{p=a+1}^{x}\lambda_{j(p)}} Y^i_-(\theta)^{-1}_{(a+1)}\dots Y^i_-(\theta)^{-1}_{(x)} Y^i_+(\theta)^{-1}_{(a+1)}\dots Y^i_+(\theta)^{-1}_{(x)}\right\} Y^i_+(\theta)
\end{aligned}
$$

Notice that the factors outside the curly bracket are independent of the subset of x chosen since $Y^i_\pm$ are expressed by exponentiating (4.1.21). We shall relate the terms inside the curly brackets to the first term on the right hand side of (4.5.78) for a suitable new choice of species and rapidity. Consider a typical factor in the curly brackets in (4.5.80) associated with one of the species whose tensor product is taken, with label p, say, namely

$$
e^{2\pi i\lambda_i\cdot\lambda_{j(p)}} Y^i_-(\theta)^{-1}_{(p)} Y^i_+(\theta)^{-1}_{(p)} \quad . \qquad (4.5.80)
$$

We shall temporarily drop the label (p) and verify the following identity for this factor:

$$
e^{2\pi i\lambda_i\cdot\lambda_j}(Y^i_-(\theta))^{-1}(Y^i_+(\theta))^{-1} = \hat{F}^{\bar{\imath}}(\theta+i\pi) \quad . \qquad (4.5.81)
$$

Thus, the contents of the curly brackets in (4.5.80) can be written

$$
\hat{F}^{\bar{\imath}}_{(a+1)}(\theta+i\pi)\hat{F}^{\bar{\imath}}_{(a+2)}(\theta+i\pi)\cdots\hat{F}^{\bar{\imath}}_{(x)}(\theta+i\pi) \quad . \qquad (4.5.82)
$$

To prove (4.5.81) note that, by (4.2.28),

$$
(Y^i_-)^{-1} = \exp\left(-\sum_{N>0}\frac{\gamma_i\cdot q([N])z^N\hat{E}_{-N}}{N}\right) \quad , \qquad (4.5.83)
$$

where

$$
z = ie^{-\theta}e^{\left(-\frac{i\pi}{2h}(1+c(i))\right)} \qquad (4.5.84)
$$

Now, in (3.3.46) we obtained that $-\gamma_i$ lies on the orbit of σ whose representative element is defined as $\gamma_{\bar{\imath}}$. As observed in the study of the quantum particle excitations, $\bar{\imath}$ is interpreted as the antispecies of i [23]. Recalling (3.2.23) we see that

$$
Y^i_-(\theta)^{-1} = \exp\sum_{n>0}\frac{\gamma_{\bar{\imath}}\cdot q([N])\left[ze^{\frac{-2\pi i}{h}\left(\frac{h}{2}-\frac{c(i)-c(\bar{\imath})}{4}\right)}\right]^N\hat{E}_{-N}}{N} \quad . \qquad (4.5.85)
$$

Now, by (4.5.84), the argument of the square brackets is equal to

$$
ie^{-\theta}e^{-\frac{i\pi(1+c(i))}{2h}}e^{-\pi i}e^{\frac{i\pi(c(i)-c(\bar{\imath}))}{2h}} = ie^{-(\theta+i\pi)}e^{-\frac{i\pi(1+c(\bar{\imath}))}{2h}} = -\bar{z}. \qquad (4.5.86)
$$

where $\bar{z}$ differs from z defined by (4.5.84) only in the species $\bar{\imath}$ replacing its antispecies i. Hence, $Y^i_\pm(\theta)^{-1} = Y^{\bar{\imath}}_\pm(\theta+i\pi)^{-1}$. Finally, in order to prove (4.5.81) we need to show

$$
e^{2\pi i\lambda_i\cdot\lambda_j} = e^{-2\pi i\lambda_{\bar{\imath}}\cdot\lambda_j} \qquad (4.5.87)
$$

and this relies on the fact that the map $i \to \bar{\imath}$ is a symmetry τ, say, of the ordinary Dynkin diagram of $\mathfrak{g}$. Hence this can be lifted to a linear automorphism of the root system, also called τ, which is either outer or equal to the identity, such that $\alpha_{\bar{\imath}} = \tau(\alpha_i)$. Comparing with (3.3.46), it follows that $\tau = -\sigma_0$, where σ_0 can be calculated but is known as that special element of the Weyl group of $\mathfrak{g}$ which maps the positive Weyl chamber into its negative. As a consequence,

$$\lambda_{\bar{\imath}} = \tau(\lambda_i) = -\sigma_0(\lambda_i) \quad . \tag{4.5.88}$$

(4.5.87) follows from this as $\sigma_0(\lambda_i)$ differ from λ_i by an element of the root lattice. Thus, (4.5.81) is proven and, using (4.5.80) and (4.5.82), we conclude

$$\frac{(\hat{F}^i(\theta))^a}{a!} = e^{-2\pi i \lambda_i \cdot \lambda} Y_-^i \frac{\left(\hat{F}^{\bar{\imath}}(\theta + i\pi)\right)^{x-a}}{(x-a)!} Y_+^i \quad . \tag{4.5.89}$$

If we take expectation values of (4.5.89) with respect to the highest weight state $|\Lambda\rangle$, the dependence on the rapidity drops out as we explained previously, leaving

$$\frac{\langle\Lambda|(\hat{F}^i)^a|\Lambda\rangle}{a!} = e^{-2\pi i \lambda_i \cdot \lambda} \frac{\langle\Lambda|(F^{\bar{\imath}})^{x-a}|\Lambda\rangle}{(x-a)!} \quad . \tag{4.5.90}$$

This relates coefficients in the single soliton solution. As it relates high powers of $\hat{F}^i(\theta)$ to low powers, it simplifies calculations of these solutions.

Since $Y_\pm^i(\theta)$ appearing in (4.5.89) are independent of the integer a, we can deduce the following relation between group elements:

$$e^{Q\hat{F}^i(\theta)} = e^{-2\pi i \lambda_i \cdot \lambda} Q^x Y_-^i(\theta) e^{\frac{1}{Q}\hat{F}^{\bar{\imath}}(\theta + i\pi)} Y_+^i(\theta) \quad . \tag{4.5.91}$$

This relation can be rewritten as (4.5.76) and therefore we finished the prove of the claimed result.

We shall now use the crossing identity (4.5.76) inside the expression for the multisoliton solution in order to see explicitly how analytic continuation through $i\pi$ in the rapidity of one soliton of species i yields the multisoliton solution in which the soliton of species i is replaced by its antisoliton with species $\bar{\imath}$. The rapidities are unchanged but the complex numbers Q_1, Q_2, … associated with each soliton undergo a transformation which will be determined.

Consider the n-soliton solution created by the group element

$$g(0) = e^{Q_1 \hat{F}^{i_1}(\theta_1)} \ldots e^{Q_{k-1}\hat{F}^{i_{k-1}}(\theta_{k-1})} e^{Q\hat{F}^{\bar{\imath}}(\theta)} e^{Q_{k+1}\hat{F}^{i_{k+1}}(\theta_{k+1})} \ldots e^{Q_n \hat{F}^{i_n}(\theta_n)}. \tag{4.5.92}$$

The k^{th} factor creates the antisoliton of species i to be crossed into an soliton of species i and is labelled in the simple way indicated. Inserting the crossing identity (4.5.76)

$$g(0) = e^{-2\pi i \lambda_{\bar{\imath}} \cdot \lambda} Q^x Y_-^{\bar{\imath}}(\theta) \tilde{g}(0) Y_+^{\bar{\imath}}(\theta) \tag{4.5.93}$$

where

$$\tilde{g}(0) = e^{\tilde{Q}_1 \hat{F}^{i_1}(\theta_1)} \dots e^{\tilde{Q}_{k-1} \hat{F}^{i_{k-1}}(\theta_{k-1})} e^{\tilde{Q} \hat{F}^{i}(\theta+i\pi)} e^{\tilde{Q}_{k+1} \hat{F}^{i_{k+1}}(\theta_{k+1})} \dots e^{\tilde{Q}_n \hat{F}^{i_n}(\theta_n)}. \quad (4.5.94)$$

is similar to $g(0)$ but with the k'th factor now of the form $e^{\tilde{Q}\hat{F}^{i}(\theta+i\pi)}$, being the analytic continuation of the factor for the creation of a soliton of species i. The quantities Q have all changed:

$$\begin{cases} \tilde{Q} = 1/Q \\ \tilde{Q}_r = Q_r X_{i_r \bar{\imath}} \ , \quad r \neq k \end{cases} . \quad (4.5.95)$$

We have used the normal ordering relations

$$\begin{aligned} Y_+^i(z)\hat{F}^j(z_j) &= \hat{F}^j(z_j) Y_+^i(z) X_{ij}(z, z_j) \\ \hat{F}^j(z_j) Y_-^i(z) &= Y_-^i(z) \hat{F}^j(z_j) X_{ij}(z, z_j) \end{aligned} \quad (4.5.96)$$

Finally we use

$$e^{-\mu \hat{E}_1 x^+} Y_-^{\bar{\imath}}(\theta)\tilde{g}(0) Y_+^{\bar{\imath}}(\theta) e^{-\mu \hat{E}_{-1} x^-} = (W_{\bar{\imath}}(\theta))^{-x} Y_-^{\bar{\imath}}(\theta) e^{-\mu \hat{E}_1 x^+} \tilde{g}(0) e^{-\mu \hat{E}_{-1} x^-} Y_+^{\bar{\imath}}(\theta) \quad (4.5.97)$$

which follows from (3.4.62) and the definitions. Putting all this together

$$\langle \Lambda_k | e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} | \Lambda_k \rangle = e^{-2\pi i \lambda_{\bar{\imath}} \cdot \lambda_k} \left(\frac{Q}{W_{\bar{\imath}}} \right)^{m_k} \langle \Lambda_k | e^{-\mu \hat{E}_1 x^+} \tilde{g}(0) e^{-\mu \hat{E}_{-1} x^-} | \Lambda_k \rangle \ . \quad (4.5.98)$$

When we insert this into (4.1.10), the factors $(Q/W_{\bar{\imath}})^x$ cancel leaving the soliton created by $g(0)$ equalling that created by $\tilde{g}(0)$, apart from a factor $e^{-2\pi i \lambda_{\bar{\imath}} \cdot \lambda_k}$ which merely indicates that the vacua are interpolated in reverse order. This then demonstrates that a soliton solution crosses into another solution with one soliton replaced by the antispecies under the analytic continuation $\theta \to \theta + i\pi$. This procedure is accompanied by the transformation (4.5.95) on the Q's. Since they describe not only the coordinate of the soliton, but also its internal degrees of freedom, this is to be viewed as the effect of the crossing matrix. It is intriguing that for the soliton which is crossed, $Q \to 1/Q$ which reminiscent of a modular transformation.

Chapter 5

Soliton Time Delays

5.1 Kinematics of scattering of two particles in two dimensions

In this chapter we shall see that the vertex operators determine yet more detail of the asymptotic behaviour of the soliton solutions[25]. We shall show that the N-soliton solutions given in the previous section behave like N one-soliton solution in the far past and in the far future. Moreover, the only effect of the "collision" of these one-soliton solutions is a lateral displacement which can be expressed in terms of the logarithms of $X_{ik}(z_i, z_k)$, extending the results of Hollowood [31] for the case $\mathfrak{g}^{(k)} = a_n^{(1)}$. It will be shown that this displacement of the soliton trajectories due to the scattering with several other solitons is independent of the temporal order in which the collisions take place. This is because the displacement is simply additive. The result constitutes the classical analogue of the Yang-Baxter equation [61] for the quantum scattering matrix featuring the factorisation property of the n particle S-matrix into two particle S-matrices.

Before obtaining these results let us start considering a generic two dimensional integrable theory and analyse the consequences of the collision of two or more particles. For these theories when two particles collide the outcome consists of two particles with the same masses as the original particles. If the two masses differ, the corresponding energy-momentum vectors are unchanged. If the two masses are equal, even though the particles are distinct, it is kinematically possible for the energy-momentum vectors to interchange.

Classically the particles describe trajectories in space-time which are straight except near the collision. The collision may displace the trajectories laterally but as the energy-momentum is unchanged, the final direction coincides with the initial direction. We now consider alternative descriptions of the displacements of the trajectories and show how the conservation laws correlate the displacement of the trajectories of the two colliding particles.

Consider first a single particle with velocity v, energy E, and hence momentum vE. Before collision the equation of the trajectory in space-time is

$$x = vt + x(I), \tag{5.1.1}$$

whereas afterwards it is

$$x = vt + x(F), \tag{5.1.2}$$

as the velocity is unchanged. Only the intercept with the x-axis changes. So

$$\Delta(x) = x(F) - x(I) \tag{5.1.3}$$

measures the lateral displacement at fixed time. This is not Lorentz invariant but the combination $E\Delta(x)$ is. Since E is always positive it follows that $\Delta(x)$ has the same sign in all Lorentz frames of reference, even though its magnitude varies. The intercepts of the trajectories (5.1.1) and (5.1.2) with the time axis are given by

$$t(I) = -\frac{x(I)}{v}, \qquad t(F) = -\frac{x(F)}{v}. \tag{5.1.4}$$

Then, we define the "time delay"

$$\Delta(t) = t(F) - t(I) = -\frac{\Delta(x)}{v}. \tag{5.1.5}$$

Again

$$E\Delta(x) = -p\Delta(t) \tag{5.1.6}$$

is Lorentz invariant. As the sign of p can be changed by a Lorentz transformation so can that of $\Delta(t)$.

Now consider both particles participating in the collision, labelling them 1 and 2. Consider the "centre of energy" coordinate

$$X = \frac{E_1 x_1 + E_2 x_2}{E_1 + E_2}.$$

Then

$$\frac{dX}{dt} = \frac{p_1 + p_2}{E_1 + E_2}$$

is constant throughout time so that

$$X = \frac{p_1 + p_2}{E_1 + E_2} t + X_0.$$

Now compare the results of inserting the trajectories (5.1.1) before the collision with the result of inserting (5.1.2) after the collision. As the results must agree

$$E_1\Big(x_1(F) - x_1(I)\Big) + E_2\Big(x_2(F) - x_2(I)\Big) = 0$$

or, denoting $\Delta_{12}(x) = x_1(F) - x_1(I)$ and similarly for $\Delta_{21}(x)$

$$E_1\Delta_{12}(x) + E_2\Delta_{21}(x) = 0. \tag{5.1.7}$$

Thus the spatial displacement of the two trajectories must have opposite signs. By (5.1.6) we have, equally,

$$p_1\Delta_{12}(t) + p_2\Delta_{21}(t) = 0, \tag{5.1.8}$$

where $\Delta_{12}(t) = t_1(F) - t_1(I)$ is the time delay sustained by particle 1 colliding with particle 2. Notice that, in the centre of momentum frame $p_1 + p_2 = 0$, the two time delays are equal:

$$\Delta_{12}(t) = \Delta_{21}(t). \tag{5.1.9}$$

We see from (5.1.6), (5.1.7) and (5.1.8) that if particle 1 moves faster than particle 2, then the three quantities (5.1.9), $\Delta_{21}(x)$ and $-\Delta_{12}(x)$ all have the same sign. This common sign has a physical interpretation. Suppose the force between the particles is attractive. Then particle 1 will accelerate as it approaches particle 2 and afterwards decelerate. As a result $\Delta_{12}(x)$ will be positive and the common sign negative. Thus an attractive force implies a negative time delay, in other words a time advance, in the centre of momentum frame. A repulsive force implies a time delay, if there is transmission. There is also the additional possibility of a reflection with either a delay or advance if the two masses are equal.

The preceding discussion of relativistic particles colliding classically applies also to relativistic solitons and, in particular, to the solitons of affine Toda field theory. In the subsequent sections we shall show that when soliton 1 collides with soliton 2 the resultant displacements are given by

$$E_1\Delta_{12}(x) = -p_1\Delta_{12}(t) = -\mathrm{sign}(\theta_1 - \theta_2)\frac{2h}{|\beta|^2}\ln X_{12}(\theta_1 - \theta_2) \tag{5.1.10}$$

This equation generalises the well known result for the time delay in sine-Gordon theory [51, 53] when

$$X_{12}(\theta) = \tanh^2\left(\frac{\theta}{2}\right).$$

Thus $X_{12}(\theta)$ has acquired a new physical interpretation whose viability requires it to possess various properties not hitherto apparent. For example we shall show that $X_{12}(\theta)$ is real when θ is real and that it satisfies the symmetry property

$$X_{12}(\theta) = X_{21}(\theta)$$

demanded by (5.1.7). Furthermore it takes values between 0 and 1. Hence, in the centre of momentum frame, the time delay is always negative by (5.1.10) whatever the velocities concerned. As explained, this suggests that affine Toda solitons exert attractive forces on each other. The only possible exception to this is when two identical solitons are considered, with the same species and topological quantum number. Then it is possible to interpret the scattering as a reflection rather than a transmission. In this case it is possible for the force to be repulsive. This is the accepted point of view in sine-Gordon theory which furnishes a special case of our result (5.1.10) [33].

5.2 Space-time trajectories of two colliding solitons

We have seem in the previous chapter that the N-solitons solution takes the form

$$e^{-\beta\lambda_j\cdot\phi} = \frac{\langle\Lambda_j|g(t)|\Lambda_j\rangle}{\langle\Lambda_0|g(t)|\Lambda_0\rangle^{m_j}}. \tag{5.2.11}$$

with

$$g(t) = g_N(t)g_{N-1}(t)\cdots g_1(t) \tag{5.2.12}$$

where

$$\begin{aligned} g_m(t) &= e^{Q_m W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)} , \\ W_m &= e^{\mu_{i(m)}\cosh\theta_m(x-v_m t)} , \\ Q_m &= e^{i\psi_m}e^{-\mu_{i(m)}x_m^0\cosh\theta_m} . \end{aligned} \tag{5.2.13}$$

For simplicity we consider that the associated Kac-Moody algebra is untwisted and simply laced. We have seen that in the representation with highest weight Λ_j and level m_j,

$$e^{QW_i(\theta)\hat{F}^i(\theta)} = 1 + \ldots + (QW_i(\theta))^{m_j}\, e^{-2\pi i\lambda_i\cdot\lambda_j}Y_-^iY_+^i. \tag{5.2.14}$$

First we consider solutions with two solitons and want to determine the asymptotic form of their trajectories in space-time and hence the lateral displacements. In the next section we consider collisions of more solitons, finding that the two-soliton result is the fundamental block, as expected in an integrable theory.

The appropriate group element contains only two factors

$$g(t) \;=\; e^{Q_2W_{i(2)}(\theta_2)\hat{F}^{i(2)}(\theta_2)}e^{Q_1W_{i(1)}(\theta_1)\hat{F}^{i(1)}(\theta_1)}, \tag{5.2.15}$$

where we shall consider $\theta_1 > \theta_2$. This inequality means that soliton 1, of species $i(1)$, moves faster than soliton 2, of species $i(2)$. It must therefore start to the left of soliton 2 and eventually overtake it, causing a collision whose outcome we wish to study.

We shall do this by *"tracking"* each soliton in time. By tracking the faster soliton 1 we mean to remain in the vicinity of the soliton, which is near $x = v_1 t + x_1^0$. Along this line

$$\begin{aligned} Q_1 W_{i(1)} &= e^{i\psi_1} = \text{const.} , \\ Q_2 W_{i(2)} &= e^{i\psi_2} e^{\mu_{i(2)} \cosh\theta_2 \left[(x_1^0 - x_2^0) + (v_1 - v_2)t\right]} , \end{aligned} \tag{5.2.16}$$

Hence, in the past, $t \to -\infty$, $Q_1 W_{i(1)}$ is held fixed but $Q_2 W_{i(2)}(\theta_2)$ tends to 0 as $v_1 > v_2$. So,

$$e^{Q_2 W_{i(2)}(\theta_2) \hat{F}^{i(2)}(\theta_2)} \to 1 \ . \tag{5.2.17}$$

Thus, by (4.1.10), in the past

$$e^{-\beta\lambda_j \cdot \phi} \to \frac{\langle \Lambda_j | \ e^{Q_1 W_{i(1)}(\theta_1) \hat{F}^{i(1)}(\theta_1)} \ |\Lambda_j\rangle}{\langle \Lambda_0 | \ e^{Q_1 W_{i(1)}(\theta_1) \hat{F}^{i(1)}(\theta_1)} \ |\Lambda_0\rangle^{m_j}} \tag{5.2.18}$$

which we recognise as a single soliton solution of species $i(1)$, velocity v_1 and phase ψ_1.

For comparison, let us track soliton 1 in the two soliton solution into the future, so $t \to \infty$. In this case $Q_1 W_{i(1)}$ remains fixed but $Q_2 W_{i(2)}$ tends to plus infinity. Now by (5.2.14) the exponential is dominated by the highest non-vanishing power. In the numerator of (5.2.12) which has level m_j this yields

$$e^{Q_2 W_{i(2)}(\theta_2) \hat{F}^{i(2)}(\theta_2)} \to e^{-2\pi i \lambda_{i(2)} \cdot \lambda_j} (Q_2 W_{i(2)})^{m_j} Y_-^{i(2)} Y_+^{i(2)} \ . \tag{5.2.19}$$

The factor $Y_-^{i(2)}$ annihilates to unity on the highest weight state $\langle \Lambda_j |$ leaving the factor $Y_+^{i(2)}$, which would likewise annihilate to unity on the right were it not for the intervening factor $e^{Q_1 W_{i(1)}(\theta_1) \hat{F}^{i(1)}(\theta_1)}$. By (5.2.14) these factors can be interchanged if Q_1 is replaced by $Q_1 X_{i(1)i(2)}(\theta_{12})$. Similar operations can be applied to eliminate the vertex operator from the denominator. The large factors $(Q_2 W_{i(2)})^{m_j}$ cancel between numerator and denominator, leaving in the future, $t \to \infty$,

$$e^{-\beta\lambda_j \cdot \phi} \to e^{-2\pi i \lambda_{i(1)} \cdot \lambda_j} \frac{\langle \Lambda_j | \ e^{Q_1 W_{i(1)}(\theta_1) X_{i(1)i(2)}(\theta_{12}) \hat{F}^{i(1)}(\theta_1)} \ |\Lambda_j\rangle}{\langle \Lambda_0 | \ e^{Q_1 W_{i(1)}(\theta_1) X_{i(1)i(2)}(\theta_{12}) \hat{F}^{i(1)}(\theta_1)} \ |\Lambda_0\rangle^{m_j}} \ . \tag{5.2.20}$$

Again we recognise a single soliton solution of species $i(1)$, rapidity θ_1 and phase ψ_1. The phase factor preceding (5.2.20) is innocuous, representing a translation of ϕ by $\frac{2\pi i}{\beta} \lambda_{i(1)}$, a symmetry of the theory. The other difference between (5.2.19) and (5.2.20) is significant. Since $X_{i(1)i(2)}(\theta_{12})$ is real and positive, it means that Q_1 has acquired a factor $X_{i(1)i(2)}(\theta_{12})$ which changes its modulus (but not the phase), and hence x_1^0 in the evolution from the past to the future. The effect is that

$$\mu_{i(1)} \cosh\theta_1 (x - v_1 t) \to \mu_{i(1)} \cosh\theta_1 (x - v_1 t) + \ln X_{i(1)i(2)}(\theta_{12}) \tag{5.2.21}$$

so that the solution (5.2.20) differs from (5.2.19) by a translation in space-time. In particular the trajectories in space-time of the outgoing soliton is translated with respect to the in-going soliton. Comparing with (5.1.3) and the subsequent discussion we see

$$E_1\Delta_{12}(x) \;=\; -\frac{M_{i(1)}}{\mu_{i(1)}}\ \ln X_{i(1)i(2)}(\theta_{12}) \tag{5.2.22}$$

as the energy of the soliton is $E_1 = \mu_{i(1)}\cosh\theta_1$. Employing the mass formula (4.1.26), this equals

$$E_1\Delta_{12}(x) \;=\; -\;\frac{2h}{|\beta^2|}\ln X_{i(1)i(2)}(\theta_{12}) \tag{5.2.23}$$

which is the announced result (5.1.10) for the faster soliton.

Now let us derive the corresponding result for the slower soliton 2, by tracking it, i.e., remaining in the vicinity of $x = v_2t + x_2^0$. Now $Q_2W_{i(2)}$ is held fixed and

$$Q_1W_{i(1)}(\theta_1) \;=\; e^{i\psi_1}e^{\mu_{i(1)}\cosh\theta_1\left[(x_2^0-x_1^0)+(v_2-v_1)t\right]}\ . \tag{5.2.24}$$

tends to ∞ and 0 in the past and future, respectively. Thus, in the past,

$$e^{-\beta\lambda_j\cdot\phi} \;\rightarrow\; e^{-2\pi i\lambda_j\cdot\lambda_{i(1)}}\frac{\langle\Lambda_j|\ e^{Q_2W_{i(2)}(\theta_2)X_{i(1)i(2)}(\theta_{12})\hat{F}^{i(2)}(\theta_2)}\ |\Lambda_j\rangle}{\langle\Lambda_0|\ e^{Q_2W_{i(2)}(\theta_2)X_{i(1)i(2)}(\theta_{12})\hat{F}^{i(2)}(\theta_2)}\ |\Lambda_0\rangle^{m_j}}\ . \tag{5.2.25}$$

using (5.2.14), whereas in the future

$$e^{-\beta\lambda_j\cdot\phi} \;\rightarrow\; \frac{\langle\Lambda_j|\ e^{Q_2W_{i(2)}(\theta_2)\hat{F}^{i(2)}(\theta_2)}\ |\Lambda_j\rangle}{\langle\Lambda_0|\ e^{Q_2W_{i(2)}(\theta_2)\hat{F}^{i(2)}(\theta_2)}\ |\Lambda_0\rangle^{m_j}}\ . \tag{5.2.26}$$

So $Q_2X_{i(1)i(2)}(\theta_{12})\rightarrow Q_2$ during the evolution and

$$E_2\Delta_{21}(x) \;=\; +\;\frac{2h}{|\beta^2|}\ln X_{i(1)i(2)}(\theta_{12}), \tag{5.2.27}$$

thereby confirming (5.1.10) for the slower soliton.

Notice that the solution considered describes only a transmission and not a reflection of solitons. The only possible exception is when the species $i(1)$ and $i(2)$ coincide as do the phases ψ_1 and ψ_2. Then we cannot tell whether the scattering is transmissive or reflective.

It is interesting to repeat the calculation with the order of the two factors in (5.2.15) reversed. The reader will find that the asymptotic results (5.2.18), (5.2.20), (5.2.25) and (5.2.26) are unchanged, as is therefore the spatial displacement.

5.3 Space-time trajectories of any number of colliding solitons

It is not difficult to extend the preceding argument from the collision of two solitons to the collision of any number of various species. The interesting result is that the total displacement of the space-time trajectories of any chosen soliton is precisely the sum of the contributions previously found for the collision of the chosen soliton with each of the others. This sum is independent of the ordered sequence in which the chosen soliton collides with the others. Hence this result is the classical analogue of the Yang-Baxter relation [61] governing quantum scattering in integrable theories.

The solution (5.2.12) describes N solitons. We shall choose to track one of these, say soliton m, by remaining close to its trajectory in space-time as the past evolves into the future. So $Q_m W_{i(m)}$ is held fixed and the other functions $Q_n W_{i(n)}$ behave as

$$Q_n W_{i(n)} \;=\; e^{i\psi_n} e^{\mu_{i(n)}\cosh\theta_n\left[(x^0_m - x^0_n)+(v_m-v_n)t\right]} \quad . \tag{5.3.28}$$

Thus if soliton n is slower than soliton m, $Q_n W_{i(n)}$ tends to 0 in the past and ∞ in the future. If soliton m is faster than soliton n, then the limits are reversed. Repeating the arguments of the preceding section we find that in the past

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; \exp\left(-2\pi i \sum_{v_k<v_m} \lambda_j\cdot\lambda_{i(k)}\right) \frac{\langle\Lambda_j|\; e^{Q_m(I)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_m(I)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_0\rangle^{m_j}} \tag{5.3.29}$$

where $Q_m(I) = Q_m \prod_{v_k<v_m} X_{i(m)i(k)}$.

In the future

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; \exp\left(-2\pi i \sum_{v_k>v_m} \lambda_j\cdot\lambda_{i(k)}\right) \frac{\langle\Lambda_j|\; e^{Q_m(F)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_m(F)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_0\rangle^{m_j}} \tag{5.3.30}$$

where $Q_m(F) = Q_m \prod_{v_k>v_m} X_{i(m)i(k)}$. This immediately yields the announced results:

$$E_m\Delta_m(x) = -p_m\Delta_m(t) = \frac{2h}{|\beta^2|}\left(\sum_{v_k>v_m} \ln X_{i(m)i(k)} - \sum_{v_k<v_m} \ln X_{i(m)i(k)}\right) \tag{5.3.31}$$

where $\Delta_m(x), \Delta_m(t)$ denote the displacement (5.1.3) and (5.1.5) for the m^{th} soliton.

Notice that this result (5.3.31) does not depend on the values of the $Q_{i(n)}$, but only on the rapidities of the solitons. Hence the temporal order of the scattering can be altered without changing the rapidities of the solitons. Thus the overall displacement of the trajectories of soliton m, (5.3.31) is independent of the order in which the collisions occurred.

The same procedure can be applied to each of the other $(N-1)$ solitons. In this way we see how the N soliton solution asymptotically contains the N single soliton solutions in both the past and future. The only alterations in time are the lateral displacement of the trajectories specified by our result (5.3.31).

5.4 Properties of the function $X_{ik}(\theta)$

As we have mentioned previously, the exponents of $X_{ik}(z_i, z_k)$, $\gamma_i \cdot \sigma^p \gamma_k$, being scalar products of roots of a simply laced Lie algebra, can only take the values $0, \pm 1, \pm 2$. Thus $X_{ik}(z_i, z_k)$ can be analytically extended to a meromorphic function of the complex variables z_i and z_k. X_{ik} only possesses a double pole if $i = \bar{k}$, while the occurrence of simple poles is governed by Dorey's fusing rule [12, 50, 23].

Using the first, (5.4.32), of the two facts that

$$\sum_{p=1}^{h} \sigma^p \gamma_k = 0 \tag{5.4.32}$$

and

$$\sum_{p=1}^{h} p \gamma_i \cdot \sigma^p \gamma_k \in h\mathbb{Z}, \tag{5.4.33}$$

we can rewrite (4.2.37) as

$$\prod_{p=1}^{h} (z_i z_k^{-1} - e^{\frac{2\pi i p}{h}})^{\gamma_i \cdot \sigma^p \gamma_k} \tag{5.4.34}$$

which means that $X_{ik}(z_i, z_k)$ depends on z_i and z_k only through the ratio $z_i z_k^{-1}$. Furthermore using both (5.4.32) and (5.4.33) we find that it exhibits the symmetry property

$$X_{ik}(z_i, z_k) = X_{ki}(z_k, z_i), \tag{5.4.35}$$

which means that the vertex operators braid trivially. This appears to be the explanation of our earlier observations that the order of the soliton factors in (5.2.12) is irrelevant. Introducing the soliton rapidity θ_k via,

$$z_k = i e^{-\theta_k} e^{-\frac{i\pi}{2h}(1+c(k))} \tag{5.4.36}$$

we find that X_{ik} can be expressed as a function of the rapidity difference $\theta = \theta_i - \theta_k$:

$$X_{ik}(z_i, z_k) = \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k} = X_{ik}(\theta). \tag{5.4.37}$$

Thus $X_{ik}(\theta)$ is Lorentz invariant since the relative rapidity is. This is in accord with our result (5.1.10).

The result (5.1.10) which was established in the preceding sections means that $X_{ik}(\theta)$ has a space-time interpretation in terms of the scattering of solitons. As a consequence it ought to be a real number (when θ is real) and exhibit some further symmetry properties.

We shall now check these properties explicitly, showing that $X_{ik}(\theta)$ has period $2\pi i$,

$$X_{ik}(\theta + 2\pi i) = X_{ik}(\theta), \tag{5.4.38}$$

is symmetric in the sense

$$X_{ik}(\theta) = X_{ki}(\theta), \tag{5.4.39}$$

is even in θ

$$X_{ik}(\theta) = X_{ik}(-\theta), \tag{5.4.40}$$

takes values in the unit interval when θ is real

$$0 \leq X_{ik}(\theta) < 1, \qquad \theta \in I\!R, \tag{5.4.41}$$

and obeys the "crossing" property

$$X_{\bar{\imath}k}(\theta) = (X_{ik}(\theta + i\pi))^{-1} \tag{5.4.42}$$

where $\bar{\imath}$ denotes the anti-species of i.

Notice that the periodic property (5.4.38) is already evident from (5.4.37). The symmetry property (5.4.39) follows using the identity [24]

$$\gamma_i \cdot \sigma^p \gamma_k = \gamma_k \cdot \sigma^{p'} \gamma_i,$$

where

$$2p + \frac{c(i) - c(k)}{2} = 2p' + \frac{c(k) - c(i)}{2}.$$

To prove the evenness property (5.4.40) note that

$$\begin{aligned} X_{ik}(-\theta) &= \prod_{p=1}^{h} \left(e^{\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(i) - c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k} \\ &= \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{-\pi i}{h}(2p + \frac{c(i) - c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k} \end{aligned}$$

by (5.4.32) and (5.4.33). Now use $\gamma_i \cdot \sigma^p \gamma_k = \gamma_k \cdot \sigma^{-p} \gamma_i$ to recognize, on changing the dummy label $p \to -p$, $X_{ki}(\theta)$ which equals $X_{ik}(\theta)$ by the symmetry property (5.4.39). To prove the reality property first note that

$$X_{ik}(\theta^*)^* = \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{-\pi i}{h}(2p + \frac{c(i) - c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k},$$

which equals $X_{ik}(-\theta)$ by (5.4.32) and (5.4.33) and hence $X_{ik}(\theta)$ by evenness. Thus $X_{ik}(\theta)$ is real when θ is.

Now let us consider the possibility that $X_{ik}(\theta)$ has zeroes or poles when θ is real. This is only possible when a factor vanishes, so that both

$$\theta = 0, \qquad \text{and} \quad p + \frac{1}{4}(c(i) - c(k)) = 0 \mod h.$$

The second condition implies that $c(i) = c(k)$ and that $p = h$. When $c(i) = c(k)$, $\gamma_i \cdot \sigma^h \gamma_k = \gamma_i \cdot \gamma_k$ vanishes unless $i = k$ when it equals 2. So for real θ, $X_{ik}(\theta)$ has no poles and the only zero occurs when $i = k$ and $\theta = 0$. We already knew that $X_{ii}(0)$ had to be zero as it implies the nilpotency condition $(\hat{F}^i(\theta))^2 = 0$ at level 1.

Now let us prove that $X_{ik}(\theta)$ takes values in the unit interval. The argument is intriguingly similar to that of section (4.5) of [24] concerned with positivity properties of the affine Toda particle scattering matrix.

Using the relation

$$\gamma_i \cdot \sigma^p \gamma_k = \lambda_i \cdot \sigma^{-p+\frac{c(k)-1}{2}} \gamma_k - \lambda_i \cdot \sigma^{-p+\frac{c(k)+1}{2}} \gamma_k,$$

we can rewrite (5.4.37) as

$$\begin{aligned} X_{ik}(\theta) &= \frac{\prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{\pi i}{h}(2p+\frac{c(i)-c(k)}{2})}\right)^{-\lambda_i \cdot \sigma^{-p+\frac{c(k)+1}{2}} \gamma_k}}{\prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{\pi i}{h}(2p+\frac{c(i)-c(k)}{2})}\right)^{-\lambda_i \cdot \sigma^{-p+\frac{c(k)-1}{2}} \gamma_k}} \\ &= \prod_{p=1}^{h} \left[\frac{\sinh \frac{1}{2}(\theta - \frac{\pi i}{h}(2p - \frac{c(i)+c(k)}{2} - 1))}{\sinh \frac{1}{2}(\theta - \frac{\pi i}{h}(2p - \frac{c(i)+c(k)}{2} + 1))}\right]^{-\lambda_i \cdot \sigma^p \gamma_k} \end{aligned}$$

on relabelling the dummy index in order to gather the factors under a common exponent. The factors can be further paired using the fact [24] that

$$\lambda_i \cdot \sigma^p \gamma_k = -\lambda_i \cdot \sigma^{p'} \gamma_k,$$

where

$$p' = h + \frac{c(i) + c(k)}{2} - p.$$

Using this we can rewrite $X_{ik}(\theta)$ as

$$X_{ik}(\theta) = \prod_{p=a}^{b} \left[\frac{\cosh(\theta) - \cos \frac{\pi}{h}(2p - \frac{c(i)+c(k)}{2} - 1)}{\cosh(\theta) - \cos \frac{\pi}{h}(2p - \frac{c(i)+c(k)}{2} + 1)}\right]^{-\lambda_i \cdot \sigma^p \gamma_k} \tag{5.4.43}$$

where $a = \frac{1+c(k)}{2}$ and $b = \frac{h-1}{2} + \frac{c(k)+c(\bar{k})}{4}$. The significance of the reduced range of p is that, in it,

$$\lambda_i \cdot \sigma^p \gamma_k \leq 0.$$

Thus all the exponents in (5.4.43) are positive. Thus in order to prove (5.4.41) it would be sufficient to show that each factor in (5.4.43) individually lies between 0 and 1. Because $\cosh\theta \geq 1$ this is ensured provided

$$1 \geq \cos\frac{\pi}{h}\Big(2p - \frac{c(i)+c(k)}{2} - 1\Big) > \cos\frac{\pi}{h}\Big(2p - \frac{c(i)+c(k)}{2} + 1\Big) > -1, \quad (5.4.44)$$

which follows from the fact that $\cos\phi$ is monotonically decreasing from 1 to -1 in the interval $0 < \phi < \pi$, providing

$$0 \leq 2p - \frac{c(i)+c(k)}{2} - 1 < 2p - \frac{c(i)+c(k)}{2} + 1 < h.$$

The smallest value of $2p - (c(i)+c(k))/2 - 1$ occurs when $p = a = (1+c(k))/2$ which is $(c(k)-c(i))/2$. This can only be negative when $c(k) = -1 = -c(i)$, so that $p = 0$. In this case the exponent $\lambda_i \cdot \sigma^p\gamma_k = \lambda_i \cdot \gamma_k = 0$, as $i \neq k$, and the factor contributes unity. A similar discussion applies to the upper limit b. Notice that expression (5.4.43) is explicitly real and even in θ.

Finally, using the relation [24],

$$\gamma_i = -\sigma^{-\frac{h}{2} - \frac{c(i)-c(\bar{\imath})}{4}}\gamma_{\bar{\imath}},$$

we obtain the "crossing" property (5.4.42)

$$\begin{aligned} X_{ik}(\theta + i\pi) &= \prod_{p=1}^{h}\Big(e^{-\theta} - e^{\frac{\pi i}{h}(2p - h + \frac{c(i)-c(k)}{2})}\Big)^{\gamma_i\cdot\sigma^p\gamma_k} \\ &= \prod_{p=1}^{h}\Big(e^{-\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(\bar{\imath})-c(i)}{2})}\Big)^{\gamma_{\bar{\imath}}\cdot\sigma^p\gamma_k} \\ &= (X_{\bar{\imath}k}(\theta))^{-1}. \end{aligned}$$

Since $X_{ik}(\theta)$ takes values between 0 and 1, it follows that the time delay experienced by any soliton in collision with any other in their centre of momentum frame is negative. This strongly suggests that the forces between any two solitons is always attractive. This further suggests that bound states (breathers) will form, though as far as we know, this can only occur when the two solitons have equal mass and are anti-species of each other. The only exception to the statement that forces are attractive is when the two solitons involved are indistinguishable. Then we can no longer recognize from the explicit solution that the scattering is only a transmission as it could be reflective in this case. If the scattering is considered to be reflective, the negative time delay can be ascribed to a repulsive core.

We should like to mention a delicate point here. We can distinguish the outgoing solitons if they have different species or, if not, possess different phases in Q. It is believed that the phase in Q, (4.1.22), is related to the topological quantum number of the soliton, (4.4.73). Unfortunately this connection has not been established in a satisfactorily general way and the correspondence is not one to one. This remains an outstanding issue. We should like to be able to say that two solitons of the same species are distinguishable only if they carry different topological quantum numbers and not just different phases, so two solitons could be indistinguishable if they have the same topological quantum number but different phases but this is not yet understood.

It is interesting to note that it is impossible for two solitons of the same species to have the same rapidities. This is because

$$\lim_{\theta_1,\theta_2\to\theta} e^{Q_1 W_1 \hat{F}^i(\theta_1)} e^{Q_2 W_2 \hat{F}^i(\theta_2)} = e^{(Q_1+Q_2) W \hat{F}^i(\theta)}$$

which creates a single soliton rather than two. This phenomenon is familiar in sine-Gordon theory where it is well known that, in the quantum theory, the solitons are the fermions in the massive Thirring model [56, 10, 42]. It appears that something like the exclusion principle is operating at the classical level. The two results, repulsive core and exclusion principle, suggest that the affine Toda solitons may also have a fermionic nature, but again much more needs to be understood.

5.5 The semi-classical limit for the soliton S-matrix

It has been shown in [33] that when the time-delay in a collision in 1+1 dimensions is known, the leading term of the S-matrix elements for transmission processes can be obtained using semi-classical approximation. They proved for a collision of particle 1 and particle 2 with total energy E in the centre of momentum frame, the transmission S-matrix element is

$$S_{12\to 12} = e^{\frac{i\delta_{12}}{\hbar} + O(\hbar)} \tag{5.5.45}$$

with

$$\frac{d\delta_{12}}{dE} = \Delta_{12} t \ . \tag{5.5.46}$$

Rewriting this derivative in terms of relative rapidity θ we obtain

$$\frac{d\delta_{12}}{d\theta} = \mathrm{sign}(\theta_1 - \theta_2) p_1 \Delta_{12} t \ . \tag{5.5.47}$$

Now consider the collision two solitons of species "i" and "j" with respective topological charges "a" and "b". Putting (5.1.10) in the last equation, it results

$$\delta_{ia,jb}(\theta) = \delta_{ia,jb}(0) + \frac{2h}{|\beta|^2}\int_0^\theta d\eta \ \ln X_{ij}(\eta) \ . \tag{5.5.48}$$

Note that since the time-delay (5.1.10) does not depend on the soliton topological charges, then only the integration constant $\delta_{ia,jb}(0)$ may depend on them. Using (4.2.36), (5.4.36) and (3.3.45) we obtain that

$$\begin{aligned}\ln X_{ij}(\eta) &= -\left\{\sum_{N>0} \frac{\gamma_i \cdot q([N])^* \gamma_j \cdot q([N])}{N} \left(\frac{z_j}{z_i}\right)^N\right\} \\ &= -2h \sum_{N>0} \overline{y}_i^{(N)} \overline{y}_j^{(N)} \frac{e^{-N\eta}}{N}\end{aligned}$$

where $\overline{y}_i^{([N]+nh)} := c(i)^n y_i^{([N])}$ and $\eta := \eta_j - \eta_i \geq 0$. Therefore putting this expression in (5.5.48) we obtain that

$$\delta_{ia,jb}(\theta) = \delta_{ia,jb}(0) + \frac{4h^2}{|\beta|^2} \sum_{N>0} \overline{y}_i^{(N)} \overline{y}_j^{(N)} \frac{\left(e^{-N\theta} - 1\right)}{N^2} \ . \tag{5.5.49}$$

Any conjectured S-matrix for the solitons of AFTF must be consistent with this semiclassical limit.

In the sine-Gordon case, the above phase will result

$$\begin{aligned}\delta_{1a,1b}(\theta) &= \delta_{1a,1b}(0) + \frac{16}{|\beta|^2} \sum_{m=0}^{\infty} \frac{\left(e^{-(2m+1)\theta} - 1\right)}{(2m+1)^2} \\ &= \delta_{1a,1b}(0) - \frac{2\pi^2}{|\beta|^2} + \frac{16}{|\beta|^2} \sum_{m=0}^{\infty} \frac{e^{-(2m+1)\theta}}{(2m+1)^2} \ .\end{aligned} \tag{5.5.50}$$

Then from the integral representation for the exact S-matrix elements obtained in [58], using the residue theorem and taking the limit $\hbar|\beta|^2 \to 0$ it does follow that (5.5.50) together with (5.5.45) gives indeed the semi-classical limit for the S-matrix elements for the transmission and identical particle processes.

Conclusions

We shall conclude with a summary of our results and a number of comments. We have analysed various aspects of particles and solitons in affine Toda field theories. In this analyses we have used Lie algebraic methods which have proved to be very powerful tools. For instance, the fact that the highest non-vanishing power of the generator $\hat{F}^i(\theta)$ could be expressed as a vertex operator, enabled us to determine the asymptotic form of the solitons solutions. Then we obtained that for a soliton of species i the topological charge satisfy

$$Q_{top} = -\frac{2\pi i}{\beta}\,(\lambda_i^{\mathrm{v}} + \Lambda_R(\mathfrak{g}^{\mathrm{v}}))$$

for an arbitrary untwisted $\mathfrak{g}^{(1)}$. The values of Q_{top} for the twisted theories is a subset of the values of the corresponding unfolded theory. The precise values that the topological charges can take at the classical level has been discussed in [31, 41] for the $a_n^{(1)}$ theories. However it has been argued that at the quantum level this spectrum can differ[32]. Therefore the determination of the complete spectrum of topological charges remains an interesting open question.

Using the vertex operator construction we were also able to show that at $t \to \pm\infty$ the N-soliton solution has the form of N isolated one-soliton solutions. The overall lateral displacement of the soliton trajectories due to the scattering with several other solitons was determined and shown to be independent of the temporal order in which the collision take place. This result constitutes a classical analogue of the Yang-Baxter equation. We obtained that the time-delay for the scattering of two solitons is always negative in the center-of-momentum frame and it depends on the function $X_{ik}(\theta)$ arising when the product of the two associated vertex operators is normal ordered. From the time-delay we obtained the leading order of the soliton S-matrix. This provides a very strong test for a conjectured exact S-matrix. This kind of test has been used by Hollowood[32], in order to check some particular elements of a conjectured S-matrix for the $a_n^{(1)}$ ATFT. However the determination of the complete soliton S-matrix for an arbitrary algebra $\mathfrak{g}^{(k)}$ is a another important problem to be solved. There is an intriguing similarity between the properties of the function $X_{ij}(\theta)$ and the S-matrix. This may indicate some relation between our soliton vertex operator and Zamolodchikov's S-matrix vertex operators[63]. This idea can be reinforced by the fact that the ATFT particle S-matrices can be obtained from the braiding of vertex operators quite similar to our vertex operator, as has been noted by Corrigan and Dorey [9]. The "crossing identity" we established in chapter 4 can also be relevant to clarify this point.

Using the folding procedure we extended various results of untwisted algebras to twisted algebras. In particular, we have obtained that the particle and soliton masses of the folded theory is a subset of the unfolded one and that the criterion of survival, $e^{-2\pi i\lambda_i\cdot\lambda_{\tau(0)}}=1$, applies equaly for both particles and solitons. This result enabled us to complete the picture of the ratio of classical masses of soliton and particles which is given by

$$\frac{\mathrm{mass(sol.}\,i)}{\mathrm{mass(par.}\,i)}=\frac{4h}{|\beta^2|\gamma_i^2}\,.$$

For simply-laced and twisted algebras $\gamma_i^2=2$ for all "i" and therefore the ratio is universal and independent of the species in question. On the other hand for theories based on non-simply laced algebras, this mass ratio is no longer universal. Recently this mass ratio has been analysed at quantum level at first order approximation[11, 40]. From the above mass ratio we can conclude that

$$\frac{\mathrm{mass(sol.}\,i)}{\mathrm{mass(par.}\,i)}\left(\mathfrak{g}^{(k)},\beta\right)=\frac{\mathrm{mass(par.}\,i)}{\mathrm{mass(sol.}\,i)}\left((\mathfrak{g}^{\mathrm{v}})^{(k)},2h/\beta\right)\,.$$

The higher spins charges of particles and solitons also satisfy the same ratio as has been shown by Freeman[22] for the untwisted algebras. These results, together with fact that the topological charges of the solitons take values in $\Lambda_W(\mathfrak{g}^{\mathrm{v}})$, indicates a possible "duality" relation between the particle and solitons of ATFT similar to the electro-magnetic dualities conjectures in four dimensions[45, 28]. We expect that ATFT can shed some light in these four dimensional theories. However there remains various points to be clarified.

Appendix A

We shall now verify, on a case by case basis, the existence of a Coxeter element σ' satisfying (3.5.85) and (3.5.86). It is instructive to start with the case $\mathfrak{g} = a_{mn-1}$. As this is the only case in which τ may have an order greater than 3, namely m, it is the most complicated.

We recall that the $mn-1$ fundamental weights λ_l can be expressed in terms of mn unit vectors ϵ_i:

$$\lambda_l = \sum_{i=1}^{l} \epsilon_i - \frac{l}{mn} \sum_{i=1}^{mn} \epsilon_i \ , \qquad l = 1, 2, \ldots, mn-1 \ .$$

The Weyl group, $W(a_{mn-1})$, is isomorphic to the group of permutation of the unit vectors, any element can be denoted by the standard permutation notation. The following element cyclically permutes the mn unit vectors and hence is a Coxeter element

$$\begin{aligned} \sigma' &= (1\ , n+1\ , 2n+1\ , \ldots, (m-1)n+1\ , 2\ , n+2\ , 2n+2\ , \ldots, \\ &\quad (m-1)n+2\ , \ldots, n\ , n+n\ , \ldots, (m-1)n+n\ ) \ . \end{aligned}$$

It is easy to check that σ' satisfies (3.5.85) and (3.5.86) so that the desired result follows.

Now we turn to the other cases. When τ has order 2, as it does unless $\mathfrak{g} = e_6$, all that has to be shown is the existence of a Coxeter element σ' conjugate to a standard one σ satisfying

$$\sigma' \lambda_{\tau(0)} = -\lambda_{\tau(0)} \ .$$

Let us now consider $\mathfrak{g} = d_r$ with its simple roots constructed out of the unit vectors in the usual way

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \ , \quad i = 1, \ldots, r-1 \\ \alpha_r &= \epsilon_{r-1} + \epsilon_r \end{aligned}$$

so that the fundamental weights are

$$\lambda_i = \sum_{i=1}^{r} \epsilon_i \qquad i = 1, 2, \ldots, r-2$$

$$\lambda_{r-1} = \frac{1}{2}\left(\sum_{i=1}^{r-1}\epsilon_i - \epsilon_r\right) , \qquad \lambda_r = \frac{1}{2}\sum_{i=1}^{r}\epsilon_i.$$

Then it is easy to see that the Coxeter element $\sigma'' = \sigma_1\sigma_2\cdots\sigma_r$ (with σ_i the reflection in α_i) has the following action on the unit vectors

$$\begin{aligned} \epsilon_1 &\rightarrow \epsilon_2 \rightarrow \ldots \rightarrow \epsilon_{r-1} \rightarrow -\epsilon_1 \rightarrow -\epsilon_2 \rightarrow \ldots \rightarrow -\epsilon_{r-1} \rightarrow \epsilon_1 , \\ \epsilon_r &\rightarrow -\epsilon_r . \end{aligned}$$

Hence as $\sigma_{\epsilon_1+\epsilon_r}(\epsilon_r) = -\epsilon_1$

$$\sigma' = \sigma_{\epsilon_1+\epsilon_r}\ \sigma''\ \sigma_{\epsilon_1+\epsilon_r}$$

has the action $\sigma'\lambda_1 = -\lambda_1$. This is the desired result if $\tau(0) = 1$.

If now r is even, equal to $2l$, say, σ'' also reverses the signs of $\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4+\cdots+\epsilon_{2l-1}$. Hence it reverses the sign of one half this plus or minus ϵ_{2l}. But this is a spinor weight Weyl conjugate to λ_{2l} or λ_{2l-1} according to the sign chosen. This establishes the existence of σ' reversing the sign of λ_{2l-1} as needed when $\tau(0) = 2l-1$.

For $\mathfrak{g} = e_6$ let us consider the element of the Weyl group

$$w = \sigma_\beta\sigma_{\alpha_1}\sigma_{\alpha_5} , \qquad \beta \equiv \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 .$$

One can check that

$$\begin{aligned} w\lambda_1 &= -\lambda_1 + \lambda_5 , \\ w\lambda_5 &= -\lambda_2 + \lambda_4 . \end{aligned}$$

Now using the bicolouration such that $c_1 = c_5 = -1$ and the identities

$$\begin{aligned} \sigma\lambda_{i_-} &= \lambda_{i_-} - \alpha_{i_-} , \\ \sigma_\pm\alpha_{i_\mp} &= \alpha_{i_\mp} - K_{i_\mp j}\alpha_j , \\ \sigma_\pm\alpha_{i_\pm} &= -\alpha_{i_\pm} , \end{aligned}$$

it can be proven that

$$\begin{aligned} (1+\sigma)\, w\lambda_1 &= w\lambda_5 , \\ \left(1+\sigma+\sigma^2\right) w\lambda_1 &= 0 , \end{aligned}$$

which correspond to (3.5.85) for $\tau(0) = 1$.

Finally, for e_7, let $w = \sigma_\beta$, where $\beta \equiv \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_5 + \alpha_6 + \alpha_7$. Then,

$$w\lambda_6 = -\lambda_4 + \lambda_6 + \lambda_7 .$$

Now, like before, using the bicolouration such that $c_4 = c_6 = c_7 = -1$ it is easy to prove that

$$(1+\sigma)w\lambda_6 = 0 .$$

This completes the claimed result.

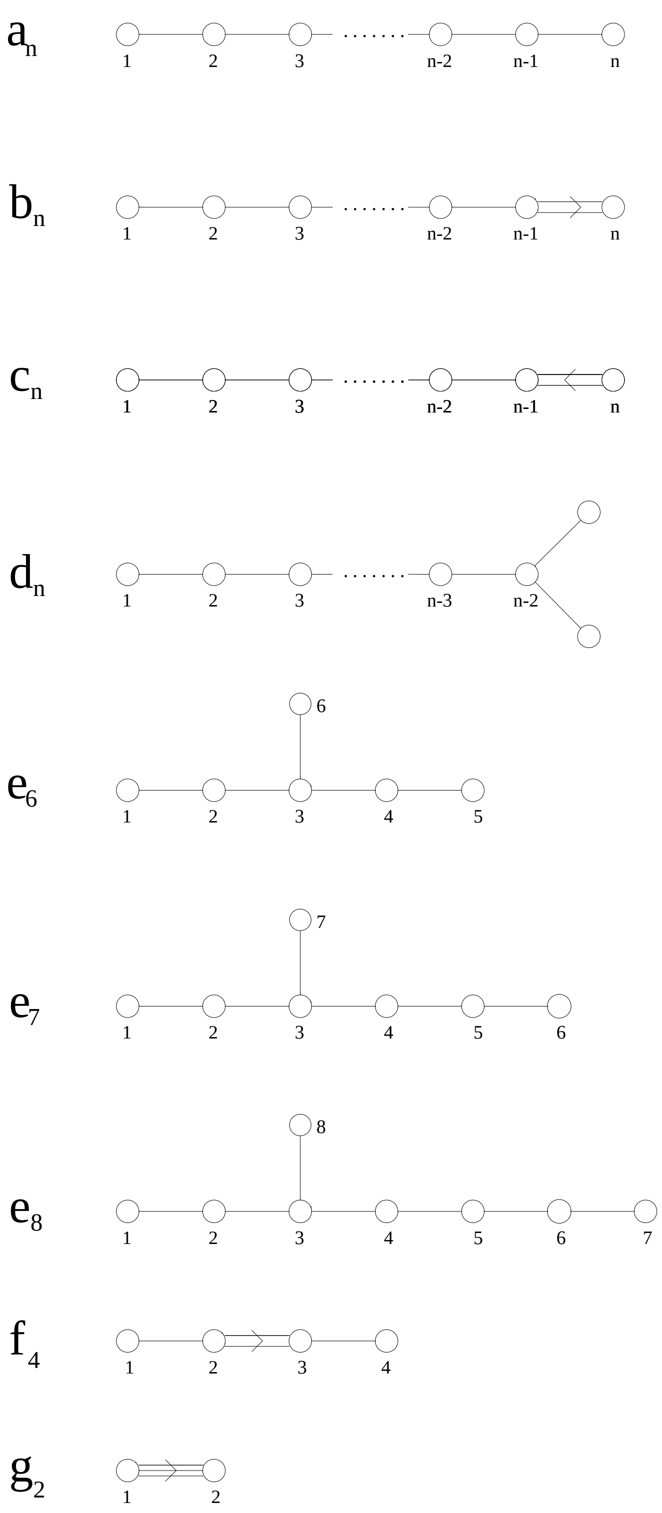

Table 1: Dynkin diagrams of Lie algebras

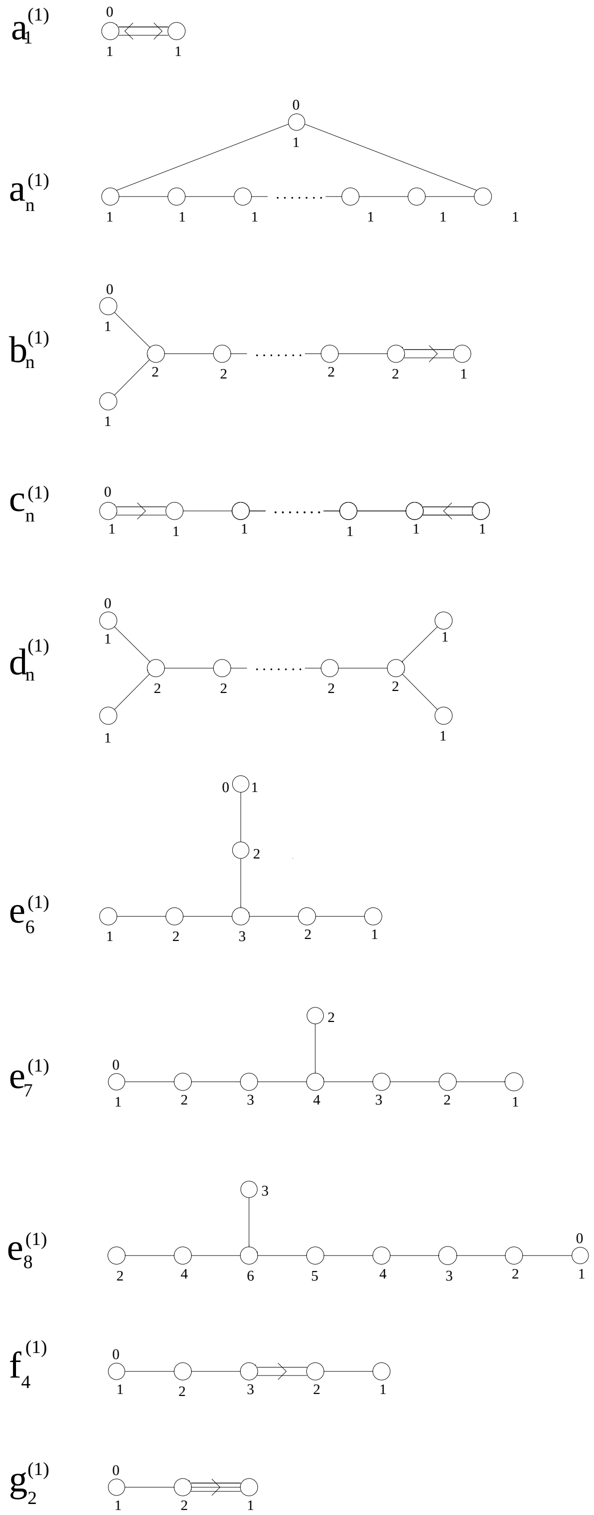

Table 2: Dynkin diagrams of untwisted Kac-Moody algebras and levels of fundamental representations

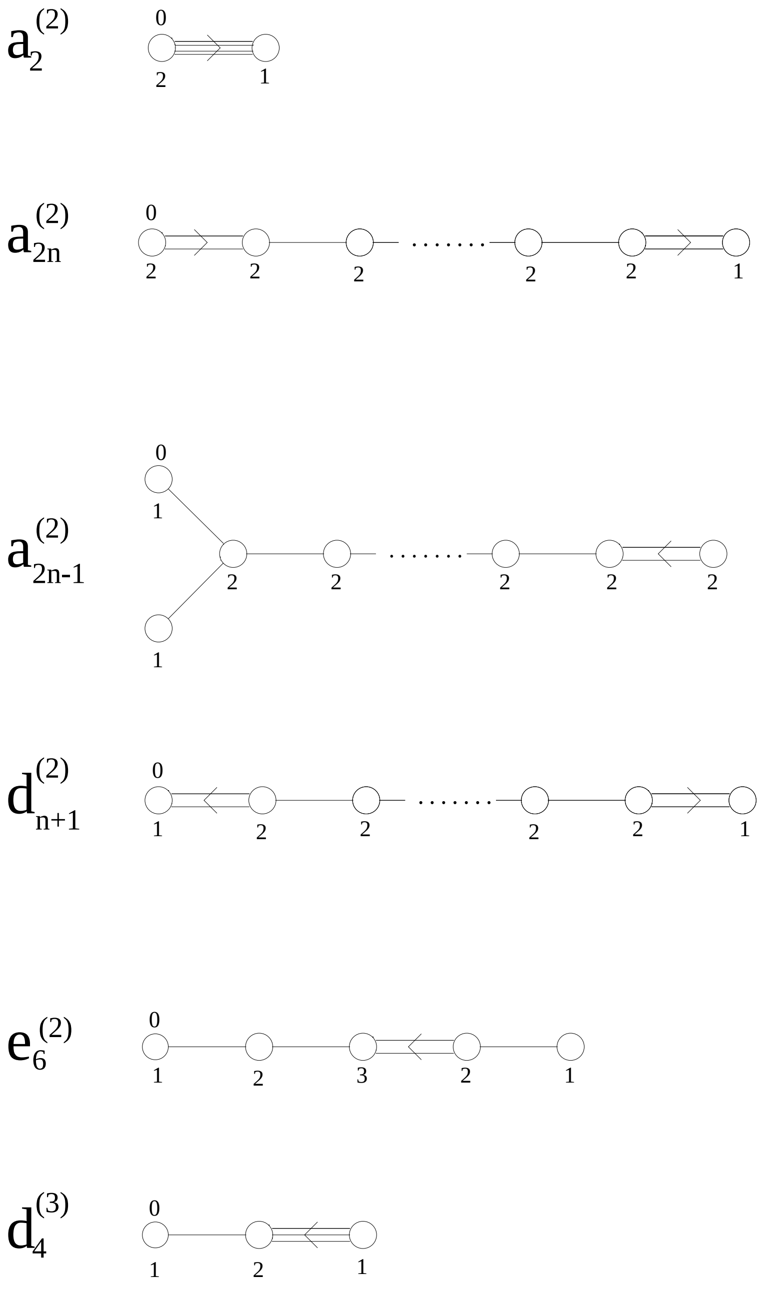

Table 3: Dynkin diagrams of twisted Kac-Moody algebras and levels of fundamental representations

Kac-Moody algebra $\mathfrak{g}^{(k)}$	Coxeter number h	Exponents mod(h)
$a_n^{(1)}$	$n+1$	$1, 2, \dots, n$
$b_n^{(1)}$	$2n$	$1, 3, \dots, 2n-1$
$c_n^{(1)}$	$2n$	$1, 3, \dots, 2n-1$
$d_n^{(1)}$	$2n-2$	$1, 3, \dots, 2n-3, n$
$e_6^{(1)}$	12	$1, 4, 5, 7, 8, 11$
$e_7^{(1)}$	18	$1, 5, 7, 9, 11, 13, 17$
$e_8^{(1)}$	30	$1, 7, 11, 13, 17, 19, 23, 29$
$f_4^{(1)}$	12	$1, 5, 7, 11$
$g_2^{(1)}$	6	$1, 5$
$a_{2n}^{(2)}$	$4n+2$	$1, 3, \dots, 2n-1, 2n+3, \dots, 4l+1$
$a_{2n-1}^{(2)}$	$4n-2$	$1, 3, \dots, 4l-3$
$d_{n+1}^{(2)}$	$2n+2$	$1, 3, \dots, 2n+1$
$d_4^{(3)}$	12	$1, 5, 7, 11$
$e_6^{(2)}$	18	$1, 5, 7, 9, 11, 13, 17$

Table 4: Exponents of Kac-Moody algebras.

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